Existence and Uniqueness of invariant measures: an approach via sectorial forms

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Abstract

We prove existence and uniqueness for invariant measures of strongly continuous semigroups on $L^2(X;\mu)$, where X is a (possibly infinite–dimensional) space. Our approach is purely analytic based on the theory of sectorial forms. The generators covered are e.g. small perturbations (in the sense of sectorial forms) of operators generating hypercontractive semigroups. An essential ingredient of the proofs is a new result on compact embeddings of weighted Sobolev spaces $H^{1,2}(\rho \cdot dx)$ on $\mathbb{R}^d$ (resp. a Riemannian manifold) into $L^2(\rho\, dx)$. Probabilistic consequences are also briefly discussed.

AMS Subject Classification
Primary: 31 C 25
Secondary: 47 D 07, 60 H 10, 47 D 06, 60 J 60
Key words and phrases: invariant measures, sectorial forms, compact embeddings, Poincare inequality, Log–Sobolev inequality
Running head: Invariant measures for semigroups

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1 Introduction

In this paper we study the question of existence and uniqueness of invariant measures for diffusions solving stochastic differential equations of type

$$dX_t = \sigma(X_t)\, dW_t + v(X_t)\, dt\ . \tag{1.1}$$

Here $(X_t)_{t\geq 0}$ takes values in a locally convex topological vector space X, $(W_t)_{t\geq 0}$ is a Brownian motion on X, and the operator valued diffusion map σ and the drift $v : X \to X$ are supposed to be merely measurable. This question has been studied by many authors. We refer to the recent book by G. Da Prato and J. Zabczyk ([13]) for references.

Our approach in this paper is analytic and based on a reformulation of the problem in terms of sectorial forms (cf. [16] and also [19]). Thus we avoid to have to solve (1.1) first and to pin down the exact meaning (strong, mild, weak etc.) of "solution" to (1.1). More precisely, the relevant sectorial forms are of type (closure of)

$$\mathcal{E}_v(f,g) := \mathcal{E}(f,g) + \int_X \Big(v(x), \nabla_H f(x)\Big)\ g(x)\ \mu(dx), \tag{1.2}$$

on $L^2(X;\mu)$ with μ a probability measure on X and

$$\mathcal{E}(f,g) := \int_X \Big(A(x)\ \nabla_H f(x), \nabla_H g(x)\Big)_H\ \mu(dx), \tag{1.3}$$

where f, g are bounded smooth cylinder functions on X and H is a separable real Hilbert space densely and continuously embedded into X. H should be thought of as a "tangent space" to X at every point $x \in X$. ∇_H denotes the corresponding gradient. We refer to Subsection 2.2 for details and the precise assumptions on the map $x \mapsto A(x)$ taking values in the space of symmetric positive definite bounded linear operators on H. We only mention here that $\mathcal{E}_v$ is assumed to be a form perturbation of $\mathcal{E}$ (i.e., v satisfies (2.12) below). In particular, $D(\mathcal{E}) = D(\mathcal{E}_v)$. In analogy with the classical case ($X = H = \mathbb{R}^d$, $A = \mathrm{Id}_H$, $\mu =$ Lebesgue measure) $D(\mathcal{E})$ is a Sobolev space of order one in $L^2(X;\mu)$ which is, therefore, also denoted by $H^{1,2}(X;\mu)$.

Let L denote the generator of the symmetric form $(\mathcal{E}, D(\mathcal{E}))$, resp. L_v that of $(\mathcal{E}_v, D(\mathcal{E}))$, and let $T := e^{tL}$, $S_t := e^{tL_v}$, $t > 0$, denote the corresponding strongly continuous semigroups on $L^2(X;\mu)$. Then μ is a $(T_t)_{t>0}$–invariant measure, i.e.,

$$\int T_t f \; d\mu = \int f \; d\mu \qquad \text{for all } t > 0 \text{ and all } f : X \to \mathbb{R}, \text{ measurable, bounded,} \tag{1.4}$$

(because for every $t > 0$, T_t is symmetric on $L^2(X;\mu)$ and $T_t 1 = 1$). The invariant measures in question are those of $(S_t)_{t>0}$. Indeed, by the results in [19] it easily follows that there exists a diffusion process associated with $(\mathcal{E}_v, D(\mathcal{E}))$ (i.e., with transition probabilities given by $(S_t)_{t>0}$). In case $A = \mathrm{Id}_H$ it solves (1.1) with $\sigma = \mathrm{Id}_H$ and for differentiable A it solves (1.1) with $\sigma = A^{1/2}$, but with a different v (e.g., if $X = H = \mathbb{R}^d$ and $A = (a_{ij})_{1\le i,j\le d}$, with v replaced by $\left(-\sum_{i=1}^d \frac{\partial a_{i,j}}{\partial x_i}\right)_{1\le j\le d} + v$). For general A (satisfying (2.9) below) the process exists (as mentioned), but can only be described in the sense of martingale problems (cf. [30], [26], and [19]). In this paper we shall, however, not make use of this probabilistic counterpart of the above sectorial forms, but instead just consider $(S_t)_{t>0}$ directly, and hence study the question of existence and uniqueness of probability measures ν on X which are (S_t)–invariant in a sense analogous to (1.4).

Let us now summerize our main results:
Concerning uniqueness we have the following general result (cf. Corollary 2.13 below).

(1) Suppose $(\mathcal{E}, D(\mathcal{E}))$ is irreducible (i.e., $f \in D(\mathcal{E})$ is constant, provided $\mathcal{E}(f,f) = 0$) and suppose there exists an (S_t)–invariant probability measure ν on X which is absolutely continuous w.r.t. μ with Radon–Nikodym density in $L^2(X;\mu)$. Then ν is the only (S_t)–invariant probability measure on X absolutely continuous w.r.t. μ.

Criteria to check irreducibility are well–known. For example, the Poincaré inequality (and hence the Log–Sobolev inequality (2.4)) implies irreducibility. For stronger more recent criteria, we refer to [4, 5], which apply to large classes of (non–Gaussian) Gibbs measures μ.

We want to emphasize that our proof of (1) is entirely analytic (and quite elementary) which is in contrast to all previous work where the corresponding proofs were based on the Girsanov transformation in an essential way (cf. e.g.

[27], [31], [32] and the very recent [15]). More precisely, there the uniqueness was obtained as a consequence, after first showing that (in the rather special case considered there) the diffusion process associated with $(\mathcal{E}_v, D(\mathcal{E}_v))$ has path space measures which are (locally in time) equivalent to those of the diffusion associated with $(\mathcal{E}, D(\mathcal{E}))$. Because of our Proposition 2.11 we can avoid all this.

As far as the existence of (S_t)–invariant measures is concerned, (as in [27]) the underlying idea here is to use classical Fredholm theory. Since, in addition, by the general results in [20] we know that $(S_t)_{t>0}$ is Markovian, we obtain the following result (cf. Theorem 3.4)

(2) Suppose the embedding $H^{1,2}(X;\mu) \subset L^2(X;\mu)$ is compact. Then there exists an (S_t)–invariant probability measure $\nu = \rho \cdot \mu$ with $\rho \in D(\mathcal{E})$ ($\subset L^2(X;\mu)$). In particular, the uniqueness result (1) applies if $(\mathcal{E}, D(\mathcal{E}))$ is irreducible.

If X is a Hilbert space, $X = H$ and μ is a Gaussian measure, the compactness of $H^{1,2}(X;\mu) \subset L^2(X;\mu)$ has been shown in [12] and [23], but is known not to hold in "most" of the examples when $\dim X = \infty$. Therefore, in the infinite dimensional case the "compact embedding"–assumption is very restrictive. This is different if $\dim X < \infty$, since we have the following new result (cf. Theorem 3.2), which is also essential for proving our main result on existence (cf. (4) below).

(3) Suppose $X = H = \mathbb{R}^d$ and that μ has a density w.r.t. Lebesgue–measure satisfying the Hamza condition (cf. (3.1) below). Assume, furthermore, that $(\mathcal{E}, D(\mathcal{E}))$ satisfies the so–called "defective" Log–Sobolev inequality (2.4)'. Then the embedding $H^{1,2}(\mathbb{R}^d;\mu) \subset L^2(\mathbb{R}^d;\mu)$ is compact.

Using this we prove our main result on existence of invariant measures in infinite dimensions avoiding the "compact embedding"–assumption (cf. Theorem 3.6).

(4) Suppose $(\mathcal{E}, D(\mathcal{E}))$ satisfies the defective Log–Sobolev inequality (2.4)' and that the projections of μ on finite dimensional subspaces have densities w.r.t. Lebesgue measure, which satisfy the Hamza condition (3.1) below. Then there exists an (S_t)–invariant probability measure $\nu = \rho \cdot \mu$ with $\rho \in D(\mathcal{E})$ ($\subset L^2(X;\mu)$). In particular, the uniqueness result (1) applies if $(\mathcal{E}, D(\mathcal{E}))$ is irreducible.

There are many well–known examples of measures μ so that $(\mathcal{E}, D(\mathcal{E}))$ with $A \equiv \mathrm{Id}_H$ satisfies the Log–Sobolev inequality (2.4), hence (2.4)' (cf. [9], [14], and, in particular, for (non–Gaussian) Gibbs measures μ see [3]). But then, if $x \mapsto A(x)$ is strictly elliptic (i.e., satifies (2.9)), the corresponding $(\mathcal{E}, D(\mathcal{E}))$ also does so. Since the condition on the finite dimensional projections is trivially checked in all these cases, result (4) applies to a large class of examples.

