OPTIMAL BOUNDS IN NON-GAUSSIAN LIMIT THEOREMS FOR U-STATISTICS

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Abstract. Let $X, X_1, X_2, \dots$ be i.i.d. random variables taking values in a measurable space $\mathcal{X}$. Let $\phi(x,y)$ and $\phi_1(x)$ denote measurable functions of the arguments $x, y \in \mathcal{X}$. Assuming that the kernel ϕ is symmetric and that $\mathbf{E}\,\phi(x, X) = 0$, for all x, and $\mathbf{E}\,\phi_1(X) = 0$, we consider U-statistics of type

$$T = N^{-1} \sum_{1 \le j < k \le N} \phi(X_j, X_k) + N^{-1/2} \sum_{1 \le j \le N} \phi_1(X_j).$$

It is known that the conditions $\mathbf{E}\,\phi^2(X, X_1) < \infty$ and $\mathbf{E}\,\phi_1^2(X) < \infty$ imply that the distribution function of T, say F, has a limit, say F_0, which can be described in terms of the eigenvalues of the Hilbert–Schmidt operator associated with the kernel $\phi(x,y)$. Under optimal moment conditions, we prove that

$$\Delta_N = \sup_x \bigl| F(x) - F_0(x) - F_1(x) \bigr| = \mathcal{O}\bigl(N^{-1}\bigr),$$

provided that at least 9 eigenvalues of the operator do not vanish. Here F_1 denotes an Edgeworth type correction. We provide explicit bounds for Δ_N and for the concentration functions of statistics of type T.

1. Introduction

Let $X, \overline{X}, X_1, \dots, X_N$ be independent identically distributed (i.i.d.) random variables taking values in an arbitrary measurable space $(\mathcal{X}, \mathcal{B})$. Let $\phi_1 : \mathcal{X} \to \mathbb{R}$

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and $\phi : \mathcal{X}^2 \to \mathbb{R}$ be real-valued measurable functions. Assume that ϕ is symmetric, that is, $\phi(x,y) = \phi(y,x)$, for all $x, y \in \mathcal{X}$. Consider the U-statistic

$$T = \frac{1}{N} \sum_{1 \le i < j \le N} \phi(X_i, X_j) + \frac{1}{\sqrt{N}} \sum_{1 \le i \le N} \phi_1(X_i), \tag{1.1}$$

assuming that

$$\mathbf{E}\, \phi_1(X) = 0, \qquad \mathbf{E}\, \phi(x, X) = 0, \qquad \text{for all } x \in \mathcal{X}. \tag{1.2}$$

Our aim is to establish an Edgeworth expansion with remainder $\mathcal{O}(N^{-1})$, for the distribution function of T. We shall show that moment conditions on ϕ_1 and ϕ together with the assumption that the limiting weighted χ^2 distribution has at least nine degrees of freedom, are sufficient for the result to hold. We shall establish as well explicit bounds of order $\mathcal{O}(N^{-1})$ for concentration functions of a somewhat larger class of statistics than (1.1).

We shall write

$$\begin{aligned} \beta_s &= \mathbf{E}\, \bigl| \phi_1(X) \bigr|^s, & \gamma_s &= \mathbf{E}\, \bigl| \phi(X, \overline{X}) \bigr|^s, \\ \sigma^2 &= \gamma_2, & \gamma_{s,r} &= \mathbf{E}\, \Bigl(\mathbf{E} \Bigl\{ \bigl| \phi(X, \overline{X}) \bigr|^s \Bigm| X \Bigr\} \Bigr)^r, \end{aligned} \tag{1.3}$$

and assume throughout that

$$\beta_2 < \infty, \qquad 0 < \sigma^2 < \infty. \tag{1.4}$$

The variance of T splits as

$$\mathbf{E}\, T^2 = \beta_2 + \frac{N-1}{2N}\, \sigma^2. \tag{1.5}$$

The statistic T is called degenerate since the condition $\sigma^2 > 0$ ensures that the quadratic part of the statistic is not asymptotically negligible and therefore T is not asymptotically normal. More precisely, the asymptotic distribution of T is non-Gaussian and is given by the distribution of the random variable

$$T_0 = \frac{1}{2} \sum_{j \ge 1} q_j \, (\eta_j^2 - 1) + \sum_{j \ge 0} a_j \, \eta_j. \tag{1.6}$$

Here η_j is a sequence of i.i.d. standard normal random variables, $a_0, a_1, \dots$ denotes a (non-random) sequence of square summable weights, and $q_1, q_2, \dots$ denote eigenvalues of the Hilbert–Schmidt operator, say $\mathbb{Q}$, associated with the kernel ϕ (see Section 2 for detailed definitions). Without loss of generality we shall assume throughout that $|q_1| \ge |q_2| \ge \dots$

Introduce the distribution functions

$$F(x) = \mathbf{P}\, \bigl\{ T \le x \bigr\}, \qquad F_0(x) = \mathbf{P}\, \bigl\{ T_0 \le x \bigr\}. \tag{1.7}$$

Write

$$\Delta_N = \sup_x \big|\Delta_N(x)\big|, \qquad \Delta_N(x) = F(x) - F_0(x) - F_1(x), \tag{1.8}$$

where $F_1(x)$ denotes an Edgeworth correction defined by (4.1). Let us notice here only that F_1 vanishes if $\phi_1 = 0$, or if

$$\mathbf{E}\,\phi_1^3(X) = \mathbf{E}\,\phi_1^2(X)\,\phi(X,x) = \mathbf{E}\,\phi_1(X)\,\phi^2(X,x) = \mathbf{E}\,\phi^3(X,x) = 0 \tag{1.9}$$

holds for all $x \in \mathcal{X}$.

We shall denote by $c, c_1, \dots$ absolute generic constants. If a constant depends on, say s, we shall write $c(s)$ or c_s.

Theorem 1.1.

i) Assume that $q_{13} \neq 0$. Then we have

$$\Delta_N \le \frac{C}{N}\Big(\frac{\beta_4}{\sigma^4} + \frac{\beta_3^2}{\sigma^6} + \frac{\gamma_3}{\sigma^3} + \frac{\gamma_{2,2}}{\sigma^4}\Big), \qquad \textit{where } C \le \exp\Big\{\frac{c\sigma}{|q_{13}|}\Big\}. \tag{1.10}$$

ii) Assume that (1.9) holds and $q_9 \neq 0$. Then we have

$$\Delta_N \le \frac{C}{N}\Big(\frac{\beta_4}{\sigma^4} + \frac{\gamma_3}{\sigma^3} + \frac{\gamma_{2,2}}{\sigma^4}\Big), \qquad \textit{where } C \le \exp\Big\{\frac{c\sigma}{|q_9|}\Big\}. \tag{1.11}$$

Notice that $\gamma_{2,2} \le \gamma_4$. Thus, for Δ_N in (1.10) and (1.11) we have

$$\Delta_N \le \frac{C}{N}\Big(\frac{\beta_4}{\sigma^4} + \frac{\beta_3^2}{\sigma^6} + \frac{\gamma_4}{\sigma^4}\Big) \quad \text{and} \quad \Delta_N \le \frac{C}{N}\Big(\frac{\beta_4}{\sigma^4} + \frac{\gamma_4}{\sigma^4}\Big) \tag{1.12}$$

respectively. See Example 9.4 for a kernel ϕ such that $\gamma_{2,2} < \infty$ and $\gamma_4 = \infty$.

