TREE ALGEBRAS WITH NON-NEGATIVE TITS FORM

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One of the seminal results of modern representation theory of algebras is Gabriel's theorem saying that a quiver algebra $A = kQ$ accepts only finitely many isoclasses of indecomposable modules if and only if the underlying quiver of Q is a Dynkin diagram. The proof may be done by introducing the *Tits form* q_A of A and showing that the conditions above are equivalent to q_A being a positive quadratic form. In fact, the isoclasses of indecomposable A-modules are in one-to-one correspondence with the positive roots of q_A. Some years later it was shown that a quiver algebra $A = kQ$ is of tame type if and only if the corresponding Tits form is non-negative. See [8].

We recall that a finite dimensional algebra over an algebraically closed field k is said to be *tame* if for each dimension, almost all the indecomposable modules occur in a finite number of one-parameter families. We are interested in the classification of tame algebras and their representations.

Let A be a basic triangular algebra with a presentation $A = kQ/I$, where Q is a finite connected quiver without oriented cycles and I is an admissible ideal of the path algebra kQ, see [8]. Let $Q_0 = \{1, \ldots, n\}$ be the set of vertices of Q. The *Tits form* $q_A \colon \mathbb{Z}^n \to \mathbb{Z}$ is defined by

$$q_A(v) = \sum_{i=1}^{n} v(i)^2 - \sum_{i \to j} v(i)v(j) + \sum_{i,j=1}^{n} r(i,j)v(i)v(j),$$

where $i \to j$ runs over all arrows of Q and $r(i,j) = \dim_k e_j(I/IJ + JI)e_i$, where J is the ideal of kQ generated by the arrows and e_i denotes the trival path at i. An algebra A is said to be strongly simply connected if for every convex subcategory B of A the first Hochschild cohomology group $H^1(B,B)$ vanishes, see [21]. Particular examples of these algebras are the *tree algebras*, that is, algebras of the form $A = kQ/I$ where the underlying graph of Q has no cycles. It has been conjectured in [13] (see [14]) that a strongly simply connected algebra A is tame if and only if q_A is weakly non-negative, i.e. $q_A(v) \geq 0$ for vectors with non-negative coordinates (notation: $v \geq 0$). In fact, every tame algebra A has q_A weakly non-negative [14]. The conjecture has been verified for various classes of algebras, see [13]. The aim of this work is to prove the following result.

Theorem 1 *Let $A = kQ/I$ be a tree algebra of global dimension at most two and assume that q_A is non-negative. Then A is tame.*

In section 1 we prove results concerning the structure of tree algebras with a weakly non-negative Tits form accepting a positive sincere isotropic vector. We show that those algebras A have $\mathrm{g}\ell \dim A \leq 2$ and if not of polynomial growth they only accept critical subcategories of type $\tilde{\mathbb{D}}_n$. In section 2 we show a construction procedure for all tree algebras satisfying the hypothesis of the Theorem. In section 3 we present the proof of the Theorem by reducing an algebra A satisfying the hypothesis to a clannish algebra [5], using tilting theory [20] and deformations [9].

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1 Structure of tree algebras.

1.1

Let $A = kQ/I$ be a connected triangular algebra. We start by recalling some general facts and definitions. As general references, see [8, 20].

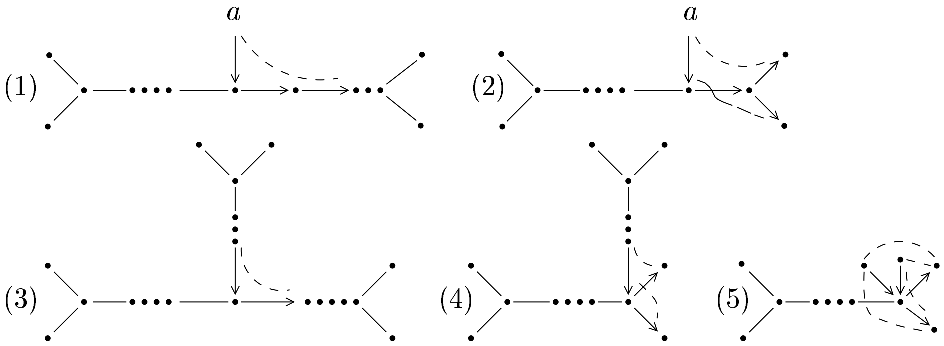
The algebra A is said to be *tame* if for every $d \in \mathbb{N}$ there is a finite family $M_1, \ldots, M_s$ of $A - k[t]$-bimodules which are free finitely generated as right $k[t]$-modules and such that almost every indecomposable A-module X with $\dim_k X = d$ is isomorphic to some $M_i \otimes_{k[t]} (k[t]/(t - \lambda))$ for $1 \leq i \leq s$ and $\lambda \in k$. The minimal possible s in the definition is denoted by $\mu(d)$. A tame algebra A is said to be *domestic* (resp. of *polynomial growth*) if there is a number $m \in \mathbb{N}$ such that $\mu(d) \leq m$ (resp. $\mu(d) \leq d^m$) for every $d \in \mathbb{N}$.

Important *examples* are the following:

(a) An algebra C is called *critical* if the Auslander-Reiten quiver Γ_A has a postprojective component and every proper quotient of C is representation-finite. Critical algebras are classified in [10], they are the concealed algebras of extended Dynkin type. The Tits form q_C is *critical*, that is, it is not weakly positive but every restriction $q_C^{(i)} = q_C(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ is weakly positive. There exists a positive vector $v \in \mathbb{N}^n$ such that $\mathbb{Z}v = \{w \in \mathbb{Z}^n : q_C(w) = 0\}$. Critical algebras are domestic.

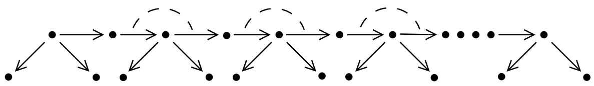
(b) Given a critical algebra A_0 and $E_1, \ldots, E_s$ a family of pairwise non-isomorphic simple regular A_0-modules with corresponding branches $K_1, \ldots, K_s$, then the branch extension $A = A_0[E_i, K_i]_{i=1}^m$ is a *tubular extension* in the sense of [20]. A tubular extension A is tame if and only if q_A is weakly non-negative [12, 20]. A tame tubular extension A is either a *domestic tubular algebra* or a *tubular algebra*. In case $A = A_0[E_i, K_i]_{i=1}^m$ is tubular, it is also a cotubular algebra that is obtained by branch coextensions from a critical algebra A_∞. The Tits form q_A is non-negative, and choosing vectors $0 \neq v_0, v_\infty \in \mathbb{N}^n$ with $q_{A_0}(v_0) = 0 = q_{A_\infty}(v_\infty)$, we get $\mathbb{Q}v_0 + \mathbb{Q}v_\infty = \{w \in \mathbb{Q}^n : q_A(w) = 0\}$ as vector spaces. Tubular algebras are of polynomial growth.

(c) A strongly simply connected tame algebra which is minimal not of polynomial growth is called a pg-*critical algebra*. These algebras have been classified in [11]. If $A = kQ/I$ is a tree pg-critical algebra, it is given by one of the following frames or their duals:



If A is a pg-critical tree, there are two convex subcategories C_0, C_1 which are critical of type $\tilde{\mathbb{D}}_n$ and such that the combined supports of C_0 and C_1 cover the quiver of A. If $0 \neq v_0, v_1 \in \mathbb{N}^n$ are vectors with $q_{C_0}(v_0) = 0 = q_{C_1}(v_1)$, then $\mathbb{Q}v_0 + \mathbb{Q}v_1 = \{w \in \mathbb{Q}^n : q_A(w) = 0\}$ as vector spaces.

