Reflections and almost concealed canonical algebras

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ABSTRACT. Given a tilting sheaf $\mathcal{T}$ over a weighted projective line $\mathbb{X} = \mathbb{X}(\underline{\lambda}, \underline{\mathbf{p}})$ (cf. [**4**]), its endomorphism ring $\Sigma = \mathrm{End}(\mathcal{T})$ is (by definition) an almost concealed canonical algebra obtained by tilting at $\mathcal{T}$. The indecomposable direct summands T_i of $\mathcal{T}$ can be partitioned into *sources* and *sinks*, according as their corresponding source and sink maps (in add $\mathcal{T}$) are both monomorphisms or both epimorphisms (cf. Theorem 0.3). As a consequence, each almost complete tilting sheaf admits *exactly two* complements (Corollary 0.4). -

By Theorem 0.5, T_i is a source [sink, resp.] in add $\mathcal{T}$ if and only if the corresponding simple Σ-module S_i is torsion-free [torsion, resp.] with respect to the split torsion theory $(\mathrm{mod}_+ \Sigma, \mathrm{mod}_- \Sigma)$ of mod Σ obtained by tilting at $\mathcal{T}$. - The linking idea in achieving this result is the concept of *mutations*: roughly spoken, the position of simple Σ-modules within the derived category $\mathcal{D}^b\mathrm{coh}\,\mathbb{X}$ of coh $\mathbb{X}$ is obtained by the "same kind" of mutations (cf. Proposition 3.1) which also allow the construction of sink/source maps for add $\mathcal{T}$ (Proposition 2.3).

Introduction

Throughout, let k denote an algebraically closed field. -

Let coh $\mathbb{X}$ denote the category of coherent sheaves over a weighted projective line $\mathbb{X} = \mathbb{X}(\underline{\lambda}, \underline{\mathbf{p}})$ (in the sense of [**4**]) attached to a weight sequence $\underline{\mathbf{p}} = (p_1, \dots, p_t)$ of integers $p_i \geq 1$ and a parameter sequence $\underline{\lambda} = (\lambda_1, \dots, \lambda_t)$ of pairwise different points $\lambda_i \in \mathbb{P}^1(k)$. - The category coh $\mathbb{X}$ is an abelian k-category, and hereditary since $\mathrm{Ext}^2(-,-) \equiv 0$; it has a tilting object $\mathcal{T}$ whose endomorphism ring is a canonical algebra Λ in the sense of [**12**]. Hence the Grothendieck group $\mathrm{K}_0\,\mathbb{X}$ is finitely generated free; it is endowed with a bilinear form - the *Euler characteristic* - induced by

$$\langle E, F \rangle \quad = \quad \dim_k \mathrm{Hom}(E,F) - \dim_k \mathrm{Ext}^1(E,F) \qquad \text{for} \quad E, F \in \mathrm{coh}\,\mathbb{X}\ .$$

Furthermore, coh $\mathbb{X}$ is $\mathbb{L}(\underline{\mathbf{p}})$-graded, where $\mathbb{L}(\underline{\mathbf{p}})$ denotes the rank-1 abelian group generated by the t elements $\vec{x_1}, \dots, \vec{x_t}$ subject to relations

$$p_1\vec{x_1} \quad = \quad \cdots \quad = \quad p_t\vec{x_t} \qquad (=: \vec{c}\ )\ .$$

Degree-shift by the dualizing element $\vec{\omega} := (t-2)\vec{c} - \sum_{i=1}^t \vec{x_i}$ serves as the Auslander-Reiten-translation τ for coh $\mathbb{X}$; Serre-Duality $\mathrm{DExt}^1(E,F) \cong \mathrm{Hom}(F, E(\vec{\omega}))$ holds for any $E, F \in \mathrm{coh}\,\mathbb{X}$. - The representation theory of coh $\mathbb{X}$ depends on the number $\delta_\omega := (t-2) - \displaystyle\sum_{i=1}^t \frac{1}{p_i}$; namely, coh $\mathbb{X}$ is of *tame-domestic*, *tubular* or *wild* representation type depending on whether $\delta_\omega < 0, = 0$ or > 0.

DEFINITION 0.1. *A sheaf* $\mathcal{T} = T_1 \oplus \cdots \oplus T_n \in \mathrm{coh}\,\mathbb{X}$, *where* T_i *are pairwise non-isomorphic indecomposable, is called a* tilting sheaf *if*

(a): $\mathrm{Ext}^1(\mathcal{T}, \mathcal{T}) = 0$ *;*

(b): $\mathcal{T}$ *generates* coh $\mathbb{X}$ *in the sense of formation of direct sums, extensions, kernels of epimorphisms and cokernels of monomorphisms.*

Note in particular that we demand a tilting sheaf to be *multiplicity-free*. Then Condition (b) is equivalent to $n = \mathrm{rk}_{\mathbb{Z}}(\mathrm{K}_0\,\mathbb{X})$, where $\mathrm{K}_0\,\mathbb{X}$ denotes the Grothendieck-group of $\mathrm{coh}\,\mathbb{X}$ which is always free of finite rank.

Let us now define the notion of a *sink-* or *source-map* for a tilting sheaf $\mathcal{T}$ (cf. [**11**]):

DEFINITION 0.2. *Let $\mathcal{T}$ be a tilting sheaf, and F be any of its indecomposable direct summands.*

(a): *Let $N_1, \dots, N_m$ be pairwise non-isomorphic direct summands of $\mathcal{T}$, and none isomorphic to F. A morphism*

$$\rho_F : \quad F \quad \longrightarrow \quad \bigoplus_{j=1}^{m} N_j^{s_j}$$

is called a source-map for F with respect to $\mathcal{T}$ *if:*

(i): *for each $X \in \mathrm{add}(\mathcal{T} \setminus \{F\})$, each morphism from F to X factorizes over ρ_F; and*

(ii): *each endomorphism $\alpha \in \mathrm{End}(\bigoplus_{j=1}^{m} N_j^{s_j})$, for which $\alpha \circ f$ also satisfies the factorization property (i), is neccessarily an automorphism.*

The N_j are called immediate successors *of F in $\mathcal{T}$ with* multiplicities s_j.

(b): *Let $E_1, \dots, E_l$ be pairwise non-isomorphic direct summands of $\mathcal{T}$, and none isomorphic to F.*

A morphism

$$\sigma_F : \quad \bigoplus_{i=1}^{l} E_i^{r_i} \quad \longrightarrow \quad F$$

is called a sink-map for F with respect to $\mathcal{T}$ *if:*

(i): *for each $Y \in \mathrm{add}(\mathcal{T} \setminus \{F\})$, each morphism from Y to F factorizes over σ_F; and*

(ii): *each endomorphism $\beta \in \mathrm{End}(\bigoplus_{i=1}^{l} E_i^{r_i})$, for which $g \circ \beta$ also satisfies the factorization property (i), is neccessarily an automorphism.*

The E_i are called immediate predeccessors *of F in $\mathcal{T}$ with* multiplicities r_i.

Therefore property (ii) admits only modifications of sink- or source-maps by automorphisms, and demands - in rough words - a sink- or source-map to be *minimal* satisfying the appropriate factorization property. - Sink- and source-maps are thereby *uniquely* determined up to isomorphism. -

THEOREM/DEFINITION 0.3. *Let $\mathcal{T}$ be a tilting sheaf. For each arbitrary direct summand F of $\mathcal{T}$ its sink-map σ_F and its source-map ρ_F are either* both simultaneously monomorphisms *or* both simultaneously epimorphisms. *- Moreover:*

(a) *In case σ_F (and hence also ρ_F) is an epimorphism;*

$$0 \longrightarrow L \xrightarrow{\mu = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_l \end{bmatrix}} \bigoplus_{i=1}^{l} E_i^{r_i} \xrightarrow{\sigma_F = [\sigma_1, \dots, \sigma_l]} F \longrightarrow 0\ ,$$

F is called a sink *in $\mathcal{T}$;* sink-reflection at F *(i.e. exchanging F against L) yields a new tilting sheaf $\mathcal{T}'$, where the E_i will become the immediate successors of L within $\mathcal{T}'$ (with the same "multiplicities"), and μ is a source-map for L with respect to $\mathcal{T}'$ in the sense of Definition 0.2.*

(b) *In case ρ_F (and hence also σ_F) is a monomorphism;*

$$0 \longrightarrow F \xrightarrow{\rho_F = \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_m \end{bmatrix}} \bigoplus_{j=1}^{m} N_j^{s_j} \xrightarrow{\epsilon = [\epsilon_1, \dots, \epsilon_m]} R \longrightarrow 0\ ,$$

F is called a source *in $\mathcal{T}$;* source-reflection at F *(i.e. exchanging F against R) yields a new tilting sheaf $\mathcal{T}''$, where the N_j will become the immediate predeccessors of R within $\mathcal{T}''$ (with the same "multiplicities"), and ϵ is a sink-map for R with respect to $\mathcal{T}''$ in the sense of Definition 0.2.*

A sheaf $\mathcal{T} = T_1 \oplus \cdots \oplus T_{n-1}$ satisfying $\mathrm{Ext}(\mathcal{T},\mathcal{T}) = 0$ and having $n-1$ non-isomorphic indecomposable summands (where $n = \mathrm{rk}_{\mathbb{Z}}(\mathrm{K}_0\,\mathbb{X})$) is called an *almost complete* tilting sheaf (lacking only one indecomposable summand to complete to a tilting sheaf). - As a consequence of Theorem 0.3 we obtain:

COROLLARY 0.4. *Each* almost complete *tilting sheaf admits* exactly two *non-isomorphic complements.*

Let $\Sigma := \mathrm{End}(\mathcal{T})$ for an arbitrary tilting sheaf $\mathcal{T} = T_1 \oplus \cdots \oplus T_n$ (with indecomposable T_i). - An algebra arising in such way as the endomorphism ring of a tilting sheaf on $\mathrm{coh}\,\mathbb{X}$ is called an *almost concealed canonical algebra.* (Cf. [**8**]).