The above described results of this paper completely cover and strictly extend the corresponding ones in the previous work [27], [31], [32] and [15], however, w.r.t. the latter reference only the case $p = 2$ (since our approach is solely within an L^2– (and not L^p–) frame work). In these references only the special situation where μ is mean–zero Gaussian measure (more precisely where (X, H, μ) forms an abstract Wiener space) is studied and only in [15] the case of unbounded (namely exponentially integrable) drifts v is included.

We emphasize that our results appear to be new even in the finite dimensional case $X = \mathbb{R}^d$. We discuss generalizations to Riemannian manifolds as well.

One further result is worth mentioning: as a by–product of our considerations we obtain a simple example of a Dirichlet form on $\mathbb{R}^d$, $d \geq 1$, which satisfies the Poincaré inequality (2.3), but not the defective Log–Sobolev inequality (2.4)' (cf. Remark 3.5 (ii) and Example 2.6 (ii)).

The organization of this paper is as follows: In Section 2 we describe our precise framework and recall all necessary definitions. In addition, we include two new general results on sectorial forms which are also of their own interest. (One of them is a generalization of the classical *Duhamel formula* to sectorial forms). The last part of Section 2 is devoted to the proof of the uniqueness result (1) for invariant measures.

The first part of Section 3 is about the result (3) on compact embeddings in the case $X = \mathbb{R}^d$ (or a Riemmannian manifold). The second part contains the two main results (2) and (4) on existence of invariant measures.

2 Preliminaries

In this section we present our framework and some general facts to be used later.

2.1 Two general results on sectorial forms

Let $(X, \mathcal{B}, \mu)$ be a probability space. Let $(\mathcal{E}, D(\mathcal{E}))$ be a *coercive closed* form on (real) $L^2(X, \mathcal{B}, \mu) =: L^2(\mu)$ in the sense of [19], i.e., $D(\mathcal{E})$ is a dense linear subspace of $L^2(\mu)$ and $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is bilinear, nonnegative definite such that its symmetic part $(\widetilde{\mathcal{E}}, D(\mathcal{E}))$ is *closed* (i.e., $D(\mathcal{E})$ is complete w.r.t. the norm $\| \ \|_{\mathcal{E}_1}$ associated with $\widetilde{\mathcal{E}}_1$, where for $\alpha \in \mathbb{R}_+$, $\widetilde{\mathcal{E}}_\alpha$ is the symmetric part of $\mathcal{E}_\alpha(f, g) := \mathcal{E}(f, g) + \alpha(f, g)_{L^2(\mu)}$ for $f, g \in D(\mathcal{E})$) and $(\mathcal{E}, D(\mathcal{E}))$ satisfies the *weak sector condition*, i.e., there exists $K \in \mathbb{R}_+$ such that

$$|\mathcal{E}(f, g)| \leq K \ \mathcal{E}_1(f, f)^{1/2} \ \mathcal{E}_1(g, g)^{1/2} \text{ for all } f, g \in D(\mathcal{E}). \tag{2.1}$$

Let $(L, D(L))$ denote the generator of $(\mathcal{E}, D(\mathcal{E}))$ (see [19]) and $(L^*, D(L^*))$ its adjoint on $L^2(\mu)$. We have, in particular, that $(L, D(L))$ is negative definite, that

$$D(L) \subset D(\mathcal{E}) \text{ and } \mathcal{E}(f, g) = (-Lf, g)_{L^2(\mu)} \text{ for all } f \in D(L), \ g \in D(\mathcal{E}). \tag{2.2}$$

Furthermore, $(L, D(L))$ generates a strongly continuous semigroup of contractions on $L^2(\mu)$ denoted by $T_t := e^{tL}$, $t > 0$.

A bilinear form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mu)$ is called a *sectorial* (or *m–sectorial*, cf. [16]) form if there exists $\alpha \in [0, \infty)$ such that $\mathcal{E}_\alpha$ with $D(\mathcal{E}_\alpha) := D(\mathcal{E})$ is a coercive closed form on $L^2(\mu)$. If $(L_\alpha, D(L_\alpha))$ is the generator of $(\mathcal{E}_\alpha, D(\mathcal{E}))$ with corresponding semigroup $(T_t^{(\alpha)})_{t>0}$, then

$$\begin{aligned} L &:= \alpha + L_\alpha \ , \quad D(L) := D(L_\alpha), \\ T_t &:= e^{\alpha t} T_t^{(\alpha)} \ , \quad t > 0, \end{aligned}$$

are called the generator resp. semigroup on $L^2(\mu)$ associated with $(\mathcal{E}, D(\mathcal{E}))$ and (2.2) still holds. In particular, the above definitions do not change when $\alpha' > \alpha$ replaces α. Clearly, a sectorial form is a coercive closed form if and only if it is nonnegative definite.

Below we fix a sectorial form $(\mathcal{E}, D(\mathcal{E}))$ on $L^2(\mu)$.

Remark 2.1 Let $f \in L^2(\mu)$. Then the following assertions are obviously equivalent:
(i) $f \in D(L^*)$ and $L^* f = 0$.
(ii) $f \in D(\mathcal{E})$ and $\mathcal{E}(g, f) = 0$ for all $g \in D(\mathcal{E})$.

In analogy to the classical Duhamel formula for semigroups and their generators, we have the following

Proposition 2.2 *(Duhamel formula for sectorial forms).*
Let $(\mathcal{E}^{(i)}, D(\mathcal{E}^{(i)}))$ be sectorial forms on $L^2(\mu)$ with corresponding semigroups $(T_t^{(i)})_{t>0}$, $i = 1, 2$. Suppose that $D(\mathcal{E}^{(1)}) = D(\mathcal{E}^{(2)})$ and that for some $\alpha, c \in (0, \infty)$

$$\mathcal{E}_\alpha^{(1)}(u,u) \le c\ \mathcal{E}_\alpha^{(2)}(u,u) \text{ for all } u \in D(\mathcal{E}^{(1)}).$$

Then for all $t > 0$ and all $f, g \in D(\mathcal{E}^{(1)})$

$$\begin{aligned}\int \left(T_t^{(1)} f - T_t^{(2)} f \right) g\ d\nu \ &= \ \int_0^t \Big[\mathcal{E}^{(2)}(T_s^{(1)} f, T_{t-s}^{(2)*} g) \\ &\qquad\qquad -\mathcal{E}^{(1)}(T_s^{(1)} f, T_{t-s}^{(2)*} g) \Big]\ ds,\end{aligned}$$

where $T_t^{(2)*}$ denotes the adjoint of $T_t^{(2)}$ on $L^2(\mu)$, $t > 0$.

Proof. Obviously, we may assume that both $(\mathcal{E}^{(1)}, D(\mathcal{E}^{(1)}))$, and $(\mathcal{E}^{(2)}, D(\mathcal{E}^{(2)}))$ are nonnegative definite, i.e., are coercive closed forms and that $\alpha = 1$. For $t > s > 0$ and $f, g \in D(\mathcal{E}^{(1)}) = D(\mathcal{E}^{(2)})$ we have

$$\begin{aligned}&\frac{d}{ds} \int T_s^{(1)} f\ T_{t-s}^{(2)*} g\ d\mu \\ = \ &\lim_{\varepsilon \to 0} \Big[\frac{1}{\varepsilon} \int (T_{s+\varepsilon}^{(1)} f - T_s^{(1)} f)\ T_{t-(s+\varepsilon)}^{(2)*} g\ d\mu \\ + \ &\frac{1}{\varepsilon} \int T_s^{(1)} f\ (T_{t-(s+\varepsilon)}^{(2)*} g - T_{t-s}^{(2)*} g)\ d\mu \Big] .\end{aligned}$$

By [2] we conclude that the linear functionals

$$u \mapsto \frac{1}{\varepsilon} (T_{s+\varepsilon}^{(1)} f - T_s^{(1)} f, u)_{L^2(\mu)},$$

and

$$u \mapsto \frac{1}{\varepsilon} (u, T_{t-(s+\varepsilon)}^{(2)*} g - T_{t-s}^{(2)*} g)_{L^2(\mu)},$$

which are continuous on $D(\mathcal{E}^{(1)})$ w.r.t. the norm $(\mathcal{E}_1^{(1)})^{1/2}$ resp. $(\mathcal{E}_1^{(2)})^{1/2}$, converge pointwise as $\varepsilon \to 0$ to the linear functionals

$$u \mapsto -\mathcal{E}^{(1)}(T_s^{(1)} f, u),$$

respectively

$$u \mapsto \mathcal{E}^{(2)}(u, T^{(2)*}_{t-s} g),$$

$u \in D(\mathcal{E}^{(1)})$. Since by [2]

$$T^{(2)*}_{t-(s+\varepsilon)} g \underset{\varepsilon\to 0}{\longrightarrow} T^{(2)*}_{t-s} g$$

in $D(\mathcal{E}^{(2)})$ w.r.t. $(\mathcal{E}^{(2)}_1)^{1/2}$, hence w.r.t. $(\mathcal{E}^{(1)}_1)^{1/2}$ by assumption, we conclude that

$$\begin{aligned} \frac{d}{ds} \int T^{(1)}_s f \; T^{(2)*}_{t-s} g \; d\mu \quad &= \quad \mathcal{E}^{(2)}(T^{(1)}_s f, T^{(2)*}_{t-s} g) \\ &\quad -\mathcal{E}^{(1)}(T^{(1)}_s f, T^{(2)*}_{t-s} g). \end{aligned}$$

Integrating from 0 to t w.r.t. ds yields the assertion. □

Remark 2.3 In the situation described in the following subsection the right hand side of the Duhamel formula in Proposition 2.2 can be calculated explicitly, if g is bounded. As a consequence one can prove that each $T^{(1)}_t$ is then continuous on $L^2(\nu)$ for $\nu := \rho \cdot \mu$, where $\rho \in L^2(\mu)$ such that $\rho = T^{(2)*}_t \rho$ ($= S_t \rho$ in Subsection 2.2) for all $t > 0$, provided the vector field v (cf. Subsection 2.2) is bounded.

