DIFFEOMORPHISM GROUPS AND CURRENT ALGEBRAS: configuration space analysis in quantum theory

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Abstract

The constuction of models of non-relativistic quantum fields via current algebra representations is presented using a natural differential geometry of the configuration space Γ of particles, the corresponding classical Dirichlet operator associated with a Poisson measure on Γ, being the free Hamiltonian. The case with interactions is also discussed together with its relation to the problem of unitary representations of the diffeomorphism group on $\mathbb{R}^d$.

Contents

1 INTRODUCTION

The fundamental concept that quantum theories are described by means of unitary representations of diffeomorphism groups has been very well elaborated in the study of certain quantum systems having infinitely many degrees of freedom. We refer to [GoMo 1,2] for an excellent explanation of the underlying physical ideas and annotated references. Historically this concept is closely related to the suggestion of physicists to formulate field theory in terms of local currents instead of canonical field operators, see e.g. [AHKMTT],[DaSh],[GoGrPoSh], [MeSh] and the references therein.

The nonrelativistic local current algebra and diffeomorphism groups appear in this contents, in particular, in the following model situation: Let $\psi(x), x \in I\!R^d$, be a second-quantized, nonrelativistic Bose field satisfying the canonical commutation relations (at fixed time):

$$[\psi(x), \psi^*(y)] = \delta(x-y),$$

$$[\psi(x), \psi(y)] = [\psi^*(x), \psi^*(y)] = 0, \; x, y \in I\!R^d.$$

(For example, we can use the well-known Fock representation for the free nonrelativistic Bose field (Bose gas at zero temperature)). Of course, we should consider such a field as an operator-valued generalized function (e.g., over the Schwartz space $\mathcal{D} := C_0^\infty(I\!R^d)$).

Let us introduce the momentum density operator $J(x)$ and the particle density operator $\rho(x)$ defined in terms of the fields as

$$J(x) = \frac{1}{2i}\{\psi^*(x)\nabla\psi(x) \; - \; \nabla\psi^*(x)\psi(x)\},$$

$$\rho(x) \; = \; \psi^*(x)\psi(x).$$

These operators are considered again as operator-valued generalized functions for which the averaged operators have sense. Namely, for any

$$v \in \mathrm{V}ect_0(I\!R^d) := V_0(I\!R^d)$$

(= the set of all C^∞ vector fields on $I\!R^d$ with compact support) we define

$$J(v) = \int < J(x), v(x) >_{I\!R^d} dm(x)$$

and for any $f \in \mathcal{D} = C_0^\infty(I\!R^d)$

$$\rho(f) \;=\; \int \rho(x) f(x) dm(x),$$

where m denotes Lebesgue measure on $I\!R^d$ and the integrals in the above formulas are meant in the sense of pairings between distributions and test functions resp. test vector fields. The canonical commutation relations for the fields imply the following commutation relations:

$$[J(v_1), J(v_2)] = -iJ([v_1, v_2]) \tag{1.1}$$

(where $[v_1, v_2] = v_1 \cdot \nabla v_2 - v_2 \cdot \nabla v_1$ is the Lie bracket of vector fields),

$$[\rho(f_1), \rho(f_2)] = 0, \tag{1.2}$$

$$[\rho(f), J(v)] = i\rho(\nabla_v f) \;\; (= i\rho(< v, \nabla f >)). \tag{1.3}$$

These relations show that the operators J and ρ form an infinite-dimensional Lie algebra, the so-called *Lie algebra of currents* (local, nonrelativistic).

Let us describe the group obtained by exponentiating this Lie algebra. To this end denote the group of all diffeomorphisms of R^d by Diff($I\!R^d$) and by $\mathrm{Diff}_0(I\!R^d)$ its subgroup of all diffeomorphisms $\phi : I\!R^d \to I\!R^d$ with compact support, i.e., which are equal to the identity outside a compact set (depending on ϕ). Any $v \in V_0(I\!R^d)$ generates a flow of diffeomorphisms $\phi_t^v \in \mathrm{Diff}_0(I\!R^d), t \in I\!R$. More precisely, for any $x \in I\!R^d$ the curve

$$I\!R \ni t \mapsto \phi_t^v(x) \in I\!R^d$$

is defined as the solution to the following Cauchy problem

$$\left\{ \begin{array}{rcl} \frac{d}{dt}\phi_t^v(x) & = & v(\phi_t^v(x)) \\ \phi_0^v(x) & = & x. \end{array} \right. \tag{1.4}$$

For any $f \in \mathcal{D}$ and $v \in V_0(I\!R^d)$ we introduce unitary operators

$$U(f) := \exp[i\rho(f)], \quad V(\phi_t^v) := \exp[itJ(v)]. \tag{1.5}$$

The commutation relations for currents imply that $V(\phi)$ is a unitary representation of the group $\mathrm{Diff}_0(\mathbb{R}^d)$:

$$V(\phi_1)V(\phi_2) = V(\phi_2 \circ \phi_1), \ \phi_1, \phi_2 \in \mathrm{Diff}_0(\mathbb{R}^d). \tag{1.6}$$

Analogously, the operators $U(f)$ form a unitary representation of the additive group $\mathcal{D}$:

$$U(f_1)U(f_2) \ = \ U(f_1 + f_2), \ f_1, f_2 \in \mathcal{D}.$$

Then for the unitary operators

$$W(f, v) \ := \ U(f)V(v), \quad f \in \mathcal{D}, \ v \in V_0(\mathbb{R}^d),$$

we have

$$W(f_1, \phi_1)W(f_2, \phi_2) = W(f_1 + f_2 \circ \phi_1, \phi_2 \circ \phi_1),$$

i.e., the operators W provide a unitary representation of an infinite-dimensional Lie group $G := G(\mathbb{R}^d) = \mathcal{D} \wedge \mathrm{Diff}_0(\mathbb{R}^d)$ which is called the *semidirect product* of the groups $\mathcal{D}$ and $\mathrm{Diff}_0(\mathbb{R}^d)$, see e.g. [VeGeGr].

Due to physical iterpretation, the currents $\rho(f)$ and $J(v)$ (and the group G) can be taken as *fundamental structures* of quantum mechanics. This point of view was elaborated in early works [DaSh], [Go 1,2], [GoSh 1,2], [GrSh]. Later on it was developed in many directions. Between others we would like to mention the following aspects which have been intensively discussed in recent publications and preprints:

- Unitary inequivalent representations of G describe the possible, *physically distinct* quantum mechanical systems.
- Such systems include ones different from the canonical second-quantized fields from which the current algebra was first obtained.
- The same Lie algebra and Lie group are obtained if one starts with fermions.
- A characterization of particle statistics in terms of representations. The diffeomorphism group approach to anyons.
- The description of particle spin.

We refer to the papers [GoMo 1,2], [GoSh 3,4], [GoMeSh] for related discussions, reviews and more references.