(d) The following example A was called ‘dancing girls’ by Sheila Brenner.



It is obtained by 'glueing' s critical trees of type $\tilde{\mathbb{D}}_6$. The Tits form q_A is non-negative and it accepts a sincere positive isotropic vector. The algebra is tame. If $s \geq 2$, then A is not of polynomial growth and contains $s-1$ pg-critical convex subcategories.

(e) Finally, we recall that by a Theorem of Drozd, an algebra which is not tame is *wild*, that is, the classification of its indecomposable modules up to isomorphism contains the well-known unsolved problem of classification of pairs of matrices under simultaneous conjugation.

Notation: Let $A = kQ/I$ as above, for a module X, its class in the Grothendieck group $K_0(A) = \mathbb{Z}^n$ is denoted by $\mathbf{dim}\, X$. The *Euler bilinear form* is defined by $\langle \mathbf{dim}\, X, \mathbf{dim}\, Y\rangle_A = \sum\limits_{i=0}^{\infty}(-1)^i \dim_k \mathrm{Ext}_A^i(X,Y)$. The corresponding quadratic form is denoted by χ_A.

1.2

Let $A = kQ/I$ be a triangular algebra and q_A its Tits form.

We denote by $\Sigma^0(q_A) = \{0 \leq v \in K_0(A) \colon q_A(v) = 0\}$ the set of non-negative isotropic vectors. If $v \in \Sigma^0(q_A)$ we call it a 0-*root.*

Denote by $q_A(x,y) = \frac{1}{2}[q_A(x+y) - q_A(x-y)]$ the symmetric bilinear form associated with q_A. If q_A is non-negative, then for $v \in \Sigma^0(q_A)$ we have $q_A(v,-) = 0$ and hence $\Sigma^0(q_A)$ is a semi-group. In this case, the *corank* of q_A (resp. the *positive corank* of q_A) is $\operatorname{corank} q_A = \dim_{\mathbb{Q}}\{v \in K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q} \colon q_A(v) = 0\}$ (resp. $\operatorname{corank}^+ q_A =$ dimension of the $\mathbb{Q}$-vector space generated by $\Sigma^0(q_A)$).

Lemma. *Let q_A be a weakly non-negative form and assume that $v \in \Sigma^0(q_A)$ is a sincere 0-root. Then the following holds:*

a) *q_A is non-negative.*

b) $\operatorname{corank} q_A = \operatorname{corank}^+ q_A$.

Proof: (a): Let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{Q}^n = K_0(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Observe that $0 \le q_A(v, e_i)$, since otherwise $q_A(2v + e_i) = 1 + 2q_A(v, e_i) < 0$. Since $0 = q_A(v, v) = \sum_{i=1}^{n} v(i) q_A(v, e_i)$, we get $q_A(v, -) = 0$.

Let $w \in \mathbb{Z}^n$. Choose $m \in \mathbb{N}$ such that $w + mv \ge 0$. Then $q_A(w) = q_A(w + mv) \ge 0$ because q_A is weakly non-negative.

(b): Let $v_1, \dots, v_s$ be a basis of $\{v \in \mathbb{Q}^n : q_A(w) = 0\}$. Choose numbers $m_i \in \mathbb{N}$ such that $w_i = m_i v + v_i \ge 0$, $i = 1, \dots, s$ and $w_1, \dots, w_s$ is a linearly independent set in the space generated by $\Sigma^0(q_A)$. Hence $\operatorname{corank} q_A \le \operatorname{corank}^+ q_A$. But obviously, $\operatorname{corank}^+ q_A \le \operatorname{corank} q_A$. $\square$

1.3

Proposition [6]. Let $A = kQ/I$ be a strongly simply connected algebra with weakly non-negative Tits form q_A. Then there is a family $C_1, \dots, C_s$ of critical convex subcategories of A with corresponding 0-roots $0 \ne v_i \in \Sigma^0(q_{C_i})$ satisfying the following:

(a) For each $v \in \Sigma^0(q_A)$, there are numbers $\mu_1, \dots, \mu_s \in \mathbb{Q}^+$ such that $v = \sum_{i=1}^{s} \mu_i v_i$.

(b) If q_A is non-negative, then $q_A(v_i, v_j) = 0$ for all $1 \le i, j \le s$ and $\operatorname{corank}^+ q_A$ is the maximal number of linearly independent vectors in $\{v_i : 1 \le i \le s\}$.

(c) If $Q_0 = \bigcup_{i \in L} \operatorname{supp} v_i$ for some $L \subset \{1, \dots, s\}$ such that $q_A(v_i, v_j) = 0$ for $i, j \in L$, then q_A is non-negative. $\square$

The minimal vectors v_i above are called the *critical roots* of q_A.

We observe that if A is a critical, tubular or pg-critical algebra (as defined in 1.1), then q_A is non-negative and it accepts a sincere 0-root.

1.4

Theorem. *Let $A = kQ/I$ be a tree algebra such that q_A is non-negative with a sincere 0-root. Then:*

(a) $\operatorname{g\ell\,dim} A \le 2$

(b) *If A is of polynomial growth, then A is either a critical or a tubular algebra.*

(c) *If A is not of polynomial growth, the following statements hold:*

(i) *A does not accept convex subcategories which are tubular algebras;*

(ii) *there is a family of convex subcategories $B_1, \dots, B_s$ of A which are* pg-*critical algebras such that* $\bigcup_{i=1}^{s} \operatorname{supp} B_i = Q_0$*;*

(iii) *if C is a critical convex subcategory of A, then C is of type $\tilde{\mathbb{D}}_n$; moreover, C is contained in some B_i.*

Part (a) easily follows from (b) and (c) and it is shown in (1.7). Part (b) is shown in [17] in the more general situtation of strongly simply connected algebras (for a shorter proof in the case of tree algebras, see [16]). Parts (i) and (iii) of (c) were shown in [18] under the assumption that A is tame with a sincere indecomposable module (in [19] this result is generalized assuming that q_A is weakly non-negative instead of A tame). Here we give an ad-hoc proof in the case of tree algebras. The proof of (i) is given in (1.5), the proof of (ii) in (1.6). Part (iii) follows immediately from (ii).

1.5

Proof of (1.4) (c.i): Let $A = kQ/I$ be a tree algebra such that q_A is non-negative with a positive sincere vector $v \in \Sigma^0(q_A)$. Assume that B is a convex subcategory of A.

First, let B be a tubular algebra and $A \neq B$. Let $a \in Q_0$ be a vertex not in B but joined by an arrow a —— b with $b \in B$. We may assume that $a \to b$. Then the restriction M of $\operatorname{rad} P_a$ to B is indecomposable and $E = B[M]$ is a tree algebra with q_E non-negative.