The exceptional direct summands $T_1, \ldots, T_n$ of $\mathcal{T}$ constitute a complete set of representatives of *indecomposable projective* Σ-modules. For $i = 1, \ldots, n$, let S_i be the *simple* Σ-module corresponding to T_i.

By [**4**], Σ has global dimension at most two; moreover, the categories $\mathrm{coh}\,\mathbb{X}$ and $\mathrm{mod}\,\Sigma$ are *derived equivalent*, in particular we have the following equivalences

$$\mathrm{Hom}_{\mathbb{X}}(\mathcal{T},-) : \mathrm{coh}_+\,\mathcal{T} \longrightarrow \mathrm{mod}_+\,\Sigma; \qquad F \mapsto \mathrm{Hom}_{\mathbb{X}}(\mathcal{T},F)$$

$$\mathrm{Ext}^1_{\mathbb{X}}(\mathcal{T},-) : \mathrm{coh}_-\,\mathcal{T}[1] \longrightarrow \mathrm{mod}_-\,\Sigma; \qquad F[1] \mapsto \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{T},F)$$

Here,

$$\mathrm{coh}_+\,\mathcal{T} := \{\, F \in \mathrm{coh}\,\mathbb{X} \mid \mathrm{Ext}^1_{\mathbb{X}}(\mathcal{T},F) = 0 \,\};$$

$$\mathrm{coh}_-\,\mathcal{T} := \{\, F \in \mathrm{coh}\,\mathbb{X} \mid \mathrm{Hom}_{\mathbb{X}}(\mathcal{T},F) = 0 \,\};$$

the Σ-modules from $\mathrm{mod}_+\,\Sigma$ [$\mathrm{mod}_-\,\Sigma$ resp.] are called *torsion-free* [*torsion*] modules; the pair $(\mathrm{mod}_+\,\Sigma, \mathrm{mod}_-\,\Sigma)$ yields a *split torsion theory* for $\mathrm{mod}\,\Sigma$:

each indecomposable Σ-module, *in particular each simple Σ-module S_i*, lies in exactly one of the two subcategroies.

THEOREM 0.5. *An indecomposable direct summand T_i of a tilting sheaf $\mathcal{T}$ is a* source *[*sink*] in $\mathcal{T}$* if and only if *the simple* $\Sigma = \mathrm{End}(\mathcal{T})$*-module S_i corresponding to (the indecomposable Σ-projective) T_i lies in* $\mathrm{mod}_+\,\Sigma$ *(i.e. S_i is a* torsion-free *Σ-module) [*$\mathrm{mod}_-\,\Sigma$ *(i.e. S_i is a* torsion *Σ-module)].*

The proof of this theorem will be obtained invoking the concept of mutations on the *exceptional sequence* $(T_1, \ldots, T_n)$ underlying $\mathcal{T}$:

in Section 1, we will define a partition for the members from this exceptional sequence by attaching a *label* "+" or "-" to each member (cf. Definition 1.10);

in Section 2, the *sink- and source-map* for any indecomposable direct summand T_i of $\mathcal{T}$ will be constructed by mutations applied to T_i; as a consequence, the sink-source-partition of the T_i is shown to coincide with the label-partition mentioned above (cf. Corollary 2.6);

in Section 3, the *position* of the *simple Σ-module S_i* (corresponding to T_i) in $\mathcal{D}^\flat\mathrm{coh}\,\mathbb{X}$ is obtained by mutating T_i; with this knowledge, the partition of the S_i given by the split torsion theory $(\mathrm{mod}_+\,\Sigma, \mathrm{mod}_-\,\Sigma)$ also coincides with the label-partition for the corresponding T_i (cf. Proposition 3.2).

1. The concept of mutations

As a consequence of heredity of the category $\mathrm{coh}\,\mathbb{X}$, any exceptional object (originally defined in the derived category $\mathcal{D}^\flat\mathrm{coh}\,\mathbb{X}$, cf. Definition 1.1) is always isomorphic to a stalk complex $E[n]$ for an exceptional sheaf $E \in \mathrm{coh}\,\mathbb{X}$. Following the definition of mutations for exceptional pairs (and sequences) by triangles, each left [or right] mutation corresponds exactly to one of three short exact sequences (1.5) of "type" (E), (M), (X) within $\mathrm{coh}\,\mathbb{X}$. It will turn out that during successive left [right] mutation of any object from a complete exceptional sequence *at most one* left [right] mutation of type (E) [(M)] can occur (cf. Lemma 1.9). - Modifying mutations by composing with a translation yields "free" mutations within $\mathcal{D}^b \operatorname{coh} \mathbb{X}$ (cf. Definition 1.11) which mutate an exceptional sequence from $\operatorname{coh} \mathbb{X}[0]$ within two neighboured "copies" of $\operatorname{coh} \mathbb{X}$ in $\mathcal{D}^b \operatorname{coh} \mathbb{X}$.

Througout this section we will use $\mathcal{D}\mathbb{X}$ as a short-hand notation for the derived category $\mathcal{D}^b \operatorname{coh} \mathbb{X}$ of $\operatorname{coh} \mathbb{X}$.

DEFINITION 1.1. *A* sheaf $E \in \operatorname{coh} \mathbb{X}$ *is called* exceptional *if* $\operatorname{End}_{\operatorname{coh} \mathbb{X}}(E) \cong k$ *and* $\operatorname{Ext}^1_{\operatorname{coh} \mathbb{X}}(E, E) = 0$.
An object $\mathcal{E} \in \mathcal{D}\mathbb{X}$ *is called* exceptional *if* $\operatorname{End}_{\mathcal{D}\mathbb{X}}(\mathcal{E}) \cong k$ *and* $\operatorname{Hom}_{\mathcal{D}\mathbb{X}}(\mathcal{E}, \mathcal{E}[i]) = 0$ *for* $i \neq 0$.

REMARK 1.2. An object $\mathcal{E} \in \mathcal{D}\mathbb{X}$ is exceptional if and only if $\mathcal{E} \cong E[n]$ for some integer ($n \in \mathbb{Z}$) and an exceptional sheaf $E \in \operatorname{coh} \mathbb{X}$. -
As in [**9**] it follows (by heredity of $\operatorname{coh} \mathbb{X}$ and the assumption that k is algebraically closed) that for an *indecomposable* sheaf E the condition $\operatorname{Ext}^1_{\operatorname{coh} \mathbb{X}}(E, E) = 0$ automatically implies $\operatorname{End}_{\operatorname{coh} \mathbb{X}}(F) \cong k$.

The latter remark is a consequence of the following fundamental Lemma:

LEMMA 1.3. *For two* indecomposable *sheaves* E *and* F *in* $\operatorname{coh} \mathbb{X}$ *satisfying* $\operatorname{Ext}^1(F, E) = 0$, *each non-zero morphism* $u : E \longrightarrow F$ *is neccessarily a monomorphism or an epimorphism.*

PROOF. Cf. [**5**]. - Moreover we use that for an arbitrary $G \in \operatorname{coh} \mathbb{X}$ the condition $[G] = 0$ in $\mathrm{K}_0\, \mathbb{X}$ (the Grothendieck group of $\operatorname{coh} \mathbb{X}$) already implies $G = 0$. ☐

DEFINITION 1.4. *A sequence* $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ *of* exceptional *objects* $\mathcal{E}_i \in \mathcal{D}\mathbb{X}$ *is called an* exceptional sequence *if* $\operatorname{Hom}_{\mathcal{D}\mathbb{X}}(\mathcal{E}_i, \mathcal{E}_j[l]) = 0$ *for all* $l \in \mathbb{Z}$, *whenever* $i > j$. -
An exceptional sequence $(\mathcal{E}, \mathcal{F})$ *is called an* exceptional pair.
An exceptional sequence is complete *if the number of its members amounts to the* $\mathbb{Z}$*-rank of the Grothendieck-group* $\mathrm{K}_0\, \mathbb{X}$ *of* $\mathcal{D}\mathbb{X}$.

REMARK 1.5. **(a):** If $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ is an exceptional sequence with objects from $\mathcal{D}\mathbb{X}$, then so is $(\mathcal{E}_1[l_1], \ldots, \mathcal{E}_n[l_n])$ for arbitrary translations $T^{l_1}, \ldots, T^{l_n}$. - By Remark 1.2 any exceptional sequence $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is of the form $(E_1[l_1], \ldots, E_n[l_n])$ for *suitable* $l_1, \ldots, l_n \in \mathbb{Z}$ and exceptional sheaves $E_1, \ldots, E_n \in \operatorname{coh} \mathbb{X}$. Considering $(E_1, \ldots, E_n)$ in $\operatorname{coh} \mathbb{X}$ rather than $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ in $\mathcal{D}\mathbb{X}$ will be called "folding" an exceptional sequence "into" $\operatorname{coh} \mathbb{X}$.
(b): It is an easy consequence of Lemma 1.3 that for each exceptional pair (E_i, E_j) $(i \leq j)$ from an exceptional sequence $(E_1, \ldots, E_n)$ in $\operatorname{coh} \mathbb{X}$ at least one of $\operatorname{Hom}(E_i, E_j)$ or $\operatorname{Ext}^1(E_i, E_j)$ must be zero. - Similarly, for an arbitrary exceptional sequence $(\mathcal{E}_1[l_1], \ldots, \mathcal{E}_n[l_n])$ in $\mathcal{D}\mathbb{X}$ we deduce that $\operatorname{Hom}_{\mathcal{D}\mathbb{X}}(\mathcal{E}_i, \mathcal{E}_j[l]) \neq 0$ for at most one $l \in \mathbb{Z}$ (whenever $i \leq j$). - We will refer to this property by saying that any exceptional sequence in $\operatorname{coh} \mathbb{X}$ (or in $\mathcal{D}\mathbb{X}$) is *strongly exceptional.* (Note that the use of terminology differs from [**15**], [**14**].)

