Character classes in the two generator free group

V.V.Benyash-Krivetz

Institute of Mathematics,
Academy Sciences of Belarus,
Surganova str., 11
220072, Minsk
Belarus

Character classes in the two generator free group

V.V.Benyash-Krivetz

Let K be a field of characteristic 0. In this paper we consider the problem of determining when two elements in the free two generator group $F_2 =< g, h >$ have the same characters under all representations of F_2 into $\mathrm{SL}_2(K)$.

Definition 1 *The character class $\{w\}$ of an element $w \in F_2$ is the set of all elements in F_2 having the same characters as w under all representations of F_2 into* $\mathrm{SL}_2(K)$.

Clearly, any representation $\rho : F_2 \to \mathrm{SL}_2(K)$ is uniquely determined by the pair of matrices $(\rho(g), \rho(h)) \in \mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$. Therefore, we may (and we will) identify the points of $\mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$ with the corresponding representations of F_2 into $\mathrm{SL}_2(K)$. (In other words, $\mathrm{SL}_2(K) \times \mathrm{SL}_2(K)$ is the representation variety of F_2 into $\mathrm{SL}_2(K)$ (cf. [3])). For arbitrary element $w = w(g, h) \in F_2$ one can consider the function

$$\tau_w : \mathrm{SL}_2(K) \times \mathrm{SL}_2(K) \to K, \qquad (\tau_w)(A, B) = \mathrm{tr}(w(A, B))$$

It was proved by Horowitz [2] (cf. also [1]) that for any $w \in F_2$

$$\tau_w = Q_w(\tau_g, \tau_h, \tau_{gh}),$$

where $Q_w \in \mathbb{Z}[x, y, z]$ is a uniquely defined polynomial with integral coefficients. The function τ_w and the polynomial Q_w are called the Fricke character and the Fricke polynomial of the element $w \in F_2$, respectively. For arbitrary elements $u, v, w \in F_2$, the following relations for the Fricke characters follow from the corresponding relations for the traces of arbitrary matrices in $\mathrm{SL}_2(K)$:

$$1)\tau_{u^{-1}} = \tau_u; \qquad 2)\tau_{uv} = \tau_{vu}; \qquad 3)\tau_{uvu^{-1}} = \tau_v; \qquad 4)\tau_{uv} = \tau_u\tau_v - \tau_{uv^{-1}}.$$

The Fricke characters τ_g, τ_h, τ_{gh} are algebraically independent functions over K in view of the following well known proposition (cf., for example, [2]).

Proposition 1 *For any $\alpha, \beta, \gamma \in K$ there exist matrices $A, B \in \mathrm{SL}_2(K)$ such that*

$$\tau_g(A, B) = \mathrm{tr}\, A = \alpha, \qquad \tau_h(A, B) = \mathrm{tr}\, B = \beta, \qquad \tau_{gh}(A, B) = \mathrm{tr}\, AB = \gamma.$$

Now using the Fricke characters we can rewrite Definition 1 as follows.

Definition $1'$ *The character class $\{w\}$ of an element $w \in F_2$ is the set*

$$\{w\} = \{u \in F_2 \mid \tau_u = \tau_w\} = \{u \in F_2 \mid Q_u = Q_w\}.$$

Next we need to know the explicit formula for the Fricke polynomials found by Traina [5]. To formulate this result let $P_n(\lambda)$ be polynomials satisfying the recurrence relations

$$P_n(\lambda) = \lambda P_{n-1}(\lambda) - P_{n-2}(\lambda)$$

and the initial conditions

$$P_0(\lambda) = 1, \qquad P_{-1}(\lambda) = 0.$$

An easy inductive argument shows that

$$P_n(2\cos\varphi) = \sin(n+1)\varphi/\sin\varphi.$$

and the degree of $P_n(\lambda)$ in λ is n for all $n \geq 0$. Let also

$$P_n(\lambda) = -P_{|n|-2}(\lambda)$$

for $n < 0$. We need to know the zeros of the polynomials $P_n(\lambda)$, $n \geq 1$.

Lemma 1 *The polynomial $P_n(\lambda)$, $n \geq 1$, has n zeros, defined by*

$$\lambda_{n,k} = 2\cos\tfrac{k\pi}{n+1}, \qquad k = 1, 2, \ldots, n.$$

The proof of Lemma 1 follows from the above trigonometric representation of $P_n(\lambda)$.