Recall the structure of B and its Auslander-Reiten quiver Γ_B. There are two critical subcategories $B_i = kQ_i/I \cap kQ_i$ of B with associated vectors $0 \neq v_1, v_2 \in \mathbb{N}^n$ such that $q_{B_1}(v_1) = 0 = q_{B_2}(v_2)$. In Γ_B there is a postprojective component $\mathcal{P}$ formed by B_2-modules; a family of inserted tubes $\mathcal{T}_0$ whose stable tubes are formed by B_2-modules; tubular families $\mathcal{T}_\gamma$ for $\gamma \in \mathbb{Q}^+$; a tubular family $\mathcal{T}_\infty$ of coinserted tubes whose stable tubes are formed by B_1-modules and a preinjective component $\mathcal{I}$ formed by B_1-modules. If M

belongs to $\mathcal{P}$, $\mathcal{T}_0$ or $\mathcal{T}_\gamma$ ($\gamma \in \mathbb{Q}^+$), it is well-known that q_E is not weakly non-negative (see for example [16]). Hence M belongs to $\mathcal{T}_\infty$ or $\mathcal{I}$. If $\operatorname{supp} M \cap Q_2 \neq \emptyset$, then using that $\mathrm{g}\ell \dim E \leq 3$ and $q_E(v_2, -) = 0$, we get that $0 < \langle v_2, \mathbf{dim}\, M\rangle_E \leq q_E(v_2, e_a)$ and $q_E(2v_2 - e_a) < 0$. Hence $\operatorname{supp} M$ is contained in a branch of B formed by vertices in Q_1. Therefore M is a preinjective B_1-module and $0 < \langle v_1, \mathbf{dim}\, M\rangle_{B_1} = q_E(v_1, e_a)$. Then $q_E(2v_1 - e_a) > 0$. This contradiction shows that $A = B$ is tubular and hence of polynomial growth. □

1.6

Proof of 1.4 (c) (ii) *and* (iii): Let $A = kQ/I$ be a tree algebra such that q_A is non-negative with sincere $v \in \Sigma^0(q_A)$. Moreover, assume that A is not of polynomial growth. By part (i), A does not accept convex subcategories which are tubular algebras. We shall proceed by induction on the number of vertices $n(A)$ of Q.

Since A is not of polynomial growth but q_A is weakly non-negative, then Skowroński's criterion [23] implies the existence of a convex subcategory B_1 of A which is a pg-critical algebra. Let $v_1 \in \Sigma^0(q_{B_1})$ be sincere. Then there are positive integers $a, b > 0$ such that $0 \leq w = av - bv_1$ is not sincere. Observe that $w \in \Sigma^0(q_A)$.

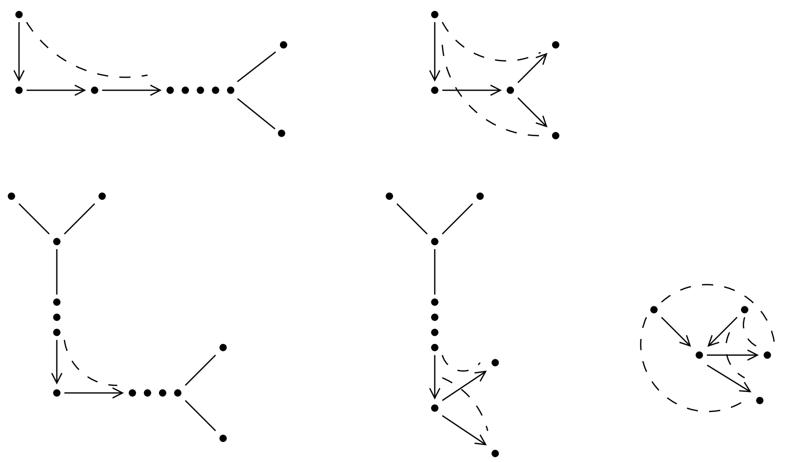
Let $D_1, \ldots, D_s$ be the convex subcategories of A corresponding to the connected components of $\operatorname{supp} w$ in Q (observe that the D_i are convex since A is a tree algebra). Let w_i be the restriction of w to D_i, $1 \leq i \leq s$. Hence $w = \sum_{i=1}^{s} w_i$ and since there are no arrows between different D_i's, we get:

$$0 = q_A(w) = \sum_{i=1}^{s} q_A(w_i) + \sum_{i<j} q_A(w_i, w_j) \geq \sum_{i=1}^{s} q_{D_i}(w_i) \geq 0.$$

Therefore, w_i is a sincere 0-root of q_{D_i}. By (1.4) (b) and induction hypothesis either D_i is critical or there are convex subcategories $B_1^{(i)}, \ldots, B_{s_i}^{(i)}$ which are pg-critical algebra such that $\bigcup_{j=1}^{s_i} \operatorname{supp} B_j^{(i)} = \operatorname{supp} D_i$, $1 \leq i \leq s$.

Assume D_1 is critical. There is an edge x —— y in A such that $x \in \operatorname{supp} D_1$ and $y \notin \operatorname{supp} D_1$. Let C_1 and C_2 be the two critical subcategories of B_1 (see 1.1 c) and suppose that $y \in \operatorname{supp} C_1$. Since A is a tree algebra, there is a convex subcategory E_1 of A with $\operatorname{supp} E_1 = \operatorname{supp} D_1 \cup \operatorname{supp} C_1$. By (b), E_1

is not of polynomial growth. If $n(E_1) < n(A)$, by induction hypothesis we get pg-critical algebras $B_1^{(1)}, \dots, B_{s_1}^{(1)}$ such that $\bigcup\limits_{j=1}^{s_1} \operatorname{supp} B_j^{(1)} = \operatorname{supp} E_1$ and this completes the proof of (ii). Therefore, we may assume that $n(E_1) = n(A)$. In particular, $\emptyset \neq (\operatorname{supp} C_2 \setminus \operatorname{supp} C_1) \subset \operatorname{supp} D_1$. Hence there is a convex and connected subcategory E_2 of A whose support is $\operatorname{supp} D_1 \cup \operatorname{supp} C_2$. As above, we may assume that $n(E_2) = n(A)$. Hence $(\operatorname{supp} C_1 \setminus \operatorname{supp} C_2) \subset \operatorname{supp} D_1$. By the structure of pg-critical trees (1.1), D_1 contains a convex subcategory of one of the following shapes:



Checking out the possible frames of critical algebras D_1 (there are only 11 of them), we find a contradiction. This shows (ii).

For (iii), let C be a convex critical subcategory of A. Let B be a pg-critical algebra which is a convex subcategory of A and such that $\operatorname{supp} C \cap \operatorname{supp} B \neq \emptyset$. If $\operatorname{supp} C \subset \operatorname{supp} B$, then by (1.1), C is of type $\tilde{\mathbb{D}}_n$. Assume that $\operatorname{supp} C \not\subset \operatorname{supp} B$. Then as above we find a critical subcategory C_1 of B such that there is a convex and connected subcategory E of A with $\operatorname{supp} E = \operatorname{supp} C \cup \operatorname{supp} C_1$ and $n(E) < n(A)$. By induction hypothesis, C is contained in a pg-critical algebra and hence it is of type $\tilde{\mathbb{D}}_n$. This shows (iii). $\square$

1.7

We require the following simple fact.

Claim: Let $A = kQ/I$ be a tree algebra such that q_A is non-negative. Let C be a critical convex subcategory of A and $i \in Q_0$ be joined by a path $i = j_0 \to j_1 \to \cdots \to j_s = j$ to j in C such that $r(i, j) > 0$. Then all j_t $(1 \leq t \leq s)$ are in C.

Proof: Let v be a sincere 0-root of q_C. Since q_A is non-negative by (1.2), $q_A(v, -) = 0$. In particular, $0 = q_A(v, e_i) = 2v(i) - \sum_{i \,-\, x \in C} v(x) + \sum_{y \in C} [r(i, y) + r(y, i)]v(y)$. We may assume that i is not in C. Since $r(i, j) > 0$, there is an arrow i —— $x \in C$. Since A is a tree, then $i \to x$ and hence $x = j_1$ and all j_t $(1 \leq t \leq s)$ are in C. □

Proof of (1.4) (a): Let $A = kQ/I$ be a connected tree algebra such that q_A is non-negative with a sincere 0-root v. If A is of polynomial growth, then by (b), A is either a critical or a tubular algebra. In both cases $\mathrm{g}\ell \dim A \leq 2$. Assume A is not of polynomial growth. By (c), there are convex subcategories $C_1, \ldots, C_s$ of A which are critical of type $\tilde{\mathbb{D}}_n$ and $\bigcup_{i=1}^{s} \operatorname{supp} C_i = Q_0$. Let v_i be a sincere 0-root of q_{C_i}, $1 \leq i \leq s$.