Just as in [**15**], [**14**], [**1**] and [**10**], for an *exceptional pair* $(\mathcal{E}, \mathcal{F})$ in $\mathcal{D}\mathbb{X}$, we define the corresponding left- and right mutation:
For this let

$$\operatorname{Hom}^\bullet(\mathcal{E}, \mathcal{F}) \quad := \quad \bigoplus_{l \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{D}\mathbb{X}}(\mathcal{E}, \mathcal{F}[l])[-l]$$

be the complex of k-vector spaces with zero differential. *Left mutation of* $\mathcal{F}$ *over* $\mathcal{E}$ will be defined by the following triangle:

(1.1)
$$\begin{tikzcd} \mathcal{L}_{\mathcal{E}}\mathcal{F} \arrow[rr] & & \operatorname{Hom}^\bullet(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \arrow[dl, "\text{can}"] \\ & \mathcal{F} \arrow[ul] & \end{tikzcd}$$

Dually, *right mutation of* $\mathcal{E}$ *over* $\mathcal{F}$ will be defined by the triangle

(1.2)
$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\mathrm{can}"] & & \mathrm{Hom}^\bullet(\mathcal{E},\mathcal{F})^* \otimes \mathcal{F} \arrow[dl] \\ & \mathcal{R}_{\mathcal{F}}\mathcal{E} \arrow[ul] & \end{tikzcd} .$$

(The complex $\mathrm{Hom}^\bullet(\mathcal{E},\mathcal{F})^*$ is obtained from $\mathrm{Hom}^\bullet(\mathcal{E},\mathcal{F})$ by dualizing; note that this changes the signs of degrees!)

"can" denotes the so-called "universal morphism"; its defining property being that application of the functor $\mathrm{Hom}_{\mathcal{D}\mathbb{X}}(\mathcal{E},-)$ to the triangle (1.1) [or application of $\mathrm{Hom}_{\mathcal{D}\mathbb{X}}(-,\mathcal{F})$ to the triangle (1.2)] induces *isomorphisms* along "can" within the resulting long-exact sequence.

We have $\mathcal{L}_{\mathcal{E}}(\mathcal{F}[l]) = (\mathcal{L}_{\mathcal{E}}\mathcal{F})[l]$ for $l \in \mathbb{Z}$ (i.e. translation of $\mathcal{F}$ commutes with left mutation of $\mathcal{F}$ over $\mathcal{E}$); whereas translation of $\mathcal{E}$ does not change the left mutant at all; $\mathcal{L}_{\mathcal{E}[l]}\mathcal{F} = \mathcal{L}_{\mathcal{E}}\mathcal{F}$; since translation of $\mathcal{E}$ does not change the complex $\mathrm{Hom}^\bullet(\mathcal{E},\mathcal{F})$.

Dually we have: translation of $\mathcal{E}$ commutes with right mutation of $\mathcal{E}$ over $\mathcal{F}$; whereas translation of $\mathcal{F}$ does not change the right mutant of $\mathcal{E}$ over $\mathcal{F}$ at all.

The left mutant $\mathcal{L}_{\mathcal{E}}\mathcal{F}$ [the right mutant $\mathcal{R}_{\mathcal{F}}\mathcal{E}$, resp.] is in turn exceptional and $(\mathcal{L}_{\mathcal{E}}\mathcal{F},\mathcal{E})$ [$(\mathcal{F},\mathcal{R}_{\mathcal{F}}\mathcal{E})$, resp.] is an exceptional pair again. The triangle (1.1) at the same time describes right mutation of $\mathcal{L}_{\mathcal{E}}\mathcal{F}$ over $\mathcal{E}$ (obtaining $\mathcal{F}$ back); the triangle (1.2) describes left mutation of $\mathcal{R}_{\mathcal{F}}\mathcal{E}$ over $\mathcal{F}$, which yields $\mathcal{E}$ again.

For an exceptional sequence $(\mathcal{E}_1,\dots,\mathcal{E}_n)$, left mutation $\mathcal{L}_i$ and right mutation $\mathcal{R}_i$ for $i \in \{1,\dots,n\}$ are defined by

$$\begin{aligned} \mathcal{L}_i\ (\mathcal{E}_1,\dots,\mathcal{E}_n) &:= (\mathcal{E}_1,\dots,\mathcal{E}_{i-1},\mathcal{L}_{\mathcal{E}_i}\mathcal{E}_{i+1},\mathcal{E}_i,\mathcal{E}_{i+2}\dots,\mathcal{E}_n) \qquad \text{and} \\ \mathcal{R}_i\ (\mathcal{E}_1,\dots,\mathcal{E}_n) &:= (\mathcal{E}_1,\dots,\mathcal{E}_{i-1},\mathcal{E}_{i+1},\mathcal{R}_{\mathcal{E}_{i+1}}\mathcal{E}_i,\mathcal{E}_{i+2},\dots,\mathcal{E}_n)\ . \end{aligned}$$

(So $\mathcal{L}_i$ mutates the $(i+1)$-st object $\mathcal{E}_{i+1}$ left-ways over the i-th object $\mathcal{E}_i$; while $\mathcal{R}_i$ mutates the i-th object $\mathcal{E}_i$ right-ways over the $(i+1)$-st object $\mathcal{E}_{i+1}$.) Setting

$$T_i^m\ (\mathcal{E}_1,\dots,\mathcal{E}_n) \quad := \quad (\mathcal{E}_1,\dots,\mathcal{E}_{i-1},\mathcal{E}_i[m],\mathcal{E}_{i+1},\dots,\mathcal{E}_n)$$

- for $i = 1,\dots,n$ and $m \in \mathbb{Z}$ - the remark following the defining triangles (1.1) and (1.2) will become ($i,j = 1,\dots,n$ and $m \in \mathbb{Z}$):

(1.3)
$$\begin{aligned} &\mathcal{L}_i \circ T_j^m = T_j^m \circ \mathcal{L}_i \quad (j \neq i, i+1); \qquad \mathcal{L}_i \circ T_i^m = T_{i+1}^m \circ \mathcal{L}_i; \quad \mathcal{L}_i \circ T_{i+1}^m = T_i^m \circ \mathcal{L}_i \\ &\mathcal{R}_i \circ T_j^m = T_j^m \circ \mathcal{R}_i \quad (j \neq i, i+1); \qquad \mathcal{R}_i \circ T_i^m = T_{i+1}^m \circ \mathcal{R}_i; \quad \mathcal{R}_i \circ T_{i+1}^m = T_i^m \circ \mathcal{R}_i\ . \end{aligned}$$

Moreover mutations $\mathcal{L}_i$ and $\mathcal{R}_i$ satisfy (cf. for instance [**1**]):

(1.4)
$$\begin{gathered} \mathcal{L}_i \circ \mathcal{R}_i = \mathrm{id} = \mathcal{R}_{\mathrm{i}} \circ \mathcal{L}_{\mathrm{i}} \qquad \text{for i} = 1,\dots,\text{n}\ , \\ \mathcal{L}_i \circ \mathcal{L}_j = \mathcal{L}_j \circ \mathcal{L}_i \qquad \text{and} \qquad \mathcal{R}_i \circ \mathcal{R}_j = \mathcal{R}_j \circ \mathcal{R}_i \qquad \text{for } |i-j| \geq 2\ , \\ \mathcal{L}_i \circ \mathcal{L}_{i+1} \circ \mathcal{L}_i = \mathcal{L}_{i+1} \circ \mathcal{L}_i \circ \mathcal{L}_{i+1} \qquad \text{and} \qquad \mathcal{R}_i \circ \mathcal{R}_{i+1} \circ \mathcal{R}_i = \mathcal{R}_{i+1} \circ \mathcal{R}_i \circ \mathcal{R}_{i+1}\ . \end{gathered}$$

These *braid relations* imply an operation of the braid group B_n on the set of all exceptional sequences with n members. - If $n := \mathrm{rk}_{\mathbb{Z}}\,\mathrm{K}_0\,\mathbb{X}$, this braid group operation is *transitive* on the set of all *complete* exceptional sequences in $\mathcal{D}\mathbb{X}$ (up to translation). (Cf. [**10**], but also [**3**], [**13**].)

REMARK 1.6. One consequence of these braid relations is their following "virtual interpretation":

Given an exceptional sequence $(\mathcal{E}_1,\dots,\mathcal{E}_n)$ in $\mathcal{D}\mathbb{X}$, successively mutating $\mathcal{E}_{i+1}$ (for some fixed i) left-ways over the partial sequence $(\mathcal{E}_k,\dots,\mathcal{E}_i)$ (for $1 \leq k \leq i$),

$$\mathcal{L}_k \circ \dots \circ \mathcal{L}_i\ (\mathcal{E}_1,\dots,\mathcal{E}_n) \quad = \quad (\mathcal{E}_1,\dots,\mathcal{E}_{k-1},\bar{\mathcal{L}},\mathcal{E}_k,\dots,\mathcal{E}_i,\mathcal{E}_{i+2}\dots,\mathcal{E}_n)\ ,$$

the resulting mutant $\bar{\mathcal{L}}$ will be independent of mutations *within the partial sequence* $(\mathcal{E}_k,\dots,\mathcal{E}_i)$ in the following sense: performing arbitrary mutations and translations *within the partial sequence* $\mathcal{E}_k,\dots,\mathcal{E}_i$ - prior to successively mutating $\mathcal{E}_{i+1}$ left-ways over the individual objects of the partial sequence (obtained from $\mathcal{E}_k, \dots, \mathcal{E}_i$ in that way) to the k-th position - will yield the same mutant $\bar{\mathcal{L}}$ (up to isomorphism) as above. So $\mathcal{E}_{i+1}$ is really left-mutated "through the subcategory of $\mathcal{D}\mathbb{X}$ generated by $\mathcal{E}_k, \dots, \mathcal{E}_i$ in the triangulated sense", and the latter remains invariant under mutation and translation within $(\mathcal{E}_k, \dots, \mathcal{E}_i)$.