In this paper we further develop the ideas in [GoGrPoSh], [Me 1,2], [MeSh] related to constructing non-relativistic quantum field models via current algebra representations. The main new ingredient of our approach is the following: in [AKR 1-3] we identified a natural differential geometry on configuration spaces such that the corresponding Dirichlet operators (on Poisson spaces) are exactly rigorous implementations of the free Hamiltonians. Also a representation for the current algebra is obtained by the said geometrical structure (cf. Section 2). In this realization interactions can be described as potentials on Poisson space (given by generalized functions). This immediately provides two classical strategies to construct interacting models of non-relativistic quantum field theory, i.e.the *Hamiltonian* and *Euclidean strategy* (in full analogy with scalar relativistic field models). We discuss both approaches and the relations between them (cf Section 3). In particular, the cut-off Hamiltonian after ground state renormalization becomes again a Dirichlet operator on configuration space but w.r.t. a new probability measure (= vacuum measure).

In order to bypass the difficult problem of removal of the cut-off, in Section 4 we discuss a direct construction of interacting models starting directly from a suitably chosen vacuum measure. The underlying general idea here is related to the well-known canonical field theory approach which goes back to E.Schrödinger, P.Jordan, W.Heisenberg and W.Pauli, H.Araki [Ar], F.Coester and R.Haag [CoHa] (see also e.g. [AH-K]). Suitable vacuum measures μ turn out to be Ruelle type measures on configuration spaces. The corresponding canonical Hamiltonians are Dirichlet operators on the physical Hilbert space $L^2(\mu)$. The currents are infinitesimal generators of Gelfand-Vilenkin type representations of $\mathrm{Diff}_0(\mathbb{R}^d)$ on $L^2(\mu)$. Both Hamiltonians and currents are given by explicit formulas in terms of the interaction potentials (obtained by computing te corresponding divergence w.r.t. μ).

We would like to emphasize explicitly that the contents of this paper is a programme that on the one hand we have realized partially and on the other hand propose as future work. In particular, we hope that our approach will lead to more detailed information in concrete cases of physically interesting interactions (as e.g. concerning spectral properties of Hamiltonians, quasi-particle descriptions, scattering theory etc.).

The results of this paper have been presented in invited talks at various occasions since the beginning of 1997, e.g. at conferences in Rome, Edinburgh, Kiev, Oberwolfach, Crete and seminars in Bielefeld (BiBoS and SFB

343), Erlangen, Moscow, Kiev, Göttingen and Bonn.

2 MODELS OF NON-RELATIVISTIC QFT

2.1 FORMAL CONSIDERATIONS

In this subsection (which has essentially a heuristic character) we will express the Hamiltonian of a physical systems in terms of currents. We follow [DaSh], [Me 1] in our considerations.

The Hamiltonian for a system of non-relativistic Bose particles with two-body interaction potential V is given (heuristically) in terms of canonical field operators by

$$H_V = \frac{1}{2}\int < \nabla\psi^*(x), \nabla\psi(x) > dm(x) + \tag{2.1}$$

$$\frac{1}{2}\int\int \psi^*(x)\psi^*(y)V(x-y)\psi(x)\psi(y)dm(x)dm(y). \tag{2.2}$$

This Hamiltonian is a sum of the kinetic energy part (2.1) and the potential energy (2.2):

$$H_V = H_0 + H_I.$$

We can express the Hamiltonian in terms of currents. Let us introduce

$$K(x) = \nabla\rho(x) + 2iJ(x).$$

(with ρ and J expressed in terms of ψ, ψ^* as indicated in Section 1). Then formally

$$H_V = \frac{1}{8}\int K^*(x)\frac{1}{\rho(x)}K(x)dm(x) + \tag{2.3}$$

$$\frac{1}{2}\int\int \rho(x)[\rho(y) - \delta(x-y)]V(x-y)dm(x)dm(y). \tag{2.4}$$

The Hamiltonian H_V has, of course, only a heuristic sense. To give a rigorous meaning to this operator we should start with a representation of the canonical field (or currents). For example, using the Fock space realization of the free Bose field we can rigorously define a bilinear form in this space which corresponds to the formal expression for H_V, see e.g. [He]. Nevertheless, this form is non-closable in the Fock space for any $V \neq 0$. The latter is related to the translation invariance of the interaction term. From the physical point of view this situation is quite typical and a standard equipment for the study of such formal Hamiltonians is given by the renormalization theory, see e.g.[He]. This way a renormalized version of the Hamiltonian H_V can be analyzed in many details (at least on the theoretical physics level of rigor) including spectral properties, scattering problem etc.

On the other hand, this "canonical approach" cannot be considered as a universal prescription to work with. The point is that the Fock space representation as a starting object is not enough even for a description of the free Bose gas: it is well-known that the Fock space representation describes only the case of the zero density gas. An alternative approach consists in using representations of the nonrelativistic current algebra suitable for describing physical systems which are associated with a given formal Hamiltonian. As a result, we can formulate a system of natural assumptions a resulting theory must satisfy, see [Me 1]. The latter can be interpreted as a system of "axioms" for nonrelativistic quantum field theory.

We assume that there is a representation $V(\phi), U(f)$ of the group G (and corresponding representation $\rho(f),\ J(v)$ of the current algebra) on a (physical) Hilbert space $\mathcal{H}$ along with a Hamiltonian H satisfying the following conditions:

(i) There is a normalized ground state Ω of lowest energy. We require $H \geq 0$. Thus the zero energy is chosen such that

$$\Omega \in \mathcal{H}\ :\ H\Omega = 0 \tag{2.5}$$

(ii) The set $Span\{U(f)\Omega\,|f \in \mathcal{D}\} \subset Dom(H)$ is dense in $\mathcal{H}$.

(iii) Current conservation:

$$\forall f \in \mathcal{D}:\quad [H, \rho(f)] = -iJ(\nabla f), \tag{2.6}$$

on some domain $D \subset Dom(H)$, dense in $\mathcal{H}$.

(iv) There exists an anti-unitary time reversal operator T such that

$$T\rho(f)T^{-1} = \rho(f), \tag{2.7}$$

$$TJ(v)T^{-1} = -J(v),\quad T\Omega = 0. \tag{2.8}$$

(on D)

(v) Translation invariance : the representation of $G(I\!\!R^d)$ is translation invariant in the following sense. For a unitary representation $Q(a), a \in I\!\!R^d$,

$$Q(a)U(f)Q(-a) = U(f(\cdot - a)), \;\; Q(a)\Omega = \Omega, \tag{2.9}$$

$$Q(a)V(\phi)Q(-a) = V(\phi_a), \; \phi_a(x) = \phi(x-a) + a. \tag{2.10}$$

Remark 2.1 *Note that as a consequence of conditions (i)-(iv) above we obtain "a priori properties" of the Hamiltonian H and the currents. Namely, it is easy to compute for vectors $e(f) = \exp[i\rho(f)]\Omega = U(f)\Omega \in \mathcal{H}$ that, e.g.,*

$$(e(f_1), He(f_2))_{\mathcal{H}} = \frac{1}{2}(e(f_1), \rho(< \nabla f_1, \nabla f_2 >_{I\!\!R^d})e(f_2))_{\mathcal{H}}, \tag{2.11}$$

$$(\rho(f)\Omega, H\rho(g)\Omega)_{\mathcal{H}} = \frac{1}{2}(\Omega, \rho(< \nabla f, \nabla g >_{I\!\!R^d})\Omega)_{\mathcal{H}}. \tag{2.12}$$

