ALGORITHMIC ORBIT CLASSIFICATION FOR PARABOLIC SUBGROUPS OF SIMPLE ALGEBRAIC GROUPS

U. Jürgens and G. Röhrle*

Abstract. For an algebraic group R acting morphically on an algebraic variety X the modality of the action, $\operatorname{mod}(R : X)$, is the maximal number of parameters upon which a family of R-orbits on X depends.

Let G be a reductive algebraic group defined over an algebraically closed field K. Let P be a parabolic subgroup of G. Then P acts on its unipotent radical P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of P is defined as $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$.

In this paper we discuss an algorithm which is used to compute upper bounds for $\operatorname{mod} P$ along with some results obtained by this algorithm. One is a classification of parabolic subgroups P of simple algebraic groups G of semisimple rank 2 and modality 0. For parabolic subgroups of semisimple rank 3 we present some partial results. This extends results from [11] and [14] where the cases of semisimple rank 0 and 1 are handled. For exceptional groups G we show that $P \subset G$ has modality zero provided the class of nilpotency of P_u is at most two. The analogous result for classical groups is proved in [20, Theorem 1.3].

For Borel subgroups B of simple groups we are able to determine the value for $\operatorname{mod} B$ in some small rank cases by combining lower bounds for $\operatorname{mod} B$ from [21] with upper bounds provided by the algorithm.

1. Introduction

Throughout, G is a simple algebraic group over an algebraically closed field K of characteristic $p \geq 0$ and P is a parabolic subgroup of G. Then P acts on its unipotent radical P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of the action of P on $\mathfrak{p}_u$, denoted by $\operatorname{mod}(P : \mathfrak{p}_u)$, is the maximal number of parameters upon which a family of P-orbits on P_u depends; likewise for the action of P on P_u. The modality of P is defined as $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$; see Section 2 for a precise definition.

In this paper we present a number of results which are obtained by an algorithm that is used to calculate upper bounds for $\operatorname{mod} P$.

In [14] the general problem was posed of determining each parabolic subgroup P of G with $\operatorname{mod} P = 0$. The cases for Borel and semisimple rank one parabolic subgroups were classified in [11] and [14] respectively. We extend these results by classifying parabolic subgroups of G of modality zero and semisimple rank two.

It is shown in [20] for classical groups that $\operatorname{mod} P = 0$ provided the nilpotency class of P_u is at most 2. Here we prove the analogous result for exceptional groups.

*Research supported in part by a grant from the Deutsche Forschungsgemeinschaft (DFG).

With the exception of Lemmas 3.11 and 3.12 the finiteness parts of each result in Section 3 and the upper bounds for $\operatorname{mod} B$ in Section 4 are obtained using our algorithm. We do not furnish additional proofs for any of these cases.

A complete classification of modality zero parabolic subgroups of classical groups is obtained in [10, Theorem 1.1]. Whence we shall focus here especially on modality zero parabolic subgroups of exceptional groups.

For Borel subgroups B of simple groups we determine $\operatorname{mod} B$ in some small rank cases by combining lower bounds for $\operatorname{mod} B$ from [21] with upper bounds provided by the algorithm.

Since the modalities of a linear action and its dual agree (this is an immediate consequence of the proof in [16], e.g., see [12, §2] or [22]), in particular we have $\operatorname{mod}(P : \mathfrak{p}_u) = \operatorname{mod}(P : \mathfrak{p}_u^*)$, where $\mathfrak{p}_u^*$ is the dual of $\mathfrak{p}_u$. Thus we get analogous results for the coadjoint action of P on $\mathfrak{p}_u^*$.

Our program generalizes the algorithm outlined in [7] which was designed to analyze the orbit structure of a Borel subgroup B for the adjoint and coadjoint actions on $\mathfrak{b}_u$ and on $\mathfrak{b}_u^*$. It is based on the usual adjoint action of P on $\mathfrak{p}_u$. The results obtained apply provided $\operatorname{char} K$ does not divide the structure constants of the Chevalley commutator relations for the groups involved. In particular, if the Dynkin-diagram of G is simply laced, no restrictions on $\operatorname{char} K$ are required. For details and various aspects of the algorithm we refer the reader to Section 5 below.

After recalling some definitions and some results from [14], [19], and [22], we present our main results on parabolic subgroups of modality zero in Section 3. In the subsequent chapter we record the values of the modality of Borel subgroups in some simple groups of small rank which were obtained with the aid of this algorithm. Finally, in our last section we outline the algorithm and discuss the various reduction procedures involved in it.

The basic machinery for investigating the modality of parabolic subgroups of reductive groups was introduced in [14]. Apart from [14] itself there are several recent articles related to this subject, such as [6], [9], [10], [13], and [22].

Our general reference for algebraic groups is Borel [3] and for information on root systems we cite Bourbaki [5]. The simple roots in a base of a root system of G are indexed in accordance with *loc. cit.* (Planches I - IX).

2. Recollection, Notation and some Preliminary Results

We denote the Lie algebra of G by $\operatorname{Lie} G$ or by $\mathfrak{g}$, its identity component by G^0 and by e the identity element of G; likewise for subgroups. The Lie algebra of P_u is denoted by $\mathfrak{p}_u$.

Recall from [14] that for an algebraic group R acting morphically on some algebraic variety X the *modality* of this action, is defined as $\operatorname{mod}(R : X) = \max_Z \min_{z \in Z} \operatorname{codim}_Z R^0 \cdot z$, where Z runs through all irreducible R^0-invariant subvarieties of X. The invariant $\operatorname{mod}(R : X)$ of this action measures the maximal number of parameters on which a family of R-orbits on X depend upon, and in particular, its value is 0 precisely when R admits only a finite number of orbits on X. The notion of modality originates in the work of Arnold [1]; though our definition here is due to Vinberg [24] and [15, 5.2].

We call $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$ the *modality* of P, cf. [14].

Let T be a maximal torus in G and Ψ the set of roots of G with respect to T. Fix a Borel subgroup B of G containing T and let Π be the set of simple roots of Ψ defined by B. We may assume that every parabolic subgroup of G under consideration contains B, i.e., is standard. For a subset J of Π we denote by P_J the standard parabolic subgroup corresponding to J such that $P_\varnothing = B$.

Throughout, ϱ denotes the highest root in Ψ. Recall that a prime is called *bad* for G if it divides the coefficients of a simple root in ϱ [23, §I, 4], else it is called *good* for G. Furthermore, we say that a prime is *very bad* for G in case it divides one of the structure constants of the Chevalley commutator relations for G.

Let H be a closed subgroup of G normalized by T (i.e., H is a *regular* subgroup of G, cf. [8]); likewise for subalgebras of $\mathfrak{g}$. In that case the root spaces of $\mathfrak{h}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of H with respect to T, $\Psi(H)$, is a subset of Ψ. If $\operatorname{char} K$ is zero or a prime which is not very bad for G, then $\Psi(H)$ is closed under addition in Ψ. Furthermore, if H is reductive and regular, then $\Psi(H)$ is a semisimple subsystem of Ψ. For a root α of G we denote by U_α the corresponding one-parameter unipotent subgroup of G. For every root α we choose a generator x_α of the corresponding root space $\operatorname{Lie} U_\alpha = \mathfrak{g}_\alpha$ of $\mathfrak{g}$.