A dual statement holds for successive right mutation of an object $\mathcal{E}_i$ right-ways over a partial sequence $(\mathcal{E}_{i+1}, \dots, \mathcal{E}_l)$; $\quad (l \geq i+1)$.

Moreover:

Proposition 1.7. *[cf. [**1**], [**10**]] Given a* complete *exceptional sequence $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ in $\mathcal{D}\mathbb{X}$,*

$$\mathcal{L}_1 \circ \cdots \circ \mathcal{L}_{n-1}\ (\mathcal{E}_1, \dots, \mathcal{E}_n) \quad = \quad (\mathcal{E}_n(\vec{\omega})[2-n], \mathcal{E}_1, \dots, \mathcal{E}_{n-1})\ ,$$
$$\mathcal{R}_{n-1} \circ \cdots \circ \mathcal{R}_1\ (\mathcal{E}_1, \dots, \mathcal{E}_n) \quad = \quad (\mathcal{E}_2, \dots, \mathcal{E}_n, \mathcal{E}_1(\vec{\omega})[n-2])\ .$$

(Here $\vec{\omega}$ denotes the dualizing element of the grading group $\mathbb{L}(\underline{\mathbf{p}})$; grade-shift by $\vec{\omega}$ amounts to the Auslander-Reiten-translation τ in $\operatorname{coh}\mathbb{X}$*.)*

Next we consider *mutations within* $\operatorname{coh}\mathbb{X}$:

Let $(\mathcal{E}, \mathcal{F})$ be an exceptional pair in $\mathcal{D}\mathbb{X}$. - By Remark 1.5 the complexes $\operatorname{Hom}^\bullet(\mathcal{E}, \mathcal{F})$ and $\operatorname{Hom}^\bullet(\mathcal{E}, \mathcal{F})^*$ each carry at most *one* non-zero entry. Denoting by $E, F, L_E F$ the sheaves in $\operatorname{coh}\mathbb{X}$ obtained from $\mathcal{E}, \mathcal{F}, \mathcal{L}_\mathcal{E}\mathcal{F}$ by "folding into $\operatorname{coh}\mathbb{X}$" (i.e. those sheaves in $\operatorname{coh}\mathbb{X}$ which coincide to the upper objects from $\mathcal{D}\mathbb{X}$ up to translation), the triangle (1.1) defining left mutation will correspond to *exactly one of* the following three short-exact sequences in $\operatorname{coh}\mathbb{X}$:

(1.5)

(E) $$0 \longrightarrow L_E F \rightarrowtail \operatorname{Hom}(E,F) \otimes E \overset{\text{can}}{\twoheadrightarrow} F \longrightarrow 0$$

(M) $$0 \longrightarrow \operatorname{Hom}(E,F) \otimes E \overset{\text{can}}{\rightarrowtail} F \twoheadrightarrow L_E F \longrightarrow 0$$

(X) $$0 \longrightarrow F \rightarrowtail L_E F \twoheadrightarrow \operatorname{Ext}^1(E,F) \otimes E \longrightarrow 0\ ;$$

or else $\operatorname{Hom}^\bullet(\mathcal{E}, \mathcal{F}) = 0$, whence $\mathcal{L}_\mathcal{E}\mathcal{F}[1] \cong \mathcal{F}$, so $F \cong L_E F$. - In the latter case left mutation amounts to a mere exchange (up to translation).

The reason for the appearence of the short-exact sequences in (1.5) lies in the fact that *all* three complexes from the triangle (1.1) are *stalk complexes*; for $\mathcal{E}$ and $\mathcal{L}_\mathcal{E}\mathcal{F}$ this follows by heredity of $\operatorname{coh}\mathbb{X}$, while for the complex $\operatorname{Hom}^\bullet(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E}$ it is a consequence of Remark 1.5. - Passing from the triangle (1.1) [defining left mutation of the exceptional pair $(\mathcal{E}, \mathcal{F})$ in $\mathcal{D}\mathbb{X}$] to the *long-exact cohomology sequence* immediately yields the corresponding short-exact sequence from (1.5): in case $\operatorname{Ext}^1(E,F) \neq 0$, the corresponding short-exact sequence is a *universal extension*; while in case $\operatorname{Hom}(E,F) \neq 0$ the *universal morphism* "can" is an epimorphism or a monomorphism, depending on whether $\mathcal{L}_\mathcal{E}\mathcal{F} = L_E F[i]$ or $\mathcal{L}_\mathcal{E}\mathcal{F} = L_E F[i-1]$ (where $\mathcal{F} = F[i]$).

The analogue statements hold for right mutation of an exceptional pair $(\mathcal{E}, \mathcal{F})$.

Definition 1.8. *A left mutation (1.1) [right mutation (1.2), resp.] is said to be* of type (E), (M), or (X)*, depending on which of the three types listed in (1.5) above coincides with the short-exact sequence in* $\operatorname{coh}\mathbb{X}$ *obtained by "folding" the universal morphism. - In case* $\operatorname{Hom}^\bullet(\mathcal{E}, \mathcal{F}) = 0$*, the mutation is said to be* of type (0).

We observe: the *inverse* to left mutation of a pair $(\mathcal{E}, \mathcal{F})$ of type (E) [(M), (X), (0)] is a right mutation of type (M) [(X), (E), (0)].

Given an exceptional sequence $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ in $\mathcal{D}\mathbb{X}$, and letting $(E_1, \dots, E_n)$ be the corresponding exceptional sequence in $\operatorname{coh}\mathbb{X}$ obtained by "folding", we can perform mutations of $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ within the derived category $\mathcal{D}\mathbb{X}$ on the one hand; or, on the other hand, perform the "corresponding" mutations of $(E_1, \dots, E_n)$ within $\operatorname{coh}\mathbb{X}$ (each followed by eventual "folding"). - The resulting sequences will merely differ from each other by translation of the individual members.

LEMMA 1.9. *Let* $(E_1, \dots, E_n)$ *be an exceptional sequence in* $\mathrm{coh}\,\mathbb{X}$. *For each of its members* E_i *the following holds:*

(a): *Successive left mutations of* E_i *(left-ways over* $E_1, \dots, E_{i-1}$ *to the very beginning) involve* at most one *left mutation of type (E). - Denoting the resulting mutant by* L, *this will be the case* if and only if $\mathrm{Hom}(E_i, L) = 0$ *and* $\mathrm{Ext}^1(E_i, L) \cong k$; otherwise $\mathrm{Hom}(E_i, L) \cong k$ *and* $\mathrm{Ext}^1(E_i, L) = 0$.

(b): *Successive right mutations of* E_i *(right-ways over* $E_{i+1}, \dots, E_n$ *to the very end) involve* at most one *right mutation of type (M). - Denoting the resulting mutant by* R, *this will be the case* if and only if $\mathrm{Hom}(R, E_i) = 0$ *and* $\mathrm{Ext}^1(R, E_i) \cong k$; otherwise $\mathrm{Hom}(R, E_i) \cong k$ *and* $\mathrm{Ext}^1(R, E_i) = 0$.

(c): *In case (a) involves a left mutation of type (E), then* no *right mutation of type (M) will occur in (b). - And vice versa: in case (b) involves a right mutation of type (M), then* no *left mutation of type (E) will occur in (a).*

(d): *Given a* complete *exceptional sequence, then* either *(a) involves one left mutation of type (E)* or *(b) involves one right mutation of type (M).*

PROOF. We will consider all mutations performed within $\mathrm{coh}\,\mathbb{X}$ (i.e. including "folding", if neccessary).

(a): Let us start with the following considerations:

Assume $E, F, G \in \mathrm{coh}\,\mathbb{X}$ exceptional, such that (E, G) and (E, F) each are exceptional pairs. In case $\mathrm{Hom}(G, F) \cong k$; $\mathrm{Ext}^1(G, F) = 0$, and left mutation of (E, F) (i.e. left mutation of F over E) is of type (M) or (X) (or (0)), application of $(G, -)$ to the corresponding short-exact sequence (cf. (1.5)) yields $\mathrm{Hom}(G, L) \cong \mathrm{Hom}(G, F) \cong k$; $\mathrm{Ext}^1(G, L) = 0$. Should however left mutation of (E, F) be of type (E), then application of $(G, -)$ to (1.5) yields $\mathrm{Hom}(G, L) = 0$; $\mathrm{Ext}^1(G, L) \cong k$.

Now assume $\mathrm{Hom}(G, F) = 0$; $\mathrm{Ext}^1(G, F) \cong k$. If left mutation of (E, F) were of type (E), then application of $(G, -)$ would now yield $\mathrm{Ext}^1(G, F) = 0$, contradictory to our assumption; therefore under the assumption above left mutation of (E, F) *cannot* be of type(E). For type (M) or (X) (or (0)) application of $(G, -)$ now yields $\mathrm{Hom}(G, L) = 0$; $\mathrm{Ext}^1(G, L) \cong \mathrm{Ext}^1(G, F) \cong k$.

With the help of the preceeding consideration we will construct our proof by induction over the number of successively performed left mutations in the sense of (a). -

Put $G := E_i$; let F denote the mutant resulting from successive left mutation of G left-ways over $E_m, \dots, E_{i-1}$ (starting the induction with $F := G = E_i$); and let $E := E_{i-1}$. By the inductive assumption the left mutations performed so far involve at most one of type (E), and moreover this is the case if and only if $\mathrm{Hom}(G, F) = 0$; $\mathrm{Ext}(G, F) \cong k$; otherwise $\mathrm{Hom}(G, F) \cong k$; $\mathrm{Ext}(G, F) = 0$. By the preceeding considerations, the next left muation (of F over E) can be of type (E) only in the latter case; the induction claim now also follows from the preceeding considerations.