Let $w = g^{u_1}h^{v_1}\ldots g^{u_s}h^{v_s} \in F_2$ be a cyclically reduced word and let $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. Consider the Fricke polynomial $Q_w(x, y, z)$ as a polynomial in z. Let

$$Q_w(x, y, z) = M_n(x, y)z^n + M_{n-1}(x, y)z^{n-1} + \ldots + M_0(x, y).$$

Theorem 1 (C. Traina [5]) *The degree of the Fricke polynomial $Q_w(x, y, z)$ in z equals $s :=$ the number of blocks $g^{u_i}h^{v_i}$ in w. The leading coefficient $M_s(x, y)$ of $Q_w(x, y, z)$ is of the form*

$$M_s(x, y) = \prod_{i=1}^{s} P_{u_i-1}(x)P_{v_i-1}(y).$$

The remaining coefficients can be determined recursively.

The following result was obtained partially by Horowitz [2] (cf. also [5]).

Theorem 2 *Let u, v be elements of F_2 of the form*

$$u = g^{\alpha_1}h^{\beta_1}\ldots g^{\alpha_s}h^{\beta_s},$$
$$v = g^{\gamma_1}h^{\delta_1}\ldots g^{\gamma_t}h^{\delta_t},$$

where $s, t > 0$ and α_i, β_i, γ_i, δ_i are nonzero integers for all i, j. If $v \in \{u\}$, i.e. $\tau_u = \tau_v$, then we have

1) $s = t$;

2) the numbers $|\alpha_1|, \ldots, |\alpha_s|$ are a rearrangement of the numbers $|\gamma_1|, \ldots, |\gamma_s|$ and the numbers $|\beta_1|, \ldots, |\beta_s|$ are a rearrangement of the numbers $|\delta_1|, \ldots, |\delta_s|$;

3) $|\alpha_1 + \ldots + \alpha_s| = |\gamma_1 + \ldots + \gamma_s|$;

4) $|\beta_1 + \ldots + \beta_s| = |\delta_1 + \ldots + \delta_s|$;

5) $|\alpha_1 + \ldots + \alpha_s + \beta_1 + \ldots + \beta_s| = |\gamma_1 + \ldots + \gamma_s + \delta_1 + \ldots + \delta_s|$.

Proof. The equality $s = t$ follows immediately from Theorem 1. Furthermore, if $\tau_u = \tau_v$, i.e. $Q_u(x,y,z) = Q_v(x,y,z)$, where $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$ then by Theorem 1 we have

$$\prod_{i=1}^{s} P_{\alpha_i - 1}(x) = \prod_{i=1}^{s} P_{\gamma_i - 1}(x), \tag{1}$$

$$\prod_{i=1}^{s} P_{\beta_i - 1}(y) = \prod_{i=1}^{s} P_{\delta_i - 1}(y). \tag{2}$$

Let $|\alpha_{s_1}| = \max_i\{|\alpha_i|\}$, $|\gamma_{s_2}| = \max_i\{|\gamma_i|\}$. If $|\alpha_{s_1}| > |\gamma_{s_2}|$ then by Lemma 1 the number $x_0 = 2\cos\frac{\pi}{\alpha_{s_1}}$ is not a root of the polynomial $P_{\gamma_i - 1}(x)$ for all $i = 1, \ldots, s$. On the other hand, x_0 is a root of the polynomial $P_{s_1 - 1}(x)$; therefore, (1) implies that $\prod_{i=1}^{s} P_{\gamma_i - 1}(x_0) = 0$ — a contradiction. Analogously, one can show that the inequality $|\alpha_{s_1}| < |\gamma_{s_2}|$ does not hold. Thus $|\alpha_{s_1}| = |\gamma_{s_2}|$, and hence $P_{\alpha_{s_1} - 1}(x) = \pm P_{\gamma_{s_2} - 1}(x)$. Now dividing both sides of (1) on the polynomial $P_{\alpha_{s_1} - 1}(x)$ and repeating this procedure s times we obtain that the numbers $|\alpha_1|, \ldots, |\alpha_s|$ are a rearrangement of the numbers $|\gamma_1|, \ldots, |\gamma_s|$. Analogously, the numbers $|\beta_1|, \ldots, |\beta_s|$ are a rearrangement of the numbers $|\delta_1|, \ldots, |\delta_s|$.