For the proof we refer to [Me 1, Theorem 1]

2.2 POISSON REPRESENTATION FOR CURRENTS

Let Γ denote the set of all locally finite configurations in $I\!\!R^d$:

$$\Gamma := \{\gamma \subset I\!\!R^d \mid |\gamma \cap K| < \infty \text{ for every compact} K \subset I\!\!R^d\}.$$

We identify $\gamma \in \Gamma$ with the (positive integer-valued) measure:

$$\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} \varepsilon_x \in \mathcal{M}(R^d) \subset \mathcal{D}^{'}(I\!\!R^d).$$

Then for any $f \in C_0(I\!\!R^d)$ we have a functional

$$\Gamma \ni \gamma \mapsto < f, \gamma >= \int f(x)\gamma(dx) = \sum_{x \in \gamma} f(x).$$

The Poisson measure with intensity measure $\lambda m(dx)$, $\lambda > 0$, is defined via the Laplace transform

$$\int_\Gamma e^{<f,\gamma>}\pi^\lambda(d\gamma) = \exp[\lambda \int_{I\!\!R^d} (e^{f(x)} - 1)m(dx)],$$

$f \in \mathcal{D}$. For any $\phi \in \mathrm{Diff}_0(I\!\!R^d)$ we define

$$\phi(\gamma) := \{\phi(x) \mid x \in \gamma\} \in \Gamma,$$

and therefore

$$< f, \phi(\gamma) >= \int f(x)\ \phi(\gamma)(dx) =< f \circ \phi, \gamma > .$$

A theorem due to Skorokhod (see e.g. [Sk], [Ta]) gives

$$\frac{d\pi^\lambda(\phi(\gamma))}{d\pi^\lambda(\gamma)} =$$

$$\prod_{x\in\gamma} j_\phi(x) \exp[\lambda \int (1 - j_\phi(x) m(dx)], \tag{2.13}$$

where $j_\phi(x) = \det(\nabla\phi(x))$ (=Jacobian). We introduce a Gelfand-Vilenkin type unitary representation of $\mathrm{Diff}_0(I\!\!R^d)$ in the (physical) Hilbert space $\mathcal{H}_\lambda = L^2(\pi^\lambda)$ by the formula

$$(V_\lambda(\phi)F)(\gamma) = F(\phi(\gamma))\sqrt{\frac{d\pi^\lambda(\phi(\gamma))}{d\pi^\lambda(\gamma)}}, \tag{2.14}$$

and for the commutative group $\mathcal{D}$ we define

$$(U_\lambda(f)F)(\gamma) = \exp[i < f, \gamma >]F(\gamma), f \in \mathcal{D}.$$

Then $W_\lambda(f, \phi) := U_\lambda(f)V_\lambda(\phi)$ gives a unitary representation of the infinite-dimensional Lie group $G = \mathcal{D} \wedge \mathrm{Diff}_0(I\!\!R^d)$ in $L^2(\pi^\lambda)$, see [VeGeGr] for a detailed study of its properties (see also [Is]).

Let us mention that in fact the Poissonian representations of $\mathrm{Diff}_0(I\!\!R^d)$ appeared in [GoGrPoSh], but the role of the Poisson measure was not clarified until the work [VeGeGr]. From the physical point of view these representations are related to the so-called N/V-limit for the free Bose gas.

Consider N bosons in a box of volume $V \subset I\!\!R^d$ (a cube identified with a torus via periodic boundary conditions). The N-particle representation of G is given by the operators

$$U_{N,V}(f)\Psi(x_1, \ldots, x_N) =$$

$$\exp[i \sum_{j=1}^{N} f(x_j)]\Psi(x_1, \ldots, x_N),$$

$$V_{N,V}(\phi)\Psi(x_1,\dots,x_N) =$$
$$\Psi(\phi(x_1),\dots,\phi(x_N))\sqrt{\prod_{k=1}^{N} j_\phi(x_k)},$$
where ϕ is a diffeomorphism on the torus and $\Psi \in L^2(V)^{\otimes N}$ is a symmetric function. Then (in a proper sense) we have in the limit $N \to \infty, V \to \infty$ and $N/V \to \lambda$
$$V_{N,V} \to V_\lambda,\ U_{N,V} \to U_\lambda,$$
see [GoGrPoSh] for detailed considerations.

To describe currents in the Poisson space we need some facts from analysis and geometry on configuration spaces from [AKR 1], [AKR 3] to which we refer for the corresponding proofs and details.

For $F : \Gamma \to I\!R$ we define the *directional derivative* along $v \in V_0(I\!R^d)$ as
$$(\nabla_v^\Gamma F)(\gamma) := \frac{d}{dt} F(\phi_t^v(\gamma))|_{t=0}$$
(if this exists). The *tangent space*
$$T_\gamma(\Gamma) := L^2(I\!R^d \to I\!R^d, \gamma)$$
consists of the measurable sections
$$V_\gamma : I\!R^d \to I\!R^d$$
(after the natural factorization) with the scalar product
$$< V_\gamma^1, V_\gamma^2 >_{T_\gamma} = \int_{I\!R^d} < V_\gamma^1, V_\gamma^2 >_{I\!R^d} \gamma(dx).$$

The *gradient* is defined as the mapping
$$\Gamma \ni \gamma \mapsto (\nabla^\Gamma F)(\gamma) \in T_\gamma(\Gamma)$$
such that
$$(\nabla_v^\Gamma F)(\gamma) = < \nabla^\Gamma F, v >_{T_\gamma}.$$

The logarithmic derivative of π^λ along $v \in V_0(I\!R^d)$ is the following function on Γ:
$$\Gamma \ni \gamma \mapsto B_v^{\pi^\lambda}(\gamma) := \int_{I\!R^d} \mathrm{div}(v(x))\gamma(dx).$$

All these definitions are applicable to the so-called *smooth cylinder functions* $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ on Γ, i.e., $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ consists of all functions

$$\Gamma \ni \gamma \mapsto F(\gamma) = g_F(< f_1,\gamma >,\ldots,< f_N,\gamma >),$$

$$f_1,\ldots,f_N \in \mathcal{D},\ g_F \in C_b^\infty(I\!R^N),\ N \in I\!N.$$

The *current operators* in $L^2(\pi^\lambda)$ have the form:

$$(\rho(f)F)(\gamma) :=< f,\gamma > F(\gamma),$$

$$(J(v)F)(\gamma) := \frac{1}{i}\nabla_v^\Gamma F(\gamma) + \frac{1}{2i}B_v^{\pi^\lambda}(\gamma)F(\gamma),$$

where $f \in \mathcal{D},\ v \in V_0(I\!R^d),\ F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$.

Remark 2.2 *The expressions for the current operators on $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ are the same for all Poisson measures π^λ (in contrast to the group representations W_λ defined above).*

3 CONSTRUCTIVE APPROACH

Let $\lambda > 0$ be a fixed density. We start with the Poisson space representation of the current algebra in the physical Hilbert space $L^2(\pi^\lambda)$. First of all we describe the free Bose gas.