(b): dual to (a). - Please observe the "direction of looking" to understand in which sense we dualize here: dealing with successive left mutations of an object E_i over its predeccessors (in the given exceptional sequence), we are looking *from the "old"* E_i *"forward" into the "new"* developing left mutant L; whereas, in dealing with successive right mutations of E_i over its successors (in the given sequence) we are looking *from the "new"* developed right mutant R *"back" to the "old"* E_i.

(c): We first mutate E_i (as in (b)) right-ways to the very end of the exceptional sequence obtaining

$$E_1, \dots, E_{i-1}, E_{i+1}, \dots, E_n, R\ , \tag{1.6}$$

and now apply (a) to *this* sequence by mutating R left-ways to the very beginning. The first of these left mutations will invert the right mutations performed before (and so, "in between", E_i will show up as the developing left mutant of R). Now right mutation of type (M) is inverse to left mutation of type (E); therefore, if those right mutations building up R involved one of type (M), then already the left mutations leading from R (back) to E_i will involve one of type (E) (namely the inverse of the right mutation of type (M) mentionend above); whence by (a), during *further* left mutation of E_i to the very beginning, there cannot be another one of type (E).

(d): Proceeding as in the proof of (c), we consider the sequence (1.6), and successively mutate R left-ways to the very beginning, obtaining an object L,

$$L, E_1, \dots, E_{i-1}, E_{i+1}, \dots, E_n \ ,$$

where (as in (c)) "in between" E_i will show up as an intermediate (developing) mutant. Since the sequence is complete by assumption, we have $L \cong \tau R$ (cf. Proposition 1.7). Should the right mutations performed (starting with E_i) *not* involve any of type (M), *nor* the left mutations performed (starting also with E_i) involve any of type (E), then by (a) for the sequence above this implies $\mathrm{Hom}(R, L) \cong k$ and $\mathrm{Ext}^1(R, L) = 0$; contradicting

$$\mathrm{DExt}^1(R, L) \quad \cong \quad \mathrm{Hom}(L, \tau R) \quad \cong \quad \mathrm{Hom}(L, L) \quad \cong \quad k$$

(where the first isomorphism is obtained from Serre-duality). □

Definition 1.10. *Given an exceptional sequence $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ in $\mathcal{D}\mathbb{X}$, we assign to each member $\mathcal{E}_i$ a* label *"-" [or "+"] depending on whether successive mutation of $\mathcal{E}_i$* left-ways *to the beginning* does *[or* does not*] involve a left mutation of type (E).*

It is an easy observation (using Lemma 1.9 and Remark 1.6) that *transpositions* (i.e. exchanges of neighboured pairs $(\mathcal{E}_i, \mathcal{E}_{i+1})$ satisfying $\mathrm{Hom}^{\bullet}_{\mathcal{D}\mathbb{X}}(\mathcal{E}_i, \mathcal{E}_{i+1}) = 0$) do not bear any influence on the label of each member.

Therefore, given any tilting sheaf $\mathcal{T} = T_1 \oplus \dots \oplus T_n \in \mathrm{coh}\,\mathbb{X}$, "admissably" ordered (i.e. such that $T_1, \dots, T_n$ constitute an exceptional sequence), the label of each T_i does not depend on the actual numbering chosen. (Any two admissable numberings can be obtained from each other by a series of transpositions.) - In this sense, we will loosely speak of "the" exceptional sequence underlying a tilting sheaf.

So far we have seen that left mutation of type (E), or right mutation of type (M), respectively, play a special role. We will now modify left- and right mutation (1.1) and (1.2) defined in the derived category; for this let $(\mathcal{E}, \mathcal{F})$ be an exceptional pair in $\mathcal{D}\mathbb{X}$. We modify the corresponding left mutation of $\mathcal{F}$ over $\mathcal{E}$ [right mutation of $\mathcal{E}$ over $\mathcal{F}$, resp.] by

$$\tilde{\mathcal{L}}_{\mathcal{E}}\ (\mathcal{E}, \mathcal{F}) \quad := \quad (\ \mathcal{L}_{\mathcal{E}}\mathcal{F}[+1], \mathcal{E}\ ) \qquad \text{and} \qquad \tilde{\mathcal{R}}_{\mathcal{F}}\ (\mathcal{E}, \mathcal{F}) \quad := \quad (\ \mathcal{F}, \mathcal{R}_{\mathcal{F}}\mathcal{E}[-1]\ )\ .$$

In the same sense we modify mutations $\mathcal{L}_i$ and $\mathcal{R}_i$ defined for an exceptional sequence $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ in $\mathcal{D}\mathbb{X}$, namely

$$\tilde{\mathcal{L}}_i \ := \ T_i \ \circ \ \mathcal{L}_i \qquad \text{bzw.} \qquad \tilde{\mathcal{R}}_i \ := \ T_{i+1}^{-1} \ \circ \ \mathcal{R}_i\ .$$

Taking in account formulae (1.3) it is easily seen that all relations from (1.3) remain valid, when replacing $\mathcal{L}_i$ by $\tilde{\mathcal{L}}_i$ and $\mathcal{R}_i$ by $\tilde{\mathcal{R}}_i$; moreover, this modification clearly does not bear any influence on the corresponding mutations when performed within $\mathrm{coh}\,\mathbb{X}$ (using "folding").

Definition 1.11. *In the sequel we will speak of mutations in $\mathcal{D}\mathbb{X}$ modified in the sense above as* free mutations in $\mathcal{D}\mathbb{X}$ *(in contrast to mutations performed within* $\mathrm{coh}\,\mathbb{X}$ *by "folding").*

- For free mutations we easily observe:

Remark 1.12. Given an exceptional pair $(\mathcal{E}, \mathcal{F})$ with $\mathcal{F}$ concentrated in position i (i.e. being isomorphic to $F[-i]$ for some $F \in \mathrm{coh}\,\mathbb{X}$), then, if left mutation (1.1) is of type (E), $\tilde{\mathcal{L}}_{\mathcal{E}}\mathcal{F}$ is concentrated in position $i-1$; if it is of type (M) or (X) or (0), $\tilde{\mathcal{L}}_{\mathcal{E}}\mathcal{F}$ is concentrated in position i.

Dually: Again given an exceptional pair $(\mathcal{E}, \mathcal{F})$ with now $\mathcal{E}$ concentrated in position i (i.e. being isomorphic to $E[-i]$ for some $E \in \mathrm{coh}\,\mathbb{X}$), then, if right mutation (1.2) is of type (M), $\tilde{\mathcal{R}}_{\mathcal{F}}\mathcal{E}$ is concentrated in position $i+1$; if it is of type (E) or (X) or (0), $\tilde{\mathcal{R}}_{\mathcal{F}}\mathcal{E}$ is concentrated in position i.

We actually defined free mutations in order to obtain the following Lemma, which will bear significance in Section 3 (where we investigate the tilting process from sheaves to modules, and determine simple modules by successive left mutations):

LEMMA 1.13. *Let* $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ *be an exceptional sequence in* $\operatorname{coh}\mathbb{X}$ *(i.e. all members concentrated in position 0), and fix any member* $\mathcal{E}_i$.

(a): *Successively* freely *mutating* $\mathcal{E}_i$ *left-ways to the beginning, the mutant* $\mathcal{L}$ *finally resulting will lie either in* $\operatorname{coh}\mathbb{X}$ *or in* $\operatorname{coh}\mathbb{X}[1]$, *the latter case occuring* if and only if *the left mutations performed did involve one of type (E).*

(b): *Successively* freely *mutating* $\mathcal{E}_i$ *right-ways to the end, the mutant* $\mathcal{R}$ *finally resulting will lie either in* $\operatorname{coh}\mathbb{X}$ *or in* $\operatorname{coh}\mathbb{X}[-1]$, *the latter case occuring* if and only if *the right mutations performed did involve one of type (M).*

PROOF. Immediate from Remark 1.12 and Lemma 1.9. □

2. Reflections for a tilting sheaf $\mathcal{T}$

The aim of this section is to construct sink- and source-maps for indecomposable direct summands of a tilting sheaf $\mathcal{T}$ (cf. Definition 0.2) by *glueing mutations* applied to the exceptional sequence underlying $\mathcal{T}$. - This technique yields a proof of Theorem 0.3: a partition of the indecomposable direct summands of $\mathcal{T}$ into *sources* and *sinks*. - As a contribution to the proof of Theorem 0.5, we will observe that the sink-source-partition agrees with the label-partition given by Definition 1.10 for the members of the underlying exceptional sequence.

Let $\mathcal{T} = T_1 \oplus \cdots \oplus T_n \in \operatorname{coh}\mathbb{X}$ be a (multiplicity-free) tilting sheaf.
We repeat the following easy consequence of Lemma 1.3:

LEMMA 2.1. *Subject to suitable re-ordering, the indecomposable direct summands* $T_1, \ldots, T_n$ *of* $\mathcal{T}$ *constitute an exceptional sequence in* $\operatorname{coh}\mathbb{X}$.