Consider now the representation

$$\rho : F_2 \to \mathrm{SL}_2(K), \qquad g \mapsto A = \mathrm{diag}(d, d^{-1}),\ h \mapsto E,$$

where $d \in K^*$ and E is the identity matrix. Then $\rho(u) = A^m$, $\rho(v) = A^{m_1}$ where $m = \alpha_1 + \ldots + \alpha_s$, $m_1 = \gamma_1 + \ldots + \gamma_s$. Since $\tau_u = \tau_v$, we have $\mathrm{tr}\, A^m = \mathrm{tr}\, A^{m_1}$, i.e.

$$d^m + d^{-m} = d^{m_1} + d^{-m_1}$$

for any $d \in K^*$ implying $|m| = |m_1|$. Analogously, $|\beta_1 + \ldots + \beta_s| = |\delta_1 + \ldots + \delta_s|$. To complete the proof consider the representation

$$\varphi : F_2 \to \mathrm{SL}_2(K), \qquad g \mapsto A = \mathrm{diag}(d, d^{-1}),\ h \mapsto A,$$

where $d \in K^*$. Then $\varphi(u) = A^l$, $\varphi(v) = A^{l_1}$, where $l = |\alpha_1 + \ldots + \alpha_s + \beta_1 + \ldots + \beta_s|$, $l_1 = |\gamma_1 + \ldots + \gamma_s + \delta_1 + \ldots + \delta_s|$ and arguing as above we obtain $|l| = |l_1|$. Theorem 1 is proved. $\square$

For any $w \in F_2$ the character class $\{w\}$ is clearly the disjouint union of sets of conjugacy classes, since conjugate elements have the same characters for all representations. The following corollary follows immediately from Theorem 2.

Corollary 1 *Let $w = g^{\alpha_1} h^{\beta_1} \ldots g^{\alpha_s} h^{\beta_s}$ be a cyclically reduced word in F_2. Then there exist at most $2^{2s}(s-1)!s!$ conjugacy classes in the character class $\{w\}$.*

The following example of Horowitz [2] shows that the number of conjugacy classes in a character class can be arbitrarily large.

Example 1 (Horowitz [2]). Let $\varepsilon_i = \pm 1$ for $i = 1, 2, \ldots$. Define a sequence W_m, $m = 0, 1, 2, \ldots$, of words in F_2 by the recursion

$$\begin{aligned} W_0 &= a \qquad (3) \\ W_m &= W_{m-1}^{-\varepsilon_m} b^{2m} W_{m-1}^{\varepsilon_m} b^{2m-1} W_{m-1}^{-\varepsilon_m} b^{2m} W_{m-1}^{\varepsilon_m}. \qquad (4) \end{aligned}$$

Clearly, for all m the word W_m is freely reduced and it begins and ends with $a^{\pm 1}$. For each integer m, the word W_m given by (4) depends on the choice of $\varepsilon_j = \pm 1$ for $j = 1, 2, \ldots, m$ only. So we will indicate this dependence by writing

$$W_m = W_m(\varepsilon_1, \ldots, \varepsilon_m).$$

Horowitz proved that if $W_m(\varepsilon_1, \ldots, \varepsilon_m)$ is conjugate to $W_m(\varepsilon'_1, \ldots, \varepsilon'_m)$ then $\varepsilon_i = \varepsilon'_i$ for $i = 1, 2, \ldots, m$. Furthermore, he proved that for all choices of $\varepsilon_i, \varepsilon'_i = \pm 1$, $i = 1, \ldots, m$, we have

$$W_m(\varepsilon'_1, \ldots, \varepsilon'_m) \in \{W_m(\varepsilon_1, \ldots, \varepsilon_m)\}.$$

So there are 2^m different words $W_m(\varepsilon_1, \ldots, \varepsilon_m)$, $\varepsilon_i = \pm 1$, $i = 1, \ldots, m$, and all of them are not conjugate and have the same characters. This means that for any number m there exists the character class in F_2 which contains at least 2^m conjugacy classes.