Theorem 1.1 follows from more general bounds for Δ_N in Section 4.

Unfortunately, the bound (1.10) is not applicable when $q_{13} = 0$. In Section 9 we prove Theorem 9.2 which holds for finite dimensional statistics such that $q_9 \neq 0$. However, comparing with (1.10), the bound of Theorem 9.2 depends on the smallest non-zero eigenvalue. Furthermore, in Section 9 we extend Theorem 1.1 to von Mises statistics, see Theorem 9.1.

Let us formulate a related result for concentration functions of statistics

$$T_* = \sum_{1 \le j < k \le N} \phi(X_j, X_k) + f_1 + f_2, \tag{1.13}$$

where $f_1 = f_1(X_1, \dots, X_M)$ is an arbitrary statistic depending only on $X_1, \dots, X_M$, and $f_2 = f_2(X_{M+1}, \dots, X_N)$ is as well arbitrary but independent of $X_1, \dots, X_M$, for some $1 \le M \le N/2$. Note that the class of statistics (1.13) is slightly more general than the class of statistics (1.1). In applications the most interesting case is $M \ge cN$, with some $c > 0$ independent of N, and in this case the bound of Theorem 1.2 has the order $\mathcal{O}(N^{-1})$. Write

$$Q(T_*; \lambda) = \sup_x \mathbf{P}\big\{x \le T_* \le x + \lambda\big\}, \qquad \lambda \ge 0. \tag{1.14}$$

Theorem 1.2. *Let $q_9 \neq 0$. Then*

$$Q(T_*;\lambda) \le \frac{C}{M}\,\max\Big\{\frac{\lambda}{\sigma};\ \frac{\gamma_3^2}{\sigma^6}\Big\}, \qquad \text{where } C \le \exp\Big\{\frac{c\sigma}{|q_9|}\Big\}. \tag{1.15}$$

See Section 3 for refinements of Theorem 1.2.

In particular cases Theorems 1.1, 1.2, 9.1 and 9.2 are closely related to the well known lattice point problem in number theory. Let $\mathbb{A} : \mathbb{R}^d \to \mathbb{R}^d$ denote a positive linear operator. The relative lattice point remainder for ellipsoids $E_r = \big\{x \in \mathbb{R}^d : \langle \mathbb{A}x, x\rangle \le r^2\big\}$ is defined as

$$\delta(r) = \big(\operatorname{vol}_{\mathbb{Z}} E_r - \operatorname{vol} E_r\big) / \operatorname{vol} E_r,$$

where $\langle x, y\rangle$ denotes the standard scalar product in $\mathbb{R}^d$, and $\operatorname{vol}_{\mathbb{Z}} E_r$ and $\operatorname{vol} E_r$ are the number of points of the standard lattice $\mathbb{Z}^d$ in E_r and the volume of E_r respectively. Let $Y, Y_1, Y_2, \dots$ be a sequence of i.i.d. bounded centered lattice random vectors taking values in $\mathbb{Z}^d$ and such that Y has the covariance $\mathbb{A}^{-1}$. Yarnold (1972) showed that

$$\mathbf{P}\big\{\langle \mathbb{A}S_N, S_N\rangle \le t\big\} = F_0(t) + e^{-t/2}\,\delta(\sqrt{t\,N}) + \mathcal{O}(N^{-1}), \tag{1.16}$$

where $S_N = N^{-1/2}(Y_1 + \dots + Y_N)$ and $F_0(t)$ denotes the χ^2 distribution function with d degrees of freedom. The statistic $\langle \mathbb{A}S_N, S_N\rangle$ is a degenerate U-statistic of a very special type since

$$\langle \mathbb{A}S_N, S_N\rangle - \frac{d}{N} = \frac{2}{N}\sum_{1\le j<k\le N}\langle \mathbb{A}Y_j, Y_k\rangle + \frac{1}{N}\sum_{1\le j\le N}\big(\langle \mathbb{A}Y_j, Y_j\rangle - d\big).$$

Therefore a comparison of (1.16) with expansions of Theorems 1.1 and 9.2 shows that $\delta(r) = \mathcal{O}(r^{-2})$, as $r \to \infty$, for $d \ge 9$. See Bentkus and Götze (1995, 1997) for directly proved optimal and explicit bounds for the lattice remainder. This result solves the lattice remainder problem for general ellipsoids, which was open since 1915 when E. Landau obtained the upper bound $\delta(r) = \mathcal{O}(r^{-2+2/(d+1)})$. Esseen's (1945) bound $\mathcal{O}(N^{-d/(d+1)})$ in the CLT for ellipsoids in $\mathbb{R}^d$ in conjunction with (1.16) provides an alternative proof of Landau's result.

The bounds of Theorems 1.1, 1.2, 9.1 and 9.2 are optimal with respect to the dependence on N in view of the following example.

Example 1.3. Assume that $\phi_1 = 0$ and that the kernel ϕ is bounded and takes integer values only. The convergence of T in distribution to a limit T_0 implies

$$\mathbf{P}\big\{|T| \le 1\big\} = \mathbf{P}\big\{|NT| \le N\big\} \ge c > 0, \tag{1.17}$$

for sufficiently large N. The statistic NT assumes integer values only and the interval $[-N, N]$ contains $2N+1$ integers. Hence, there exists an integer $|j| \le N$ such that $\mathbf{P}\big\{|NT| = j\big\} \ge c_1/N$, and

$$\sup_t \mathbf{P}\big\{T = t\big\} \ge c_1/N. \tag{1.18}$$

The estimate (1.18) indicates that an $\mathcal{O}(N^{-1})$ bound for the concentration function in Theorem 1.2 is optimal. Similarly, the dependence on N in Theorems 1.1, 9.1 and 9.2 is optimal since the function F_0+F_1 is continuous. Thus, in order to obtain bounds $o(N^{-1})$ we have either to impose additional smoothness conditions on the distributions involving ϕ, ϕ_1 and X or to replace F_1 by some highly distribution dependent discontinuous function.

Notice that the construction of the example extends to the case of bounded ϕ_1 taking integer values only. Here (1.17) and (1.18) hold for sufficiently large $N = n^2$, $n \in \mathbb{N}$. Furthermore, one can choose ϕ such that the associated Hilbert–Schmidt operator $\mathbb{Q}$ has a given number, say s, $1 \le s \le \infty$, of non-zero eigenvalues.

Probably the dimensional dependence (that is, the dependence on the eigenvalues) of our results is not optimal. To prove $\mathcal{O}(N^{-1})$ we require that $q_9 \ne 0$. An optimal condition would be $q_5 \ne 0$. To prove (or disprove) that $q_5 \ne 0$ is sufficient for rates $\mathcal{O}(N^{-1})$, seems to be a difficult problem, since its solution combined with (1.16) would imply a solution of the corresponding unsolved problem for the lattice point remainder. The question about precise convergence rates in lower dimensions when $q_d \ne 0$ and $q_{d+1} = 0$, for some $2 \le d \le 4$, still remains completely open. For instance, in the case $q_2 \ne 0$ and $q_3 = 0$, a precise convergence rate would imply a solution of the circle problem. Known lower bounds in the circle problem correspond to $\mathcal{O}(N^{-3/4} \log^{\delta} N)$ in our setup. A famous conjecture by Hardy (1916) says that this is up to logarithmic factors the true order.