Assume that $\mathrm{g}\ell \dim A > 2$. Then there is a path in Q of the form

$$a \to \cdots \to b \to \cdots \to c \to \cdots \to d$$

with $b \neq c$, $r(a, c) > 0$ and $r(b, d) > 0$. Assume that a is in C_1, then by the claim, b is also in C_1. But then again the claim implies that c in C_1, which is impossible because C_1 is a hereditary algebra. □

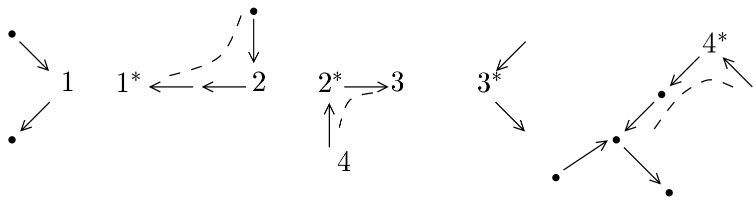
2 Structure of non-polynomial growth tree algebras.

2.1

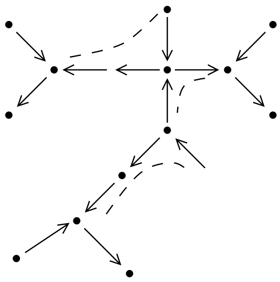
Let $A = kQ/I$ be a tree algebra with $\mathrm{g}\ell \dim A \leq 2$ and q_A non-negative. Since we want to prove that A is tame, we may assume that A is not of polynomial growth.

In this section we shall describe a method of construction of the trees A as above. The idea follows Brüstle's definition of a 'kit' [4]. Let $T_1 \ldots, T_s$ be a finite set of basic and connected algebras and S a set of vertices of the quivers of $T_1, \ldots, T_s$. We say that A is *obtained as a glueing* of $T_1, \ldots, T_s$ along S if there exists an involution $\sigma\colon S \to S$ (i.e. $\sigma^2 = \sigma$) such that A is obtained from the disjoint union $\coprod_{i=1}^{s} T_i$ by identifying a and $\sigma(a)$ for all $a \in S$. If A is a tree, observe that a and $\sigma(a)$ belong to different trees T_i.

In the following examples we denote $a^* = \sigma(a)$.

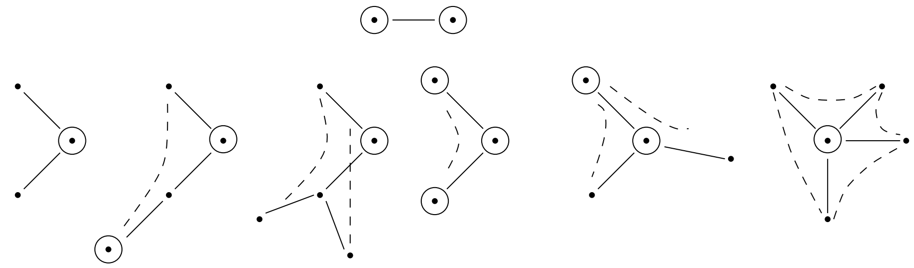


A:



Our first structure result deals with the important class of algebras considered in Theorem (1.4).

Proposition. *Let $A = kQ/I$ be a tree algebra with q_A non-negative and such that there exists a sincere vector $v \in \Sigma^0(q_A)$. Moreover, assume that A is not of polynomial growth. Then A is obtained by glueing trees isomorphic to some of the following list along a subset of the encircled vertices, where the edges may have any possible orientation.*



List 1

In particular, any minimal path $x_1 \to \cdots \to x_s$ in I has $s \leq 4$.

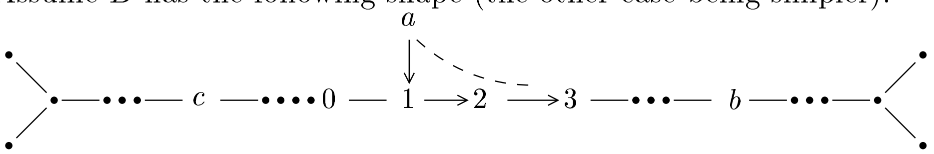
Proof: By (1.4), there is a pg-critical convex subcategory B of A.

(1) Assume that B corresponds to one of the frames (1) or (2) in (1.1). Let a be a source in B as in (1.1). We shall prove that a has a unique neighbour in Q. Indeed, let v_1 and v_2 be the two critical vectors of B, let $v_1(a) \neq 0 = v_2(a)$. Assume b is a vertex in A which is a neighbour of a not in B. Then $0 = q_A(v_1, e_b) = -v_1(a) + \sum\limits_i v_1(i) r(b,i)$. Hence there is a path $b \to a \to \cdots \to i \in B$ belonging to I. Then $v_2(i) \neq 0$ and there are no arrows from b to $\operatorname{supp} v_2$, hence $q_A(v_2, e_b) > 0$ which contradicts that q_A is non-negative.

Let $v \in \Sigma^0(q_A)$ be sincere. Since $v_1(a) = 1$, then the vector $w = v - v(a)v_1 + v(a)v_2 \in \Sigma^0(q_A)$ and has $\operatorname{supp} w = Q_0 \setminus \{a\}$. Hence the tree algebra $\bar{A}$ obtained as the full subcategory of A with vertices $Q_0 \setminus \{a\}$ has a sincere vector $w \in \Sigma^0(q_{\bar{A}})$. By (1.4), $\bar{A}$ is either critical of type $\tilde{\mathbb{D}}_n$ or not of polynomial growth. In the first case, $\bar{A}$ is obviously obtained by glueing at most 3 trees of our list.

In the second situation, we may apply induction to get trees $T_1, \ldots, T_s$ isomorphic to some trees in the List 1, a subset S of the encircled vertices and an involution $\sigma\colon S \to S$ such that $\bar{A}$ is obtained by glueing $T_1, \ldots, T_s$ along S.

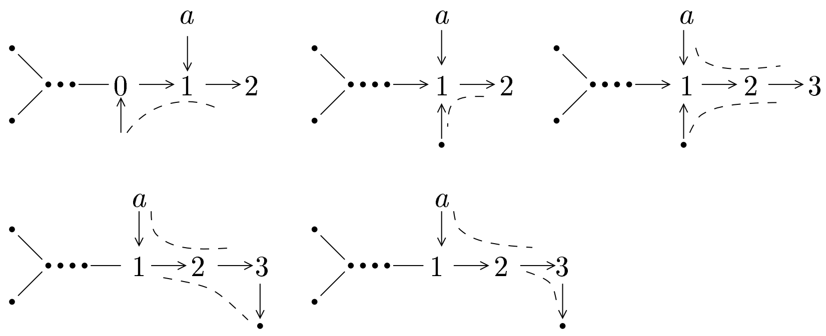
Assume B has the following shape (the other case being simpler):

B : 

Observe that 2 has only two neighbours (1 and 3) in Q. Otherwise, we get either $2 \leftarrow x \neq 1$ (and then $\operatorname{supp} v_1 \cup \{x\}$ yields a wild subcategory of A) or

$2 \to x \neq 3$ (and then $q_A(v_2, e_x) = 0$ implies $q_A(v_1, e_x) > 0$, a contradiction). Let 2 be a vertex of the tree T_1, then we claim that T_1 is linear.