3.1 FREE HAMILTONIAN

The free Hamiltonian H_0 has the following representation in terms of currents, see (2.3):

$$H_0 = \frac{1}{8}\int_{I\!R^d} K^*(x)\frac{1}{\rho(x)}K(x)m(dx),$$

where $K(x) = \nabla\rho(x) + 2iJ(x)$ or

$$K(v) = -\rho(\mathrm{div}v) + 2iJ(v),\ v \in V_0(I\!R^d).$$

As in [AKR 2,3] we introduce the *intrinsic Dirichlet form* associated to the Poisson measure π^λ:

$$\forall F,G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$$

$$\mathcal{E}^{\Gamma}_{\pi^\lambda}(F,G) = \frac{1}{2}\int_\Gamma < \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) >_{T_\gamma} \pi^\lambda(d\gamma).$$

Let $H^\Gamma := -\frac{1}{2}\mathrm{div}^\Gamma \nabla^\Gamma$ denote the operator in $L^2(\pi^\lambda)$ associated with (the closure of) this form [AKR 2,3].

Theorem 3.1 *The operator H^Γ is a positive essentially self-adjoint operator such that for any $f \in \mathcal{D}, f > -1$, we have*

$$\exp(-tH^\Gamma)\exp(< \log(1+f), \gamma >= \tag{3.1}$$

$$\exp(< \log(1+e^{\frac{t}{2}\triangle} f), \gamma >), t \geq 0, \gamma \in \Gamma.$$

Corollary 3.1 *H^Γ is unitary equivalent to the second quantization of the operator $-\frac{1}{2}\triangle$ under a unitary isomorphism of the Poisson space $L^2(\pi^\lambda)$ and the Fock space $\mathcal{F}(L^2(I\!R^d, \lambda m))$.*

The Fock space realization of Poisson spaces is known since classical work of K.Ito on 1956 [It], see also [IK], [KSS] for related discussions and more references.

Now we can identify the free Hamiltonian.

Theorem 3.2 *In the Poisson space representation of the current algebra we have*

$$H_0 = H^\Gamma$$

(i.e., H^Γ is a rigorous version of the heuristic Hamiltonian H_0).

As a heuristic explanation we have

$$iJ(x) = \rho(x)\nabla^\Gamma - \frac{1}{2}\nabla\rho(x).$$

Then

$$K(x) = \rho(x) + 2iJ(x) = 2\rho(x)\nabla^\Gamma$$

and

$$\int_\Gamma \int_{I\!R^d} < K(x)F, \frac{1}{\rho(x)}K(x)G >_{I\!R^d} m(dx)\pi^\lambda(d\gamma)$$

$$= 4\int_\Gamma\int_{I\!R^d} < \nabla^\Gamma F, \nabla^\Gamma G >_{I\!R^d} \rho(x)m(dx)\pi^\lambda(d\gamma)$$

$$= 4\int_\Gamma < \nabla^\Gamma F, \nabla^\Gamma G >_{T_\gamma} \pi^\lambda(d\gamma).$$

Remark 3.1 *The configuration space* Γ *can be considered as an infinite dimensional differentiable manifold with a "Riemannian volume" measure* π^λ*. Then the operator* H^Γ *is the Laplace-Beltrami operator on* Γ*, see [AKR 1-3] for discussions of this concept.*

3.2 FREE EUCLIDEAN MEASURE

We have a positive self-adjoint operator H^Γ in $L^2(\pi^\lambda)$ which creates the *heat semigroup* $T_t^\Gamma := \exp(-tH^\Gamma), t \geq 0$. It is a *Markov semigroup* which generates a stationary Markov process (time homogeneous) with invariant distribution π^λ. This process can be considered as the Brownian motion on the manifold Γ with stationary Poisson measure π^λ. Let us set

$$\mathcal{C}_\Gamma := C(I\!R \to \Gamma)$$

(the path space). Let W^{π^λ} be the probability measure on $\mathcal{C}_\Gamma$ which corresponds to this process (Wiener measure on $\mathcal{C}_\Gamma$), i.e.,

$$\forall F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma),\ \forall t, s \in I\!R$$

$$\int_{\mathcal{C}_\Gamma} F(X(t))G(X(s))W^{\pi^\lambda}(dX(\cdot)) =$$

$$\int_\Gamma F(\gamma)(e^{-|t-s|H^\Gamma}G)(\gamma)\pi^\lambda(d\gamma).$$

The path space measure W^{π^λ} will be called the *free Euclidean measure.*

Set $\mathcal{C} := C(I\!R \to I\!R^d)$ and introduce $\Gamma_{\mathcal{C}}$ as the corresponding configuration space. Let W^λ be the Wiener measure on $\mathcal{C}$ with stationary measure $\lambda m(dx)$. Then we can construct a measure π_{W^λ} as the Poisson measure on $\Gamma_{\mathcal{C}}$ with intensity measure W^λ.

Theorem 3.3 *We can canonically identify the Euclidean measure* W^{π^λ} *with* π_{W^λ}.

Remark 3.2 . *The Markov process* Ξ_t *on the probability space* $(\Gamma_{\mathcal{C}}, \pi_{W^\lambda})$ *which is given by*

$$I\!R \ni t \mapsto \Xi_t(X) = X(t) \in \mathcal{M}_p(I\!R^d)$$

is called also Doob's equilibrium process, see [D]. This process was studied in details by R.L.Dobrushin [Do 1], see also [M-L],[ST]. An alternative description: we take any configuration $\gamma \in \Gamma$ and consider a collection of independent Brownian motions $\xi^x(t), t \geq 0,\ x \in \gamma$, started at $x \in I\!R^d$. Set

$$\Xi_t^\gamma = \sum_{x\in\gamma} \varepsilon_{\xi^x(t)} \in \mathcal{M}_p(I\!R^d).$$

We obtain the above process considering random initial date γ distributed w.r.t. the measure π^λ.

The equilibrium process gives the following useful heat semigroup representation:

Proposition 3.1 *For all $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we have*

$$(T_t^\Gamma F)(\gamma) = E[F(\Xi_t^\gamma)],\ \gamma \in \Gamma.$$

As a sketch of the proof we consider the heat semigroup on the exponentials. Then by Theorem 3.1 we have

$$(T_t^\Gamma e^{<f,\cdot>})(\gamma) = \prod_{x\in\gamma} (e^{\frac{t}{2}\triangle} e^f)(x).$$

But we obtain the same by a direct computation with the equilibrium process:

$$E[\exp(< f, \Xi_t^\gamma >] = E[\exp(\sum_{x\in\gamma} f(\xi^x(t))] =$$

$$\prod_{x\in\gamma} E[e^{f(\xi^x(t))}].$$

For more details we refer to [AKR 3, Section 6].

3.3 MODELS WITH INTERACTIONS: Hamiltonian strategy

The interaction H_I in terms of currents has the form (see (2.4))

$$\frac{1}{2}\int\int \rho(x)[\rho(y) - \delta(x-y)]V(x-y)m(dx)m(dy).$$

There exist several versions of the theory of generalized functions on Poisson spaces, see e.g. [KSS] and the references therein. As a technical tool of such a theory one can use the Fock space isomorphism we have mentioned above (see [IK]) or a general approach based on canonical Appell biorthogonal systems associated with given measures [ADKS], [KSWY]. In the following proposition and in considerations below we have in mind the construction of distribution spaces on the Poisson space (Γ, π^λ) presented in [KSS].