For the Krull-Schmidt-category $\operatorname{add}\mathcal{T}$ - where $\mathcal{T} = T_1 \oplus \cdots \oplus T_n$ continues to be a tilting sheaf with pairwise non-isomorphic exceptional summands $T_1, \ldots, T_n$, representing all indecomposable objects in $\operatorname{add}\mathcal{T}$ - we define in the usual way (cf. [**12**]) the notion of an *irreducible map with respect to* $\operatorname{add}\mathcal{T}$:

Let E, F be two *indecomposable* (and thus *exceptional*) objects in $\operatorname{add}\mathcal{T}$, and let

$$\operatorname{rad}_{\mathcal{T}}(E,F) \quad := \quad \begin{cases} \operatorname{Hom}(E,F) & \text{if } E \ncong F \\ 0 & \text{if } E \cong F \end{cases} .$$

For $E \ncong F$ let $\operatorname{rad}^2_{\mathcal{T}}(E,F) \subseteq \operatorname{rad}_{\mathcal{T}}(E,F)$ denote the k-vectorspace generated by all morphisms $g \circ f \in \operatorname{Hom}(E,F)$ - where $f \in \operatorname{rad}_{\mathcal{T}}(E,Z)$ and $g \in \operatorname{rad}_{\mathcal{T}}(Z,F)$ for some $Z \in \operatorname{add}\mathcal{T}$. - Finally we denote by

$$\operatorname{irr}_{\mathcal{T}}(E,F) \quad := \quad \operatorname{rad}_{\mathcal{T}}(E,F) \;/\; \operatorname{rad}^2_{\mathcal{T}}(E,F)$$

the k-vectorspace of *irreducible morphisms* from E to F *with respect to* $\operatorname{add}\mathcal{T}$ [or shorthand: *with respect to* $\mathcal{T}$].

The *existance* of sink- and source-maps for each indecomposable direct summand F of $\mathcal{T}$ can be shown in all generality (cf. [**12**]): roughly spoken, a source-map [sink-map, resp.] for F can be composed from a k-bases of $\operatorname{irr}_{\mathcal{T}}(F,N)$ [$\operatorname{irr}_{\mathcal{T}}(E,F)$, resp.] (for all indecomposable direct summands N of $\mathcal{T}$ [E of $\mathcal{T}$, resp.]).

In the sequel we will explicitely construct the sink- and source-maps for an *arbitrary* exceptional summand of a tilting sheaf by *glueing successive mutations*. - The following criterion allows to recognize sink- and source-maps more easily:

LEMMA 2.2. *Using the notation of Definition 0.2 we have:*

(a): ρ_F *is a source-map for* F *with respect to* $\mathcal{T}$ *if application of* $\operatorname{Hom}(-,Z)$ *along* ρ_F *induces an epimorphism*

$$(2.1) \qquad - \circ \rho_F : \qquad \operatorname{Hom}(\textstyle\bigoplus_{j=1}^m N_j^{s_j}, Z) \longrightarrow\!\!\!\!\!\rightarrow \operatorname{Hom}(F,Z)$$

for each exceptional summand $Z \ncong F$ *of* $\mathcal{T}$, *and if moreover this epimorphism becomes an* isomorphism *for the summands* $Z \cong N_j$ *occuring on the right-hand-side of* ρ_F.

(b): *σ_F is a sink-map for F with respect to $\mathcal{T}$ if application of* $\mathrm{Hom}(A,-)$ *along σ_F induces an epimorphism*

$$\sigma_F \circ - : \qquad \mathrm{Hom}(A, \bigoplus_{i=1}^{l} E_i^{r_i}) \longrightarrow\!\!\!\!\rightarrow \mathrm{Hom}(A,F) \tag{2.2}$$

for each exceptional summand $A \ncong F$ of $\mathcal{T}$, and if moreover this epimorphism becomes an isomorphism *for the summands $A \cong E_i$ occuring on the left-hand-side of σ_F.*

Proof. **(a):** The factorization property is a consequence of the isomorphism obtained in (2.1). - Now let $\alpha \in \mathrm{End}(\bigoplus_{j=1}^{m} N_j^{s_j})$ be an epimorphism such that $\alpha \circ \rho_F$ satisfies the appropriate factorization property, too. We write ρ_F as a colummn matrix having $s_1 + \cdots + s_m$ entries, and apply the factorization property of $\alpha \circ \rho_F$ to each entry of that column corresponding to ρ_F. Since by assumption application of $\mathrm{Hom}(-, N_j)$ induces *isomorphisms* in (2.1) (and thus the factorization of a morphism from $\mathrm{Hom}(F, N_j)$ over ρ_F has to be unique), we obtain - for the factorization of each entry f_ν over ρ_F - a morphism $\tilde{\rho_F}_\nu \in \mathrm{Hom}(\bigoplus_{j=1}^{m} N_j^{s_j}, N_j)$ such that $\tilde{\rho_F}_\nu \circ \alpha \in \mathrm{Hom}(\bigoplus_{j=1}^{m} N_j^{s_j}, N_j)$ will be given by the corresponding unit vector. Collecting the $\tilde{\rho_F}_\nu$ yields a right-inverse to the endomorphism α. - Since $\mathrm{End}(\bigoplus_{j=1}^{m} N_j^{s_j})$ is finite-dimensional as a k-vectorspace, α needs to be an automorphism.

(b): completely dual. □

Proposition 2.3. *Let $\mathcal{T}$ be a tilting sheaf; F an (arbitrary) exceptional summand of $\mathcal{T}$.*

(a) *Successive left mutations over the immediate predeccessors $E_1, \ldots, E_l$ of F can be glued to a sink-map*

$$\sigma = \sigma_F = [\sigma_1, \ldots, \sigma_l] : \bigoplus_{i=1}^{l} E_i^{r_i} \longrightarrow F$$

such that $\sigma_i = [\sigma_i^{(1)}, \ldots, \sigma_i^{(r_i)}]$ consists of a basis of the k-vectorspace $\mathrm{irr}_{\mathcal{T}}(E_i, F)$.

σ is either a monomorphism or an epimorphism. - In case σ is a monomorphism, $\mathrm{Hom}(A, L) = 0 = \mathrm{Ext}^1(A, L)$ *for all predeccessors A of its cokernel L.*

(b) *Successive right mutations over the immediate successors $N_1, \ldots, N_m$ of F can be glued to a source-map*

$$\rho = \rho_F = \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_m \end{bmatrix} : F \longrightarrow \bigoplus_{j=1}^{m} N_i^{s_j}$$

such that $\rho_j = \begin{bmatrix} \rho_j^{(1)} \\ \vdots \\ \rho_j^{(s_j)} \end{bmatrix}$ *consists of a basis of the k-vectorspace* $\mathrm{irr}_{\mathcal{T}}(F, N_j)$.

ρ is either a monomorphism or an epimorphism. - In case ρ is an epimorphism, $\mathrm{Hom}(R, Z) = 0 = \mathrm{Ext}^1(R, Z)$ *for all successors Z of its kernel R.*

Proof. **(a):** After the h-th step, let

$$[\sigma_1, \ldots, \sigma_h] : \bigoplus_{i=1}^{h} E_i^{r_i} \longrightarrow F \tag{2.3}$$

be the morphism built up so far, such that $(E_i, -)$ induces an isomorphism along (2.3) $(i = 1, \ldots, h)$. By assumption, (2.3) is mono or epi; its cokernel [or kernel] L resulting from successively mutating F over its predeccessors $E_1, \ldots, E_h$.

In case (2.3) is mono, i.e.

$$0 \longrightarrow \bigoplus_{i=1}^h E_i^{r_i} \stackrel{\sigma}{\rightarrowtail} F \stackrel{\epsilon}{\twoheadrightarrow} L \longrightarrow 0 \ ,$$

we have $\mathrm{Ext}^1(A,F)=0$ for each predeccessor A of L; let E denote the next predeccessor of L satisfying $\mathrm{Hom}(E,F) \neq 0$. Let $r := \dim_k \mathrm{Hom}(E,L)$ and $\bar{\sigma^{(1)}},\dots,\bar{\sigma^{(r)}}$ be a k-basis of (E,L); $\sigma^{(1)},\dots,\sigma^{(r)}$ counter-images of these basis elements under

$$(E,F) \xrightarrow{\epsilon\circ -} (E,L) \longrightarrow 0 \ \cdot$$

Setting $\sigma := [\sigma^{(1)},\dots,\sigma^{(r)}]$ and $\bar{\sigma} := [\bar{\sigma^{(1)}},\dots,\bar{\sigma^{(r)}}]$ we obtain the following diagram as the "pull-back" along $\bar{\sigma}$ and ϵ; due to $\mathrm{Ext}^1(E,\bigoplus_{i=1}^h E_i^{r_i})=0$ the upper horizontal sequence splits:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=1}^h E_i^{r_i} \arrow[r, tail] \arrow[d, equal] & \bigoplus_{i=1}^h E_i^{r_i} \oplus E^r \arrow[r, two heads] \arrow[d, "{[\sigma_1,\dots,\sigma_h,\sigma]}"] & E^r \arrow[r] \arrow[d, "\bar{\sigma}"] & 0 \\
0 \arrow[r] & \bigoplus_{i=1}^h E_i^{r_i} \arrow[r, tail, "{[\sigma_1,\dots,\sigma_h]}"] & F \arrow[r, two heads, "\epsilon"] & L \arrow[r] & 0
\end{tikzcd}$$

Now $\bar{\sigma}$ (and hence also $[\sigma_1,\dots,\sigma_h,\sigma]$) is mono [or epi], having isomorphic cokernels [or kernels] $\bar{L}$; the right vertical short-exact sequence describes left mutation of the pair (E,L); by construction $(E,-)$, applied to

$$\bigoplus_{i=1}^h E_i^{r_i} \oplus E^r \xrightarrow{[\sigma_1,\dots,\sigma_h,\sigma]} F \ ,$$

induces an isomorphism, and due to $\mathrm{Hom}(E_i,E)=0$ for $i=1,\dots,h$ this remains to be the case for $(E_i,-)$.

In case (2.3) is epi, i.e.

$$\eta: \qquad 0 \longrightarrow L \stackrel{\mu}{\rightarrowtail} \bigoplus_{i=1}^h E_i^{r_i} \xrightarrow{[\sigma_1,\dots,\sigma_h]} F \longrightarrow 0 \ ,$$

for each predeccessor A of L either $\mathrm{Hom}(A,L)=0$ or $\mathrm{Ext}^1(A,L)=0$. Let E denote the next predeccessor of L satisfying $\mathrm{Ext}^1(E,L) \neq 0$ (if there still exists such one). After suitable re-ordering we may assume that there exist *no* morphisms from objects A, lying between E and L, into L.