The dependence on moments in Theorem 1.1 seems to be optimal since this theorem implies optimal bounds in the CLT for conic sections in $\mathbb{R}^d$, $d \ge 13$, and Hilbert spaces (see Section 9).

For degenerate U-statistics rates of convergence and Edgeworth expansions were proved by Götze (1979, 1984) and by Koroljuk and Borovskich (1994). Assuming infinitely many non-zero eigenvalues and certain moment conditions, Götze (1979) proved a bound $\mathcal{O}(N^{-1+\varepsilon})$, $\varepsilon > 0$, using a Weyl type symmetrization inequality (Weyl 1915/16). In their monograph Koroljuk and Borovskich prove bounds of order $o(N^{-1/2})$. Götze and Zitikis (1995) obtained approximations with remainders of order $\mathcal{O}(N^{-1})$ and better under certain smoothness assumptions related to the kernel ϕ and the distribution of X.

Degenerate U-statistics (1.1) include as partial cases a number of other statistics (see, for example, Lee 1990). In particular, they include quadratic forms of sums of independent random vectors in finite dimensional and Hilbert spaces. A special choice of these random vectors leads to empirical processes and, in particular, to the uniform empirical process on $[0,1]$ and its L^2-norm, the so called ω^2-statistics of Cramér and von Mises. For the ω^2-statistic the relevant kernel is given by

$$\phi(x,y) = \frac{1}{2}\,(x^2+y^2) - \max\,\{x;\,y\} + \frac{1}{3}, \qquad \text{for } 0 \le x, y \le 1,$$

with eigenvalues $\lambda_k = (k\pi)^{-2}$. Related examples are Watson's goodness of fit statistics on a circle. The limiting distribution of such statistics has been studied by

Smirnov (1937) and Anderson and Darling (1952). Rates of convergence have been proved, to name a few results, starting with rates like $\mathcal{O}\big((\log n)^{-1}\big)$ (Kandelaki 1965) to rates $\mathcal{O}(n^{-\alpha})$: $\alpha = 1/6$, Sazonov (1969); $\alpha = 1/4$, Kiefer (1972); $\mathcal{O}(n^{-1/2}\log n)$ via the KMT-approximation, Csőrgő (1976). For random vectors in Hilbert spaces, Paulauskas (1976) obtained $\alpha = 1/6$. Assuming infinitely many positive eigenvalues, Götze (1979, 1984) proved $\alpha = 1-\varepsilon$, $\varepsilon > 0$. Eigenvalue and moment conditions were improved, e.g., by Yurinskii (1982), Bentkus (1984), Nagaev and Chebotarev (1986), Zalesskii, Sazonov and Ulyanov (1988), Senatov (1989). For ω^2-statistics bounds of order $\mathcal{O}(N^{-1})$ and better were obtained by Bentkus, Götze and Zitikis (1993). The rate $\alpha = 1$ for diagonal quadratic forms was proved in Bentkus and Götze (1996a) using a discretization method and the Hardy–Littlewood circle method from analytic number theory. In Bentkus and Götze (1996b) the result was extended to the case of non-diagonal quadratic forms again via discretization techniques and a useful extension of Weyl's inequality (henceforth called multiplicative inequality) for characteristic functions of lattice distributions.

The class of U-statistics is much more general than that of quadratic forms. Using inequalities of the form $|\phi(x,y)| \le c\,\mathcal{N}(x)\,\mathcal{N}(y)$, where $\mathcal{N}$ is a norm function satisfying $\mathbf{E}\,\mathcal{N}^4(X) \le c\gamma_4$, one would be able to reduce estimates for U-statistics to estimates for quadratic forms like those derived in Bentkus and Götze (1996b). Unfortunately, such inequalities for *general* U-statistics do not hold (e.g., for the kernel ϕ of Example 9.4). Therefore we had to develop new versions of random selections, discretization and multiplicative inequalities. The bounds in terms of the given stochastic setup involving moment and eigenvalue conditions only are more natural compared with those for quadratic forms in Bentkus and Götze (1996a,b), where geometric conditions are used. In spite of the large generality, our results are sharp enough to recover optimal moment conditions in the case of quadratic forms, see Section 9. The methods developed turned out to be useful for investigations of Poissonian approximations of stable laws (Bentkus, Götze and Paulauskas 1996), and for infinite divisible approximations of sums of independent random vectors (see Bentkus, Götze and Zaitsev 1997).

The paper is organized as follows.

In Section 2 we introduce the notation used throughout and describe a representation of T as a sum of a quadratic form and a linear functional of random vectors taking values in $\mathbb{R}^\infty$. This simplifies the notation and certain algebraic operations like symmetrization.

In Section 3 we derive explicit bounds for the concentration functions of statistics (1.13). The proofs in Section 3 are simpler than those of bounds for Δ_N, although they already involve the main principal techniques.

In Section 4 we obtain the main result—explicit bounds for Δ_N, using some auxiliary results, which are proved in Sections 5–8. In particular, these explicit bounds are applicable in cases when random variables and the kernels ϕ_1 and ϕ depend on N and other parameters.

In Section 9 we extend the bounds to von Mises statistics and prove Theorem 9.2. An application to the distributions of quadratic forms of sums of random vectors in Euclidean spaces shows that our bounds are indeed sharp, up to certain constants.

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2. Special representations of bivariate U-statistics; notation and auxiliary results

Let us introduce notation related to the operator $\mathbb{Q}$. Consider the measurable space $(\mathcal{X}, \mathcal{B}, \mu)$ with measure $\mu = \mathcal{L}(X)$. Let $L^2 = L^2(\mathcal{X}, \mathcal{B}, \mu)$ denote the real Hilbert space of square integrable real functions. The Hilbert–Schmidt operator $\mathbb{Q} : L^2 \to L^2$ is defined via

$$\mathbb{Q}f(x) = \int_{\mathcal{X}} \phi(x,y)\, f(y)\, \mu(dy) = \mathbf{E}\, \phi(x,X)\, f(X), \qquad f \in L^2. \tag{2.1}$$

Let $\{e_j : j \geq 1\}$ denote an orthonormal complete system of eigenfunctions of $\mathbb{Q}$ ordered by decreasing absolute values of the corresponding eigenvalues $q_1, q_2, \dots$, that is, $|q_1| \geq |q_2| \geq \dots$ Then

$$\sigma^2 = \mathbf{E}\, \phi^2(X, \overline{X}) = \sum_{j \geq 1} q_j^2 < \infty, \qquad \phi(x,y) = \sum_{j\geq 1} q_j\, e_j(x)\, e_j(y) \tag{2.2}$$

since $\mathbb{Q}$ is the Hilbert–Schmidt operator and the kernel ϕ is degenerate. The series in (2.2) converges in $L^2(\mathcal{X}^2, \mathcal{B}^2, \mu \times \mu)$. Consider the subspace $L^2(\phi_1, \phi) \subset L^2$ generated by ϕ_1 and eigenfunctions e_j corresponding to non-zero eigenvalues $q_j \neq 0$. Introducing, if necessary, a normalized eigenfunction, say e_0, such that $\mathbb{Q}e_0 = 0$, we can assume that $e_0, e_1, \dots$ is an orthonormal basis of $L^2(\phi_1, \phi)$. Thus, we can write

$$\phi_1(X) = \sum_{j\geq 0} a_j\, e_j(X) \ \text{ in } \ L^2, \qquad \beta_2 = \mathbf{E}\, \phi_1^2(X) = \sum_{j \geq 0} a_j^2, \tag{2.3}$$

with $a_j = \mathbf{E}\, \phi_1(X)\, e_j(X)$. It is easy to see that $\mathbf{E}\, e_j(X) = 0$, for all j. Therefore $\big(e_j(X)\big)_{j \geq 0}$ is an orthonormal system of mean zero random variables.