We will say that a word $u \in F_2$ is obtainable from a word $w \in F_2$ by a **standard transformation** if one of the following holds:

i) $u = wvw^{-1}$ for some $w \in F_2$;

ii) $u = v^{-1}$;

iii) if $v = g^{\alpha_1} h^{\beta_1} \ldots g^{\alpha_s} h^{\beta_s}$ is a cyclically reduced word then $u = h^{\beta_s} g^{\alpha_s} \ldots h^{\beta_1} g^{\alpha_1}$.

Definition 2 *We say that two words $u, v \in F_2$ are equivalent, if u is obtainable from v by a sequence of standard transformations.*

The following proposition can be easily verified.

Proposition 2 *Let $v = g^{\alpha_1} h^{\beta_1} \ldots g^{\alpha_s} h^{\beta_s}$ be a cyclically reduced word in F_2 and u is equivalent to v. Then u is conjugate to one of the following words: v, v^{-1}, v_1, v_1^{-1}, where $v_1 = h^{\beta_s} g^{\alpha_s} \ldots h^{\beta_1} g^{\alpha_1}$.*

Proposition 3 *Let $u, v \in F_2$ be equivalent elements. Then $\tau_u = \tau_v$, i.e. $u \in \{v\}$.*

Proof. We need only to check that if

$$\begin{aligned} u = u(g, h) &= g^{\alpha_1} h^{\beta_1} \ldots g^{\alpha_s} h^{\beta_s} \\ v = v(g, h) &= h^{\beta_s} g^{\alpha_s} \ldots h^{\beta_1} g^{\alpha_1} \end{aligned}$$

then $\tau_u = \tau_v$. Since $u^{-1} = h^{-\beta_s} g^{-\alpha_s} \ldots h^{-\beta_1} g^{-\alpha_1}$, we have $u^{-1}(g, h) = v(g^{-1}, h^{-1})$. Let $Q_v(\tau_g, \tau_h, \tau_{gh})$ be the Fricke polynomial of v. Then one holds

$$\tau_{v(g^{-1}, h^{-1})} = Q_v(\tau_{g^{-1}}, \tau_{h^{-1}}, \tau_{g^{-1}h^{-1}}) = Q_v(\tau_g, \tau_h, \tau_{gh}) = \tau_{v(g,h)},$$

whence $\tau_{u(g,h)} = \tau_{u^{-1}(g,h)} = \tau_{v(g^{-1},h^{-1})} = \tau_{v(g,h)}$, as required. $\square$

Thus, two equivalent words lie in the same character class. Let us put the question the other way around: *does the fact $u \in \{v\}$ imply that u and v are equivalent?* Example of Horowitz and Proposition 2 imply that there exist character classes containing not equivalent words. However, we have the following

Theorem 3 *Let $u = g^{\alpha_1}h^{\beta_1}\dots g^{\alpha_s}h^{\beta_s}$ be a cyclically reduced word in F_2 and $v \in \{u\}$. If $s \le 3$ then u and v are equivalent.*

For $s = 1$ this result was obtained by Horowitz [2]. More precisely Horowitz proved the following proposition.

Proposition 4 (Horowitz [2]) *Let $u \in F_2$. If $\tau_u = \tau_{a^lb^m}$, where l, m are integers and a, b is a pair of primitive elements in F_2, then u is conjugate to either a^lb^m or $b^{-m}a^{-l}$.*

Proof of Theorem 3. Let $s = 2$, i.e. $u = g^{\alpha_1}h^{\beta_1}g^{\alpha_2}h^{\beta_2}$, and v be a cyclically reduced word such that $v \in \{u\}$. By Theorem 2, the word v can be written in the form

$$v = g^{\alpha'_1}h^{\beta'_1}g^{\alpha'_2}h^{\beta'_2},$$

where $\{|\alpha_1|, |\alpha_2|\} = \{|\alpha'_1|, |\alpha'_2|\}$, $\{|\beta_1|, |\beta_2|\} = \{|\beta'_1|, |\beta'_2|\}$ and

$$\begin{aligned} |\alpha_1 + \alpha_2| &= |\alpha'_1 + \alpha'_2|, \\ |\beta_1 + \beta_2| &= |\beta'_1 + \beta'_2|, \\ |\alpha_1 + \alpha_2 + \beta_1 + \beta_2| &= |\alpha'_1 + \alpha'_2 + \beta'_1 + \beta'_2|. \end{aligned}$$