The *support* of a subset S of $\mathfrak{g}$, denoted by $\operatorname{supp} S$, is the set of all roots α such that the projection from $\mathfrak{g}$ onto $\mathfrak{g}_\alpha$ is non-trivial when restricted to S.

We denote the semisimple rank of P by $\operatorname{rank}_{ss} P$ and the Weyl group of some Levi subgroup of P by W_P.

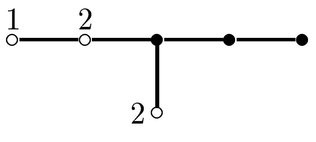
Let $\beta \in \Psi^+$. Write $\beta = \sum_{\alpha\in\Pi} c_\alpha(\beta)\alpha$, with $c_\alpha(\beta) \in \mathbb{Z}_0^+$ for each $\alpha \in \Pi$. Recall that $\sum_{\alpha\in\Pi} c_\alpha(\beta)$ is the *height* of β. Let J be a subset of Π. We call $\operatorname{ht}_J \beta := \sum_{\alpha\in\Pi\setminus J} c_\alpha(\beta)$ the *J-height* of β, also see [2].

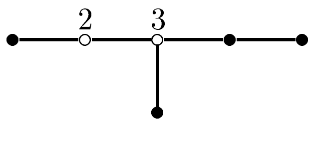
First we recall Theorem 3.1 from [19] which is valid in any characteristic by [22, Remark 3.13]. Here and later on $\ell(P_u)$ denotes the class of nilpotency of P_u.

Proposition 2.1. *Let G be a simple algebraic group and P a parabolic subgroup of G. Then* $\operatorname{mod} P > 0$ *provided one of the following holds:*

(i) *G is of type A_r, B_r, C_r, E_8, F_4, or G_2, and $\ell(P_u) \geq 5$;*
(ii) *G is of type D_r and $\ell(P_u) \geq 6$;*
(iii) *G is of type D_r, $\ell(P_u) = 5$, and P is stable under the automorphism of G of order 2 stemming from the interchange of the simple roots α_{r-1} and α_r;*
(iv) *G is of type E_6 or E_7, and $\ell(P_u) \geq 6$;*
(v) *G is of type E_6 or E_7, $\ell(P_u) = 5$, and P is not conjugate to one of the parabolic subgroups listed in Table 1 below.*

In the subsequent table we indicate the standard Levi components of P in each of the diagrams by coloring the corresponding nodes in the Dynkin-diagram for G. Furthermore, the coefficients attached to the simple roots outside the Levi subgroup sum up to $\ell(P_u)$, according to Remark 2.5 below.

1. 

2. 

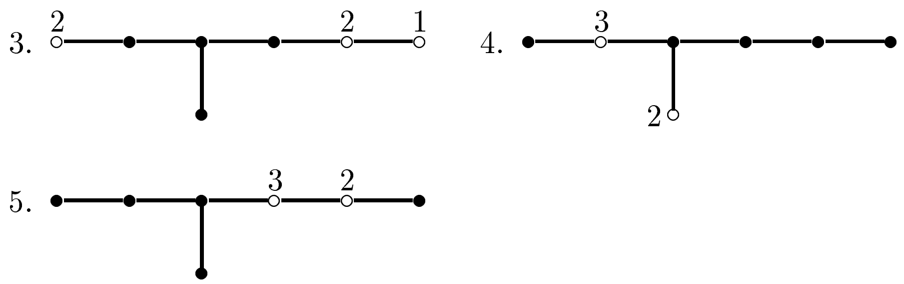


Table 1: The undecided cases for exceptional groups with $\ell(P_u) = 5$.

We are able to resolve the cases 1 and 3 from Table 1 in this paper.

We require Theorem 1.2 from [22], see also [13, Theorem 4], [14, Remark 2.14]:

Proposition 2.2. *Suppose that* $\operatorname{char} K$ *is zero or a good prime for* G*. Let* H *be a closed reductive subgroup of* G *normalized by the maximal torus* T *of* G *in* P*. Set* $Q := P \cap H$*. Then* $\operatorname{mod} Q \leq \operatorname{mod} P$*.*

Note that there are no characteristic restrictions in Proposition 2.2, while the corresponding results in [13] and [14] are only valid in characteristic 0.

The next lemma [22, Theorem 1.3] yields analogous results for $\operatorname{mod}(P : P_u)$.

Lemma 2.3. *Suppose that* $\operatorname{char} K$ *is either zero or a good prime for* G*. Then* $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$*.*

We need an elementary observation, cf. [19, Lemma 2.8].

Lemma 2.4. *Let* $Q \subseteq P$ *be parabolic subgroups of* G*. Then* $\operatorname{mod} P \leq \operatorname{mod} Q$*.*

We say that $P \subset G$ is *minimally finite*, if $\operatorname{mod} P = 0$ and for any parabolic subgroup R of G which is properly contained in P, we have $\operatorname{mod} R > 0$. In order to classify parabolic subgroups of modality zero it suffices to determine the minimally finite ones, because of Lemma 2.4.

Remark 2.5. Suppose that $\operatorname{char} K$ is 0 or else not a very bad prime for G. Then there are no degenerate commutators in G and for $P = P_J$ a standard parabolic subgroup of G, the class of nilpotency of P_u can be determined as follows, cf. [4, Prop. 4.7(iii)]. Let $\beta \in \Psi^+$ and write $\beta = \sum_{\alpha \in \Pi} c_\alpha(\beta)\alpha$, with $c_\alpha(\beta) \in \mathbb{Z}_0^+$ for each $\alpha \in \Pi$. Then $\ell(P_u) = \sum_{\alpha \in \Pi \setminus J} c_\alpha(\varrho)$, i.e., $\ell(P_u)$ is precisely the J-height of ϱ.

3. Parabolic subgroups of modality zero

We maintain the notation from the previous sections. Here we present some results on parabolic subgroups of modality zero. In each of these the upper bound zero for $\operatorname{mod} P$ is established using our algorithm with the exception of Lemmas 3.12 and 3.13. Furthermore, in each of the Theorems 3.1 through 3.3 and in Proposition 3.4 below, the fact that $\operatorname{mod} P$ is positive, whenever P does not satisfy any of the given conditions, follows directly from Proposition 2.1.

In [10, Theorem 1.1] a complete classification of modality zero parabolic subgroups of classical groups is obtained, and thus the parts of the subsequent results relating to classical groups can be derived directly from *loc. cit.*

The first result here is due to Kashin [11] in case $\operatorname{char} K = 0$.

Theorem 3.1. *Let $B \subset G$ be a Borel subgroup of G. Then* $\operatorname{mod} B = 0$ *if and only if one of the following holds:*

(i) *G is of type A_r for $r \leq 4$;*
(ii) *G is of type B_2.*

The fact that $\operatorname{mod} B = 0$ in the cases (i) and (ii) of Theorem 3.1 is also proved in [7] assuming that p is not very bad for G.