For a random variable X, we shall denote by $\overline{X}$, X_j, $\overline{X}_j$, its independent copies. Throughout we shall assume as well that all random variables and vectors are independent in aggregate, if the contrary is not clear from the context.

A special representation of U-statistics. Let $\mathbb{R}^\infty$ denote the space of all real sequences $x = (x_0, x_1, x_2, \dots)$, $x_j \in \mathbb{R}$. The Hilbert space $l_2 \subset \mathbb{R}^\infty$ consist of $x \in \mathbb{R}^\infty$ such that

$$|x|^2 \stackrel{\text{def}}{=} \langle x, x\rangle, \qquad |x| < \infty, \qquad \big\langle x, y \big\rangle = \sum_{j \geq 0} x_j\, y_j.$$

Consider the random vector

$$\mathbf{X} \stackrel{\text{def}}{=} \big(e_0(X), e_1(X), e_2(X), \dots\big) \tag{2.4}$$

which takes values in $\mathbb{R}^\infty$. Since $\big\{e_j(X)\big\}_{j\geq 0}$ is a system of mean zero uncorrelated random variables with variances 1, the random vector $\mathbf{X}$ has identity covariance and mean zero. Using (2.2) and (2.3), we can write

$$\phi(X, \overline{X}) = \big\langle \mathbb{Q}\mathbf{X}, \overline{\mathbf{X}} \big\rangle, \qquad \phi_1(X) = \big\langle a, \mathbf{X} \big\rangle, \tag{2.5}$$

where we define $\mathbb{Q}x = (0, q_1 x_1, q_2 x_2, \dots)$, for $x \in \mathbb{R}^\infty$, and $a = (a_j)_{j\ge 0} \in \mathbb{R}^\infty$. The equalities (2.5) allow to assume throughout that the measurable space $\mathcal{X}$ is $\mathbb{R}^\infty$, the random variable X is a random vector taking values in $\mathbb{R}^\infty$ with mean zero and identity covariance, and that

$$\phi(X, \overline{X}) = \langle \mathbb{Q}X, \overline{X} \rangle, \qquad \phi_1(X) = \langle a, X \rangle. \tag{2.6}$$

In particular, we shall assume throughout that the kernels $\phi(x, y)$ and $\phi_1(x)$ are linear functions in each of their arguments.

Introduce the Gaussian random vector $G = (\eta_0, \eta_1, \dots)$ with values in $\mathbb{R}^\infty$, where $\eta_0, \eta_1, \dots$ denote i.i.d. standard normal random variables. The random vectors G and X have mean zero and equal covariances, that is,

$$\begin{aligned} &\mathbf{E}\,\phi_1(G) = \mathbf{E}\,\phi(G, x) = 0, && \mathbf{E}\,\phi_1^2(G) = \mathbf{E}\,\phi_1^2(X), \\ &\mathbf{E}\,\phi_1(G)\,\phi(G, x) = \mathbf{E}\,\phi_1(X)\,\phi(X, x), && \mathbf{E}\,\phi(G, x)\,\phi(G, y) = \mathbf{E}\,\phi(X, x)\,\phi(X, y). \end{aligned} \tag{2.7}$$

Furthermore, we have

$$\mathbf{E}\,\bigl|\phi_1(G)\bigr|^s \le c_s\,\mathbf{E}\,\bigl|\phi_1(X)\bigr|^s, \qquad \mathbf{E}\,\bigl|\phi(G, x)\bigr|^s \le c_s\,\mathbf{E}\,\bigl|\phi(X, x)\bigr|^s, \qquad s \ge 2. \tag{2.8}$$

Indeed, the random variable $\phi_1(G)$ (or $\phi(G, x)$) is Gaussian and therefore

$$\mathbf{E}\,\bigl|\phi_1(G)\bigr|^s \le c_s \bigl(\,\mathbf{E}\,\phi_1^2(G)\bigr)^{s/2} = c_s \bigl(\,\mathbf{E}\,\phi_1^2(X)\bigr)^{s/2} \le c_s\,\mathbf{E}\,\bigl|\phi_1(X)\bigr|^s.$$

Introduce the statistic

$$T(G_1, \dots, G_N) = \frac{1}{N} \sum_{1\le i<j\le N} \phi(G_i, G_j) + \frac{1}{\sqrt{N}} \sum_{1\le i\le N} \phi_1(G_i) \tag{2.9}$$

with

$$\phi_1(G) = \langle a, G \rangle, \qquad \phi(G, \overline{G}) = \langle \mathbb{Q}G, \overline{G} \rangle.$$

Define the random variable

$$\phi_0(G, G) = \sum_{j\ge 1} q_j\,(\eta_j^2 - 1). \tag{2.10}$$

The series converges in L^2 since $\sum_j q_j^2 < \infty$. Furthermore, using the finite dimensional approximations and substituting $\sqrt{N}\,G \overset{\mathcal{D}}{=} G_1 + \dots + G_N$ in (2.10), we see that

$$\phi_0(G, G) \overset{\mathcal{D}}{=} \frac{2}{N} \sum_{1\le i<j\le N} \phi(G_i, G_j) + \frac{1}{N} \sum_{1\le j\le N} \phi_0(G_j, G_j). \tag{2.11}$$

Consequently, for the statistic T_0 defined by (1.6), we have

$$\begin{aligned} T_0 &= \frac{1}{2}\sum_{j\geq 1} q_j\,(\eta_j^2-1) + \sum_{j\geq 0} a_j\,\eta_j = \frac{1}{2}\,\phi_0(G,G)+\phi_1(G) \\ &\overset{\mathcal{D}}{=} T(G_1,\dots,G_N) + \frac{1}{2\,N}\sum_{1\leq k\leq N}\phi_0(G_k,G_k). \end{aligned} \tag{2.12}$$

Let us introduce some additional notation which will be used throughout the paper. We write $[b]$ for the integer part of a number b. We denote $\mathrm{e}\{x\}=\exp\{i\,x\}$. We write $A\ll B$ (resp. $A\ll_s B$) if $A\leq cB$ (resp. $A\leq c_sB$). Furthermore, we write $A\asymp B$ (resp. $A\asymp_s B$) if $A\ll B\ll A$ (resp. $A\ll_s B\ll_s A$).

For a matrix $\mathbb{A}=(a_{ij})$, $1\leq i,j\leq s$, (resp. for the related linear operator), we denote by $|\mathbb{A}|=\sup_{|x|=1}|\mathbb{A}x|$ its operator norm. We shall use as well the norms

$$|\mathbb{A}|_2^2=\sum_{1\leq i,j\leq s} a_{ij}^2, \qquad |\mathbb{A}|_\infty=\max_{1\leq i,j\leq s}|a_{ij}|.$$

Throughout $\mathbb{I}$ stands for the identity matrix (resp. identity operator).