We recall that $(\mathcal{E}, D(\mathcal{E}))$ is said to satisfy the *Poincaré inequality* resp. the *Log–Sobolev inequality* (cf. [9], [14]) if there exists $\lambda \in (0, \infty)$ such that for all $f \in D(\mathcal{E})$

$$\|f - \int f \; d\mu\|^2_{L^2(\mu)} \leq \lambda \; \mathcal{E}(f,f) \tag{2.3}$$

resp.

$$\int f^2 \; \log(|f|/\|f\|_{L^2(\mu)}) \; d\mu \leq \lambda \mathcal{E}(f,f) \; . \tag{2.4}$$

We shall also use a weaker form of (2.4), namely that for some $\lambda \in (0, \infty)$ and all $f \in D(\mathcal{E})$

$$\int f^2 \log(|f|/\|f\|_{L^2(\mu)}) \; d\mu \leq \lambda \mathcal{E}_1(f,f). \tag*{(2.4)'}$$

(2.4)' is called the *defective Log–Sobolev inequality.*

Clearly, (2.3), (2.4), and (2.4)' are properties of the symmetric part of $(\mathcal{E}, D(\mathcal{E}))$ It is well–known that (2.4) implies (2.3) (cf. [14] and the references therein). We note that (2.4) implies that $(\mathcal{E}, D(\mathcal{E}))$ is nonnegative definite, hence that it is a coercive closed form.

As we shall see, the classical theory of Fredholm operators will give us the following result (to be applied later):

Proposition 2.4 *Assume that $(\mathcal{E}, D(\mathcal{E}))$ satisfies the Poincaré inequality (2.3) for some $\lambda > 0$ and that $1 \in D(L)$ with $L1 = 0$. Let $\widetilde{L} : D(L) \to L^2(\mu)$ be a linear operator which is continuous provided $D(L)$ is equipped with the graph norm with $\widetilde{L}1 = 0$. If there exist constants $\kappa \in (0, \lambda^{-\frac{1}{2}})$ and $K > 0$ such that*

$$(Lf - \widetilde{L}f, f)_{L^2(\mu)} \geq -\kappa\sqrt{\mathcal{E}(f,f)}\ \|f\|_{L^2(\mu)} \quad \forall f \in D(L),$$

and

$$|(Lf - \widetilde{L}f, g)_{L^2(\mu)}| \leq K\|f\|_{\mathcal{E}_1}\ \|g\|_{\mathcal{E}_1}\ \forall f, g \in D(\mathcal{E}),$$

then dim Ker $\widetilde{L}^* = 1$ (where $\widetilde{L}^*$ denotes the adjoint of $\widetilde{L}$ on $L^2(\mu)$).

Proof. Let $\varepsilon > 0$ be such that $1 - \kappa\sqrt{\lambda + \varepsilon} =: \delta > 0$. Set $M := 1 + \varepsilon^{-1}\kappa\sqrt{\lambda + \varepsilon}$ and consider the operator A on $D(L)$ defined by $Af := \widetilde{L}f - M(f, 1)_{L^2(\mu)}$. Let $g \in L^2(\mu)$ be fixed. We shall prove that there is an element $f \in D(L)$ such that $Af = g$. To this end, let us consider the bilinear form

$$Q(\varphi, \psi) = \mathcal{E}(\varphi, \psi) + ((L - \widetilde{L})\varphi, \psi)_{L^2(\mu)} + M(\varphi, 1)_{L^2(\mu)}(\psi, 1)_{L^2(\mu)}$$

on $D(L) \times D(L)$. Since by assumption $\mathcal{E}(\psi, 1) = 0$ for all $\psi \in D(\mathcal{E})$, we have by (2.3) that

$$\|\varphi\|^2_{L^2(\mu)} \leq \lambda\mathcal{E}(\varphi, \varphi) + (\varphi, 1)^2_{L^2(\mu)}. \tag{2.5}$$

Hence one gets for all $\varphi \in D(L)$

$$Q(\varphi, \varphi) \geq \mathcal{E}(\varphi, \varphi) - \kappa\sqrt{\mathcal{E}(\varphi, \varphi)}\sqrt{\lambda\mathcal{E}(\varphi, \varphi) + (\varphi, 1)^2_{L^2(\mu)}} + M(\varphi, 1)^2_{L^2(\mu)}.$$

If $(\varphi, 1)^2_{L^2(\mu)} \leq \varepsilon\mathcal{E}(\varphi, \varphi)$, then

$$Q(\varphi, \varphi) \geq \delta\mathcal{E}(\varphi, \varphi) + M(\varphi, 1)^2_{L^2(\mu)}.$$

If $(\varphi,1)^2_{L^2(\mu)} \geq \varepsilon\mathcal{E}(\varphi,\varphi)$, then by the definition of M

$$Q(\varphi,\varphi) \geq \mathcal{E}(\varphi,\varphi) + (\varphi,1)^2_{L^2(\mu)}.$$

Therefore, by virtue of (2.5) there exists $c > 0$ such that

$$Q(\varphi,\varphi) \geq c\|\varphi\|^2_{\mathcal{E}_1} \qquad \forall\varphi \in D(L). \tag{2.6}$$

On the other hand, by assumption there exists $K_1 > 0$ such that

$$|Q(\varphi,\psi)| \leq K_1\|\varphi\|_{\mathcal{E}_1}\|\psi\|_{\mathcal{E}_1} \qquad \forall\varphi,\psi \in D(L).$$

In particular, since $D(L)$ is dense in $D(\mathcal{E})$ with respect to the norm $\|\cdot\|_{\mathcal{E}_1}$, Q extends uniquely to a continuous bilinear form on $D(\mathcal{E})$ denoted again by Q. Clearly, this extension satisfies (2.6).

Since the functional $\psi \mapsto -(g,\psi)_2$ is continuous on $D(\mathcal{E})$, by virtue of the Lax–Milgram theorem there is an element $f \in D(\mathcal{E})$ such that

$$Q(f,\psi) = -(g,\psi)_{L^2(\mu)} \quad \forall\psi \in D(\mathcal{E}).$$

Hence, the functional

$$\psi \mapsto \mathcal{E}(f,\psi) = -((L-\widetilde{L})f,\psi)_{L^2(\mu)} - M(f,1)_{L^2(\mu)}(\psi,1)_{L^2(\mu)} - (g,\psi)_{L^2(\mu)}$$

is continuous on $D(\mathcal{E})$ w.r.t. $L^2(\mu)$-norm. This means that f is in the domain of L, hence

$$-(g,\psi)_{L^2(\mu)} = Q(f,\psi) = (-Af,\psi)_{L^2(\mu)} \text{ for all } \psi \in D(\mathcal{E}),$$

i.e., $Af = g$. Thus, A is surjective on $D(L)$. In addition, A is injective. Indeed, if $Af = 0$ for some $f \in D(L)$, then

$$Q(f,f) = -(Af,f)_{L^2(\mu)} = 0,$$

whence by virtue of (2.6) we get $f = 0$. Since the operator $f \mapsto M(f,1)_2$ is compact on $D(L)$, the operator $\widetilde{L}$ is Fredholm with index zero. This means that $\mathrm{codim}\widetilde{L}(D(L)) = \dim \mathrm{Ker}\widetilde{L} \geq 1$, since $\widetilde{L}1 = 0$. The latter also implies that for $\varphi \in \mathrm{Ker}\widetilde{L}$

$$0 = \widetilde{L}(\varphi - (\varphi,1)_{L^2(\mu)} = A(\varphi - (\varphi,1)_{L^2(\mu)}).$$

Hence $\varphi = (\varphi,1)_{L^2(\mu)}$, i.e., $\dim \mathrm{Ker}\widetilde{L} = 1$. Since $\dim \mathrm{Ker}\widetilde{L}^* = \widetilde{L}(D(L))^\perp$, the assertion follows. $\square$

Remark 2.5 (i) Proposition 2.4 implies the existence (and uniqueness) of invariant measures "in case of small perturbations" (cf. Remark 3.5 (iii) below).
(ii) As Example 2.6 (iii) below shows, in Proposition 2.4 the restriction on κ cannot be dropped in general.