This classification appears in [14, Corollary 1.4] in case that $\operatorname{char} K = 0$:

Theorem 3.2. *Suppose* $\operatorname{rank}_{ss} P = 1$. *Then* $\operatorname{mod} P = 0$ *if and only if one of the following holds:*

(i) *G is of type A_r for $r \leq 5$;*
(ii) *G is of type B_r or C_r for $r \leq 3$;*
(iii) *G is of type D_4, or G_2.*

Next is the classification of semisimple rank two parabolics of modality zero.

Theorem 3.3. *Suppose* $\operatorname{rank}_{ss} P = 2$. *Then* $\operatorname{mod} P = 0$ *if and only if one of the following holds:*

(i) *G is of type A_r for $r \leq 6$;*
(ii) *G is of type B_r or C_r for $r \leq 4$;*
(iii) *G is of type D_r for $r \leq 5$ and if $r = 5$, then P is not conjugate to $P_{\{\alpha_4,\alpha_5\}}$;*
(iv) *G is of type F_4 and P is conjugate to $P_{\{\alpha_2,\alpha_3\}}$.*

For parabolic subgroups of semisimple rank three we present some partial results.

Proposition 3.4. *Suppose G is not of type D_r or F_4 and $P = LP_u \subset G$ satisfies* $\operatorname{rank}_{ss} P = 3$. *Then* $\operatorname{mod} P = 0$ *if and only if one of the following holds:*

(i) *G is of type A_r for $r \leq 7$;*
(ii) *G is of type B_r or C_r for $r \leq 5$;*
(iii) *G is of type E_6 and L' is of type A_3, i.e., L' is simple.*

Remarks 3.5. The case in Proposition 3.4(iii) when $J = \{\alpha_4, \alpha_5, \alpha_6\}$ is precisely the first one from Table 1 above. This is the only known event in exceptional groups when $\ell(P_u) = 5$ and $\operatorname{mod} P = 0$. In the classical situation such instances only occur for groups of type D_r, cf. [10].

Observe that, in view of Proposition 2.1, the parabolic subgroups in Theorems 3.2 through 3.4 are all minimally finite when r is chosen to be maximal. This also applies to the F_4 incident of Theorem 3.3 and to the E_6 cases of Proposition 3.4.

For the remainder of this section we focus on two results which provide natural extensions of known statements for classical groups. For these the first one was proved in [20, Theorem 1.4] and the second one in [14, Theorem 5.4], see also [10].

Theorem 3.6. *Let G be a simple algebraic group and $P \subset G$ a parabolic subgroup of G with $\ell(P_u) \leq 2$. Then* $\operatorname{mod} P = 0$.

Corollary 3.7. *Let G be a simple algebraic group and $P = LP_u$ a maximal parabolic subgroup of G such that L' is simple. In case G is of type E_8 suppose that P is not conjugate to $P_{\Pi\setminus\{\alpha_2\}}$. Then* $\operatorname{mod} P = 0$.

For exceptional groups Corollary 3.7 follows from Theorems 3.2, 3.3, 3.6 and Lemma 2.4.

Proof of Theorem 3.6. We only need to address the exceptional cases here. If $\ell(P_u) = 1$, then P_u is abelian and the desired finiteness result follows from the subsequent Lemma 3.9. The case when $\dim P_u' = 1$ was treated in [14, Proposition 5.1]. The argument in *loc. cit.* applies in any characteristic.

For one of the two remaining cases for E_7 we present an argument by direct analysis in Lemma 3.11 below. In E_8 there are two classes of parabolic subgroups P with $\ell(P_u) = 2$, in one $\dim P_u' = 1$ and in the other P is conjugate to $P_{\Pi\setminus\{\alpha_1\}}$. In this second case we also give a direct proof for the finiteness property in Lemma 3.12 below. In each of the remaining cases the result follows from Lemma 2.4, Theorems 3.2 and 3.3, Proposition 3.4 and Lemma 3.8.

Lemma 3.8. *Let G be of type E_7. Then the number of P-orbits on P_u is finite if $P = P_J$ and J is $\{\alpha_1, \alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$, $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$, or $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_7\}$.*

For our further analysis we require a special case of a general theorem due to R.W. Richardson [17, Theorem E].

Lemma 3.9. *Let $P = LP_u \subset G$ be a parabolic subgroup of G. Then L operates on each of the quotients of successive members of the lower central series of $\mathfrak{p}_u$ with a finite number of orbits.*

Remark 3.10. For a given parabolic subgroup P of G we aim to prove that $\mathrm{mod}\, P = 0$ as follows. Say, $P = P_J$ for some $J \subset \Pi$. Suppose that $\mathrm{mod}(P : \mathfrak{p}_u') = 0$ is known inductively. By Lemma 3.9 there are only finitely many P-orbits on $\mathfrak{p}_u/\mathfrak{p}_u'$. So $\mathfrak{p}_u$ is a finite union of subsets of the form $P \cdot x + \mathfrak{p}_u'$ for x in $\mathfrak{p}_u$. We may assume that all roots in the support of x have J-height 1. It suffices to show that the subset $x + \mathfrak{p}_u'$ is contained in a finite union of P-orbits on $\mathfrak{p}_u$. So, we need to study orbits passing through elements of the form $y = x + x'$, where each root in $\mathrm{supp}\, x$ has J-height 1, and x' runs through $\mathfrak{p}_u'$. We can conclude that $\mathrm{mod}\, P = 0$, once we have shown that either y is P-conjugate to x, i.e., that x' can be eliminated, or $\mathrm{supp}\, y$ is a linearly independent set of cardinality at most $\mathrm{rank}\, G$. In the latter case we can use the action of the maximal torus T in P to scale all coefficients of x_γ in y to 1. So that $\sum x_\gamma$ is a representative of the P-orbit passing through y, where the sum is taken over all γ in $\mathrm{supp}\, y$.

Subsequently, we use the Bourbaki notation for elements in the root systems of the simple groups of type E_7 and E_8.

Lemma 3.11. *Let G be of type E_7 and let $P = P_{\Pi\setminus\{\alpha_2\}}$. Then* $\mathrm{mod}\, P = 0$.

Proof. Let P be as above. Since $\ell(P_u) = 2$, the subalgebra $\mathfrak{p}_u'$ is abelian, and thus $\mathrm{mod}(P : \mathfrak{p}_u') = 0$, by Lemma 3.9. Thus it suffices to show that for an arbitrary element x in $\mathfrak{p}_u \setminus \mathfrak{p}_u'$ the set $x + \mathfrak{p}_u'$ is contained in a finite union of P-orbits on $\mathfrak{p}_u$. Let $y = x + x'$ be in $\mathfrak{p}_u \setminus \mathfrak{p}_u'$, where each root in $\mathrm{supp}\, x$ is of J-height 1, and $x' \in \mathfrak{p}_u'$. Note that $\dim \mathfrak{p}_u' = 7$ in this instance. We proceed as indicated in Remark 3.10.