These conditions imply that up to standard transformations v has one of the following forms:

1) $\alpha'_i = \alpha_i$, $\beta'_i = \beta_i$ for $i = 1, 2$, i.e. $v = u$;

2) $\alpha'_i = -\alpha_i$, $\beta'_i = -\beta_i$ for $i = 1, 2$, i.e. $v = g^{-\alpha_1}h^{-\beta_1}g^{-\alpha_2}h^{-\beta_2}$;

3) $\alpha'_1 = \alpha_1 = -\alpha'_2 = -\alpha_2$, $\beta'_1 = \beta_1 = -\beta'_2 = -\beta_2$, i.e. $u = g^{\alpha_1}h^{\beta_1}g^{-\alpha_1}h^{-\beta_1}$, $v = g^{\alpha_1}h^{-\beta_1}g^{-\alpha_1}h^{\beta_1}$

If $v = g^{-\alpha_1}h^{-\beta_1}g^{-\alpha_2}h^{-\beta_2}$ then the following sequence of standard transformations

$$v \to v^{-1} = h^{\beta_2}g^{\alpha_2}h^{\beta_1}g^{\alpha_1} \to g^{\alpha_1}h^{\beta_1}g^{\alpha_2}h^{\beta_2} = u.$$

implies that v is equivalent to u. In the third case, obviously, u and v are equivalent as well.

Let now $s = 3$, $u = g^{\alpha_1}h^{\beta_1}g^{\alpha_2}h^{\beta_2}g^{\alpha_3}h^{\beta_3}$ and v be a cyclically reduced word such that $v \in \{u\}$. Again by Theorem 2, the word v is conjugate to

$$v' = g^{\alpha'_1}h^{\beta'_1}g^{\alpha'_2}h^{\beta'_2}g^{\alpha'_3}h^{\beta'_3},$$

where $\{|\alpha_1|, |\alpha_2|, |\alpha_3|\} = \{|\alpha'_1|, |\alpha'_2|, |\alpha'_3|\}$, $\{|\beta_1|, |\beta_2|, |\beta_3|\} = \{|\beta'_1|, |\beta'_2|, |\beta'_3|\}$ and

$$\begin{aligned} |\alpha_1 + \alpha_2 + \alpha_3| &= |\alpha'_1 + \alpha'_2 + \alpha'_3|, \\ |\beta_1 + \beta_2 + \beta_3| &= |\beta'_1 + \beta'_2 + \beta'_3|, \\ |\alpha_1 + \alpha_2 + \alpha_3 + \beta_1 + \beta_2 + \beta_3| &= |\alpha'_1 + \alpha'_2 + \alpha'_3 + \beta'_1 + \beta'_2 + \beta'_3|. \end{aligned}$$

The direct computation shows that the word v satisfying the above conditions, up to standard transformations has one of the following forms:

1) $v = u$;

2) $v = v_1 = g^{\alpha_1} h^{\beta_1} g^{\alpha_2} h^{\beta_3} g^{\alpha_3} h^{\beta_2}$;

3) $v = v_2 = g^{\alpha_1} h^{-\beta_1} g^{\alpha_2} h^{-\beta_2} g^{\alpha_3} h^{-\beta_3}$;

4) $v = v_3 = g^{\alpha_1} h^{-\beta_1} g^{\alpha_2} h^{-\beta_3} g^{\alpha_3} h^{-\beta_2}$.