Introduce the function

$$\mathcal{M}(t;N)=1/\sqrt{|t|\,N},\ \text{ for } |t|\leq N^{-1/2}, \qquad \mathcal{M}(t;N)=\sqrt{|t|},\ \text{ for } |t|\geq N^{-1/2}. \tag{2.13}$$

Notice that, for $s>0$,

$$2^{-1}\Big(|t/N|^{-s/2}+|t|^{s/2}\Big)\leq\mathcal{M}^s(t;N)\leq|t/N|^{-s/2}+|t|^{s/2} \tag{2.14}$$

and

$$\inf_{t\in\mathbb{R}}\mathcal{M}^s(t;N)=N^{-s/4}. \tag{2.15}$$

For positive constants $c_1(s)$ and $c_2(s)$, the inequalities (2.14) imply

$$\mathcal{M}^s\big(c_1(s)\,t;c_2(s)\,N\big)\ll_s\mathcal{M}^s(t;N)\ll_s\mathcal{M}^s\big(c_1(s)\,t;c_2(s)\,N\big). \tag{2.16}$$

For subsets of the set $\mathbb{Z}$ of integer numbers, we write

$$[1,N]=\{1,\dots,N\}, \qquad (a,b]=\{j\in\mathbb{Z}:\ a<j\leq b\},\ \text{ etc.}$$

By $|A|$ we denote the number of elements of a set $A\subset\mathbb{Z}$. We denote by $\mathbb{N}$ the set of natural numbers, and $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$.

If a random vector assumes values in a linear space, then $\widetilde{X}=X-\overline{X}$ denotes its symmetrization. For conditional expectations we shall use the following notation (and its natural extensions)

$$\mathbf{E}\,\big\{\,f(X,Y)\,\big|\,Y\,\big\}=\mathbf{E}_X\,f(X,Y)=\mathbf{E}^Y\,f(X,Y).$$

Let $\xi_1, \dots, \xi_N$ denote independent real random variables which have mean zero. Then the following Rosenthal type inequality holds,

$$\mathbf{E}\Big|\sum_{j=1}^N \xi_j\Big|^q \ll_q \sum_{j=1}^N \mathbf{E}\,|\xi_j|^q + \Big(\sum_{j=1}^N \mathbf{E}\,|\xi_j|^2\Big)^{q/2}, \quad 2 \le q < \infty. \tag{2.17}$$

The inequality (2.17) combined with Hölder's inequality implies that

$$\mathbf{E}\Big|\sum_{j=1}^N \xi_j\Big|^q \ll_q N^{-1+q/2} \sum_{j=1}^N \mathbf{E}\,|\xi_j|^q, \qquad 2 \le q < \infty. \tag{2.18}$$

Consider a sufficiently smooth function $f : \mathbb{E} \to \mathbb{B}$, where $\mathbb{E}$ and $\mathbb{B}$ denote real Banach spaces. Let $\tau \in [0,1]$ be a random variable uniformly distributed on $[0,1]$ and independent of all other random variables. For $r = 0, 1, \dots,$ we shall use the Taylor formula

$$f(x) = f(0) + f'(0)\,x + \dots + \frac{1}{r!}\,f^{(r)}(0)\,x^r + \frac{1}{r!}\,\mathbf{E}\,(1-\tau)^r f^{(r+1)}(\tau x)\,x^{r+1}. \tag{2.19}$$

3. Bounds for concentration functions

In this section we shall consider the statistic T_* defined by (1.13). In the case $M \ge cN$, with some $c > 0$, the bounds of Theorem 3.1 are of order $\mathcal{O}(N^{-1})$.

We shall use the following non-degeneracy condition $\mathcal{N}(p, \delta, s, Z)$ for the distribution of a random vector Z, a kernel ϕ, parameters $0 < p \le 1$, $\delta \ge 0$ and $s \in \mathbb{N}$. Define the (random) matrix

$$\mathbb{A} = \mathbb{A}(Z) = (a_{ij})_{1 \le i,j \le s} \qquad \text{with entries} \qquad a_{ij} = \phi(Z_i, \overline{Z}_j), \tag{3.1}$$

where Z_i and $\overline{Z}_j$ are independent copies of Z. We say that Z and ϕ satisfy the non-degeneracy condition $\mathcal{N}(p, \delta, s, Z)$ if

$$\mathbf{P}\left\{|\mathbb{A}(Z) - \mathbb{I}|_\infty \le \delta\right\} \ge p, \tag{3.2}$$

where $\mathbb{I}$ denotes the identity matrix. A Gaussian random vector G satisfies this condition for any $s \ge 1$ and $\delta > 0$ with some $p > 0$, provided that at least s eigenvalues of the operator $\mathbb{Q}$ are non-zero (see Lemma 5.3). Another example of a random vector which satisfies the condition is

$$Y = (2\,m)^{-1/2}(\widetilde{X}_1 + \dots + \widetilde{X}_m) \tag{3.3}$$

provided that $q_s \ne 0$ and m is sufficiently large. For the random variable Y defined by (3.3), the condition $\mathcal{N}(p, \delta, s, Y)$ can be expressed for sufficiently large N in terms of ϕ and $X, X_1, \dots, X_N$ as follows. Let $X_i, \overline{X}_j$, $1 \le i,j \le s$, denote independent copies of X. Consider $2m$ independent copies $X_{ik}, \overline{X}_{jl}$ and $U_{ik}, \overline{U}_{jl}$, $1 \le k,l \le m$, of the sequence $X_i, \overline{X}_j$, $1 \le i,j \le s$. Then the entries a_{ij} of the random matrix $\mathbb{A}(Y)$ can be written as

$$a_{ij} = \frac{1}{2\,m} \sum_{k,l=1}^m \Big(\phi(X_{ik}, \overline{X}_{jl}) - \phi(X_{ik}, \overline{U}_{jl}) - \phi(U_{ik}, \overline{X}_{jl}) + \phi(U_{ik}, \overline{U}_{jl})\Big).$$

Theorem 3.1. *Let $s \geq 9$ and $\lambda \geq 0$. Then for the statistic T_* defined by* (1.13) *we have:*

i) Let $q_s \neq 0$. Then

$$Q(T_*;\,\lambda) \ll_s \exp\bigl\{c_s\,\sigma/|q_s|\bigr\}\,\frac{\max\,\{\lambda/\sigma;\,m_0\}}{M}, \qquad m_0 = \frac{\gamma_3}{\sigma^3} + \frac{\gamma_{2,3/2}^2}{\sigma^6}; \tag{3.4}$$

ii) Assume that the condition $\mathcal{N}(p,(4\,s)^{-1},s,G)$ holds. Then

$$Q(T_*;\,\lambda) \ll_s \frac{\max\,\{\lambda;\,m_0\}}{p\,M} \qquad \text{with } m_0 = \frac{\gamma_3}{p} + \frac{\gamma_{2,3/2}^2}{p^2}; \tag{3.5}$$

iii) Let $m \in \mathbb{N}$. Assume that the random vector Y defined by (3.3) *satisfies the condition $\mathcal{N}(p,(2\,s)^{-1},s,Y)$. Then we have*

$$Q(T_*;\,\lambda) \ll_s \frac{\max\,\{\lambda;\,m\}}{p\,M}. \tag{3.6}$$

Remark. The bound (3.6) of Theorem 3.1 is valid without any moment assumptions. The bounds (3.4) and (3.5) imply $Q(T_*;\,\lambda) = \mathcal{O}\Bigl(\dfrac{\lambda+1}{M}\Bigr)$ provided that $\gamma_3 < \infty$. An inspection of the proofs shows that the condition $\gamma_3 < \infty$ can be relaxed to $\gamma_{2+\varepsilon} < \infty$, for some $\varepsilon > 0$.