Because all roots of J-height 1 are conjugate under W_P, cf. [2, Lemma 1], we may assume that α_2 is in $\mathrm{supp}\, x$. Then all roots of J-height 1 which are of the form

$\alpha_2 + \alpha$ can be removed from $\operatorname{supp} x$. In particular, we may suppose that all roots with coefficient 1 at α_4 have been removed from the support of x. Furthermore, by applying suitable elements from the root subgroups relative to $\begin{smallmatrix}123210\\1\end{smallmatrix}$, $\begin{smallmatrix}123211\\1\end{smallmatrix}$, $\begin{smallmatrix}123221\\1\end{smallmatrix}$, and $\begin{smallmatrix}123321\\1\end{smallmatrix}$, we can remove the four roots from the support of x' which are the sums of these with α_2. We distinguish several cases.

(**I**) If there is no other root of J-level 1 in the support of x, then $|\operatorname{supp} y| \leq 4$ and all the roots in $\operatorname{supp} y$ are linearly independent. Thus we can use the action of T to obtain a finite set of representatives.

(**II**) So we consider the case when $|\operatorname{supp} x| \geq 2$. First suppose that the only roots left in $\operatorname{supp} x$ have coefficient 3 at α_4. Then after possibly applying elements from the Weyl group W_P we may assume that $\gamma = \begin{smallmatrix}123210\\1\end{smallmatrix}$ is in the support of x. We can remove the remaining roots of J-level 1, all of which have coefficient 3 at α_4. This does not reintroduce roots which have already been removed from $\operatorname{supp} y$ and fixes the summand x_{α_2} of y. By operating with suitable elements of the root subgroups relative to the roots $\begin{smallmatrix}001111\\1\end{smallmatrix}$, $\begin{smallmatrix}011111\\1\end{smallmatrix}$, and $\begin{smallmatrix}111111\\1\end{smallmatrix}$, we can remove the remaining three roots from $\operatorname{supp} x'$. That is, we can eliminate x'. Thus we have $y = x_{\alpha_2} + x_\gamma$ as a representative of this P-orbit.

(**III**) If there is a root of J-level 1 in the support of x with coefficient 2 at α_4, then after possibly applying elements from W_P which fix x_{α_2}, we may assume that $\delta = \begin{smallmatrix}012100\\1\end{smallmatrix}$ is in the support of x. This does not reintroduce roots into $\operatorname{supp} y$ that have already been removed. Then acting with suitable elements from the root subgroups relative to $\begin{smallmatrix}112221\\1\end{smallmatrix}$ and $\begin{smallmatrix}122221\\1\end{smallmatrix}$, we can remove two additional roots from $\operatorname{supp} x'$, so that $x' \in K x_\varrho$, where ϱ is the highest root of G. We may suppose that the coefficient of x_ϱ is nonzero, as else y is conjugate to x. If $|\operatorname{supp} x| = 2$, then we are done using the action of T again.

(**IV**) So we may suppose that $|\operatorname{supp} x| \geq 3$ and also that all roots of the form $\delta + \alpha$ of J-level 1 are removed from $\operatorname{supp} x$ where $\alpha_2 + \alpha$ is not a root of Ψ. This leaves a list of 15 roots which possibly remain in $\operatorname{supp} x$. Twelve of these sum up with a unique root to ϱ. Thus acting with an appropriate root group element allows us to eliminate x' while fixing that part of x already chosen. We leave the details to the reader. So, we may assume that none of these particular twelve roots is in $\operatorname{supp} x$. Set $\eta = \begin{smallmatrix}012221\\1\end{smallmatrix}$, $\epsilon = \begin{smallmatrix}123210\\1\end{smallmatrix}$, and $\mu = \begin{smallmatrix}123211\\1\end{smallmatrix}$. These are the only possibilities for additional roots in $\operatorname{supp} x$ apart from α_2 and δ. Thus, we may assume that y is in the linear subspace $Kx_{\alpha_2} + Kx_\delta + Kx_\eta + Kx_\epsilon + Kx_\mu + Kx_\varrho$ of $\mathfrak{p}_u$. Unfortunately, the support of this subspace is not linearly independent. We consider two subvarieties. First the one with coefficient 0 at the coordinate μ which is then supported by five linearly independent roots. Thus by the action of T we arrive at a finite list of orbit representatives in this instance. The other subvariety consists of all vectors in this subspace with a non-zero coordinate at μ. Let y be an element of this subvariety. By applying a suitable element from the root subgroup relative to $\tau = \begin{smallmatrix}111110\\1\end{smallmatrix}$ we can remove $\varrho = \tau + \mu$ from the support of y for the expense of reintroducing the root $\tau + \delta = \alpha_2 + \gamma$ into its support. However, this root can be removed again by applying a suitable element from the root subgroup relative to γ, arriving at an element which is again supported by five linearly independent roots. So, ultimately, we also get a finite list of orbit representatives in this instance. This completes the proof of the lemma.

Lemma 3.12. *Let G be of type E_8 and $P = P_{\Pi\setminus\{\alpha_1\}}$. Then* $\operatorname{mod} P = 0$.

Proof. Let P be as in the statement. Since $\ell(P_u) = 2$, the subalgebra $\mathfrak{p}_u'$ is abelian, and thus $\operatorname{mod}(P : \mathfrak{p}_u') = 0$, by Lemma 3.9. Let $y = x + x'$ be an arbitrary element in $\mathfrak{p}_u \setminus \mathfrak{p}_u'$, where each root in $\operatorname{supp} x$ is of J-height 1 and $x' \in \mathfrak{p}_u'$. Observe that $\dim \mathfrak{p}_u' = 14$. We argue again as indicated in Remark 3.10. Because all roots of J-height 1 are conjugate under W_P, cf. [2, Lemma 1], we may assume that α_1 is in $\operatorname{supp} x$. Then all roots of J-height 1 which are of the form $\alpha_1 + \alpha$ can be removed from $\operatorname{supp} x$. In particular, there is no root remaining in the support of x with coefficient 1 at α_3. In the same fashion all 7 roots in $\Psi(\mathfrak{p}_u')$ with coefficient 3 at α_3 can be removed from $\operatorname{supp} x'$ resulting in $|\operatorname{supp} x'| \leq 7$.

(**I**) Suppose that there are no further roots in $\operatorname{supp} x$, and suppose that $\operatorname{supp} x'$ is not empty, as otherwise, we are done. Without loss and possibly after applying elements from W_P fixing α_1, we may assume that $\beta = \genfrac{}{}{0pt}{}{2454321}{2}$ is in $\operatorname{supp} x'$. Now one easily checks that all the remaining roots in $\operatorname{supp} x'$ can be removed arriving at $x_{\alpha_2} + x_\beta$ as a representative of this P-orbit.

(**II**) Now we are in the case when $|\operatorname{supp} x| \geq 2$. Suppose that the only other roots in $\operatorname{supp} x$ have coefficient 3 at α_3. Then we may suppose that $\gamma = \genfrac{}{}{0pt}{}{1343210}{2}$ is in $\operatorname{supp} x$. Now we can remove all other remaining roots in $\operatorname{supp} x$ with operators fixing x_{α_1} and x_γ. Likewise we can remove all of the remaining seven roots in $\Psi(\mathfrak{p}_u')$ with coefficient 4 at α_3 from $\operatorname{supp} x'$ with the exception of the highest root ϱ. Using the action of the maximal torus T we get $x_{\alpha_1} + x_\gamma$ and $x_{\alpha_1} + x_\gamma + x_\varrho$ as orbit representatives in this instance depending on the coefficient of x_ϱ.