2.2 The framework

Let X be a locally convex topological vector space (over $\mathbb{R}$) which for simplicity is assumed to be Souslinean (i.e., the continuous image of a Polish space) with dual X^* and Borel σ–algebra $\mathcal{B} := \mathcal{B}(X)$. Let H be a separable Hilbert space which is continuously embedded into X. Define

$$\mathcal{F}C_b^\infty := \{\varphi(l_1, \dots, l_m) \mid m \in \mathbb{N}, l_i \in X^*, \varphi \in C_b^\infty(\mathbb{R}^m)\},$$

i.e., the space of bounded smooth cylinder functions on X. Here $C_b^\infty(\mathbb{R}^m)$ denotes the set of all infinitely differentiable real–valued functions on $\mathbb{R}^m$ with partial derivatives of all order bounded. For $f \in \mathcal{F}C_b^\infty$, $x \in X$, define $\nabla_H f(x)$ to be the (unique) element in H such that for all $h \in H$

$$(\nabla_H f(x), h)_H = \partial_h f(x) := \frac{d}{ds} f(x + sh)_{|s=0}. \tag{2.7}$$

Let μ be a probability measure on $(X, \mathcal{B})$ such that $f = g$ μ–a.e. implies $\nabla_H f = \nabla_H g$ μ–a.e. for all $f, g \in \mathcal{F}C_b^\infty$ and such that

$$\mathcal{E}_\mu(f, g) = \int_X (\nabla_H f, \nabla_H g)_H \, d\mu \ ; \ f, g \in \widetilde{\mathcal{F}C_b^\infty}^\mu \tag{2.8}$$

is closable on $L^2(\mu)$. (Here $\widetilde{\mathcal{F}C_b^\infty}^\mu$ denotes the set of μ–classes determined by $\mathcal{F}C_b^\infty$). Then for any strongly measurable map $A : X \to \mathcal{L}_s(H)$ (:= set of all symmetric nonnegative definite bounded linear operators on H) such that

$$C^{-1} I_H \leq A \ \mu\text{–a.e. for some } C > 0 \text{ and } \int_X \|A(x)\|_{\mathcal{L}(H)} \mu(dx) < \infty, \tag{2.9}$$

(where $\|A(x)\|_{\mathcal{L}(H)} := \sup_{|h|_H = 1} |A(x)h|_H$), also the form

$$\mathcal{E}_{\mu,A}(f, g) := \int_X (A(x)\nabla_H f(x), \nabla_H g(x))_H \ \mu(dx); \ f, g \in \widetilde{\mathcal{F}C_b^\infty}^\mu \tag{2.10}$$

is closable on $L^2(\mu)$ (see [8, Theorem 3.1] and also [7] for sufficient conditions for this to hold). Let $(\mathcal{E}_{\mu,A}, D(\mathcal{E}_{\mu,A}))$ denote the corresponding closure. Below μ, A are fixed, so we set

$$(\mathcal{E}, D(\mathcal{E})) := (\mathcal{E}_{\mu,A}, D(\mathcal{E}_{\mu,A})). \tag{2.11}$$

∇_H extends (via $L^2(\mu)$–limits) to every $f \in D(\mathcal{E})$. This extension is also denoted by ∇_H or ∇_H^μ if we want to express its dependence on μ. Clearly, $(\mathcal{E}, D(\mathcal{E}))$ is a coercive closed form which is symmetric. Let $(L, D(L))$ denote the corresponding (self–adjoint) generator.

Let $v : X \to H$ be $\mathcal{B}(X)/\mathcal{B}(H)$–measurable with $|v|_H \in L^2(\mu)$ such that there exist $\varepsilon \in (0,1)$, $a, b \in \mathbb{R}_+$ such that for all $f, g \in \mathcal{F}C_b^\infty$

$$\left| \int (v, \nabla_H f)_H \; g \; d\mu \right| \leq b \; \mathcal{E}_1(f,f)^{1/2} \mathcal{E}_1(g,g)^{1/2} \tag{2.12a}$$

and

$$\int (v, \nabla_H f)_H \; f \; d\mu \geq -\varepsilon \; \mathcal{E}(f,f) - a \; (f,f)_{L^2(\mu)} \; . \tag{2.12b}$$

Then

$$\mathcal{E}_v(f,g) := \mathcal{E}(f,g) + \int (v, \nabla_H f)_H \; g \; d\mu; \; f, g \in \widetilde{\mathcal{F}C_b^\infty}$$

is closable on $L^2(\mu)$. Let $(\mathcal{E}_v, D(\mathcal{E}_v))$ denote its closure. Clearly, for some $\alpha > 0$ the norms $\| \; \|_{\mathcal{E}_1}$ and $\| \; \|_{\mathcal{E}_{v,\alpha}}$ are equivalent. Hence $D(\mathcal{E}_v) = D(\mathcal{E})$ and it is easy to check that

$$\begin{aligned} \mathcal{E}_v(f,g) &= \mathcal{E}(f,g) + \int_X (v, \nabla_H f)_H \; g \; d\mu, \\ & \forall \; f, g \in D(\mathcal{E}_v) = D(\mathcal{E}), \; g \text{ bounded.} \end{aligned} \tag{2.13}$$

$(\mathcal{E}_v, D(\mathcal{E}))$ is a sectorial form on $L^2(\mu)$. In view of (2.11) we also write $\mathcal{E}_{\mu,A,v}$ instead of $\mathcal{E}_v$ if we want to express the dependence on A, μ. Let $(L_v, D(L_v))$ and $S_t := e^{tL_v}$, $t > 0$, denote the corresponding generator resp. strongly continuous semigroup.

Example 2.6 (i) Typical examples for v above are bounded vector fields from X to H or such with sufficiently good integrability properties. If $(\mathcal{E}, D(\mathcal{E}))$ satisfies the classical Sobolev inequality

$$\|f\|_{\frac{2d}{d-2}} \leq \text{const.} \; \mathcal{E}_1(f,f)^{1/2}$$

it is enough to have $|v| \in L^p(\mu)$ with $p \geq d$ (cf. e.g. [24, Sect. 1]). If $\mathcal{E}$ satisfies the Log–Sobolev inequality (2.4) we need the exponential integrability of $\alpha|v|_H^2$ for sufficiently large (depending on the Log–Sobolev constant λ) $\alpha > 0$ (see [29, Sect. 2.4]).

(ii) Let $X = H = \mathbb{R}^d$ and $\mu := \rho\, dx$ where dx denotes Lebesgue measure on $\mathbb{R}^d$ and $\rho(x) := C\exp(-|x|)$, $x \in \mathbb{R}^d$. Then $(\mathcal{E}, D(\mathcal{E}))$ as defined above (i.e., $A = I_{\mathbb{R}^d}$) is known to satisfy the Poincaré inequality (2.3). Take $v(x) := -x/|x|$, $x \in \mathbb{R}^d$, and $\widetilde{L} := L_v$. Then as is easily checked all assumptions in Proposition 2.4 except for the "smallness condition" on κ are fulfilled. But on the other hand $\widetilde{L} = \Delta$. Hence the restriction on κ cannot be dropped, because otherwise Proposition 2.4 would imply that there exists $f \in L^2(\mu)$, $f \not\equiv 0$, such that

$$\int f\ \Delta\varphi\ \rho\ dx \quad \text{for all } \varphi \in C_b^\infty(\mathbb{R}^d).$$

Hence $f \cdot \rho$ is (weakly) harmonic. Since $f \cdot \rho \in L^1(dx)$, it follows that $f\rho \equiv 0$, contradicting the fact that $f \not\equiv 0$.

2.3 Uniqueness of invariant measures

We recall that e.g. by [20, Chap. 2] both semigroups are *positivity preserving* (i.e., $T_t f$, $S_t f \geq 0$ for all $t > 0$, provided $f \in L^2(\mu)$, $f \geq 0$). But both are also *Markovian*, since in addition $T_t 1 = S_t 1 = 1$ for all $t > 0$, because $1 \in D(L) \cap D(L_v)$ and $L1 = L_v 1 = 0$. To formulate and prove our uniqueness result we need some preparations:

Definition 2.7 *Let $\rho \in L^1(\mu)$. The signed measure $\nu := \rho \cdot \mu$ is called (S_t)–*invariant*, if*

$$\int S_t f\ d\nu = \int f\ d\nu \text{ for } f \in L^\infty(\mu).$$

Let $(L_v^*, D(L_v^*))$, S_t^*, $t > 0$, be the adjoint operators of $(L_v, D(L_v))$, S_t, $t > 0$, respectively. We have that $D(L_v^*) \subset D(\mathcal{E}_v)$ $(= D(\mathcal{E}))$. (This follows e.g. from [19, Chap.I, Sect. 2] since $(\mathcal{E}_v + \alpha(\ ,\ )_{L^2(\mu)}, D(\mathcal{E}))$ is a coercive closed form for large enough $\alpha > 0$.) Hence the following is entirely straightforward (see [27] and cf. Remark 2.1).

Lemma 2.8 *Let $\rho \in L^2(\mu)$. Then the following assertions are equivalent:*

(i) ρ *is* (S_t)*–invariant.*

(ii) $S_t^*\rho = \rho$ *for all* $t > 0$*.*

(iii) $\rho \in D(\mathcal{E})$ *and* $\mathcal{E}_v(f, \rho) = 0$ *for all* $f \in D(\mathcal{E})$*.*

(iv) $\rho \in D(L_v^*)$ *and* $L_v^*\rho = 0$*.*

Remark 2.9 Obviously, Lemma 2.8 (iii) is equivalent to

$$\mathcal{E}_v(f, \rho) = 0 \text{ for all } f \in \mathcal{F}C_b^\infty \ .$$

If $\mathcal{F}C_b^\infty \subset D(L_v)$, and if $\nu := \rho \cdot \mu$ this can be written equivalently

$$\int L_v f \ d\nu = 0 \text{ for all } f \in \mathcal{F}C_b^\infty$$

This means that ν satisfies the equation

$$L_v^* \nu = 0$$

as defined in [1]. In particular, all existence results of (S_t)–invariant measures yield also existence of solutions to such equations in [1]. However, no result in this paper implies that ν is the only probability measure satisfying $L_v^*\nu = 0$. This question is addressed and answered in many cases in [1]. In particular, in these cases uniqueness among *all* probability measures (not necessarily absolutely continuous w.r.t. μ) is proved.