Indeed, otherwise we get a convex subcategory of A of one of the following types:



This contradicts either that q_A is non-negative or that $g\ell \dim A \leq 2$. Therefore T_1 is identified in A with the walk

$$c \;\text{——}\; \cdots \;\text{——}\; 1 \;\text{——}\; 2 \;\text{——}\; 3 \;\text{——}\; \cdots b$$

(possibly $c = 1$ or $b = 3$). Define the trees:

T_1': c ——•••—— 1 $\qquad$ T_2': 3 ——•••—— b $\qquad$ T_0:
$$\begin{tikzcd} & a \arrow[d] & \\ & 1' \arrow[r] & 2 \arrow[r] & 3' \end{tikzcd}$$

Then A is obtained by glueing the trees $T_0, T_1', T_2', T_2, \ldots, T_s$ along $S \cup \{1, 1', 3, 3'\}$ with $\sigma(1) = 1'$, $\sigma(3) = 3'$.

(2) We may assume that all the pg-critical subcategories of A have frames of type (3), (4) or (5) in the notation of (1.1).

Let $a_1, \ldots, a_t$ be all vertices in Q such that for $i = 1, \ldots, t$, there are neighbours b_i, c_i, d_i of a_i such that the full subcategory of A with vertices a_i, b_i, c_i, d_i is:

$$\begin{tikzcd} & b_i \arrow[d] & \\ d_i \arrow[r, dash] & a_i \arrow[r] & c_i \end{tikzcd}$$

Let D be the minimal convex and connected subcategory of A containing all $a_1, \ldots, a_t$. Since D is a tree, we may assume that a_1 is extremal in D. Let D' (resp. D'') be the component of $Q \setminus \{a_1\}$ containing b_1 (resp. c_1), then either D' or D'' does not contain vertices a_i, $i = 1, \ldots, t$. Without loss of generality, we assume that D' has this property. Let a be a extremal vertex of Q belonging to D' and set $\bar{A}$ as the full subcategory of A with vertices not in D'. Let C_1 be the critical subcategory of A containing a. Then the following happens:

(i) The subcategory D' has the form

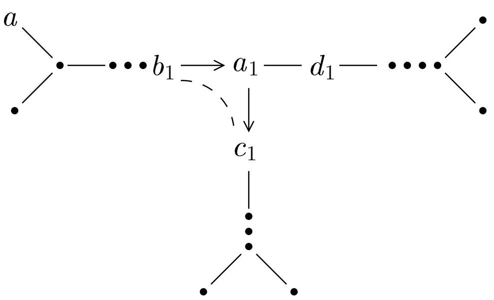
$$\begin{tikzcd}[arrows=dash]
 & & & b_{t+1} \arrow[d] & \\
b_1 \arrow[r] & b_2 \; \cdots \arrow[r] & & b_t \arrow[r] & a
\end{tikzcd}$$

(ii) Any critical subcategory C of A containing a contains all of D'. In particular, any vector $w \in \Sigma^0(q_A)$ with $w(a) = 0$ vanishes in D'.

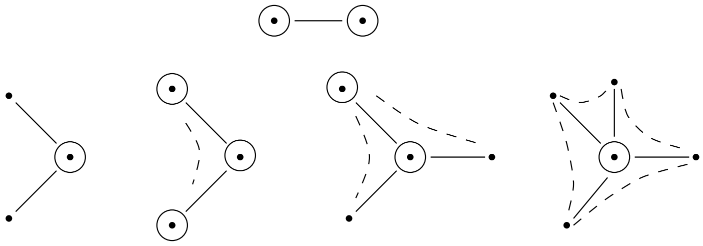
In fact, (i) is easily shown using that $q_A(v_1, -) = 0$, $g\ell \dim A \leq 2$ and the assumption that no pg-critical subcategory of A has frame of type (1) or (2). The statement (ii) follows from (i) and (1.3).

Consider B a pg-critical subcategory of A containing a and such that v_1, v_2 are the critical vectors in $\Sigma^0(q_B)$. Then by (ii) above, the vector $w = v - v(a)v_1 + v(a)v_2$ is a sincere vector in $\Sigma^0(q_{\bar{A}})$. We may assume that the shape of B is as in the following drawing (we assume shape (3) of (1.1), the other cases being simpler).

B:



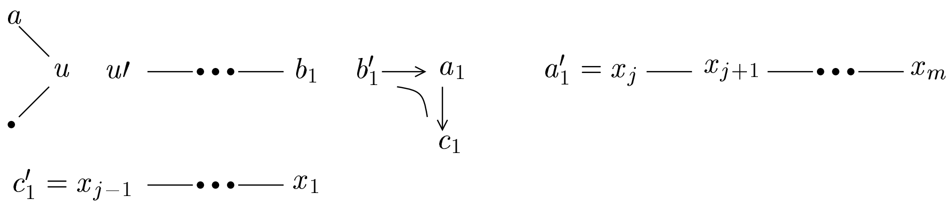
As in (1), we get trees $T_1, \ldots, T_s$ isomorphic to some trees in the List 1, a subset S of the encircled vertices and an involution $\sigma: S \to S$ such that $\bar{A}$ is obtained by glueing $T_1, \ldots, T_s$ along S. Since there are no pg-critical subcategories of A with frames of types (1) or (2), then all the trees T_i are of one of the types:



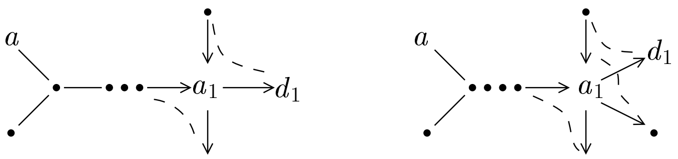
Let T_1 be a tree such that its image in $\bar{A}$ contains a_1. If T_1 is the line

$$x_1 \text{ --- } x_2 \text{ --- } \cdots \text{ --- } x_m$$

and $a_1 = x_j$ with $2 \leq j \leq m-1$, then A may be obtained by glueing the trees $T_2, \ldots, T_s$ and those below along an appropiate set of vertices.



Hence we may assume that T_1 is not linear. By the non-negativity of q_A, it is easy to see that A contains a convex subcategory of one of the following shapes:



In any case, we may define easily a new set of trees $T'_1, \ldots, T'_t$ such that A is obtained by glueing $T'_1, \ldots, T'_t$ along an appropiate set of vertices. We leave the details to the reader. $\square$

2.2

We prove the main structure result of the work.

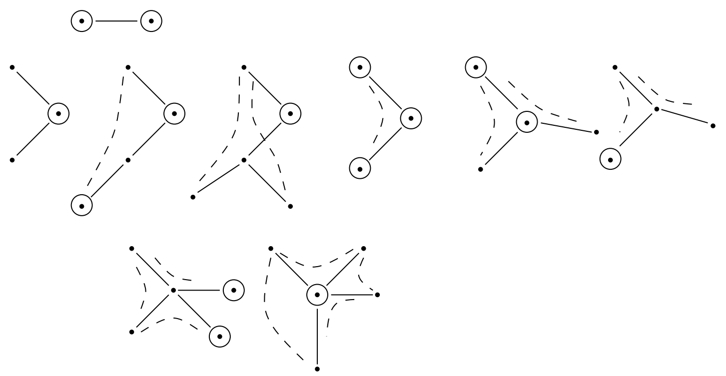
Theorem 2 *Let $A = kQ/I$ be a tree algebra satisfying the following conditions:*

(i) $g\ell \dim A \leq 2$;

(ii) q_A *is non-negative;*

(iii) A *is not of polynomial growth.*

Then A *is a quotient of an algebra obtained by glueing trees isomorphic to some of the list below along a subset of the encircled vertices where the edges may have any possible orientation.*



List 2

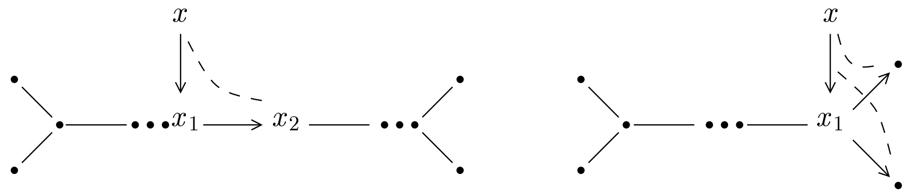
Proof: Let B of a maximal connected and convex subcategory of A such that there is a sincere vector in $\Sigma^0(q_B)$. By (iii), we may assume that B is not of polynomial growth. By Proposition (2.1), B is obtained by glueing some of the trees in the above List 1 along a subset of the encircled vertices.