For the integration of characteristic functions we shall use the following Lemma 3.2 (see Bentkus and Götze 1996b). This lemma slightly extends Lemma 6.1 in Bentkus and Götze (1997).

Lemma 3.2. *Let $\varphi(t)$, $t \geq 0$, denote a continuous function such that $\varphi(0) = 1$ and $0 \leq \varphi \leq 1$. Assume that*

$$\varphi(t)\,\varphi(t+\tau) \leq \Theta\,\mathcal{M}^s(\tau;N), \qquad \text{for all} \quad t \geq 0 \quad \text{and} \quad \tau \geq 0,$$

with some $\Theta \geq 1$ independent of t and τ. Then, for any $A \geq 1$, $0 < B \leq 1$ and $N \geq 1$,

$$\int_{B/\sqrt{N}}^{A} \varphi(t)\,\frac{dt}{t} \ll_s \frac{\Theta^2\,(1+\log A)}{N} + \Theta^2\,B^{-s/2}\,N^{-s/4}, \qquad \text{for} \quad s > 8.$$

For $A \geq t_0, t_1 \geq 0$, define the integrals

$$I_0 = \int_{-t_1}^{t_1} \bigl|\,\widehat{\Psi}(t)\,\bigr|\,dt, \qquad I_1 = \int_{t_0 \leq |t| \leq A} \bigl|\,\widehat{\Psi}(t)\,\bigr|\,\frac{dt}{|t|}, \tag{3.7}$$

where $\widehat{\Psi}$ denotes the Fourier–Stieltjes transform of $\Psi(x) = \mathbf{P}\bigl\{T_* \leq x\bigr\}$.

Lemma 3.3. *Assume that condition $\mathcal{N}(p, \delta, s, Y)$ holds with some $0 \le \delta \le 1/(2s)$ and $s \ge 9$. Let*

$$k = \frac{pM}{m}, \qquad t_0 = \frac{c_0(s)}{m}\, k^{-1+2/s}, \qquad t_1 = \frac{c_1(s)}{m}\, k^{-1/2}, \qquad \frac{c_2(s)}{m} \le A \le \frac{c_3(s)}{m}$$

with some positive constants $c_j(s)$, $0 \le j \le 3$. Then

$$I_0 \ll_s (pM)^{-1}, \qquad I_1 \ll_s m\,(pM)^{-1}. \tag{3.8}$$

Proof. Without loss of generality we shall assume that $k \ge c_s$, for a sufficiently large constant c_s. Indeed, if $k \le c_s$, then we can derive (3.8) using $|\widehat{\Psi}| \le 1$. Another consequence of $k \ge c_s$ is that $1/(km) \le t_0 \le t_1 \le A$.

Let us prove (3.8) for I_0. By Theorem 6.2 we have $\bigl|\widehat{\Psi}(t)\bigr| \ll_s \mathcal{M}^s(tm; k)$. Using the obvious inequality $|\widehat{\Psi}| \le 1$, we get $\bigl|\widehat{\Psi}(t)\bigr| \ll_s \min\bigl\{1;\, \mathcal{M}^s\bigl(tm; k\bigr)\bigr\}$. Furthermore, denoting $t_2 = m^{-1} k^{-1/2} \max\{1; c_1(s)\}$ and using the definition of the function $\mathcal{M}$, we obtain

$$I_0 \ll_s \int\limits_0^{1/(km)} dt + \int\limits_{1/(km)}^{\infty} \frac{dt}{(tmk)^{s/2}} + \int\limits_0^{t_2} (tm)^{s/2}\, dt = \frac{1}{km} + \frac{c_s}{km} + \frac{c_s}{k^{(s+2)/4} m} \ll_s \frac{1}{km},$$

thus proving (3.8) for I_0.

It remains to estimate I_1. We shall reduce the proof to an application of Lemma 3.2. Using Lemma 6.3, we have

$$I_1 = \int\limits_{t_0 \le |t| \le A} \bigl|\widehat{\Psi}(t)\bigr|\, \frac{dt}{|t|} \le \mathbf{E} \int\limits_{t_0 \le |t| \le A} \psi(t)\, \frac{dt}{|t|}, \tag{3.9}$$

where a random function $\psi = \bigl|\mathbf{E}_\vartheta\, \mathrm{e}\{T^\vartheta\}\bigr|$ (defined in Lemma 6.3) may depend on $X_1, \dots, X_N$. By Lemma 7.1, there exists a (random) event D such that its complement D^c satisfies

$$\mathbf{P}\{D^c\} \ll_{s,d} k^{-d}, \qquad d \ge 0,$$

and

$$\mathbf{1}\{D\}\, \psi(t-\gamma)\psi(t+\gamma) \ll_{s,d} \mathcal{M}^s(\gamma m; k), \quad \text{for all } t, \gamma \in \mathbb{R}. \tag{3.10}$$

Thus, (3.9) yields

$$I_1 \ll_{s,d} k^{-d} \log(A/t_0) + \mathbf{E}\, I, \qquad I = \int\limits_{t_0 \le |t| \le A} \varphi_0(t)\, \frac{dt}{|t|}, \qquad \varphi_0 = \mathbf{1}\{D\}\, \psi, \tag{3.11}$$

Clearly, $k^{-d}\log(A/t_0) \ll_s k^{-d}\log k \ll k^{-1}$ provided that we choose $d=2$. We have

$$I = \int\limits_{m\,t_0 \le |t| \le m\,A} \varphi(t)\,\frac{dt}{|t|}, \quad \text{where} \quad \varphi(t) = \varphi_0(t/m).$$

The inequality (3.10) yields $\varphi(t-\gamma)\,\varphi(t+\gamma) \ll_{s,d} \mathcal{M}^s(\gamma;k)$. Hence, in order to estimate I in (3.11) we can use Lemma 3.2. Replacing in this lemma N by k and A by $m\,A$ respectively, choosing $B=\sqrt{k}\,m\,t_0$, we obtain $I \ll_s 1/k$. Thus, (3.11) implies $I_1 \ll_s k^{-1}$, proving (3.8) for I_1. $\square$

Proof of Theorem 3.1. First we shall prove *iii*) and then *iii*) $\Longrightarrow$ *ii*) $\Longrightarrow$ *i*).

Let us prove *iii*). Without loss of generality we shall assume that $m < p\,M$, since otherwise the result follows from the trivial estimate $Q(T_*;\lambda) \le 1$.