Proposition 3.2 *In the Poisson space representation the interaction H_I is realized as a potential given by the (generalized) function*

$$\mathbf{V}(\gamma) = \sum_{\alpha \subset \gamma} \Phi(\alpha),$$

where $\Phi(\alpha) = V(x-y)$ for $\alpha = \{x, y\}$.

Remark 3.3 *Using an embedding $\Gamma \subset \mathcal{D}'(I\!\!R^d)$ and a (generalized) kernel*

$$V^{(2)}(x,y) = V(x-y)$$

we can rewrite this potential in terms of the so-called Wick powers on Poisson space (see e.g. [IK]):

$$\frac{1}{2} < V^{(2)}, : \gamma^{\otimes 2} :> .$$

The latter can be expressed in terms of generalized Charlier polynomials, [KSS]. As a result, the potential $\mathbf{V}$ can be considered as a second order generalized polynomial on the Poisson space. Together with an interpretation of H_0 as the Laplace operator on Γ this gives the possibility to interpret the Hamiltonian $H_0 + H_I$ as an infinite dimensional "harmonic oscillator" Hamiltonian on Γ.

The situation which appears now is very similar to the one in polynomial models of constructive quantum field theory. In these models free Hamiltonians also can be realized as Dirichlet operators (on Gaussian spaces instead of Poissonian ones) and $P(\varphi)$ perturbations (cf. [Si], [GlJ], [AH-K]) have an interpretation as potentials on Gaussian spaces given by generalized functions (in fact, generalized polynomials), see e.g. [BK] for such interpretation and an annotated list of references. Having in the mind the analogy with

$P(\varphi)$ models, we can propose two closely related constructive approaches to the problem of an operator realization of the formal Hamiltonian $H_0 + \mathbf{V}$. Below we describe both strategies in a short form.

The singularity of the potential $\mathbf{V}$ is, of course, produced by the translation invariance of the kernel $V^{(2)}$ (even in the case of a non-singular V which in not identically zero). Let us take a family of "nice" kernels

$$V_\varepsilon^{(2)}(x,y), \quad ,\varepsilon > 0,$$

which approximate $V^{(2)}$ when $\varepsilon \to 0$ (in the weak sense). A typical cutoff is e.g.

$$V_\varepsilon^{(2)}(x,y) := \sigma_\varepsilon(x)\sigma_\varepsilon(y)V(x-y),$$

where $\sigma_\varepsilon \in \mathcal{D}$, $\sigma_\varepsilon \to 1$ uniformly on compacts in $I\!R^d$, when $\varepsilon \to 0$. To any such kernel there corresponds an *approximating potential*

$$\mathbf{V}_\varepsilon(\gamma) := \frac{1}{2} < V_\varepsilon^{(2)}, : \gamma^{\otimes 2} :>, \ \gamma \in \Gamma.$$

and an *approximating Hamiltonian*

$$H_{V_\varepsilon} := H_0 + \mathbf{V}_\varepsilon$$

as an essentially self-adjoint semibounded operator in $L^2(\pi^\lambda)$. The latter needs regularity assumptions about V which will be discussed separately, see Section 4 for typical conditions on V. We assume that H_{V_ε} has a ground state G_ε corresponding to the eigenvalue $E_\varepsilon = \inf \mathrm{sp}(H_{V_\varepsilon})$. Then $G_\varepsilon > 0$ π^λ-a.s. due to the ergodic property of the heat semigroup [AKR 3].

Now we apply to the approximating Hamiltonian the (both in quantum mechanics and constructive field theory) well-known *ground state renormalization*, see [GlJ], [Si] and e.g. [BK] (including the bibliographical notes in this book). To this end we introduce the vacuum measure

$$d\mu_\varepsilon(\gamma) := G_\varepsilon^2(\gamma) d\pi^\lambda(\gamma)$$

and construct the renormalized Hamiltonian

$$H_{V_\varepsilon, ren} = G_\varepsilon^{-1}(H_{V_\varepsilon} - E_\varepsilon)G_\varepsilon$$

as a positive self-adjoint operator in $L^2(\mu_\varepsilon)$.

Theorem 3.4 *The renormalized Hamiltonian is associated with the intrinsic Dirichlet form of the vacuum measure, i.e.,*

$$\forall\, F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma) \qquad (H_{V_\varepsilon}F,F)_{L^2(\mu_\varepsilon)} =$$

$$\frac{1}{2}\int_\Gamma < \nabla^\Gamma F, \nabla^\Gamma F >_{T(\Gamma)} d\mu_\varepsilon .$$

With the vacuum measure μ_ε there is associated a canonical Gelfand-Vilenkin type unitary representation of the current group in the space $L^2(\mu_\varepsilon)$ (which can be obtained also from the Poissonian one by using the same unitary isomorphism):

$$(G_\varepsilon^{-1}V_\lambda(\phi)G_\varepsilon F)(\gamma) = (V_{\mu_\varepsilon}F)(\gamma) =$$

$$F(\phi(\gamma))\sqrt{\frac{d\mu_\varepsilon(\phi(\gamma))}{d\mu_\varepsilon(\gamma)}}, \phi \in \mathrm{Diff}_0(I\!R^d),$$

$$(U_{\mu_\varepsilon}(f)F)(\gamma) \;=\; e^{i<f,\gamma>}F(\gamma),\; f \in \mathcal{D}.$$

As a result we have for any $\varepsilon > 0$ an *approximating system* in the physical Hilbert space $\mathcal{H}_\varepsilon = L^2(\mu_\varepsilon)$ with the energy operator $H_{\mu_\varepsilon} \geq 0$ and currents

$$\rho_\varepsilon(f), J_\varepsilon(v),\; f \in \mathcal{D}, v \in V_0(I\!R^d).$$

The quantum theory we have obtained satisfies all our "axioms" (i)-(iv) except for the translation invariance property (v).

The problem we should solve to complete the program is the following: how to remove the "cutoff" (i.e., $\varepsilon \to 0$)?

The main observation in the "Hamiltonian strategy" discussed, can be formulated as: "all one needs" is the limit vacuum measure, because (at least on a heuristic level) this measure reconstructs in a canonical way the rest of the theory (the Hamiltonian and the currents). This observation is closely related to the canonical fields approach (see the works of H.Araki [Ar], F.Coester and R.Haag [CoHa], S.Albeverio and R.Høegh-Krohn [AH-K]). We shall go back to this point of view in Section 4.

3.4 MODELS WITH INTERACTIONS: Euclidean strategy

First of all we describe the construction of the Euclidean measure for the renormalized Hamiltonian. Starting with the vacuum measure μ_ε and renormalized Hamiltonian $H^\Gamma_{\mu_\varepsilon}$ we introduce a Markov semigroup

$$T^\Gamma_{t,\varepsilon} := e^{-tH^\Gamma_{\mu_\varepsilon}},\ t \geq 0,$$

in the physical Hilbert space $\mathcal{H}_\varepsilon = L^2(\mu_\varepsilon)$ and corresponding to this semigroup and invariant measure μ_ε the renormalized Euclidean measure ν_ε on the path space $\Gamma_{\mathcal{C}}$.