We may assume that there is an arrow $x \to y$ with x not in B and y in B. We shall show that the following happens.

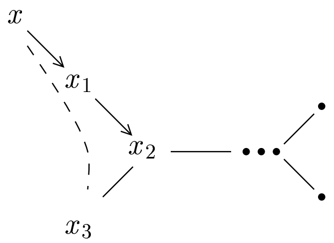
(α) There are at most two minimal paths $x = x_0 \to y = x_1 \to x_2 \to \cdots \to x_s$ in I with x_i ($1 \leq i \leq s$) in B. If $s \geq 3$, the path is unique and $s = 3$.

($\alpha\alpha$) Let C be a critical subcategory of B containing y. Then the convex subcategory of A containing C and x has one of the following shapes (or their duals):

(a)



(b)



Moreover, in case (b), x_3 (resp. x_1) has exactly one (resp. two) neighbour(s) in Q.

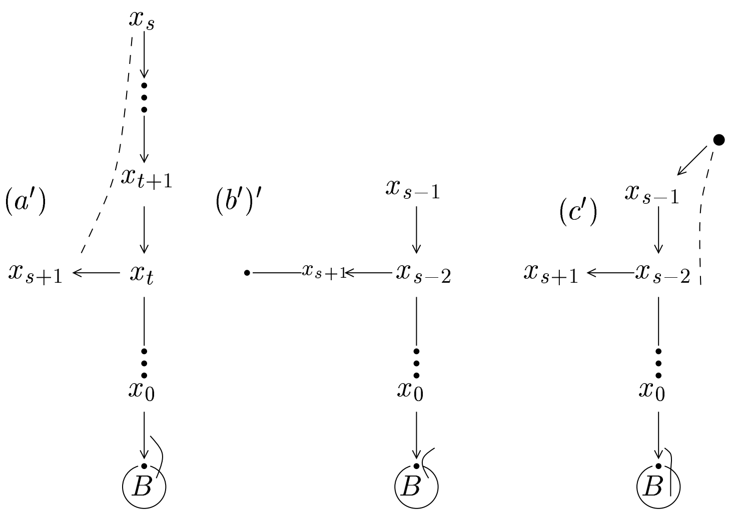
Indeed, for the proof of this claim we recall that for any critical vector v of A, we shall have $q_A(v, e_x) = 0$. First, using that there is a convex pg-critical subcategory D of B containing y (1.4), a simple inspection shows the first part of claim (α). The maximality of B implies that $s \leq 3$ and in case $s = 3$, then y is extremal in C. Let $z \neq x_2$ be a neighbour of x_3. Since $g\ell$ dim $A = 2$, there is no relation from x_1 to z and therefore A contains a wild convex subcategory. A contradiction. Similarly, x_1 has exactly two neighbours in Q (namely x and x_2). Hence only situations (a) and (b) above are possible.

We get that the convex subcategory of A with vertices in $B \cup \{x\}$ may be obtained by glueing trees in the List 2 as indicated in the statement of the Theorem (as we did in (2.1)). Let B_1 be a convex subcategory of A containing B and a maximal linear subcategory of the form

$x = x_0 \;\text{——}\; x_1 \;\text{——}\; \cdots \;\text{——}\; x_s$

without subpaths in I. We may assume that there is a path $\gamma\colon x_t \leftarrow \cdots \leftarrow x_s \leftarrow x_{s+1}$ in I. Moreover, we may assume that x_j does not lie on paths in I for $1 \leq j \leq t-1$.

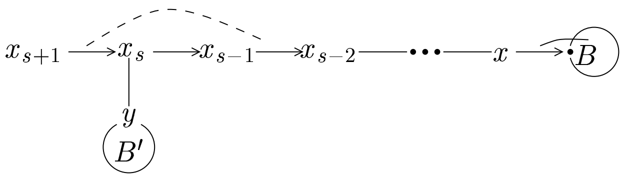
We claim that $s-2 \leq t$ and if $s-2 = t$, then x_{s+1} (resp. x_{s-1}) has exactly one (resp. two) neighbour(s) in Q. Indeed, the Tits form of the convex subcategory B_2 of A formed by $B_1 \cup \{x_{s+1}\}$ is equivalent by conjugation to that of the algebra (a') below (consider the conjugation $e_i \to e_i$ if $i \neq s+1$ and $e_{s+1} \to -e_{s+1}$).



By the structure of B, in the algebra (a'), the vertices x_{s+1} and x_{t+1} are extremal in a critical subcategory. Therefore the Tits form of such an algebra is not non-negative if $s \geq t+3$. In case $t = s-2$, if x_{s+1} (resp. x_{s-1}) has more than one (resp. two) neighbour(s) in Q, then we get a convex subcategory of A with Tits form equivalent to that of the algebra (b') (resp. (c')) above. This contradicts again the non-negativeness of q_A. We get that B_2 may also be obtained by glueing trees in the form indicated in the statement.

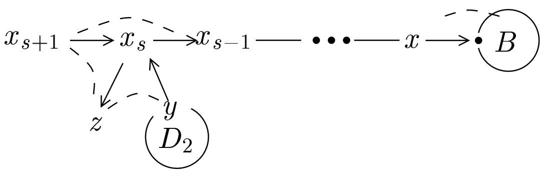
Let $D_1, \cdots, D_m$ be all the connected components of the tree obtained from A by deleting x. Assume D_1 contains B. Then consider B^x be the convex subcategory of A formed by B, x and $D_2, \cdots, D_m$. We shall prove that B^x may be obtained by glueing trees in the way indicated in the statement of the Theorem.

Clearly B_2 as defined above is contained in B^x. We may assume that B_2 is a proper subcategory of B^x. Assume first that γ has length 3 (that is, $s = t+2$). We claim that B^x has the following shape:

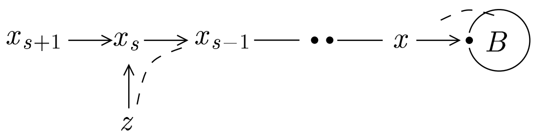


Hence by induction (since the subcategory of A with vertices $Q \setminus \{x_{s+1}\}$ satisfies the conditions (i), (ii) and (iii)) B^x can be obtained in the desired way. To show this claim, first observe that, by the choice of γ and B, the vertices x_i for $0 \leq i \leq s-1$ have exactly two neighbours in Q. Moreover observe that there is a vector $v \in K_0(B_2)$ with $q_A(v) = 0$ and $v(x_{s-2}) = 2$, $v(x_{s-1}) = 1$, $v(x_s) = 0$, $v(x_{s+1}) = -1$. Hence there are no relations between vertices in B' and vertices in $B_2 \setminus \{x_s\}$.