Let $v = v_1$. If $\alpha_1 = \alpha_2$ then v_1 is equivalent to u. Let $\alpha_1 \neq \alpha_2$ and $\tau_u = \tau_{v_1}$. Then we have

$$\tau_u = \tau_{g^{\alpha_1} h^{\beta_1}} \tau_{u_1} - \tau_{u_2} = \tau_v = \tau_{g^{\alpha_1} h^{\beta_1}} \tau_{w_1} - \tau_{w_2},$$

where

$$u_1 = g^{\alpha_2} h^{\beta_2} g^{\alpha_3} h^{\beta_3}, \qquad u_2 = g^{\alpha_2 - \alpha_1} h^{\beta_2} g^{\alpha_3} h^{\beta_3 - \beta_1},$$
$$w_1 = g^{\alpha_2} h^{\beta_3} g^{\alpha_3} h^{\beta_2}, \qquad w_2 = g^{\alpha_2 - \alpha_1} h^{\beta_3} g^{\alpha_3} h^{\beta_2 - \beta_1}.$$

Clearly, w_1 and u_1 are equivalent so we have $\tau_{u_1} = \tau_{w_1}$ and hence

$$\tau_{u_2} = \tau_{w_2}.$$

As we have already proved u_2 and w_2 are equivalent, whence $\beta_2 \in \{\beta_3, \beta_2 - \beta_1\}$. Since $\beta_2 \neq \beta_2 - \beta_1$ then $\beta_2 = \beta_3$. Therefore $v_1 = u$.

The cases $v = v_2$ and $v = v_3$ are considered by a similar way. Let, for example, $v = v_2$. Let $Q_u(x, y, z)$ and $Q_{v_2}(x, y, z)$ be the Fricke polynomials of the words u and v_2 respectively, where $x = \tau_g$, $y = \tau_h$, $z = \tau_{gh}$. By Theorem 1, we have

$$Q_u(x, y, z) = A_0 z^3 + A_1 z^2 + A_2 z + A_3,$$
$$Q_{v_2}(x, y, z) = B_0 z^3 + B_1 z^2 + B_2 z + B_3,$$

where

$$A_0 = \prod_{i=1}^{3} P_{\alpha_i - 1}(x) P_{\beta_i - 1}(y)$$
$$B_0 = \prod_{i=1}^{3} P_{\alpha_i - 1}(x) P_{-\beta_i - 1}(y) = -\prod_{i=1}^{3} P_{\alpha_i - 1}(x) P_{\beta_i - 1}(y) = -A_0.$$

It follows that $Q_u(x, y, z) \neq Q_{v_2}(x, y, z)$, hence $v_2 \notin \{u\}$. Analogously, one can prove that $v_3 \notin \{u\}$. Theorem 3 is proved. $\square$

Theorem 3 is in a sense an optimal result for F_2. If $s = 4$ we will construct an example of a cyclically reduced word for which the assertion of Theorem 3 is no longer true. To begin with, consider a set

$$M = \{(u, v) \in F_2 \times F_2 \mid \tau_u = \tau_v\}.$$

Let

$$M_1 = \{(u, v) \in M \mid u \text{ and } v \text{ are equivalent}\}.$$

Example 2. For any $w = w(g, h) \in F_2$ consider the function $\varphi_w : M \to M$ defined by

$$(u, v) \mapsto (w(u, v), w(v, u)).$$

First we check that $\varphi_w(u,v) \in M$. Indeed, let $Q_w(\tau_g, \tau_h, \tau_{gh})$ be the Fricke polynomial of the word $w(g,h)$. Then

$$\tau_{w(u,v)} = Q_w(\tau_u, \tau_v, \tau_{uv})$$

and

$$\tau_{w(v,u)} = Q_w(\tau_v, \tau_u, \tau_{vu}).$$

Since $(u,v) \in M$, we have $\tau_u = \tau_v$. Clearly, $\tau_{uv} = \tau_{vu}$, therefore $\tau_{w(u,v)} = \tau_{w(v,u)}$ and $\varphi_w(u,v) \in M$.

Furthermore, we claim that there exists a word $w \in F_2$ such that $\varphi_w(M_1) \neq M_1$. To show it let

$$w = gh^2, \quad u = g, \quad v = (h^l g^s h^t)g(h^l g^s h^t)^{-1},$$

where l, s, t are integer numbers not equal 0 and $l \neq t$. Since u and v are conjugate we have $\tau_u = \tau_v$. Futhermore,

$$\varphi_w(u,v) = (gh^l g^s h^t g^2 h^{-t} g^{-s} h^{-l}, h^l g^s h^t g h^{-t} g^{-s} h^{-l} g^2) = (u_1, v_1).$$

The words u_1 and v_1 are cyclically reduced and it is easy to see that u_1 is conjugate to neither v_1 nor v_1^{-1}. Let v_2 be the word obtained from v_1 by the third standard transformation, i.e.

$$v_2 = g^2 h^{-l} g^{-s} h^{-t} g h^t g^s h^l.$$

Since $t \neq l$, it is easy to see that the word u_1 is conjugate to neither v_2 nor v_2^{-1}. Thus, u_1 and v_1 are not equivalent, i.e. $(u_1, v_1) \notin M_1$.