It is well known that

$$Q(T_*;\lambda) \le 2\,\max\left\{\lambda;\,\frac{1}{A}\right\}\int_{-A}^{A}\big|\,\widehat{\Psi}(t)\,\big|\,dt, \tag{3.12}$$

for any $A>0$ (see Petrov (1975), Lemma 3 of Ch. 3). Choose

$$A = c_s\,m^{-1}, \qquad t_1 = m^{-1}\,(p\,M/m)^{-1/2}, \qquad t_0 = c_0(s)\,m^{-1}(p\,M/m)^{-1+2/s}.$$

We can assume that $t_0 \le t_1$ since otherwise *iii*) follows from $Q(T_*;\lambda) \le 1$. The estimate (3.12) yields *iii*) provided that we show that

$$J = \int_{|t|\le A}\big|\,\widehat{\Psi}(t)\,\big|\,dt \ll_s (p\,M)^{-1}, \tag{3.13}$$

assuming that the condition $\mathcal{N}(p,(2\,s)^{-1},s,Y)$ is fulfilled. Using the obvious inequality $1 \ll_s |m\,t|^{-1}$, for $|t| \le c_s/m = A$, we have

$$J \ll_s I_0 + m^{-1} I_1 \quad\text{with}\quad I_0 = \int\limits_{-t_1}^{t_1}\big|\,\widehat{\Psi}(t)\,\big|\,dt, \quad I_1 = \int\limits_{t_0\le|t|\le A}\big|\,\widehat{\Psi}(t)\,\big|\,\frac{dt}{|t|} \tag{3.14}$$

since $t_0 \le t_1$. Lemma 3.3 yields $I_0 \ll_s (p\,M)^{-1}$ and $I_1 \ll_s m\,(p\,M)^{-1}$, proving (3.13) and *iii*).

Let us prove that *iii*) $\Longrightarrow$ *ii*). By Lemma 5.4, the condition $\mathcal{N}(2p,\delta,s,G)$ implies $\mathcal{N}(p,2\,\delta,s,Y)$, provided that

$$m \ge \frac{c_s}{\delta^3\,p}\,\Big(\frac{\gamma_{2,3/2}^2}{\delta^3\,p} + \gamma_3\Big). \tag{3.15}$$

Choosing in (3.15) the minimal possible m, we derive ii).

Let us prove that $ii) \Longrightarrow i)$. Write $\varepsilon^2 = \delta/(4\,s\,\sigma)$ and $\delta = (4\,s)^{-1}$. Then

$$Q(T_*;\,\lambda) = \sup_x \mathbf{P}\,\big\{\varepsilon^2\,x \le \varepsilon^2\,T_* \le \varepsilon^2\,x + \varepsilon^2\,\lambda\big\}. \tag{3.16}$$

The statistic $\varepsilon^2\,T_*$ is a statistic of type (3.1) with kernel $\varepsilon^2\,\phi$. By Lemma 5.3, the condition $\mathcal{N}(p, \delta, s, \varepsilon\,G)$ is fulfilled with $p = \exp\big\{-c_s/(\varepsilon^2\,|q_s|)\big\}$. Therefore we can apply ii) to the statistic $\varepsilon^2\,T_*$. Using (3.16) and $p \le 1$, we get

$$Q(T_*;\,\lambda) = Q(\varepsilon^2\,T_*;\,\varepsilon^2\,\lambda) \ll_s \frac{\max\,\{\varepsilon^2\,\lambda;\,m_0\}}{p^3\,M} \qquad \text{with } m_0 = \frac{\gamma_3}{\sigma^3} + \frac{\gamma_{2,3/2}^2}{\sigma^6}. \tag{3.17}$$

The bound (3.17) implies i) since by to our choice of ε and δ we have $p = \exp\Big\{-\frac{c_s\,\sigma}{|q_s|}\Big\}$. Finally, $c(s) \le \exp\big\{c_1(s)\,\sigma/|q_s|\big\}$ since $\sigma/|q_s| \ge 1$. $\square$

Proof of Theorem 1.2. The result follows from i) of Theorem 3.1 choosing $s = 9$, using $\gamma_3/\sigma^3 \ge 1$ as well as the inequality $\gamma_{2,3/2} \le \gamma_3$. $\square$

4. Proof of Theorem 1.1

The Edgeworth correction $F_1(x) = F_1(x; \mathcal{L}(X), \phi_1, \phi)$ is defined as a function of bounded variation (provided $|q_s| > 0$, for some $s \ge 9$; see (4.14) and Lemma 8.5) satisfying $F_1(-\infty) = 0$ and with the Fourier–Stieltjes transform given by

$$\widehat{F}_1(t) = \frac{(it)^3}{6\,\sqrt{N}}\,\mathbf{E}\,\big(\phi_1(X) + \phi(X, G)\big)^3\,\mathrm{e}\big\{t\,T_0\big\}. \tag{4.1}$$

Lemma 4.1. *Assume that the condition $\mathcal{N}(p, (4\,s)^{-1}, s, G)$ is fulfilled.*

i) Let $s \ge 13$. Then

$$\Delta_N \ll_s \frac{1}{N}\,\Big(\frac{1}{|q_s|^s} + \frac{1}{|q_s|^6} + \frac{1}{p^6}\Big)\,\big(\beta_4 + \beta_3^2 + \sigma^2 + \gamma_3 + \sigma^2\,\gamma_3 + \gamma_{2,2} + \sigma^2\,\gamma_{2,2}\big). \tag{4.2}$$

ii) Assume that the condition (1.9) *holds and that $s \ge 9$. Then*

$$\Delta_N \ll_s \frac{1}{N}\,\Big(\frac{1}{|q_s|^s} + \frac{1}{|q_s|^6} + \frac{1}{p^6}\Big)\,(\beta_4 + \sigma^2 + \gamma_3 + \gamma_{2,2}). \tag{4.3}$$

Proof of Theorem 1.1. We shall derive the result from Lemma 4.1. We shall prove (1.10) only. The proof of (1.11) is similar provided we choose $s = 9$ and use (4.3) instead of (4.2). Write

$$\varepsilon^2 = 1/\big(16\,s^2\,\sigma\big), \qquad \text{and} \qquad \Delta_N = \sup_x\big|\,F(x/\varepsilon^2) - F_0(x/\varepsilon^2) - F_1(x/\varepsilon^2)\,\big|. \tag{4.4}$$

Let us estimate Δ_N in (4.4) using Lemma 4.1. We have $F(x/\varepsilon^2) = \mathbf{P}\{T^{(\varepsilon)} \le x\}$, where the statistic $T^{(\varepsilon)}$ is defined similarly as T just replacing ϕ_1 and ϕ by $\varepsilon^2\phi_1$ and $\varepsilon^2\phi$ respectively. Similar representations are valid for $F_0(x/\varepsilon^2)$ and $F_1(x/\varepsilon^2)$ as well. Notice that $\varepsilon^2\phi(x,y) = \phi(\varepsilon x, \varepsilon y)$. By Lemma 5.3 the condition $\mathcal{N}(p,(4s)^{-1},s,\varepsilon G)$ holds with $p = \exp\{-c_s/(\varepsilon^2|q_s|)\}$. Therefore we can use (4.2). Choosing $s = 13$ and replacing moments and eigenvalues by those corresponding to the kernel $\varepsilon^2\phi$ and $\varepsilon^2\phi_1$, we obtain

$$\Delta_N \ll \frac{1}{N}\Big(\frac{\sigma^{13}}{|q_{13}|^{13}} + \frac{\sigma^6}{|q_{13}|^6} + \exp\Big\{\frac{c\sigma}{|q_{13}|}\Big\}\Big)\Big(\frac{\beta_4}{\sigma^4} + \frac{\beta_3^2}{\sigma^6} + 1 + \frac{\gamma_3}{\sigma^3} + \frac{\gamma_{2,2}}{\sigma^4}\Big). \tag{4.5}$$

The bound (4.5) implies (1.10) since $1 \le \gamma_3/\sigma^3$ and $\sigma^{13}/|q_{13}|^{13} \le \exp\{c\sigma/|q_{13}|\}$, in view of $\sigma/|q_{13}| \ge 1$. $\square$