Lemma 2.10 *Let* $\rho \in L^2(\mu)$ *such that* $\nu := \rho \cdot \mu$ *is* (S_t)*–invariant and let* $\rho_n := (\rho \wedge n) \vee (-n)$*. Then for all* $f \in D(\mathcal{E})$

$$\mathcal{E}(f, \rho) = -\lim_{n \to \infty} \int (v, \nabla_H f)_H \ \rho_n \ d\mu.$$

In particular,

$$\mathcal{E}(\rho, \rho) \leq \frac{a}{1 - \varepsilon} \|\rho\|_{L^2(\mu)}^2 \ , \tag{2.14}$$

where a, ε *are as in (2.12b).*

Proof. Note that since $\rho \in D(\mathcal{E})$ by Lemma 2.8, it follows from [19, Chap. I, Proposition 4.17 (i)] and (2.12), (2.13) that for all $f \in D(\mathcal{E})$

$$\begin{aligned}\mathcal{E}(f,\rho) &= \lim_{n\to\infty}\mathcal{E}(f,\rho_n) = \lim_{n\to\infty}(\mathcal{E}_v(f,\rho_n) - \int (v,\nabla_H f)_H\ \rho_n\ d\mu)\\ &= \mathcal{E}_v(f,\rho) - \lim_{n\to\infty}\int (v,\nabla_H f)_H\ \rho_n\ d\mu\\ &= -\lim_{n\to\infty}\int (v,\nabla f)_H\ \rho_n\ d\mu.\end{aligned}$$

Taking $f=\rho$ we obtain by (2.12b) that

$$\begin{aligned}\mathcal{E}(\rho,\rho) &\le \varepsilon\ \lim_{n\to\infty}\mathcal{E}(\rho_n,\rho_n) + a\ \lim_{n\to\infty}\|\rho_n\|^2_{L^2(\mu)}\\ &= \varepsilon\ \mathcal{E}(\rho,\rho) + a\ \|\rho\|^2_{L^2(\mu)},\end{aligned}$$

and the last part of the assertion follows also. $\square$

The following result is the heart of our proof of uniqueness of invariant measures for $(S_t)_{t>0}$.

Proposition 2.11 *Let $\rho \in L^2(\mu)$ be such that $\nu := \rho\cdot\mu$ is an (S_t)–invariant probability measure.*

(i) If $N \in \mathcal{B}(X)$ such that $\{\rho = 0\} \subset N$, and $\nu(N) = 0$. Then $T_t 1_N = 1_N$ μ–a.e. for all $t > 0$.

(ii) If $f \in L^2(\mu)$ with $\int f\ d\nu = 0$, then $T_t f = 0$ ν–a.e. for all $t > 0$.

Proof. (i): Let $Z := X \setminus N$. Since $S_t 1_N = 0$ ν–a.e. for all $t > 0$, it follows that $S_t 1_Z = S_t 1 - S_t 1_N = 1 = 1_Z$ ν–a.e. Since $(\mathcal{E}_v + \alpha(\ ,\ )_{L^2(\mu)}, D(\mathcal{E}))$ for $\alpha > 0$ sufficiently large is a coercive closed form, we conclude by [2] (cf. the proof of Proposition 2.2 above) that $1_Z \in D(\mathcal{E})$. (Indeed, since $\{\rho = 0\} \subset N$, $\mu \sim \nu$ on Z, hence

$$\begin{aligned}&\sup_{t>0}\frac{1}{t}((1-e^{-\alpha t}S_t)\,1_Z, 1_Z)_{L^2(\mu)}\\ =\ &\sup_{t>0}\frac{1}{t}(1-e^{\alpha t})\ (1_Z,1_Z)_{L^2(\mu)}\\ =\ &\alpha\ (1_Z,1_Z)_{L^2(\mu)}.\ )\end{aligned}$$

Since then also $1_N = 1 - 1_Z \in D(\mathcal{E})$, we have by the product rule

$$0 = \nabla_H(1_N 1_Z) = 1_N \nabla_H 1_Z + 1_Z \nabla_H 1_N \quad \mu\text{–a.e.},$$

which implies that

$$\nabla_H 1_Z = 0 \quad \text{on } N \ \mu\text{–a.e.}$$

and

$$\nabla_H 1_N = 0 \quad \text{on } Z \ \mu\text{–a.e.}$$

Therefore,

$$(A^{1/2}\nabla_H 1_Z, A^{1/2}\nabla_H 1_N)_H = 0 \ \mu\text{–a.e.},$$

i.e.,

$$\mathcal{E}(1_Z, 1_N) = 0.$$

It follows that

$$\begin{aligned}
\mathcal{E}(1_Z, 1_Z) &= \mathcal{E}(1_Z, 1) - \mathcal{E}(1_Z, 1_N) \\
&= \mathcal{E}(1_Z, 1) \\
&= 0.
\end{aligned}$$

Consequently, $1_Z \in D(L)$ and $L1_Z = 0$, which in turn implies that for all $t > 0$

$$T_t 1_Z - 1_Z = \int_0^t T_s \ L1_Z \ ds = 0 \ \mu\text{–a.e.} \ ,$$

and thus for all $t > 0$

$$T_t 1_N = T_t 1 - T_t 1_Z = 1 - 1_Z = 1_N \ \mu\text{–a.e.}$$

(ii): Obviously, we may assume $f = 1_N$, $N \in \mathcal{B}(X)$, $\nu(N) = 0$, and $\{\rho = 0\} \subset N$ (otherwise consider $N \cup \{\rho = 0\}$). Then by (i) for all $t > 0$

$$T_t 1_N = 1_N = 0 \quad \nu\text{–a.e.}$$

$\square$

Remark 2.12 Note that in the previous literature (cf. e.g. [27], [15]), in more particular cases Proposition 2.11 was established using probabilistic tools, i.e., Girsanov's theorem. Our proof above is, however, purely analytic.

Corollary 2.13 *Suppose $\nu_i := \rho_i \cdot \mu$, $i = 1, 2$, are probability measures which are (S_t)–invariant such that at least one ρ_1 or ρ_2 is in $L^2(\mu)$. If $(\mathcal{E}, D(\mathcal{E}))$ is* irreducible *(i.e., $f \in D(\mathcal{E})$ is constant, provided $\mathcal{E}(f,f) = 0$), then $\nu_1 = \nu_2$.*

Proof. Let $t > 0$. Since S_t^* is positivity preserving (see e.g. [20]), we have for all $f \in L^2(\mu)$

$$\int |S_t^* f| \, d\mu \leq \int S_t^* |f| \, d\mu = \int S_t 1 \; |f| \, d\mu = \int |f| \, d\mu,$$

i.e., S_t^* extends to $L^1(\mu)$. We denote this extension again by S_t^*. Furthermore, for $\rho \in L^1(\mu)$, $\rho_k := (\rho \vee (-k) \wedge k$, $k \in \mathbb{N}$, and all $f \in L^\infty(\mu)$

$$\begin{aligned} \int S_t f \; \rho \; d\mu &= \lim_{k\to\infty} \int S_t f \; \rho_k \; d\mu \\ &= \lim_{k\to\infty} \int f \; S_t^* \rho_k \; d\mu \\ &= \int f \; S_t^* \rho \; d\mu \end{aligned}$$

Hence $\rho \cdot \mu$ is (S_t)–invariant if and only if

$$\rho \in V := \{g \in L^1(\mu) \mid S_t^* g = g \text{ for all } t > 0\} \, .$$

Claim 1. The linear space V is a lattice.

Indeed, if $\rho \in V$, $t > 0$, then $S_t^* \rho^+ \geq S_t^* \rho = \rho$ and $S_t^* \rho^+ \geq 0$, i.e., $S_t^* \rho^+ \geq \rho^+$. In addition,

$$\int (S_t^* \rho^+ - \rho^+) \, d\mu = \int S_t 1 \; \rho^+ \; d\mu - \int \rho^+ \; d\mu = 0$$

hence $S_t^* \rho^+ = \rho^+$, i.e., $\rho^+ \in V$, and Claim 1 is proven.

Claim 2. Let $\nu := \rho \cdot \mu$ be an (S_t)–invariant probability measure with $\rho \in L^2(\mu)$. Then $\rho > 0$ μ–a.e.

To prove this claim let $Z := \{\rho > 0\}$, $N := X \setminus Z$, $t > 0$. Then $1_N = 0$ ν–a.e., hence by Proposition 2.11 (i) we know that $T_t 1_N = 1_N$ μ–a.e. Consequently, $T_t 1_Z = T_t 1 - T_t 1_N = 1 - 1_N = 1_Z$ μ–a.e. and the irreducibility of $(\mathcal{E}, D(\mathcal{E}))$ implies that $1_Z =$const. μ–a.e. (cf. [4, Proposition 2.3]), i.e., $\mu(Z) = 1$ and Claim 2 is proven.

Claim 3. If ρ_1, ρ_2 are both in $L^2(\mu)$, then $\rho_1 = \rho_2$.