(**III**) Next we consider the situation when there are roots in $\operatorname{supp} x$ with coefficient 2 at α_3. After applying elements from W_P that fix α_1 we may suppose that $\delta = \genfrac{}{}{0pt}{}{1221000}{1}$ is in $\operatorname{supp} x$. First we remove all other roots in $\operatorname{supp} x$ of the form $\delta + \alpha$ where $\alpha_1 + \alpha$ is not a root. Further we can eliminate four additional roots from $\operatorname{supp} x'$, namely $\genfrac{}{}{0pt}{}{2454321}{2}$, $\genfrac{}{}{0pt}{}{2454321}{3}$, $\genfrac{}{}{0pt}{}{2464321}{3}$, and $\genfrac{}{}{0pt}{}{2465321}{3}$. Thus, at the most the three roots in Ψ of maximal height possibly remain in $\operatorname{supp} x'$. If there are no further roots in $\operatorname{supp} x$, then $\operatorname{supp} y$ consists of at most 5 linearly independent roots. So this case leads to a finite list of orbit representatives as well.

(**IV**) Next we consider the situation when $\alpha_1, \delta \in \operatorname{supp} x$, $|\operatorname{supp} x| \geq 3$, and $\operatorname{supp} x'$ is a subset of of the three highest roots of G. Suppose that the only roots left in the support of x have coefficient 3 at α_3. Then, after possibly applying elements from W_P which fix α_1 and δ, we may assume that $\epsilon = \genfrac{}{}{0pt}{}{1343321}{2}$ is in $\operatorname{supp} x$. This is the unique root of minimal height remaining in $\operatorname{supp} x$ with coefficient 3 at α_3. Applying suitable elements from the root subgroups relative to the roots $\genfrac{}{}{0pt}{}{1122100}{1}$, $\genfrac{}{}{0pt}{}{1122110}{1}$, and $\genfrac{}{}{0pt}{}{1122111}{1}$, we can remove the remaining roots of the support of x' without reintroducing other elements into $\operatorname{supp} x'$. That is, we can remove x'; and thus we are done in this case.

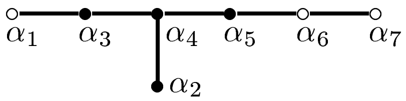
(**V**) Next we need to consider the case when there are other roots in $\operatorname{supp} x$ apart from δ with coefficient 2 at α_3. After applying elements from W_P which fix δ and α_1, we may suppose that $\eta = \genfrac{}{}{0pt}{}{1222210}{1}$ is in the support of x. We may remove those roots from $\operatorname{supp} x$ of the form $\eta + \alpha$ such that $\delta + \alpha$ and $\alpha_1 + \alpha$ are not roots of G. This leaves a list of 13 possibilities for additional roots in $\operatorname{supp} x$. By applying suitable elements from the root subgroups relative to $\genfrac{}{}{0pt}{}{1243211}{2}$ and $\genfrac{}{}{0pt}{}{1243221}{2}$ we can remove all but the highest root from $\operatorname{supp} x'$. So we are in the situation where

$x' \in Kx_\varrho$. If the coefficient of x_ϱ is zero, we are done; so assume it is nonzero. If $|\operatorname{supp} x| = 3$, then $\operatorname{supp} y$ contains at most 4 linearly independent elements. So we may suppose that $|\operatorname{supp} x| > 3$. One checks that for each of these particular 13 roots of J-height 1 – at least one of which must be in $\operatorname{supp} x$ – there exists a positive root α such that their sum equals ϱ and α does not add up with α_1, δ, or η to form a root in Ψ. Thus also in this case we can remove x'. Therefore, we arrive again at a finite set of orbit representatives in this instance. This completes the proof of the lemma.

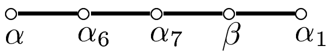
Together with Proposition 2.1 our next result proves that the last three cases of Lemma 3.8 are minimally finite. Here we show that the parabolic P from case 3 of Table 1 above is in fact of positive modality.

Lemma 3.13. *Let G be of type E_7 and $P = P_{\{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}}$. Then* $\operatorname{mod} P > 0$.

Proof. Set $\alpha := \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$ and $\beta := \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$, and let H be the regular simple subgroup of G relative to the subsystem of Ψ which is spanned by $\Pi(H) := \{\alpha, \alpha_6, \alpha_7, \beta, \alpha_1\}$. Then H is of type A_5 and $Q = H \cap P$ is the standard Borel subgroup of H relative to $\Pi(H)$. It follows from Theorem 3.1 that $\operatorname{mod} Q > 0$ and thus by Proposition 2.2 that $\operatorname{mod} P > 0$ also, as desired.



$G = E_7$, $P_{\{\alpha_2,\alpha_3,\alpha_4,\alpha_5\}}$, $\ell(P_u) = 5$



$H = A_5$, Q, $\ell(Q_u) = 5$

4. The modality of Borel subgroups of simple groups of small rank

Let G and P be as above and N a closed connected normal subgroup of P contained in P_u. Then, since $T \subset P \subseteq N_G(N)$, we see that $\mathfrak{n} = \oplus_{\beta\in\Psi(N)}\mathfrak{g}_\beta$ and thus $N = \prod U_\beta$, where the product is taken over some fixed order in $\Psi(N)$. Define $\mu(\mathfrak{n}) := 2\dim\mathfrak{n} - \dim\mathfrak{p} - \dim[\mathfrak{n},\mathfrak{n}]$. The following lower bound appears in [13].

Lemma 4.1. $\operatorname{mod} P \geq \mu(\mathfrak{n})$.

In another application of our program we determine explicit upper bounds for $\operatorname{mod} B$ in some small rank cases. In [21] lower bounds for $\operatorname{mod} B$ are derived in terms of quadratic functions in $\operatorname{rank} G$ using the μ bound from Lemma 4.1 for certain ideals $\mathfrak{n}$. In the subsequent tables we present some low rank situations where these upper and lower bounds for $\operatorname{mod} B$ coincide, and we specify ideals $\mathfrak{a}$ of $\mathfrak{b}$ whose μ-value $\mu(\mathfrak{a})$ equals $\operatorname{mod} B$.

Proposition 4.2. *The modality of a Borel subgroup of a simple group G of small rank is given as in Tables 2 and 3.*

In these tables we record the roots α such that $\mathfrak{a}$ is the B-invariant subalgebra of $\mathfrak{b}_u$ generated by the root spaces $\mathfrak{g}_\alpha$. For the E_7 and E_8 cases we only give the lower bounds $\mu(\mathfrak{a})$.