(Examples show that such a re-ordering might really be neccessary; for instance, cf. [**7**], Example 4.22.)

[If such a re-ordering of predeccessors were impossible, there would have to exist a path

$$E \xrightarrow{u_0} A_1 \xrightarrow{u_1} A_2 \xrightarrow{u_2} \cdots \xrightarrow{u_{s-1}} A_s \xrightarrow{u_s} L \ ,$$

where each single arrow represents a non-zero non-isomorphism u_k, with the objects A_k being members of our exceptional sequence lying between E and L. In fact all these morphisms need to be monomorphisms:

u_s is mono or epi, in the latter case however application of $\mathrm{Ext}^1(E,-)$ yields $\mathrm{Ext}^1(E,L) = 0$, whence u_s has to be mono. If u_{s-1} were an epi, $u_s \circ u_{s-1}$ - as a non-zero morphism - would in turn have to be mono or epi, so either u_{s-1} or u_s would be an isomorphism; whence also u_{s-1} is mono; and so on.

We obtain a monomorphism $E \hookrightarrow L$, contradicting $\mathrm{Hom}(E,L)=0$.]

Let $r := \dim_k \mathrm{Ext}^1(E,L)$ and $\bar{\varphi^{(1)}},\dots,\bar{\varphi^{(r)}}$ be a k-basis of $\mathrm{Ext}^1(E,L)$; let $\varphi^{(1)},\dots,\varphi^{(r)}$ be counter-images of these basis elements under

$$(E,F) \xrightarrow{\eta\circ -} {}^1(E,L) \longrightarrow 0 \ \cdot$$

Setting $\varphi := [\varphi^{(1)}, \dots, \varphi^{(r)}]$ and $\bar{\varphi} := [\varphi^{\bar{(1)}}, \dots, \varphi^{\bar{(r)}}]$ we obtain the following diagram as the "pull-back" along $\bar{\varphi}$ and $[\sigma_1, \dots, \sigma_h]$:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, tail] \arrow[d, equal] & \bar{L} \arrow[rr, two heads] \arrow[d] & & E^r \arrow[r] \arrow[d, "\varphi"] & 0 \\
0 \arrow[r] & L \arrow[r, tail, "\mu"] & \bigoplus_{i=1}^h E_i^{r_i} \arrow[rr, two heads, "{[\sigma_1, \dots, \sigma_h]}"] & & F \arrow[r] & 0
\end{tikzcd}$$

The upper row of this diagram describes left mutation of the pair (E, L); the mutant $\bar{L}$ is obtained as the kernel of the diagonal sequence

$$\begin{tikzcd}
0 \arrow[r] & \bar{L} \arrow[r, tail] & \bigoplus_{i=1}^h E_i^{r_i} \oplus E^r \arrow[rr, two heads, "{[\sigma_1, \dots, \sigma_h, \varphi]}"] & & F \arrow[r] & 0 \ ;
\end{tikzcd}$$

by construction, application of $(E, -)$ to

$$\begin{tikzcd}
\bigoplus_{i=1}^h E_i^{r_i} \oplus E^r \arrow[rr, two heads, "{[\sigma_1, \dots, \sigma_h, \varphi]}"] & & F \ ,
\end{tikzcd}$$

induces an isomorphism, and due to $\mathrm{Hom}(E_i, E) = 0$ for $i = 1, \dots, h$ this remains to be true for $(E_i, -)$.

The procedure will be finished, if *either* (2.3) is a monomorphism and there is no further predeccessor A of F satisfying $\mathrm{Hom}(A, L) \neq 0$, *or* if (2.3) is an epimorphism and there is no further predeccessor A of F satisfying $\mathrm{Ext}^1(A, L) \neq 0$. For both cases each morphism $A \longrightarrow F$ factorizes over (2.3); for the epi-case (and only for this) in general there will exist further morphisms from predeccessors of L into it (corresponding to *relations terminating in* F, cf. [**7**]); however there will not be any non-trivial extensions.

(b): Completely dual to (a) we now invoke successive *right mutations* of F over its immediate successors in $\mathcal{T}$. In case the morphism ρ_F in (b) is an epimorphism and R is its kernel, there will neither exist any non-trivial extensions nor morphisms from R into its successors; in case ρ_F is a monomorphism (and R is its cokernel), in general there will exist further morphisms from R into its successors (corresponding to *relations starting in* F); however there will not be any non-trivial extensions. ☐

Remark 2.4. During successive left mutation of an exceptional summand F of $\mathcal{T}$ over its immediate predeccessors E_i in $\mathcal{T}$ according to part (a) of the preceeding Proposition 2.3 it may happen that $\mathrm{Hom}(A, L) = 0 = \mathrm{Ext}^1(A, L)$ for a predeccessor A of L, although "further left" of A there still exist immediate predeccessors of F; in this case left mutation of L over A simply amounts to an exchange. This situation occurs whenever *non-immediate predeccessors* lie *between* immediate predeccessors of F. - A dual obvservation holds for part (b).

We are now in the position to prove Theorem 0.3:

Proof. (of Theorem 0.3): By the construction presented in the proof of the preceeding Proposition 2.3, σ_F [ρ_F] is an epimorphism [monomorphism] if and only if a left [right] mutation of type (E) [(M)] (cf. Definition 1.8) occurs in the process of glueing the sink-map σ_F [source map ρ_F].

If σ_F were a monomorphism *and at the same time* ρ_F were an epimorphism, by Proposition 2.3 L [R] can be mutated to the very beginning [end] by simply exchanging, since neither morphisms nor extensions exist from predeccessors of L into L [from R into successors of R]. This would contradict Lemma 1.9.

Lemma 1.9 also rules out the case that σ_F is an epimorphism while at the same time ρ_F is a monomorphism, since then left mutation of F would involve a left mutation of type (E) while at the same time right mutation of F would involve a right mutation of type (M).

(a): Assume that the sink-map σ_F belonging to F (with respect to $\mathcal{T}$) is an epimorphism.

Non-trivial extensions $\mathrm{Ext}^1(A, L)$ could only exist for predeccessors A of L (which is ruled out by the construction process invoked in the proof of Proposition 2.3 (a)); or non-trivial extensions $\mathrm{Ext}^1(L, Z)$ could exist for successors Z of L (now application of $\mathrm{Ext}^1(-, Z)$ and $\mathrm{Ext}^1(\bigoplus_{i=1}^{l} E_i^{r_i}, Z) = 0$ yield $\mathrm{Ext}^1(L, Z) = 0$).

Therefore the exchange of F against L indeed results in a tilting sheaf $\mathcal{T}'$.

Due to $\mathrm{Hom}(F, E_k) = 0 = \mathrm{Ext}^1(F, E_k)$ we know that $(-, E_k)$ induces an isomorphism

$$-\circ\mu : \qquad \mathrm{Hom}(\bigoplus_{i=1}^{l} E_i^{r_i}, E_k) \quad \cong \quad \mathrm{Hom}(L, E_k) \qquad k = 1, \ldots, l\ ;$$

since $\mathrm{Ext}^1(F, Z) = 0$ we obtain an epimorphism

$$-\circ\mu : \qquad \mathrm{Hom}(\textstyle\bigoplus_{i=1}^{l} E_i^{r_i}, Z) \quad \twoheadrightarrow \quad \mathrm{Hom}(L, Z)\ ;$$

for successors Z of L; therefore μ is indeed a source-map for L with respect to $\mathcal{T}'$ (cf. Lemma 2.2).

Applying successive right mutations of L over its immediate successors in $\mathcal{T}'$ according to Proposition 2.3 (b) (hereby taking $\mathcal{T}'$ as a tilting sheaf in its own right), the right mutations occuring - in reversed order - are the inverses of those left mutations which were invoked mutating F to L, and since the source-map μ' constructed in this way for L with respect to $\mathcal{T}'$ must be isomorphic to μ, in particular it has to be a monomorphism.

(b): completely dual to (a). ☐

PROOF. (of Corollary 0.4): for an *almost complete* tilting sheaf, the existance of *one* complement is assured by the Bongartz-complement (cf. [**2**]; [**7**], Prop. 5.1); *reflection* at this complement (which - by Theorem 0.3 - must either be a sink or a source in the tilting sheaf arising from completion) immediately yields *another* complement (non-isomorphic to the first). General arguments (as in [**6**], [**7**]) then show that no almost complete tilting sheaf can admit more than two non-isomorphic complements. ☐

REMARK 2.5. Replacing a tilting sheaf by a *partial tilting sheaf*, Theorem 0.3 will no longer hold:

As for a tilting sheaf, for an exceptional summand of a *partial tilting sheaf*, its sink-map cannot be an epimorphism while at the same time its source-map is a monomorphism. - However, in contrast to the situation for a tilting sheaf, the *converse* situation *is possible*; i.e. its sink-map can be a monomorphism while at the same time its source-map is an epimorphism. In fact, the actual *choice* of completing to a tilting sheaf will finally decide about the attribute.

COROLLARY 2.6. *An indecomposable direct summand F of a tilting sheaf $\mathcal{T}$ is a sink [source]* if and only if - *in the exceptional sequence underlying $\mathcal{T}$ - F has label "-" ["+"]. (Cf. Definition 1.10.)*

PROOF. Compare the first statement of the proof of Theorem 0.3. ☐

3. The position of the simple $\mathrm{End}(\mathcal{T})$-modules in $\mathcal{D}^b\mathrm{coh}\,\mathbb{X}$

Given any tilting sheaf $\mathcal{T}$, the position of the simple $\Sigma = \mathrm{End}(\mathcal{T})$-modules S_i can be obtained by freely mutating (cf. Section 1) the indecomposable direct summands T_i of $\mathcal{T}$ to the left of the underlying exceptional sequence (Proposition 3.1). - As a consequence, the partition of simple Σ modules given by the split torsion theory $(\mathrm{mod}_+\,\Sigma, \mathrm{mod}_-\,\Sigma)$ coincides with the partition given by the *labels* of the corresponding T_i.