Indeed, by Claim 1 we know that $\alpha(\rho_1 - \rho_2)^+ \mu$, $\beta(\rho_1 - \rho_2)^- \mu$ are both (S_t)–invariant probability measures for some $\alpha, \beta > 0$ unless they are zero. Since both are mutually singular, and by Claim 2 also both are equivalent to μ, at least one of them must be zero, which implies that ρ_1 and ρ_2 coincide. Hence Claim 3 is proven.

To finish the proof it suffices to note that by Claim 1 $(\rho_1 \wedge \rho_2) \cdot \mu$ is (S_t)–invariant and $\rho_1 \wedge \rho_2 \in L^2(\mu)$. So, if e.g. $\rho_1 \in L^1(\mu)$ then by Claim 2 we have $\rho_1 \wedge \rho_2 = \alpha\rho_1$ for some $\alpha > 0$. (Indeed, $\alpha \neq 0$, since $\rho_1 > 0$ μ–a.e., by Proposition 2.11). Since $\int \rho_1 \, d\mu = 1$, it follows that $\alpha \leq 1$, hence if $\alpha < 1$, $\rho_2 = \alpha\rho_1$ which contradicts the fact that $\int \rho_2 \, d\mu = 1$. Therefore, $\alpha = 1$, i.e., $\rho_1 \leq \rho_2$, therefore $\rho_1 = \rho_2$. □

3 Log–Sobolev Dirichlet forms and their perturbations

In this section we consider perturbations of Dirichlet forms satisfying the defective Log–Sobolev inequality (2.4)'.

3.1 The finite–dimensional case: compact embeddings

Consider the situation described in Subsection 2.2 with $X := H := \mathbb{R}^d$ equipped with the Euclidean inner product $(\ ,\ ) := (\ ,\ )_{\mathbb{R}^d}$ and fix μ, A, as defined there. Now $x \mapsto A(x)$ is a matrix–valued map. Suppose μ is absolutely continuous w.r.t. Lebesgue measure dx on $\mathbb{R}^d$ and let $\rho := \frac{d\mu}{dx}$. We recall that ρ is said to satisfy *Hamza's condition* if the set

$$R(\rho) := \{y \in \mathbb{R}^d \mid \exists \varepsilon > 0 : \int_{|x-y|<\varepsilon} |\rho(x)|^{-1} \, dx < \infty\} \tag{3.1}$$

has full μ–measure (cf. [25]). Set $\nabla^\mu := \nabla^\mu_{\mathbb{R}^d}$

$$(\mathcal{E}, D(\mathcal{E})) := (\mathcal{E}_{\mu,A}, D(\mathcal{E}_{\mu,A})).$$

Remark 3.1 (i) Byt [7, Proposition 5.1] Hamza's condition, in particular, implies closability and that $\nabla f = \nabla g$ μ–a.e. provided $f, g \in C_b^\infty(\mathbb{R}^d)$ with $f = g$ μ–a.e.

(ii) Note that in this case $\mathcal{F}C_b^\infty = C_b^\infty(\mathbb{R}^d)$. But in fact, $(\mathcal{E}, D(\mathcal{E}))$ is also the closure of $C_0^\infty(\mathbb{R}^d)$ (= all smooth functions on $\mathbb{R}^d$ with compact support). So, the role of $\mathcal{F}C_b^\infty$ can be replaced by $C_0^\infty(\mathbb{R}^d)$.

Our next result deals with an interesting property of Log–Sobolev Dirichlet forms which is also important for the sequel. We shall see that under rather mild restrictions on the weight, a Log–Sobolev Dirichlet form on $\mathbb{R}^d$ (or a Riemannian manifold) has the property that the embedding of the Sobolev class $H^{1,2}(\mu) := D(\mathcal{E})$ into $L^2(\mu)$ is compact. The problem of finding conditions on a weight ρ on $\mathbb{R}^n$ that give the compactness of the embedding $H^{1,2}(\rho\, dx) \to L^2(\varrho\, dx)$ has been studied by many authors over the last four decades (see, e.g., references in [18]). Typically the conditions obtained so far concern the behaviour of ρ at infinity and some local bounds. Theorem 3.2 below gives natural conditions that are fulfilled in many concrete examples not covered by previously known results.

Theorem 3.2 *Let ϱ satisfy Hamza's condition. Suppose that $(\mathcal{E}, D(\mathcal{E}))$ satisfies the defective Log–Sobolev inequality (2.4)'. Then the embedding $D(\mathcal{E}) \hookrightarrow L^2(\mu)$ is compact.*

Proof. It suffices to prove that for every sequence $(f_n)_{n\in\mathbb{N}}$ that converges to zero in the weak topology of the space $D(\mathcal{E})$ w.r.t. $\| \; \|_{\mathcal{E}_1}$, one has norm convergence to zero of this sequence in $L^2(\mu)$.

Assume that $(f_n)_{n\in\mathbb{N}}$ does not converge to zero in $L^2(\mu)$. Then there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ whose norms are bounded away from zero. Denoting this subsequence again by $(f_n)_{n\in\mathbb{N}}$ we thus have

$$\inf_n \|f_n\|_{L^2(\mu)} > 0 \; . \tag{3.2}$$

Since $(f_n)_{n\in\mathbb{N}}$ is bounded in $D(\mathcal{E})$, we have

$$\sup_n \int f_n^2 \, d\mu + \int (\nabla^\mu f_n, \nabla^\mu f_n) \, d\mu < \infty.$$

By the defective Log–Sobolev inequality (2.4)' it follows that

$$\sup_n \int f_n^2 \log |f_n| \, d\mu < \infty,$$

hence $\{f_n^2 \mid n \in \mathbb{N}\}$ is uniformly μ–integrable. Therefore, we get a contradiction to (3.2), if we can prove that $(f_n)_{n\in\mathbb{N}}$ has a subsequence that converges in μ–measure.

Let Ω be an open ball on which $1/\rho \in L^1(\Omega)$. Since for every smooth compactly supported function ψ one has

$$\int_\Omega |\psi| \, dx \le C(\Omega) \sqrt{\int_\Omega \psi^2 \, \mu(dx)}, \qquad \int_\Omega |\nabla\psi| \, dx \le C(\Omega) \sqrt{\int_\Omega |\nabla\psi|^2 \, \mu(dx)}$$

where $C(\Omega) = \sqrt{\int_\Omega \rho(x)^{-1}\, dx}$, it follows from the definition of $D(\mathcal{E})$ that for every element $f \in D(\mathcal{E})$, $f|_\Omega \in H^{1,1}_{loc}(\Omega)$ (i.e., the classical local Sobolev space w.r.t. Lebesgue measure of order 1 in $L^1_{loc}(dx)$) and the restriction to Ω of its gradient $\nabla^\mu f$ coincides with ∇f in the Sobolev sense. Therefore, we get

$$\sup_n \int_\Omega |f_n|\, dx + \int_\Omega |\nabla f_n|\, dx < \infty.$$

By the compactness of the embedding of $H^{1,1}(\Omega)$ to $L^1(\Omega)$ (see, e.g. [18]), we get a subsequence of $(f_n)_{n\in\mathbb{N}}$ denoted again by $(f_n)_{n\in\mathbb{N}}$ which converges in $L^1(\Omega)$, hence, in measure with respect to Lebesgue measure, and, consequently, with respect to μ on Ω. It follows that (by a standard diagonal procedure) we can find a subsequence that converges in measure with respect to the measure μ on the whole space $\mathbb{R}^d$. Its limit F, however, is zero μ–a.e., since by the Banach–Saks theorem (selecting again a subsequence if necessary) $(f_1 + \cdots + f_n)/n \underset{n\to\infty}{\longrightarrow} 0$ in $L^2(\mu)$, but also $(f_1 + \cdots + f_n)/n \underset{n\to\infty}{\longrightarrow} F$ in $L^2(\mu)$.

□

Remark 3.3 (i) Note that the Hamza condition is trivially satisfied if μ-almost every point x has a neighborhood on which $\operatorname{essinf} \rho$ is strictly positive. For example, this is true if ρ is lower semicontinuous μ-a.e.

(ii) Replacing the role of $C_b^\infty(\mathbb{R}^d)$ resp. $C_0^\infty(\mathbb{R}^d)$ by $C_0^\infty(U)$, the above theorem immediately generalizes to the case where $\mathbb{R}^d$ is replaced by an open subset U. Likewise there is a straightforward generalization to Riemannian manifolds. (This also holds for Theorem 3.4 below.) Since such situations admit finite products (see e.g. [14]), one gets a lot of nontrivial examples starting with the one-dimensional case. For more concrete examples of measures satisfying the Log–Sobolev inequality we refer to [14] and [9], as well as the references therein.

(iii) The condition that μ satisfies the defective Log–Sobolev inequality (2.4)' cannot be replaced by the Poincaré inequality (2.3) as will follow from the next theorem (cf. Remark 3.5 below).

(iv) Also in infinite dimensions there are situations where $D(\mathcal{E}) \hookrightarrow L^2(\mu)$ is compact. In these cases typically $X = H$. We refer to [12], [23] for examples.

3.2 Existence and uniqueness of invariant measures

In this subsection we shall see that by the classical Fredholm theory the compact embedding discussed above implies the existence of invariant measures. We now again consider the general situation of Subsection 2.2 and adopt the notation introduced there. In particular, $(\mathcal{E}, D(\mathcal{E}))$, $(\mathcal{E}_v, D(\mathcal{E}))$, $(L_v, D(L_v))$ are as defined there and fixed below.