For any $T > 0$ we introduce the probability measure

$$d\nu^T_\varepsilon(X) :=$$

$$\frac{1}{Z_T} \exp[- \sum_{\{\omega', \omega''\} \subset X} \int_{-T}^{T} V^{(2)}_\varepsilon(\omega'(t), \omega''(t))dt] d\pi_{W^\lambda}(X)$$

on $\Gamma_{\mathcal{C}}$.

Theorem 3.5 *The renormalized Euclidean measure ν_ε is the limit as $T \to \infty$ of the measures ν^T_ε in the sense of the convergence of integrals on bounded cylinder functions.*

Coming back to the constructive program we introduce the following two-parameter family of Euclidean measures (with cutoffs):

$$\{\nu^T_\varepsilon \mid T > 0,\ \varepsilon > 0\}$$

Then the Euclidean version of the renormalization procedure for the heuristic Hamiltonian (2.1) requires a construction of the limit Euclidean measure ($T \to \infty, \varepsilon \to 0$) and includes a "reconstruction theorem". The renormalized Hamiltonian is then the generator of the Markov semigroup associated with the limit Euclidean measure and the limit vacuum measure is nothig but an invariant measure of the limit process. Of course, all we have discussed is only a general scheme. A rigorous realization of this approach needs the development of corresponding technicqnes adapted to the problem. In our case this can be, in particular, a version of the cluster expansion method which (as we hope) can be applied at least in the small coupling constant regime. An essential additional technical difficulty in the study of the models discussed, is related to the absence of a spectral gap for the free Hamiltonian (due to Corollary 3.1).

Heuristically, the limit Euclidean measure $d\nu(X)$ has the representation:

$$\frac{1}{Z}\exp[-\sum_{\{\omega',\omega''\}\subset X}\int_{I\!R} V(\omega'(t)-\omega''(t))dt]d\pi_{W^\lambda}(X).$$

This representation produces an interesting relation to models of quantum statistical physics. Namely, let us consider a quantum continuous system with two-point interaction given by the potential V. Then the measure ν is nothing but the *Euclidean measure* corresponding to the Gibbs state at zero temperature for such a system. We refer to [KLRRS] for a general concept of Euclidean mesures in quantum statistical physics.

4 CANONICAL FIELD MODELS

4.1 CANONICAL QUANTIZATION: general scheme

Let μ be a $\mathrm{Diff}_0(I\!R^d)$ – quasi-invariant measure on Γ. We define a Gelfand-Vilenkin type unitary representation of $\mathrm{Diff}_0(I\!R^d)$ in the space $L^2(\mu)$ by the formula

$$(V_\mu(\phi)F)(\gamma) := F(\phi(\gamma))\sqrt{\frac{d\mu(\phi(\gamma))}{d\mu(\gamma)}} \tag{4.1}$$

for $\phi \in \mathrm{Diff}_0(I\!R^d)$.

The representation of the commutative group $\mathcal{D}$ is also standard:

$$(U_\mu(f)F)(\gamma) := e^{i<f,\gamma>}F(\gamma),$$

$f \in \mathcal{D}$. Then

$$W_\mu(f,\phi) := U_\mu(f)V_\mu(\phi)$$

gives a unitary representation of $\mathcal{D} \wedge \mathrm{Diff}_0(I\!R^d)$.

Consequently, their generators $\rho_\mu(f)$, $f \in \mathcal{D}$, and $J_\mu(v)$, $v \in V_0(I\!R^d)$, form a representation of the current algebra on $L^2(\mu)$.

The representation for the renormalized Hamiltonian from Section 3 motivates to define the *canonical Hamiltonian* as an operator associated to the intrinsic Dirichlet form of the measure μ:

$$\mathcal{E}_\mu^\Gamma(F,G) = \frac{1}{2}\int_\Gamma < \nabla^\Gamma F, \nabla^\Gamma G >_{T(\Gamma)} d\mu, \tag{4.2}$$

$$\mathcal{E}_\mu^\Gamma(F,G) = (H_\mu^\Gamma F, G)_{L^2(\mu)}, \tag{4.3}$$

$$H_\mu^\Gamma = -\frac{1}{2}\mathrm{div}_\mu^\Gamma \nabla^\Gamma. \tag{4.4}$$

Of course, the existence of H_μ^Γ needs regularity properties of μ, integration by parts formulas etc. (which do not hold in general).

Although it is explicitly stated in [GoMo 2] that "the construction of quasi-invariant measure when the configuration spaces are infinite dimensional is an in-general unsolved problem", there seems to be a quite large class of such measures obtainable via the classical Gibbsian approach (see the next subsections). Also the mentioned regularity properties necessary to construct H^Γ can be proved for such measures μ (cf. [AKR 4]). We shall summarize all this in Subsection 4.2 below.

Suppose that we have already solved all these problems and assume that μ is translation invariant, i.e., invariant w.r.t. the additive group $\mathbb{R}^d$ which acts on Γ in the natural way that gives unitary operators $Q_\mu(a), a \in \mathbb{R}^d$. Then the operators

$$\rho_\mu,\ J_\mu,\ H_\mu^\Gamma,\ Q_\mu$$

describe a quantum system which (formally) satisfies our axioms (i) - (v) (including currents conservation). By the results of the next subsection this program can be realized rigorously.

4.2 CANONICAL QUANTIZATION: Gibbs measures

The results of this subsection are proved in detail in [AKR 4]. Here we only summarize the framework and main statements.

Let us start with a description of the important class of so-called *Ruelle measures* on the configuration space Γ.

A pair potential is a Lebesgue measurable function $V : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $V(-x) = V(x)$. Any pair potential V defines a potential $\Phi = \Phi_V$ as follows: we set $\Phi(\gamma) := 0,\ |\gamma| \neq 2$ and $\Phi(\gamma) := V(x-y)$ for $\gamma = \{x, y\} \subset I\!R^d$. The conditional energy is defined for any $\Lambda \in \mathcal{O}_c(I\!R^d)$ (= open subsets with compact closures) by the formula

$$E_\Lambda^\Phi(\gamma) = E_\Lambda^\Phi(\gamma_\Lambda)\ +\ W_\Lambda(\gamma_\Lambda|\gamma_{X\setminus\Lambda}), \tag{4.5}$$

where

$$W_\Lambda(\gamma_\Lambda|\gamma_{X\setminus\Lambda})\ :=\ \sum_{x\in\gamma_\Lambda, y\in\gamma_{X\setminus\Lambda}} V(x-y)$$

describes the interaction energy between γ_Λ and $\gamma_{X\setminus\Lambda}$.