Assume now that γ has length 2. We claim that B^x has one of the following shapes:



(3)



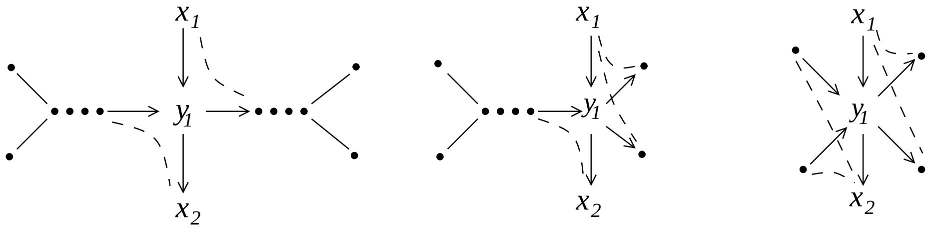
(4)

which immediately implies that B^x may be obtained in the desired way. We proceed to prove the last claim.

First observe that the maximality of B implies that there is at most one arrow y —— x_s such that y —— $x_s \to x_{s-1}$ and y —— $x_s \leftarrow x_{s+1}$ are not in I (by the assumption on γ, a path $y \to x_s \to \cdots \to x_m$ in I has length two). Assume $x_{s+1} \neq z \to x_s \to x_{s-1}$ is in I. Then there is a vector v with support in $B_2 \cup \{z\}$ satisfying $q_A(v) = 0$ and $v(x_{s-1}) = 2$, $v(x_s) = 0$, $v(x_{s+1}) = -1 = v(z)$. We easily get that B^x has the shape (4). Similarly, if $x_{s+1} \to x_s \to z \neq x_{s-1}$ is in I, there is a vector w with support in $B_2 \cup \{z\}$ satisfying $q_A(w) = 0$ and $w(x_{s-1}) = 2$, $w(x_s) = 1 = w(z)$, $w(x_{s+1}) = -1$. By conditions (i) and (ii), we get that B^x has shape (3), possibly with D_2 empty. Finally, if there is an arrow y —— x_s such that y —— x_s —— x_{s-1} and y —— $x_s \leftarrow x_{s+1}$ are not in I, then clearly, B^x has either shape (2) or (3).

To complete the proof we shall investigate the way A is obtained from the different B^x. For this purpose, let $x_1, \cdots, x_t$ be all the vertices in Q and not in B such that there are corresponding vertices $y_1, \cdots, y_t$ in B with x_i and y_i neighbours for $1 \leq i \leq t$. If the vertices $y_1, \cdots, y_t$ are pairwise different, then clearly A is a quotient of a tree obtained by glueing trees in the way indicated in the statement (there may be some zero relations between vertices a, b not in B with $a \in B_{x_1}$ and $b \in B_{x_2}$ for $x_1 \neq x_2$). Assume $y_1 = y_2$. Let C be a critical subcategory of B containing y_1. By claim $(\alpha\alpha)$, the convex subcategory of A containing C and y_i has the shape (a) or its dual, $i = 1, 2$.

By the maximality of B we get that the convex subcategory of A containing C and x_1, x_2 has the following shape (or its dual):



Moreover, the same argument shows that $y_i \neq y_1$ for all $3 \leq i \leq t$. Again, it is easy to see that A is a quotient of a tree obtained in the desired form. This completes the proof of the Theorem. □

3 Proof of the Main Theorem.

3.1

To state the result (3.4) which implies the main Theorem, we need to recall some basic concepts.

Let $d \in \mathbb{N}$ and denote by $\operatorname{alg}(d)$ the variety of k-algebras with dimension d. The group $GL_d(k)$ acts on $\operatorname{alg}(d)$ by conjugation in such a way that the orbits of the action are the iso-classes of k-algebras, see [9]. An algebra A_1 is said to be a *deformation* of A_0 if A_0 lies in the Zariski closure of the $GL_d(k)$-orbit of A_1. The main result in [9] states that deformations of tame algebras are tame.

3.2

Let Δ be a quiver admitting some loops. Let Sp be a subset of the loops in Δ. Let Z be a set of paths in $k\Delta$ not involving any loop in Sp and let J be the ideal of $k\Delta$ generated by Z and $\{\alpha(\alpha-1) \colon \alpha \in Sp\}$. Following [5], $B = k\Delta/J$ is called a *clannish algebra* provided the following conditions hold:

(a) at most two arrows start (resp. end) at any vertex;

(b) for any arrow β not in Sp, there is at most one arrow γ (resp. one arrow δ) such that $\gamma\beta$ (resp. $\beta\delta$) is not in Z.

The main result in [5] implies that a clannish algebra is always tame.

3.3

Assume that B_0 is obtained by glueing an algebra C_0 and the tree $\bullet \;\to a \to 1$ along the vertices a and $c \in C_0$ with $\sigma(a) = c$. The *reflection* D of B_0 at the vertex 1 is obtained by glueing C_0 and the tree $\bullet \;\to a_1 \leftarrow\; \bullet$ along the vertices a_1 and $c \in C_0$ with $\sigma(a_1) = c$.

Assume that $A = B_0, B_1, \ldots, B_s$ are algebras such that B_{i+1} is a reflection of B_i at some vertex. Then we say that A and B_s are *reflection-equivalent*. In this situation, it is well-known that the representation theories of A and B_s differ only by finitely many indecomposable modules. In fact, B_s is obtained from A by a splitting tilting theory, see [20]. By [15], B_s is tame if and only if so is A.

3.4

Theorem. *Let $A = kQ/I$ be a tree algebra satisfying the following conditions:*

(i) $\mathrm{g}\ell \dim A \leq 2$*;*

(ii) q_A *is non-negative;*

(iii) A *is not of polynomial growth.*

Then there are algebras B_0, B_1 and B_2 such that:

(a) *A is a quotient of a deformation of B_0;*

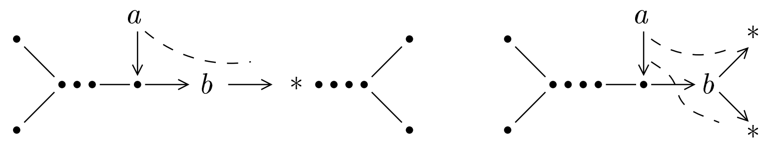
(b) *B_0 is reflection-equivalent to B_1;*

(c) mod_{B_1} *and* mod_{B_2} *are equivalent;*

(d) *B_2 is a clannish algebra.*

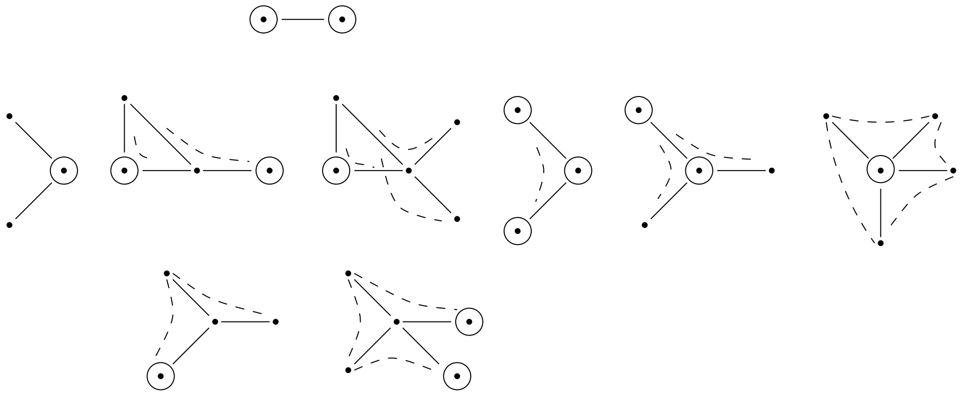
Proof: We proceed step by step defining the algebras B_0, B_1 and B_2.