Type of G	$\mathfrak{a}$	$\dim \mathfrak{a}$	$\mathrm{mod}\, B$
A_5	$\alpha_1, \alpha_3, \alpha_5$	13	1
A_6	$\alpha_1, \alpha_3, \alpha_5$	18	1
A_7	$\alpha_1, \alpha_4, \alpha_7$	22	2
A_8	$\alpha_1, \alpha_4, \alpha_7$	29	3
A_9	$\alpha_1, \alpha_4, \alpha_8$	35	4
A_{10}	$\alpha_1, \alpha_5, \alpha_9$	42	5
B_3	α_2	7	1
B_4	α_1, α_3	14	2
B_5	α_1, α_4	21	3
B_6	α_1, α_4	29	5
C_3	α_1, α_3	8	1
C_4	α_1, α_4	13	2
C_5	α_1, α_5	19	3
C_6	α_2, α_6	29	5
C_7	α_2, α_7	38	7
D_4	α_2	9	1
D_5	α_3	15	2
D_6	α_1, α_4	25	4
D_7	α_1, α_5	34	5

Table 2: the modality of Borel subgroups in classical groups of small rank.

Type of G	$\mathfrak{a}$	$\dim \mathfrak{a}$	$\mathrm{mod}\, B$
G_2	α_1	5	1
F_4	α_2	20	4
E_6	α_4	29	5
E_7	α_5	50	≥ 10
E_8	α_2	92	≥ 20

Table 3: the modality of Borel subgroups in exceptional groups.

5. The algorithm

Throughout this section, $P = P_J$ is some standard parabolic subgroup of a simple algebraic group G with respect so some maximal torus T of G for some $J \subset \Pi$. Let m be in $\mathbb{N}$. We shall describe an algorithmic procedure which can be used to prove that $\operatorname{mod} P \leq m$. We outline the various concepts involved and discuss some aspects of its implementation for actual machine computations.

5.1. This algorithm generalizes and combines ideas from [7] and [14]. However, in contrast to the methods used in these papers, our algorithm is inductive in the following sense. Let H be a proper semisimple regular subgroup of G satisfying $\Psi(H)^+ \subset \Psi^+$ and let $Q = P \cap H$. Inductively, $\operatorname{mod} Q$ is known and in particular, we may assume that $\operatorname{mod} Q$ is at most m (else, $\operatorname{mod} P > m$ by Proposition 2.2). It follows from the proof of Proposition 2.2 in [22] that $\operatorname{mod}(P : P \cdot \mathfrak{q}_u) = \operatorname{mod} Q$. Whence, we only need to consider the P-orbits in $\mathfrak{p}_u \setminus P \cdot \mathfrak{q}_u$. This applies to any such Q. Here it obviously suffices to only take those H which are maximal among such subgroups. Hence, we only take maximal rank subgroups or regular semisimple subgroups H of corank 1 in G. We form the list of all subsystems $\Psi(\mathfrak{q}_u)$, where $Q = P \cap H$, and H runs through this set of large rank regular subgroups of G. From now on we refer to this as the *stoplist of* P. So, if we study a P-orbit and find that it has a representative x which lies in $\mathfrak{q}_u$ for such a Q, i.e., when $\operatorname{supp} x \subseteq \Psi(\mathfrak{q}_u)$, where $\Psi(\mathfrak{q}_u)$ is in the stoplist of P, then we may discard that orbit.

5.2. Instead of working with individual orbits, we simultaneously consider all P-orbits passing through subvarieties S of $\mathfrak{p}_u$ of the form $S = \sum K x_\beta + \sum M x_\gamma$, where $M = K \setminus \{0\}$, and the two sums are taken over disjoint subsets of $\Psi(\mathfrak{p}_u)$, of which one may possibly be empty.

We fix a total ordering of the roots in $\Psi(\mathfrak{p}_u)$ such that $\alpha \leq \beta$ implies $\operatorname{ht}_J \alpha \leq \operatorname{ht}_J \beta$. Then such a subvariety S can be represented symbolically by a sequence of the form "000KKMM00M0K...", where a "0" indicates that the corresponding root β say, is not in the support of S, while x_β has a nonzero coefficient in case of the label "M" and an arbitrary one if "K" occurs, cf. [7] and [14, Section 6]. We refer to this representation of a subvariety S as a *string* and often use that term also for S itself. The support of S is the set of roots in $\Psi(\mathfrak{p}_u)$ whose coefficient is either M or K.

5.3. Our goal is to construct a finite collection of such subvarieties S of $\mathfrak{p}_u$ with the property that firstly, every P-orbit in $\mathfrak{p}_u$ has a representative in some S in this list, and secondly, such that $\operatorname{mod}(P : P \cdot S)$ is at most m for each such S. The desired inequality $\operatorname{mod} P \leq m$ then follows, since $\operatorname{mod} P = \max_S \operatorname{mod}(P : P \cdot S)$, where the maximum is taken over all sets S in our finite collection.

A set $\mathcal{S}$ of such subvarieties S of $\mathfrak{p}_u$ satisfying the first property is called a *stack* of P or simply a *stack of strings* of P.

5.4. We now describe two operations that we can perform on strings in $\mathcal{S}$ which preserve the first property, namely that each P-orbit in $\mathfrak{p}_u$ passes through some S in $\mathcal{S}$. These operations are aimed at reducing the support of the strings in $\mathcal{S}$, while at the same time preserving this property of $\mathcal{S}$. They are identical to the ones described in [7] and [14].

The first is a *reduction* or *elimination* operation. Fix a string S from the stack $\mathcal{S}$ with coefficient M at β and K at $\alpha + \beta$, with $\alpha \in \Psi(P)$. Furthermore, if $\gamma \neq \beta$

and $i\alpha + \gamma$ is in $\Psi(\mathfrak{p}_u)$, for some $i \geq 1$, and the coefficient of γ is nonzero, then suppose that the coefficient of $i\alpha + \gamma$ equals K, for each $i \geq 1$. Note that U_α maps $\mathfrak{g}_\gamma$ into $\sum_{i\geq 0} \mathfrak{g}_{i\alpha+\gamma}$. So, the point of this last requirement is that, while U_α may still act nontrivially on summands of the subvariety S other than Mx_β, it preserves S as a whole. Now let x be an arbitrary element in the subvariety S. Then x has a non-zero coefficient at x_β. Thus by acting on x with a suitable element from the root subgroup U_α of P we can remove $\alpha + \beta$ from its support. Consequently, every P-orbit passing through S also has a representative in the variety S' say, which is obtained from S by replacing the coefficient K at $\alpha + \beta$ by 0. By our assumption on S no other roots are introduced into the support of S' in this process, and thus its support is smaller than that of S. Now we replace S by S' in $\mathcal{S}$ and we may repeat this elimination process with this new collection of strings.

The second one is simply a *splitting* or *branching* operation. Let S be a given subvariety in $\mathcal{S}$ with K at position β. Then S is the union of the two subvarieties S' and S'', where in position β the K is replaced by 0 in S' and by M in S''. Thus S' is of smaller support than S, and in S'' we have a new position labeled with M which allows for new applications of the elimination technique described above. In this situation we remove S from the stack and replace it by S' and S''.

Observe that these two operations do in fact always yield a new stack $\mathcal{S}$ with the property that every P-orbit through $\mathfrak{p}_u$ passes through a string in $\mathcal{S}$.

The algorithm consists of an iteration and combination of these splitting and elimination operations. We have proven that $\operatorname{mod} P$ is bounded above by m, once we have arrived at a stack $\mathcal{S}$ satisfying $\operatorname{mod}(P : P \cdot S) \leq m$ for each S in $\mathcal{S}$.