Definition 4.1 *For $\Lambda \in \mathcal{O}_c(I\!R^d)$ define for $\gamma \in \Gamma, \Delta \in \mathcal{B}(\Gamma)$*

$$\Pi_\Lambda^\Phi(\gamma, \Delta)\ :=\ 1_{\{Z_\Lambda^\Phi < \infty\}}(\gamma)[Z_\Lambda^\Phi(\gamma)]^{-1} \int_\Gamma 1_\Delta(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)$$
$$\times\ \exp[-E_\Lambda^\Phi(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)]\pi^z(d\gamma'),$$

where

$$Z_\Lambda^\Phi(\gamma)\ :=\ \int_\Gamma \exp[-E_\Lambda^\Phi(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)]\pi^z(d\gamma').$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a grand canonical Gibbs measure with interaction potential Φ if

$$\mu\Pi_\Lambda^\Phi\ =\ \mu\ \textit{for all}\ \Lambda \in \mathcal{O}_c(X). \tag{4.6}$$

Let $\mathcal{G}_{gc}(z, \Phi)$ denote the set of all such probability measures μ.

We refer the reader to the books [Ge], [Pr] for detailed discussions of this concept. For every $r = (r^1, \dots, r^d) \in Z\!\!Z^d$ we define the cube

$$Q_r\ :=\ \{x \in I\!R^d \mid r^i - \frac{1}{2} \leq x^i < r^i + \frac{1}{2}, 1 \leq i \leq d\}.$$

These cubes form a partition of $I\!R^d$. For any $\gamma \in \Gamma$ we set $\gamma_r := \gamma_{Q_r} = \gamma \cap Q_r,\ r \in Z\!\!Z^d$. For $N \in I\!N$ let Λ_N be the hypercube of length $2N-1$ centered at the origin in $I\!R^d$; Λ_N is then a union of $(2N-1)^d$ unit cubes of the form Q_r. We will also regard Λ_N as a subset of $Z\!\!Z^d$ by letting Λ_N be represented by $\Lambda_N \cap Z\!\!Z^d$.

We will use the following *conditions on the interaction* which are standard in statistical physics of classical continuous systems.

(S) (*Stability*) There exists $B \geq 0$ such that for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and for all $\gamma \in \Gamma_\Lambda$

$$E_\Lambda^\Phi(\gamma) \ \geq \ - B|\gamma|. \tag{4.7}$$

A condition stronger than stability is the following.

(SS) (*Superstability*) There exist $A > 0, \ B \geq 0$ such that if $\gamma \in \Gamma_{\Lambda_N}$ for some N, then

$$E_{\Lambda_N}^\Phi(\gamma) \ \geq \ \sum_{r \in \Lambda_N} [A|\gamma_r|^2 \ - \ B|\gamma_r|]. \tag{4.8}$$

(LR) (*Lower regularity*) There exists a decreasing positive function $a : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r \in \mathbb{Z}^d} a(\|r\|) \ < \ \infty, \tag{4.9}$$

and for any Λ', Λ'' which are each finite unions of unit cubes of the form Q_r and disjoint, with $\gamma' \in \Gamma_{\Lambda'}, \ \gamma'' \in \Gamma_{\Lambda''}$,

$$W(\gamma'|\gamma'') \geq \ - \sum_{r' \in \Lambda', r'' \in \Lambda''} a(\|r' - r''\|) \, |\gamma'_{r'}| \cdot |\gamma''_{r''}|. \tag{4.10}$$

Useful sufficient conditions on the potential (ensuring (SS) and (LR)) are given by the following Dobrushin-Fisher -Ruelle criterium [Do 2],[Ru]:

Proposition 4.1 *Let* $0 < d_1 < d_2 < +\infty$ *and let*

$$s_1 : [0, d_1] \to \mathbb{R} \cup \{+\infty\}, \ \ s_2 : [d_2, +\infty) \to \mathbb{R}$$

be positive, decreasing and such that

$$\int_0^{d_1} t^{d-1} s_1(t) dt = +\infty, \ \int_{d_2}^\infty t^{d-1} s_2(t) dt < +\infty. \tag{4.11}$$

If the pair potential V *is bounded below and satisfies*

$$V(x) \ \geq \ s_1(\|x\|) \ \ for \ \ \|x\| \leq d_1,$$

$$|V(x)| \ \leq \ s_2(\|x\|) \ \ for \ \ \|x\| \geq d_2,$$

then V *is superstable and lower regular.*

Definition 4.2 *A probability measure* μ *on* $(\Gamma, \mathcal{B}(\Gamma))$ *is called tempered if* μ *is supported by*

$$S_\infty := \cup_{m=1}^{\infty} S_m,$$

where

$$S_m := \{\gamma \in \Gamma \mid \forall N \in I\!N \sum_{r \in \Lambda_N} |\gamma_r|^2 \leq m^2 |\Lambda_N| \ \}. \tag{4.12}$$

We denote by $\mathcal{G}_{gc}^t(V,z) \subset \mathcal{G}_{gc}(V,z)$ the set of all tempered grand canonical Gibbs measures (*Ruelle measures* for short). By [Ru] the set $\mathcal{G}_{gc}^t(V,z)$ is non-empty for all $z > 0$ and any superstable, low regular potential V which satisfies the *integrability condition* (I):

$$\int_{I\!R^d} |1 - e^{-V(x)}| m(dx) \ < \ +\infty. \tag{4.13}$$

Let us note that if V is semibounded from below then condition (I) is equivalent to the integrability of V on the set $I\!R^d \setminus \{V > 1\}$ if $m(\{V > 1\} < \infty$. Of course, the stability condition (S) implies such semiboundedness.

Let us introduce the following additional condition on the potential V:

(D) (*differentiability*) The weak derivative ∇V of V exists and satisfies

$$\nabla V \in L^1(I\!R^d, e^{-V} dm) \cap L^2(I\!R^d, e^{-V} dm). \tag{4.14}$$

Lemma 4.1 *Let the potential* V *satisfy conditions (SS), (LR), (I) and (D). For any vector field* $v \in V_0(I\!R^d)$ *we consider the function*

$$\Gamma \ni \gamma \mapsto L_v^V(\gamma) \ := \ - \sum_{\{x,y\} \subset \gamma} < \nabla V(x-y), v(x) - v(y) >_{I\!R^d} . \tag{4.15}$$

Then for any Ruelle measure μ *and all* $v \in V_0(I\!R^d)$ *we have*

$$L_v^V \in L^2(\mu). \tag{4.16}$$

Remark 4.1 *An analysis of the proof shows that in the definition of the differentiability property (D) it is enough to assume that* ∇V *belongs to the weak Lebesgue spaces* $L_w^1(e^{-V} dm)$ *and* $L_w^2(e^{-V} dm)$.

Theorem 4.1 *Let V be a pair potential with properties (SS), (LR), (I) and (D). For any $v \in V_0(I\!R^d)$ we introduce the following function on Γ:*

$$\Gamma \ni \gamma \mapsto B_v^V(\gamma) \ := \ L_v^V(\gamma) \ + \ B_v^{\pi^z}(\gamma). \tag{4.17}$$

Then for any Ruelle measure $\mu \in \mathcal{G}_{gc}^t(V,z)$ and all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ the following integration by parts formula holds:

$$\int_\Gamma \nabla_v^\Gamma F \; G \, d\mu \ = \ - \int_\Gamma F \; \nabla_v^\Gamma G \, d\mu \ - \ \int_\Gamma F \; G \; B_v^V \, d\mu, \tag{4.18}$$

or

$$(\nabla_v^\Gamma)^* \ = \ - \nabla_v^\Gamma \ - \ B_v^V, \tag{4.19}$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ in $L^2(\mu)$.