Theorem 3.4 *Assume $D(\mathcal{E}) \hookrightarrow L^2(\mu)$ is compact (cf. Theorem 3.2 and Remark 3.3 (iv) for sufficient conditions resp. examples). Let $(L_v^*, D(L_v^*))$ be the adjoint of $(L_v, D(L_v))$ on $L^2(\mu)$. Then:*

(i) dim Ker $L_v^* < \infty$ and there exists a non-negatve $\rho \in$ Ker $L_v^* \setminus \{0\}$. In particular, $\nu := (\int \rho\, d\mu)^{-1}\ \rho \cdot \mu$ is an (S_t)–invariant probability measure.

(ii) Suppose $(\mathcal{E}, D(\mathcal{E}))$ is irreducible. Then dim Ker $L_v^* = 1$. In particular, ν defined in (i) is the only (S_t)–invariant probability measure absolutely continuous w.r.t. μ.

Proof. (i): Since $(L_v, D(L_v))$ generates a strongly continuous semigroup $(S_t)_{t>0}$ on $L^2(\mu)$, there exists $\alpha > 0$ such that $(\alpha - L_v) : D(L_v) \to L^2(\mu)$ is a bijection. Hence $\alpha - L_v$ is a Fredholm operator with index zero. Since $\| \, \|_{\mathcal{E}_1}$ and $\| \, \|_{\mathcal{E}_{v,\alpha}}$ are equivalent and $D(\mathcal{E}) \hookrightarrow L^2(\mu)$ is compact, so is $D(L_v) \hookrightarrow L^2(\mu)$ provided $D(L_v)$ is equipped with the graph norm. Thus $\alpha - L_v$ is an L_v–compact perturbation of L_v, i.e., L_v is also a Fredholm operator with index zero (cf. [16]). Thus dim Ker $L_v^* =$ dim Ker $L_v < \infty$. Since $1 \in D(L_v)$ and $L_v 1 = 0$, we obtain that Ker$L_v^* \neq \{0\}$.

Now let $g \in$ Ker L_v^*, then by Lemma 2.8

$$S_t^* g = g \quad \text{for all } t > 0$$

where $(S_t^*)_{t>0}$ is the adjoint semigroup of $(S_t)_{t>0}$ on $L^2(\mu)$. By Claim 1 in the proof of Corollary 2.13 (note that the irreducibility assumption was not used there) we know that

$$S_t^*\, g^+ = g^+ \quad \text{for all } t \geq 0,$$

which in turn implies that $g^+ \in$Ker L_v^* and also (cf the beginning of the proof of Corollary 2.13) that $g^+ \cdot \mu$ is (S_t)–invariant.

(ii): This follows by (i) from Corollary 2.13. □

Remark 3.5 (i) As Theorem 3.2 also the previous result has a natural extension to Riemannian manifolds.

(ii) As we have seen in Example 2.6 (ii), Theorem 3.4 does not apply to the case described there. Hence there, $D(\mathcal{E}) \hookrightarrow L^2(\mu)$ cannot be compact. Since clearly Hazma's condition holds for $\rho(x) = C\exp(-|x|)$, $x \in \mathbb{R}^d$, $(\mathcal{E}_\mu, D(\mathcal{E}_\mu))$ by Theorem 3.2 cannot satisfy the defective Log–Sobolev inequality (2.4)' though it satifies the Poincaré inequality (2.3). This fact is known to experts (at least with (2.4) replacing (2.4)'), but our results here provide a new proof.

(iii) In the special situation of Theorem 3.4, the uniqueness part can be proven directly without refering to Corollary 2.13. Indeed, by straightforward analytic arguments it is easy to see that, if $(\mathcal{E}, D(\mathcal{E}))$ is irreducible, so is $(\mathcal{E}_v, D(\mathcal{E}_v))$ (see also [21, Satz 4.4] for a probabilistic proof). Hence it follows that $1 = \dim \operatorname{Ker} L_v = \dim \operatorname{Ker} L_v^*$ (cf. the proof of Theorem 3.4).

Despite of [12], [23] the number of examples where $D(\mathcal{E}) \hookrightarrow L^2(\mu)$ is compact is quite limited if $\dim X = \infty$. However, there is a large number of cases where $(\mathcal{E}, D(\mathcal{E}))$ still satisfies the defective Log–Sobolev inequality (2.4)'. We refer besides [14], [9], in particular, to [3, Sections 3 and 4]. The latter reference contains examples which include non–Gaussian (Gibbs) measures. Below we shall, therefore, prove a result on existence and uniqueness of invariant measures where instead of the compactness of $D(\mathcal{E}) \hookrightarrow L^2(\mu)$ we assume the defective Log–Sobolev inequality (2.4)' and a mild "positivity assumption" (cf. condition (i) in Theorem 3.6 below) to hold.

From now on we assume that H is dense in X. Identifying H with its dual H^* we obtain a map $j_H : X^* \to H^* \equiv H \subset X$ such that $(j_H(k), h)_H = {}_{X^*}\langle k, h\rangle_X$ for all $k \in X^*$, $h \in H$. Furthermore, (e.g. by [8, Lemma 5.6]) there exists a set $\{l_n \mid n \in \mathbb{N}\} \subset X^*$ separating the points of X such that $e_n := j_H(l_n)$, $n \in \mathbb{N}$, form an orthonormal basis of H. For $n \in \mathbb{N}$ define $\Pi_n : X \to \mathbb{R}^n$ by

$$\Pi_n(x) := (l_1(x), \dots, l_n(x)).$$

Furthermore, set $\mu_n := \mu \circ \Pi_n^{-1}$.

Theorem 3.6 *Assume that the following conditions hold:*

(i) $(\mathcal{E}, D(\mathcal{E}))$ satisfies the defective Log–Sobolev inequality (2.4)'.

(ii) There exist l_n, $n \in \mathbb{N}$, as described above, such that $\mu_n = p_n\, dx$ and p_n satisfies Hamza's condition for all $n \in \mathbb{N}$.

Then there exists $\rho \in D(\mathcal{E})$, $\rho \geq 0$, such that $\nu := \rho \cdot \mu$ is an (S_t)–invariant probability measure. If $(\mathcal{E}, D(\mathcal{E}))$ is irreducible (which is e.g. the case provided $(\mathcal{E}, D(\mathcal{E}))$ satisfies the Poincaré inequality (2.3) *or instead of (2.4)' in (i) the stronger Log–Sobolev inequality* (2.4)*) then ν is the only (S_t)–invariant probability measure, absolutely continuous w.r.t. μ.*

Proof. Fix $n \in \mathbb{N}$. For $i, j \in \mathbb{N}$, $x \in X$, set

$$\begin{aligned} a_{ij}(x) &:= (A(x)e_i, e_j)_H \\ v_i(x) &:= (v(x), e_i)_H, \end{aligned}$$

where $e_n := j_H(l_n)$, $n \in \mathbb{N}$, as above. Let $a_{ij}^{(n)} : \mathbb{R}^n \to \mathbb{R}$, $v_i^{(n)} : \mathbb{R}^n \to \mathbb{R}^n$, be $\mathcal{B}(\mathbb{R}^n)$–measurable such that

$$\begin{aligned} a_{ij}^{(n)} \circ \Pi_n &= E_\mu[a_{ij} \mid \sigma_n] \qquad \mu\text{–a.e.} \\ v_i^{(n)} \circ \Pi_n &= E_\mu[v_i \mid \sigma_n] \qquad \mu\text{–a.e.} \end{aligned}$$

Here $E_\mu[\cdot \mid \sigma_n]$ denotes the conditional expectation of μ given the σ–algebra $\sigma_n := \sigma(l_1, \ldots, l_n)$. Define for $f, g \in C_b^\infty(\mathbb{R}^n)$

$$\mathcal{E}^{(n)}(f,g) := \sum_{i,j=1}^{n} \int \partial_i f \; \partial_j g \; a_{ij}^{(n)} \; d\mu_n$$

(where ∂_i means partial derivative w.r.t. th i–th coordinate) and

$$\mathcal{E}_{v^{(n)}}^{(n)}(f,g) := \mathcal{E}^{(n)}(f,g) + \int (v^{(n)}, \nabla f)_{\mathbb{R}^n} \; g \; d\mu_n$$

where $v^{(n)} := (v_1^{(n)}, \ldots, v_n^{(n)})$.

Since $(a_{ij}^{(n)})_{1 \leq i,j \leq n} \geq C^{-1} \; I_{\mathbb{R}^n}$ μ_n–a.e. by (2.9), we know by assumption (ii) and Remark 3.1 that $(\mathcal{E}^{(n)}, \widetilde{C_b^\infty(\mathbb{R}^n)}^{\mu_n})$ is closable and its closure $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ is a (symmetric) coercive closed form on $L^2(\mathbb{R}^n; \mu_n)$.