In our analysis, we initially start out with $\mathcal{S}$ consisting only of $\mathfrak{p}_u$ itself or its corresponding string.

5.5. If a sequence of splitting and elimination operations yields a string S with $\operatorname{mod}(P : P \cdot S) \leq m$, then, instead of keeping S, we may simply delete it from $\mathcal{S}$. We have reached our goal of showing that $\operatorname{mod} P$ is at most m, precisely when all the strings which were generated in the course of this process have again been eliminated from $\mathcal{S}$, i.e., when the resulting stack is empty. We shall now discuss the various possibilities when a string S can be removed from the stack, i.e., when S satisfies $\operatorname{mod}(P : P \cdot S) \leq m$.

5.6. We describe three different *deletion criterions* which enable us to remove a string from the stack. Let S be a subvariety in $\mathcal{S}$ as above.

Suppose that $\operatorname{supp} S \subseteq \Psi(\mathfrak{q}_u)$, where $\Psi(\mathfrak{q}_u)$ is a member of the stoplist of P, i.e., $S \subseteq \mathfrak{q}_u$. Then we infer that $\operatorname{mod}(P : P \cdot S) = \operatorname{mod}(Q : Q \cdot S) \leq \operatorname{mod} Q \leq m$, by induction, and thus we can remove S from $\mathcal{S}$. The equality part of this statement follows from the proof of Proposition 2.2 in [22].

Another situation when we can delete a string S from the stack, arises in the following way. Suppose that the support of S consists of at most $\dim T + m$ roots. Let d be the number of linearly independent roots in $\operatorname{supp} S$. Note that d is at most $\dim T$. Then for an arbitrary element x in S we can apply suitable elements from T to scale as many as d coefficients of x to equal 1 with at most $|\operatorname{supp} S| - d$ coefficients of x remaining free. Now, if $|\operatorname{supp} S| - d \leq m$, then the resulting set of P-orbits passing through all the elements which are obtained by varying the entries in the remaining free coefficients depends on at most m parameters, and thus, $\operatorname{mod}(P : P \cdot S)$ is bounded above by m, as desired. Again, we can remove S from the stack $\mathcal{S}$.

The third situation which allows us to remove a string from the stack is mainly a device to avoid redundancy. Suppose that two strings of the form $0 \ldots 00KKK$ and $0 \ldots 0MKKK$ have already been analyzed. Then in fact we have already analyzed all P-orbits passing through the linear subspace represented by the string $0 \ldots 0KKKK$. We denote the largest such subspace with this property or its string by D, store its support, and update it as we go along. So, initially $\operatorname{supp} D$ is empty but it grows larger as we proceed with our study. For every newly created string S during the course of our investigation we check whether it satisfies $S \subseteq D$, and if it does, we remove it from the stack $\mathcal{S}$. As our analysis proceeds, $\operatorname{supp} D$ grows, and therefore, this criterion becomes more and more effective in the course of the investigation.

We refer to these as the *stoplist*, *rank*, and *redundancy criterion*, respectively.

5.7. In the general situation, each of these criterions is applied many times. However, the stoplist and the redundancy criterion are by far the most effective of the four. Their advantage is twofold. Firstly, they only involve a subset check of the support of the string at hand and the members of the stoplist of P, or of $\operatorname{supp} D$. In terms of computing time this is not too costly provided the stoplist is short. Secondly and more importantly, by applying these criterions we may be able to remove strings from the stack which have large support. These could take a long time to break down into smaller ones using the reduction and branching operations.

On the contrary, the rank criterion can only be applied when $\operatorname{supp} S \leq \dim T + m$ which is usually very small compared to $\dim \mathfrak{p}_u$. More importantly, it requires calculating the rank of matrices which is less favorable in terms of computing time than a simple subset check as involved in the other three criterions.

5.8. In the course of applying splitting and elimination operations to strings from the stack it rarely happens that in the end all the strings that are produced in this fashion will satisfy one of the four deletion criterions from (5.7) above.

Ultimately, strings S will remain in the stack which have the property that the coefficient of any root in $\operatorname{supp} S$ is M (i.e., no further branching operations are possible), no further elimination is possible (every root operator which removes a root from $\operatorname{supp} S$ introduces new roots in the resulting string), and finally, S does not satisfy any of the four deletion criterions above, so it cannot be removed from the stack. Generally, this situation occurs rather frequently.

Then we enter a process to which we refer to as *shuffling*. We continue to eliminate roots from $\operatorname{supp} S$ by the reduction process above for the expense of allowing new roots to be introduced in the resulting string S', say. At first, we only consider those root operators U_α in the elimination procedure that will introduce one new root in $\operatorname{supp} S'$ while removing another, so that $\operatorname{supp} S'$ and $\operatorname{supp} S$ have the same size (so the labels M are more less shuffled around among various roots). And we iterate this shuffle process by applying it to the resulting strings S'. Along the way we need to memorize the list of all strings S' produced in this fashion, as at each stage we are only interested in generating new strings which have not been encountered before in order to avoid that this process gets stuck in a circle of such strings.

Once all possible strings S' with $|\operatorname{supp} S'| = |\operatorname{supp} S|$ have been generated in this iterated shuffle process, we then generate new strings using this process by now allowing root operators in the reduction operation which will enlarge the support of the resulting string, first by one, then two, etc., and continue shuffling with these.

So, during this shuffle procedure we constantly produce new strings S' from S, such that we stay in the same set of orbits, i.e., $P \cdot S = P \cdot S'$ and such that the support of S' is slightly different at each stage. Ultimately, the aim is to find such a shuffled version S' which satisfies one of the deletion criterions above. Thus whenever a new string is generated during shuffling it is tested against any of our deletion criterions. With the exception of some small rank examples, this shuffling procedure did occur in all cases we studied. In fact, it is invoked very often. So this turns out to be quite a successful strategy.

5.9. A parameter can be set which limits the number of strings the algorithm will produce during this shuffling process starting with a single string. Mainly, this is intended to force the algorithm to end at some stage. Usually, we allowed for 10.000 steps.

For instance, let us consider the situation for a Borel subgroup B of a simple group of type C_r for $3 \leq r \leq 7$. Using the algorithm we computed upper bounds for $\operatorname{mod} B$, as shown in Table 2. For $r \leq 5$ no shuffle operations were invoked. For $r = 6$ a total of 5 strings were analyzed using shufflings with the highest number of 3235 trials occurring. While for $r = 7$ already for 65 strings shuffle operations were invoked with a maximum number of 35.856 trials occurring before a deletion criterion was satisfied. This is one of the most extreme cases that we encountered. This number tends to grow rapidly with the rank of the group.

An analysis of the various cases shows that virtually the entire computing time goes into these time consuming shufflings, as there all strings that are encountered along the way have to be memorized. This feature appears to be the limiting factor for the entire algorithm.