(a): By (2.2), A is a quotient of an algebra B which is a tree obtained by glueing some trees in the List 2 along some of the encircled vertices. If all the relations in B have length two, then we define $B_0 = B$. Otherwise, we get a convex subcategory of B of one of the following shapes (or their duals), where all the neighbours of b in Q are indicated.



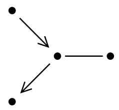
Hence there is a non-split exact sequence of B-modules $0 \to S_b \to M \to N \to 0$ with $M = \operatorname{rad} P_a$. Then B is a one-point extension $D[M]$, where D is the quotient algebra of B obtained by deleting the vertex a. Moreover, for every arrow α in Q, the matrix $M(\alpha)$ can be written as $M(\alpha) = \begin{bmatrix} S_b(\alpha) & f_\alpha \\ 0 & N(\alpha) \end{bmatrix}$ for some linear map f_α. For each $\lambda \in k$, we define the D-module M_λ such that $M_\lambda(\alpha) = \begin{bmatrix} S_b(\alpha) & \lambda f_\alpha \\ 0 & N(\alpha) \end{bmatrix}$ for arrows α in Q. Then for $\lambda \neq 0$, $B_\lambda = D[M_\lambda] \cong B$ and $B_0 = D[M_0] = D[S_b \oplus N]$, that is, B is a deformation of B_0.

Proceeding inductively along all relations of length 3 in B, we get an algebra B_0 such that B is a deformation of B_0 and B_0 is obtained by glueing algebras isomorphic to some of the following list along a subset of the encircled vertices where the edges may have any possible orientation.

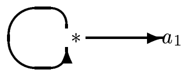


List 3

(b): Clearly, we can construct an algebra B_1 which is reflection-equivalent to B_0 and such that B_1 does not admit a convex subcategory of the form

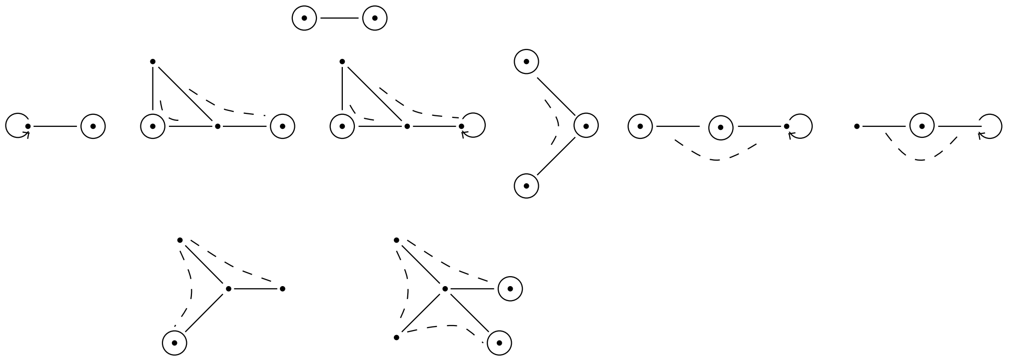


(c): Let B_1 be as above. Assume that B_1 is obtained by glueing the algebra C_0 and the tree $0 \to a \leftarrow 1$ along a and $c \in C_0$ with $\sigma(a) = c$. We form the algebra E_1 obtained by glueing the algebra C_0 and the algebra



with loop α_1 satisfying $\alpha_1(\alpha_1 - 1) = 0$, along the vertices a_1 and c with $\sigma(a_1) = c$. It is well-known that $\mathrm{mod}_B \xrightarrow{\sim} \mathrm{mod}_E$ [indeed, for $X \in \mathrm{mod}_B$, we define $Y \in \mathrm{mod}_E$ whose restriction to C_0 coincide with X and such that $Y(*) = X(0) \oplus X(1)$ and $Y(\alpha_1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$; for the inverse, let $Y \in \mathrm{mod}_{E_1}$ and define $X \in \mathrm{mod}_{B_1}$ where $Y(*) = X(0) \oplus X(1)$ is the spectral decomposition of the k-vector space $Y(*)$ corresponding to the endomorphism $Y(\alpha_1)$].

Repeating the procedure, we get an algebra B_2 such that $\mathrm{mod}_{B_1} \xrightarrow{\sim} \mathrm{mod}_{B_2}$ and B_2 is obtained by glueing algebras isomorphic to some of the following list along a subset of the encircled vertices where the edges may have any possible orientation and any loop α satisfies $\alpha(\alpha - 1) = 0$.



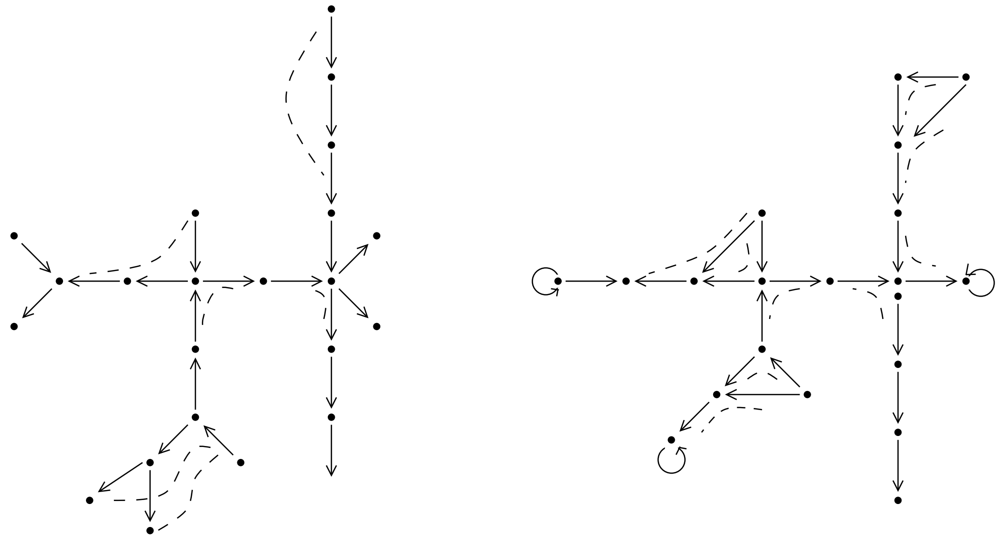
List 4

(d): An easy inspection shows that an algebra B_2 obtained as above satisfies the conditions (2.2) for being a clannish algebra. $\square$

3.5

Proof of the Main Theorem: Let $A = kQ/I$ be a tree algebra with $\mathrm{g}\ell \dim A \leq 2$ and q_A non-negative. We may assume that A is not of polynomial growth. By (3.4), A is a quotient of a deformation of an algebra B_0 which is reflection-equivalent to an algebra B_1 whose module category is equivalent to a clannish algebra B_2. By (3.2), B_2 is tame. Hence (3.3) and (3.1) imply that A is tame. $\square$

As an *example* we consider the algebra A



which satisfies the hypothesis of the Theorem. The corresponding algebra B_2 with loops α satisfying $\alpha(\alpha-1) = 0$ as constructed in (3.4) is shown above.

3.6

Remarks: (a) Probably the List 2 given in (2.2) forms a kit in the sense of Brüstle which is enough for the tameness of the algebras obtained as in the Theorem.

(b) Many of the algebras satisfying the hypothesis of Proposition (2.1) may be obtained as biextensions of algebras derived equivalent to $\tilde{\mathbb{D}}_n$ $(n > 4)$ by indecomposable modules of derived regular length 2. Hence by [7], these algebras are tame. We shall come back to this point in a forthcoming publication.

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