The indecomposable direct summands $T_1, \ldots, T_n$ of a tilting sheaf $\mathcal{T} = T_1 \oplus \cdots \oplus T_n$ constitute a complete system of indecomposable projective Σ-modules, where $\Sigma := \mathrm{End}(\mathcal{T})$ denotes the *almost concealed canonical algebra* arising as endomorphism ring of $\mathcal{T}$ (cf. [**4**]).

The following Proposition describes how the *position of simple Σ-modules* - with respect to the embedding of $\mathrm{mod}\,\Sigma$ into $\mathcal{D}^b\mathrm{coh}\,\mathbb{X}$ given by tilting at $\mathcal{T}$ - can be obtained from the exceptional sequence

$$T_1, T_2, T_3, \quad \ldots \quad, T_{n-1}, T_n \tag{3.1}$$

underlying $\mathcal{T}$ through application of *free left mutations* (cf. Definition 1.11):

Proposition 3.1. *Using the notation of above, the* simple Σ*-module* S_i *corresponding to the indecomposable projective* Σ*-module* T_i *can be obtained from the exceptional sequence (3.1) by successively applying* free *(cf. Definition 1.11) mutations to* T_i *left-ways over* $T_1, \dots, T_{i-1}$ *to the very beginning of (3.1).*

Therefore, the simple Σ*-modules* $S_n, \dots, S_1$ *(in "reversed" order) constitute an exceptional sequence; in particular they are themselves exceptional objects. - By Remark 1.5 (b), this sequence is neccessarily* strongly exceptional*; as a consequence* $\mathrm{Ext}^1_\Sigma(S_k, S_l) = 0$ *or* $\mathrm{Ext}^2_\Sigma(S_k, S_l) = 0$ *for each pair* (S_k, S_l) *of simple* Σ*-modules.*

Proof. We choose an arbitrary $i \in \{1, \dots, n\}$ and apply successive free (in the sense of Definition 1.11) left mutations to T_i, mutating it left-ways to the beginning of (3.1), and obtaining the exceptional sequence

$$S_i, T_1, \dots T_{i-1}, T_{i+1}, \dots, T_n \ . \tag{3.2}$$

Then $\mathrm{Hom}(T_k, S_i) = 0 = \mathrm{Ext}^1(T_k, S_i)$ for $k \neq i$; moreover either $S_i \in \mathrm{coh}\,\mathbb{X}$ and $\mathrm{Hom}_{\mathbb{X}}(T_i, S_i)$ [$\cong \mathrm{Hom}_\Sigma(T_i, S_i)$] $\cong k$, $\mathrm{Ext}_{\mathbb{X}}(T_i, S_i) = 0$ (in which case $S_i \in \mathrm{mod}_+ \Sigma$ is a torsion-free Σ-module); or $S_i \in \mathrm{coh}\,\mathbb{X}[1]$, so $S_i[-1] \in \mathrm{coh}\,\mathbb{X}$, and then $\mathrm{Hom}_{\mathbb{X}}(T_i, S_i[-1]) = 0$, $\mathrm{Ext}_{\mathbb{X}}(T_i, S_i[-1])$ [$\cong \mathrm{Hom}_\Sigma(T_i, S_i)$] $\cong k$ (so now $S_i \in \mathrm{mod}_- \Sigma$ is a torsion Σ-module).

Mutating in this way first T_n to obtain S_n, then T_{n-1} to obtain S_{n-1}, and so on; after consecutive free left mutation of T_i, we obtain an exceptional sequence

$$S_n, S_{n-1}, \dots, S_i, T_1, \dots T_{i-1} \ ,$$

and finally - concluding with $T_1 = S_1$ - the exceptional sequence

$$S_n, S_{n-1}, \dots, S_2, S_1$$

of simple Σ-modules, as was claimed in the Proposition. □

We are now in the position to complete the proof of Theorem 0.5 with the next Proposition:

Proposition 3.2. *For an indecomposable direct summand* F *of a tilting sheaf* $\mathcal{T}$*, the simple* $\Sigma = \mathrm{End}(\mathcal{T})$*-module* S_F *(corresponding to the indecomposable* Σ*-projective* F*) lies in* $\mathrm{mod}_- \Sigma$ *[*$\mathrm{mod}_+ \Sigma$*]* if and only if F *has label "-" ["+"] in the exceptional sequence underlying* $\mathcal{T}$*.*

Proof. Immediate from the preceeding Proposition 3.1 and Lemma 1.13 (a). □

Remark 3.3. It should be clearly noted at this place that the sink-source-partition of the indecomposable projective Σ-modules for an almost concealed canonical algebra $\Sigma = \mathrm{End}(\mathcal{T})$ does *not* reflect an *inherent* property of the algebra Σ itself, but rather depends on its actual *realization* through a chosen *tilting sheaf* $\mathcal{T}$:

Theorem 0.5 stresses that the attributes "sink" and "source" are determined by the *position* of simple Σ-modules given by the embedding of $\mathrm{mod}\,\Sigma$ into $\mathcal{D}^b\mathrm{coh}\,\mathbb{X}$, which in turn depends on the choice of $\mathcal{T}$.

In Section 2, we saw how reflections are "built up" by performing consecutive mutations; while in Section 3 consecutive mutations were invoked to obtain the (position of) simple Σ-modules from the projective ones. Linking these concepts, we are able to control the *change of simple modules* under *reflection*. - More precisely:

Proposition 3.4. **(a)** *Let* $\mathcal{T}$ *be a tilting sheaf in* $\mathrm{coh}\,\mathbb{X}$*; assume* F *is a sink of* $\mathcal{T}$*; let* $\mathcal{T}'$ *denote the tilting sheaf obtained from* $\mathcal{T}$ *by reflection at* F *(exchanging* F *against* L*); let* $\Sigma := \mathrm{End}(\mathcal{T})$; $\Sigma' := \mathrm{End}(\mathcal{T}')$.

For the simple Σ'*-modules we achieve:*

$S'_L \cong S_F[-1]$*; for each immediate predeccessor* E *of* F *in* $\mathcal{T}$ *[which becomes an immediate successor of* L *in* $\mathcal{T}'$*] the simple* Σ'*-module* S'_E *(corresponding to* E *in* $\mathcal{T}'$*) is obtained from the simple* Σ*-module* S_E *(corresponding to* E *in* $\mathcal{T}$*) by* free left mutation of S_E over S_F. *- All remaining simple* Σ*- and* Σ'*-modules coincide.*

Dually:

(b) *Let $\mathcal{T}$ be a tilting sheaf in* $\operatorname{coh}\mathbb{X}$*; assume G is a source of $\mathcal{T}$; let $\mathcal{T}''$ denote the tilting sheaf obtained from $\mathcal{T}$ by reflection at G (exchanging G against R); let* $\Sigma := \operatorname{End}(\mathcal{T})$; $\Sigma'' := \operatorname{End}(\mathcal{T}'')$.

For the simple Σ''-modules we achieve:

$S''_R \cong S_G[+1]$*; for each immediate successor E of G in $\mathcal{T}$ [which becomes an immediate predeccessor of R in $\mathcal{T}''$] the simple Σ''-module S''_E (corresponding to E in $\mathcal{T}''$) is obtained from the simple Σ-module S_E (corresponding to E in $\mathcal{T}$) by* free right mutation of S_E over S_G. *- All remaining simple Σ- and Σ''-modules coincide.*

PROOF. Cf. [**7**]. □

References

[1] A. I. Bondal, *Representations of associative algebras and coherent sheaves*, Math. USSR-Izv. **34** (1990), 23–42.

[2] K. Bongartz, *Tilted algebras*, Springer Lecture Notes Math. **903** (1981), 26–38.

[3] W. Crawley-Boevey, *Exceptional sequences of representations of quivers*, Representations of Algebras, Sixth International Conference, Ottawa 1992, CMS Conf. Proc. **14** (1993), 117–124.

[4] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, Singularities, Representation of Algebras and Vector Bundles (Lambrecht 1985), 1987, pp. 265–297.

[5] D. Happel and C.M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. **274** (1982), 399–443.

[6] D. Happel and L. Unger, *Almost complete tilting modules*, Proc. of the AMS **107** (1989), 603–610.

[7] T. Hübner, *Exzeptionelle Vektorbündel und Reflektionen an Kippgarben über projektiven gewichteten Kurven*, Dissertation (1996).

[8] ———, *Tilting sheaves and concealed-canonical algebras*, Representation Theory of Algebras, ICRA VII, Cocoyoc 1994, CMS Conf. Proc. **18** (1996), 455–473.

[9] H. Lenzing and J.A. de la Peña, *Wild canonical algebras*, Math. Z., to appear.

[10] H. Meltzer, *Exceptional sequences for canonical algebras*, Arch. Math. **64** (1995), 304–312.

[11] C. Riedtmann and A. Schofield, *On a simplicial complex associated with tilting modules*, Comment. Math. Helvetici **66** (1991), 70–78.

[12] C.M. Ringel, *Tame algebras and integral quadratic forms*, Springer Lecture Notes Math. **1099** (1984).

[13] ———, *The braid group action on the set of exceptional sequences of a hereditary Artin algebra*, Abelian group theory and related topics, Conference, Oberwolfach 1993, Contemp. Math. **171** (1994), 339–352.

[14] A. N. Rudakov, *Exceptional vector bundles on a quadric*, Math. USSR-Izv. **33** (1989), 115–138.

[15] ———, *The Markov numbers and exceptional bundles on* $\mathbb{P}^2$, Math. USSR-Izv. **32** (1989), 99–112.

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