5.10. There is another aspect of the program that we need to mention. Already in the situation of a semisimple rank one parabolic subgroup of a simple group of type D_4 the mere use of splitting and elimination procedures does not lead to a complete determination of the modality zero cases. In that instance there is one string which can not be treated in this fashion. However, this case was handled using an argument in [14, Lemma 6.2] which involves the solution of a particular quadratic equation over the field k. The proof of this fact in *loc. cit.* applies in any characteristic. In order to handle these cases using an algorithm this particular string has to be accounted for. Thus when studying a case in type D_r, we check against a special list which includes all such known finite cases which can not be treated simply by means of branching and reduction operations.

5.11. In terms of an implementation of the outlined algorithm, it is necessary to have a good and efficient way of measuring or evaluating the various branching or elimination options we have for any given string. Firstly, one needs to decide which of the two operational procedures should be applied to a given string from the stack. Secondly, and more specifically, we have to be able to determine the optimal splittings and the optimal choices for eliminations and the order in which they should be performed at any stage of our analysis. Examples show that the order in which roots are eliminated may prove to be crucial for a successful outcome of the algorithm.

One difficulty here is to come up with an optimal strategy. This should certainly incorporate that roots from the support of $S \in \mathcal{S}$ are eliminated as rapidly as possible. However, a more conservative approach by not eliminating roots from the support which allow for more elimination operators to be applied on the longer run, may ultimately prove to be more successful.

We have chosen a mixture of two obvious heuristic strategies to determine a splitting position of a string S. Suppose that β is a root in $\operatorname{supp} S$ with coefficient K such that the number of roots in $\operatorname{supp} S$ of the form $\alpha + \beta$ with $\alpha \in \Psi(P)$ and nonzero coefficient at $\alpha + \beta$ is maximal among all roots in $\operatorname{supp} S$. Splitting at β will yield a high number of eliminations for the resulting string, where M replaces the K in the β-position. If there is more than one such root yielding an optimal number of elimination candidates for the next step, then among those we choose a root for the splitting which is itself harder to remove from $\operatorname{supp} S$ than the others.

As far as elimination is concerned, we take on a rather cautious approach. Before removing a root β from $\operatorname{supp} S$, we first determine the number of relevant positions within S from which β can be eliminated using root subgroup operations from P. If this number is small, we simply delete β. However, if it is relatively large, say at least 4, then we spare β. The reason is that the K in position β may allow for other root subgroup operators to be applied to S at some later stage as part of some other elimination process. So this still eliminates all the candidates, but at a slower pace, while at the same time providing greater flexibility as far as eliminations are concerned on the long run. The parameter given here, at what stage roots should not be eliminated can be set individually for each run of the program.

5.12. Because of the inductive nature of our method, and because of the fact that we simultaneously study a collection of orbits passing through strings, we do not get the number of orbits in case we have shown that $\operatorname{mod} P = 0$.

Acknowledgments. Both authors are grateful to L. Hille and V.L. Popov for helpful comments.

The work presented in this paper was completed during a visit by the second author to the Isaac Newton Institute in Cambridge, England, in Spring of 1997. It is a pleasure to thank M. Broué, R.W. Carter, and J. Saxl, the organizers of the special half year on representations of algebraic groups and related finite groups, as well as the members of the Institute for their hospitality.

References

1. V.I. Arnold, *Critical points of smooth functions*, Proc. of ICM, Vancouver **1** (1974), 19–41.
2. H. Azad, M. Barry, and G. Seitz, *On the structure of parabolic subgroups*, Com. in Algebra **18** (1990), 551–562.
3. A. Borel, *Linear Algebraic Groups*, W. A. Benjamin, New York, 1969.
4. A. Borel, J. Tits, *Homomorphismes "abstaits" de groupes algébriques simples*, Annals of Math. **98** (1973), 499–571.
5. N. Bourbaki, *Groupes et algèbres de Lie, Chapitres 4,5 et 6*, Hermann, Paris, 1975.
6. T. Brüstle, L. Hille, *Actions of parabolic subgroups of $GL(V)$ on certain unipotent subgroups and quasi-hereditary algebras*, preprint Bielefeld/Chemnitz (1997).
7. H. Bürgstein, W. H. Hesselink, *Algorithmic orbit classification for some Borel group actions*, Comp. Math. **61** (1987), 3–41.

8. E. B. Dynkin, *Semisimple subalgebras of semisimple Lie algebras*, Amer. Math. Soc. Transl. Ser. 2 **6** (1957), 111–244.
9. L. Hille, G. Röhrle, *On parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, C. R. Acad. Sci. Paris, Série I, **325** (1997), 465–470.
10. L. Hille, G. Röhrle, *The classification of parabolic subgroups of classical groups with a finite number of orbits on the unipotent radical*, preprint 97–103, SFB 343, Bielefeld (1997).
11. V. V. Kashin, *Orbits of adjoint and coadjoint actions of Borel subgroups of semisimple algebraic groups*, (in Russian), Questions of Group Theory and Homological algebra, Yaroslavl' (1990), 141–159.
12. D. Panyushev, *The Jacobian modules of a representation of a Lie algebra and geometry of commuting varieties*, Comp. Math. **94** (1994), 181–199.
13. V.L. Popov, *A finiteness theorem for parabolic subgroups of fixed modality*, Indag. Math. **8** (1997), 125–132.
14. V. Popov, G. Röhrle, *On the number of orbits of a parabolic subgroup on its unipotent radical*, Algebraic Groups and Lie Groups; (G. I. Lehrer, ed.), Australian Mathematical Society Lecture Series, vol. 9, Cambridge University Press, 1997, pp. 297–320.
15. V. L. Popov, E. B. Vinberg, *Invariant Theory. Encyclopaedia of Math. Sci.: Algebraic Geometry IV.*, vol. 55, Springer Verlag, 1994, pp. 123–284.
16. V.S. Pyasetskii, *Linear Lie groups acting with finitely many orbits*, Funct. Anal. Appl. **9** (1975), 351–353.
17. R. W. Richardson, *Finiteness Theorems for Orbits of Algebraic Groups*, Indag. Math. **88** (1985), 337–344.
18. R. Richardson, G. Röhrle, R. Steinberg, *Parabolic subgroups with Abelian unipotent radical*, Inv. Math. **110** (1992), 649–671.
19. G. Röhrle, *Parabolic subgroups of positive modality*, Geom. Dedicata **60** (1996), 163–186.
20. G. Röhrle, *Maximal parabolic subgroups in classical groups are of modality zero*, Geom. Dedicata **66** (1997), 51–64.
21. G. Röhrle, *A note on the modality of parabolic subgroups*, Indag. Math. N. S. **8 (4)** (1997), 549–559.
22. G. Röhrle, *On the modality of parabolic subgroups of linear algebraic groups*, Preprint 97–036, SFB 343, The University of Bielefeld.
23. T.A. Springer, R. Steinberg, *Conjugacy classes*, Seminar on algebraic groups and related finite groups, vol. 131, Springer, Heidelberg, 1970.
24. E. B. Vinberg, *Complexity of actions of reductive groups*, Funct. Anal. Appl. **20** (1986), 1-11.

Fakultät für Mathematik, Universität Bielefeld, 33615 Bielefeld, Germany

ulf@mathematik.uni-bielefeld.de
roehrle@mathematik.uni-bielefeld.de