Corollary 4.1 *Let V and μ be as in Theorem 4.1. Then for any $\psi \in Diff_0(I\!R^d)$ the measures $\mu \circ \psi$ and μ are equivalent.*

Moreover, let $\Lambda \in \mathcal{O}_c(I\!R^d)$ be such that ψ is trivial outside Λ. Then

$$\frac{d(\mu \circ \psi)(\gamma)}{d\mu(\gamma)} \ = \ \exp[E_\Lambda^\Phi(\gamma) \ - \ E_\Lambda^\Phi(\psi(\gamma))] \frac{d(\pi^z \circ \psi)(\gamma)}{d\pi^z(\gamma)}, \tag{4.20}$$

where

$$\frac{d(\pi^z \circ \psi)(\gamma)}{d\pi^z(\gamma)} = \prod_{x \in \gamma} J_m^\psi(x) \exp[z \int (1 - J_m^\psi(x)) m(dx)], \tag{4.21}$$

and $J_m^\psi = det[\nabla \psi]$ is the Jacobian determinant of ψ (with respect to Lebesgue measure).

The Radon-Nikodym derivative depends explicitly on the density (i.e., activity parameter) $z > 0$ (in contrast to the integration by parts formula). It gives us the possibility to characterize grand canonical Gibbs measures as follows.

We introduce

$$E_{\mathrm{rel}}^\Phi(\psi(\gamma)|\gamma) := E_\Lambda^\Phi(\psi(\gamma)) \ - \ E_\Lambda^\Phi(\gamma) \ = \tag{4.22}$$

$$\sum_{\{x,y\} \subset \gamma} [V(\psi(x) - \psi(y)) - V(x - y)],$$

where we have used any $\Lambda \in \mathcal{O}_c(I\!R^d)$ such that ψ is trivial outside Λ. It is natural to call E^{Φ}_{rel} the *relative energy* functional. From the physical point of view this functional decsribes the variation of the (formal) potential energy functional $E^{\Phi}(\gamma) = \sum_{\{x,y\}\subset\gamma} V(x-y)$ when the configuration γ is locally deformed by the diffeomorpism ψ. From (4.22) and (4.20) we have

$$\frac{d(\mu\circ\psi)(\gamma)}{d\mu(\gamma)} = \exp[-\, E^{\Phi}_{\mathrm{rel}}(\psi(\gamma)|\gamma)]\frac{d(\pi^z\circ\psi)(\gamma)}{d\pi^z(\gamma)}, \tag{4.23}$$

Corollary 4.2 *For any Ruelle measure μ the current operators in $L^2(\mu)$ on the domain $\mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)$ have the following form*

$$(\rho_\mu(f)F)(\gamma) = <f,\gamma> F(\gamma),\ f\in\mathcal{D},$$

$$(J_\mu(v)F)(\gamma) = \frac{1}{i}\nabla^{\Gamma}_v F(\gamma) + \frac{1}{2i}B^{\mu}_v(\gamma)F(\gamma).$$

The canonical Hamiltonian associated with the intrinsic Drichlet form (cf. (4.2)) of the measure μ has the following representation on $\mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)$

$$(H^{\Gamma}_{\mu}F)(\gamma) = (H^{\Gamma}_{\pi^z}F)(\gamma) +$$

$$\frac{1}{2}\sum_{\{x,y\}\subset\gamma} <\nabla V(x-y), \nabla^{\Gamma}F(\gamma,x) - \nabla^{\Gamma}F(\gamma,y)>_{I\!R^d}.$$

Using the explicit form of the currents and the Hamiltonian we can *compute* the commutation relation

$$[\rho_\mu(f), H^{\Gamma}_{\mu}] = iJ_\mu(\nabla f)$$

for $f\in\mathcal{D}$, i.e., we have the *currents preservation property* which gives an important *conservation law* for the constructed quantum system. It is known that among all Ruelle measures there exist also *translation invariant* ones (see e.g. [Do 2], [Ru]). As a result, any such measure creates in the described way a quantum theory which satisfies axioms (i) - (v).

Note that the Euclidean measure corresponding to our quantum system is nothing but the path space measure which is generated by the stochastic dynamics of an interacting particle system associated with the considered Gibbs measure. We refer to [AKR 4], [AKMR], [Os], [Yo] for related considerations.

4.3 FROM VACUUM MEASURE TO INTERACTION

It Subsection 4.2 we have used the Gibbs measures as vacuum measures for the canonical construction of non-relativistic quantum models. There exists an interesting *inverse problem.* Namely, can we reconstruct a potential (as a generalized function on Γ) which generates a given vacuum measure?

A version of this question is well-known in quantum mechanics. Let us recall the corresponding construction: We consider a quantum mechanical system in $I\!\!R^d$ with Hamiltonian given by a Schrödinger operator

$$H = -\frac{1}{2}\Delta + V.$$

We assume that H has a ground state

$$g(x) = e^{-W(x)}.$$

The "inverse problem" here is the following question: how can one reconstruct V from the ground state?

By a direct computation we have

$$(-\frac{1}{2}\Delta g)(x) = \frac{1}{2}(\Delta W(x) - |\nabla W(x)|^2)g(x),$$

i.e.,

$$V(x) = \frac{1}{2}[|\nabla W(x)|^2 - \Delta W(x)].$$

Note that such a simple relation between the ground state and the potential for Schrödinger operators gives a possibility to construct renormalized Hamiltonians for many interesting singular potentials, see [AH-KS].

Now we would like to address the analogous question in the situation of Subsection 4.2. Let us consider a Gibbs measure μ_W on Γ which is defined as before by a two-point potential $2W$. We assume W to satisfy the conditions of Subsection 4.2 and additionally suppose $W \in C^2(I\!\!R^d \setminus \{0\})$.

Let us interpret μ_W as a vacuum measure for the Hamiltonian

$$H_0 + \mathbf{V}$$

with an unknown potential $\mathbf{V}(\gamma)$, to be determined.

An approximation of $W(x-y)$ by regular kernels (similar to the one in Subsection 3.3) gives a sequence of approximating ground state measures which are absolutely continuous w.r.t. the Poisson measure π^z. Using an "inverse ground state transformation" we can reconstruct then an approximating potential and (removing the cutoff) a formal interaction potential. Realizing this approach we obtain the following result.

Theorem 4.2 *The following representation is true for the potential* $\mathbf{V}$ *(as a generalized function on* Γ*)*

$$\mathbf{V}(\gamma) = \sum_{\{x,y\}\subset\gamma} V(x-y), \tag{4.24}$$

where

$$V(x) = \frac{1}{2}[|\nabla W(x)|^2 - \Delta W(x)]. \tag{4.25}$$

Acknowledgements

We would like to thank Robert Minlos and Ludwig Streit for their interest in this work and many valuable discussions. Financial support of the DFG through SFB 237 Bochum-Düsseldorf-Essen, SFB 343 Bielefeld, and Project AL 214/9-2, and the EC-Science Project $SCI\ CT^*92-0784$, and the Project INTAS-378 is gratefully acknowledged.

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