Example 3. For any $w_1, w_2 \in F_2$ consider the function $\psi_{(w_1,w_2)} : M \to M$ defined by

$$(u,v) \mapsto (u(w_1,w_2), v(w_1,w_2)).$$

Since $\tau_u = \tau_v$, the Fricke polynomials $Q_u(\tau_g, \tau_h, \tau_{gh})$ and $Q_v(\tau_g, \tau_h, \tau_{gh})$ coincide. Therefore,

$$\tau_{u(w_1,w_2)} = Q_u(\tau_{w_1}, \tau_{w_2}, \tau_{w_1 w_2}), \qquad \tau_{v(w_1,w_2)} = Q_v(\tau_{w_1}, \tau_{w_2}, \tau_{w_1 w_2}),$$

whence $\tau_{u(w_1,w_2)} = \tau_{v(w_1,w_2)}$, i.e. $\psi_{(w_1,w_2)}(u,v) \in M$. Thus, the function $\psi_{(w_1,w_2)}$ is well defined.

We claim that there exist words $w_1, w_2 \in F_2$ such that

$$\psi_{(w_1,w_2)}(M_1) \neq M_1.$$

Let

$$w_1 = g, \quad w_2 = h^s g^t h^{-s}, \quad u = ghg^2h^2, \quad v = gh^2g^2h,$$

where $s, t \in Z$, $s \neq 0$, $t \neq 0$. Clearly, u and v are equivalent, hence $(u,v) \in M$. Then we have

$$\psi_{(w_1,w_2)}(u,v) = (gh^s g^t h^{-s} g^2 h^s g^{2t} h^{-s}, gh^s g^{2t} h^{-s} g^2 h^s g^t h^{-s}) = (u_1, v_1).$$

The words u_1 and v_1 are cyclically reduced and it is easy to see that u_1 is conjugate to neither v_1 nor v_1^{-1}. Let v_2 be the word obtained from v_1 by the third standard transformation, i.e.

$$v_2 = h^{-s}g^t h^s g^2 h^{-s} g^{2t} h^s g.$$

Since $s \neq -s$, one can easily check that u_1 is conjugate to neither v_2 nor v_2^{-1}. Thus, u_1 is not equivalent to v_1, i.e. $(u_1, v_1) \notin M_1$.

Consider the following two sets:

$$M_2 = \cup_{w \in F_2} \varphi_w(M_1),$$

$$M_3 = \cup_{(w_1,w_2) \in F_2 \times F_2} \psi_{(w_1,w_2)}(M_1).$$

The following questions are open:

1) Does M_2 equal to M?

2) Does M_3 equal to M?

3) Does $M_2 \cup M_3$ equal to M?

Acknowledgements. This paper was written while the author was visiting the University of Bielefeld as a guest of SFB 343 "Diskrete Strukturen in der Mathematik". I gratefully acknowledge the support and the hospitality of the University of Bielefeld. The author wishes to thank Prof. H. Helling for suggesting this topic and for his interest.

References

[1] Helling H. Diskrete Untergruppen von $\mathrm{SL}_2(R)$, Invent. Math. **17**(1972), 217–229.

[2] Horowitz R. Characters of free groups represented in the two dimensional linear group, Comm. Pure. Appl. Math. **25**(1972), 635–649.

[3] Lubotzky A., Magid A. Varieties of representations of finitely generated groups, Memoirs AMS. **58**(1985), 1-116.

[4] Magnus W. The uses of 2 by 2 matrices in combinatorial group theory. Asurvey. Result. Math. **4**(1981), 171–192.

[5] Traina C. Trace polynomial for two generated subgroups of $\mathrm{SL}_2(C)$, Proc. AMS. **79**(1980), 369–372.