By (2.12a) we have that for all $f, g \in C_b^\infty(\mathbb{R}^n)$

$$\begin{aligned} &\left| \int (v^{(n)}, \nabla f)_{\mathbb{R}^n} \; g \; d\mu_n \right| = \left| \sum_{i=1}^{n} \int E_\mu[v_i \mid \sigma_n] \; (\partial_i f \circ \Pi_n)(g \circ \Pi_n) \; d\mu \right| \\ = \;& \left| \int (v, \nabla_H (f \circ \Pi_n))_H \; (g \circ \Pi_n) \; d\mu \right| \\ \leq \;& b \; \mathcal{E}_1(f \circ \Pi_n, f \circ \Pi_n)^{1/2} \; \mathcal{E}_1(g \circ \Pi_n, g \circ \Pi_n)^{1/2} \\ = \;& b \; \mathcal{E}_1^{(n)}(f,f)^{1/2} \; \mathcal{E}_1^{(n)}(g,g)^{1/2} \; . \end{aligned}$$

Likewise (2.12b) with the *same* ε and a (hence independent of n) holds and we obtain that $(\mathcal{E}^{(n)}_{v^{(n)}}, \widetilde{C_b^\infty(\mathbb{R}^n)}^{\mu_n})$ is closable on $L^2(\mathbb{R}^n;\mu_n)$ and its closure $(\mathcal{E}^{(n)}_{v^{(n)}}, D(\mathcal{E}^{(n)}_{v^{(n)}}))$ is a sectorial form with $D(\mathcal{E}^{(n)}_{v^{(n)}}) = D(\mathcal{E}^{(n)})$ satisfying (2.13). Similarly, using assumption (i) one easily checks that the defective Log–Sobolev inequality (2.4)' for $(\mathcal{E}^{(n)}, D(\mathcal{E}^{(n)}))$ holds with the *same* λ. Hence assumption (ii) and Theorems 3.2, 3.4 imply that there exists $\rho_n \in D(\mathcal{E}^{(n)})$ such that $\rho_n \geq 0$, $\int_{\mathbb{R}^n} \rho_n^2 \, d\mu_n = 1$, and

$$\mathcal{E}^{(n)}_{v^{(n)}}(f, \rho_n) = 0 \quad \text{for all } f \in D(\mathcal{E}^{(n)}). \tag{3.3}$$

By what we have just shown and (2.14) in Lemma 2.10 we conclude that

$$\mathcal{E}^{(n)}(\rho_n, \rho_n) \leq \frac{a}{1-\varepsilon} \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Hence the defective Log–Sobolev inequality implies that

$$\int (\rho_n \circ \Pi_n)^2 \log(\rho_n \circ \Pi_n) \, d\mu = \int \rho_n^2 \log \rho_n^2 \, d\mu_n \leq \lambda\left(\frac{a}{1-\varepsilon} + 1\right)$$

for all $n \in \mathbb{N}$. Therefore, $\{(\rho_n \circ \Pi_n)^2 \mid n \in \mathbb{N}\}$ is uniformly μ–integrable. Clearly, $\rho_n \circ \Pi_n \in D(\mathcal{E})$ and $\mathcal{E}(\rho_n \circ \Pi_n, \rho_n \circ \Pi_n) = \mathcal{E}^{(n)}(\rho_n, \rho_n)$, hence by (3.4)

$$\mathcal{E}(\rho_n \circ \Pi_n, \rho_n \circ \Pi_n) \leq \frac{a}{1-\varepsilon} \text{ for all } n \in \mathbb{N}.$$

In particular, a subsequence $(\rho_{n_k} \circ \Pi_{n_k})_{k\in\mathbb{N}}$ of $(\rho_n \circ \Pi_n)_{n\in\mathbb{N}}$ weakly converges to some $\rho \in D(\mathcal{E})$ in $(D(\mathcal{E}), \mathcal{E}_1)$. Since it also converges to ρ weakly in $L^2(\mu)$, it follows that $\rho \geq 0$ and that $\rho \not\equiv 0$. The latter follows since

$$\int \rho \, d\mu = \lim_{k\to\infty} \int \rho_{n_k} \circ \Pi_{n_k} \, d\mu \geq \liminf_{n\to\infty} \|\rho_n \circ \Pi_n\|_{L^1(\mu)}.$$

But the right hand side is strictly positive, because otherwise, a subsequence would converge to zero in $L^1(\mu)$, hence in μ–measure which in turn implies convergence to zero in $L^2(\mu)$ by the uniform μ–integrability of $\{(\rho_n \circ \Pi_n)^2 \mid n \in \mathbb{N}\}$. This contradicts the fact that $\int (\rho_n \circ \Pi_n)^2 \, d\mu = 1$ for all $n \in \mathbb{N}$.

Claim: $\mathcal{E}_v(f, \rho) = 0$ for all $f \in D(\mathcal{E})$.

Let $m \in \mathbb{N}$, $\psi \in C_b^\infty(\mathbb{R}^m)$, $f := \psi \circ \Pi_m$. Then considering ψ also as a function on $\mathbb{R}^n$ for $n \geq m$

$$\begin{aligned} \mathcal{E}_v(f,\rho) &= \lim_{\substack{n\to\infty \\ n\geq m}} \mathcal{E}_v(f, \rho_n \circ \Pi_n) \\ &= \lim_{\substack{n\to\infty \\ n\geq m}} \mathcal{E}^{(n)}_{v^{(n)}}(\psi, \rho_n) \\ &= 0 \end{aligned}$$

where the last equality follows from (3.3).

Since $\{\psi \circ \Pi_m \mid m \in \mathbb{N},\ \psi \in C_b^\infty(\mathbb{R}^m)\}$ is dense in $\mathcal{F}C_b^\infty$, hence in $D(\mathcal{E})$ w.r.t. $\| \ \|_{\mathcal{E}_1}$ (cf. [6, Proposition 2.10]) the Claim is proven.

By normalization we get the desired (S_t)–invariant probability measure. The uniqueness part of the assertion follows again by Corollary 2.13. □

Remark 3.7 In Theorem 3.6 (as well as in Proposition 3.9 below) assumption (i) can be replaced by any other on $(\mathcal{E}, D(\mathcal{E}))$ ensuring that boundedness w.r.t. $\| \ \|_{\mathcal{E}_1}$ implies uniform μ–square integrability. Other examples are thus provided by cases where the classical Sobolev inequality, briefly discussed in Example 2.6 (i), holds.

Example 3.8 The conditions of Theorem 3.6 are fulfilled if μ is a centered Radon Gaussian measure on a locally convex space X and H is a separable Hilbert space continuously embedded into X, containing the Cameron–Martin space $H(\mu)$ of the measure μ. Indeed, the finite dimensional projections of μ are Gaussian measures. Since the Log–Sobolev inequality holds true for $H = H(\mu)$, it holds also for any bigger Hilbert space continuously embedded into X. In fact, for H one can take any densely embedded Hilbert space leading to Log–Sobolev Dirichlet forms. (However, not every H has the latter property.) For non–Gaussian examples we refer to [3, Sections 3.4].

We conclude this section with the following proposition which is of its own interest, but is proven entirely similarly to Theorem 3.6. We, therefore, only state the result and omit the proof.

Proposition 3.9 *Suppose assumption (i) in Theorem 3.6 holds. For $n \in \mathbb{N}$ let $A_n : X \to \mathcal{L}_s(H)$, $v_n : X \to H$, satisfy the same type of assumptions of $A : X \to \mathcal{L}_s(H)$ and $v : X \to H$ as specified in Subsection 2.2. Suppose that the following conditions hold:*

(i) $A_n(x) \underset{n\to\infty}{\longrightarrow} A(x)$ *strongly on* H *for* μ*–a.e.* $x \in X$ *and* $\int_X \sup_n \|A_n(x)\|_{\mathcal{L}(H)}\, \mu(dx) < \infty$.

(ii) $|v_n - v|_H \underset{n\to\infty}{\longrightarrow} 0$ *in* $L^2(\mu)$ *and the constant* b *in the estimate (2.12a), satisfied by each* v_n*, is independent of* $n \in \mathbb{N}$.

(iii) $Ker\, L^*_{A_n,v_n} \neq \{0\}$ *for all* $n \in \mathbb{N}$*, where* $(L_{A_n,v_n}, D(L_{A_n,v_n}))$ *are the generators of* $(\mathcal{E}_{\mu,A_n,v_n}, D(\mathcal{E}_{\mu,A_n,v_n}))$ *and* $(L^*_{A_n,v_n}, D(L^*_{A_n,v_n}))$ *are their adjoints on* $L^2(\mu)$*. Then, if* $\rho_n \in Ker\, L^*_{A_n,v_n}$*,* $n \in \mathbb{N}$*, are unit vectors in* $L^2(\mu)$ *it follows that there exists* $\rho_1 \in D(\mathcal{E})$ *such that* $\rho_{n_k} \underset{k\to\infty}{\longrightarrow} \rho_1$ *weakly in* $(D(\mathcal{E}), \mathcal{E}_1)$ *for some subsequence* $(\rho_{n_k})_{k\in\mathbb{N}}$*. Furthermore,* $\rho_1 \in Ker\, L^*_v$*, and there exists an* (S_t)*–invariant probability measure* $\nu := \rho\cdot\mu$ *with* $\rho \in D(\mathcal{E})$*.* ν *is the unique* (S_t)*–invariant probability measure absolutely continuous w.r.t.* μ *provided* $(\mathcal{E}, D(\mathcal{E}))$ *is irreducible.*

Acknowledgement We would like to thank Wilhelm Stannat for some very useful discussions. Financial support of the SFB 343 (Bielefeld), DFG/RFFI–Project: No. 436 RUS 113/343/1, EU–TMR–Project: No. ERB–FMRX–CT96–0075, and the Russian Foundation of Fundamental Research (Grant No. 97–01–00932) is gratefully acknowledged.

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