Write $l = \big[(N-2)/20\big]$ and introduce the following bound

$$\varkappa(t) = \varkappa\big(t, N, \phi, \mathcal{L}(X)\big) = \varkappa_1(t) + \varkappa_2(t) \tag{4.6}$$

of characteristic functions, where

$$\begin{aligned}\varkappa_1(t) &= \sup_L \Big|\mathbf{E}\,\mathrm{e}\Big\{t N^{-1}\sum_{1\le j<k\le l}\phi(X_j,X_k) + L(X_1,\dots,X_l)\Big\}\Big|,\\ \varkappa_2(t) &= \sup_L \Big|\mathbf{E}\,\mathrm{e}\Big\{t N^{-1}\sum_{1\le j<k\le l}\phi(G_j,G_k) + L(G_1,\dots,G_l)\Big\}\Big|,\end{aligned} \tag{4.7}$$

where supremum is taken over all linear statistics L, that is, over all functions L which can be represented as $L(x_1,\dots,x_l) = \sum_{j=1}^{l} f_j(x_j)$ with some functions $f_1,\dots,f_l$.

For $r \in \mathbb{Z}_+$ and functions f_i, introduce the statistic

$$T^{(r)} = \frac{1}{N}\sum_{1\le i<j\le N}\phi(Z_i,Z_j) + \sum_{1\le i\le N} f_i(Z_i), \tag{4.8}$$

where

$$Z_j = X_j, \ \text{for } 1 \le j \le r, \qquad \text{and} \qquad Z_j = G_j, \ \text{for } r < j \le N.$$

Notice that any of statistics T, $T(G_1,\dots,G_N)$ can be represented as $T^{(r)}$ in (4.8), with some r and f_i. For the statistic T_0 we have $\mathcal{L}(T_0) = \mathcal{L}(T^{(0)})$ with some $T^{(0)}$ (to see this combine (2.9) and (2.12)).

Lemma 4.2. *Let $m \in \mathbb{N}$, $s \ge 9$ and $t_0 = m^{-1}(pN/m)^{-1+2/s}$. Assume that the random vector Y defined by (3.3) satisfies the condition $\mathcal{N}(p,(2s)^{-1},s,Y)$. Furthermore, assume that $\mathcal{N}(p,(4s)^{-1},s,G)$ holds. Then, for $pN > m$ and $m^{-1} \ge t_* \ge t_0$, the distribution function $F^{(r)}$ of $T^{(r)}$ satisfies*

$$F^{(r)}(x) = \frac{1}{2} + \frac{i}{2\pi}\,\mathrm{V.P.}\int_{-Nt_*}^{Nt_*}\mathrm{e}\{-xt\}\,\widehat{F}^{(r)}(t)\,\frac{dt}{t} + R \tag{4.9}$$

where $|R| \ll_s m/(pN)$.

Proof. We shall use the following approximate formula for the Fourier–Stieltjes inversion. For any $B > 0$ and any distribution function F with characteristic function $\widehat{F}$ we have (see, for example, Bentkus and Götze 1996b, Section 4)

$$F(x) = \frac{1}{2} + \frac{i}{2\pi}\,\mathrm{V.P.}\int_{-B}^{B} \mathrm{e}\{-xt\}\,\widehat{F}(t)\,\frac{dt}{t} + R, \tag{4.10}$$

where

$$|R| \le \frac{1}{B}\int_{-B}^{B} \bigl|\widehat{F}(t)\bigr|\,dt.$$

Here $\mathrm{V.P.}\int f(t)\,dt = \lim_{\varepsilon\to 0}\int_{|t|>\varepsilon} f(t)\,dt$ denotes Cauchy's Principal Value of the integral.

First let us consider the case $r \ge N/2$. Choosing $B = N/m$ and applying the approximate Fourier inversion formula (4.10), we have

$$F^{(r)}(x) = \frac{1}{2} + \frac{i}{2\pi}\,\mathrm{V.P.}\int\limits_{-N t_*}^{N t_*} \mathrm{e}\{-xt\}\,\widehat{F}^{(r)}(t)\,\frac{dt}{t} + J_1 + J_2 \tag{4.11}$$

with

$$J_1 = \frac{m}{N}\int\limits_{-N/m}^{N/m} \bigl|\widehat{F}^{(r)}(t)\bigr|\,dt, \qquad J_2 = \int\limits_{N t_0 \le |t| \le N/m} \bigl|\widehat{F}^{(r)}(t)\bigr|\,\frac{dt}{|t|},$$

Let us show that

$$J_1 \ll_s m/(pN) \qquad \text{and} \qquad J_2 \ll_s m/(pN). \tag{4.12}$$

Conditioning on $Z_{r+1}, \dots, Z_N$, changing variables in the integral and rearranging summands in (4.8), we have

$$J_1 \le m\,\mathbf{E}\int\limits_{-1/m}^{1/m} \Bigl|\mathbf{E}_{X_1,\dots,X_r}\,\mathrm{e}\bigl\{t T_*^{(r)}\bigr\}\Bigr|\,dt \tag{4.13}$$

with

$$T_*^{(r)} = \sum_{1\le i<j\le r} \phi(X_i, X_j) + \sum_{1\le i\le r} f_i^*(X_i),$$

with some f_j^* which can depend on Z_j, $j > r$. The statistic $T_*^{(r)}$ is a statistic of type T_* (see (1.13)) with N replaced by $r \ge N/2$. For $T_*^{(r)}$ the number M in the definition (1.13) of T_* may be chosen arbitrary such that $M \le N/2$. Let us take $M \asymp N/4$. Now an application of (3.13) to the right hand side of (4.13) shows that J_1 satisfies (4.12).

Arguments similar to those for J_1 show that J_2 satisfies (4.12); however instead of (3.13) one has to apply an analog of the second inequality in (3.8). Combining (4.11) and (4.12), we conclude the proof of the lemma in the case $r \ge N/2$.

In the case $r < N/2$ the statistic $T^{(r)}$ depends on at least $N/2$ Gaussian random vectors G_j. Therefore we can repeat the proof for the case $r \ge N/2$ replacing X_j by G_j. Instead of $\mathcal{N}(p/2, (2\,s)^{-1}, s, Y)$ we shall use the condition $\mathcal{N}(p, (2\,s)^{-1}, s, Y')$ with $Y' = (2\,m)^{-1/2}(\widetilde{G}_1 + \dots + \widetilde{G}_m)$. But this condition holds since $Y' \overset{\mathcal{D}}{=} G$ and $\mathcal{N}(p, (4\,s)^{-1}, s, G)$ is fulfilled. $\square$

Proof of Lemma 4.1. We shall represent the Edgeworth correction by the following Fourier–Stieltjes inversion formula. For any function $F : \mathbb{R} \to \mathbb{R}$ of bounded variation such that $F(-\infty) = 0$ and $2\,F(x) = F(x+) + F(x-)$, for all $x \in \mathbb{R}$, we have (see, e.g., Chung 1974)

$$F(x) = \frac{1}{2}\,F(\infty) + \frac{i}{2\,\pi}\,\lim_{M\to\infty} \mathrm{V.\,P.} \int_{|t|\le M} \mathrm{e}\{-x\,t\}\,\widehat{F}(t)\,\frac{dt}{t}. \tag{4.14}$$

The formula is well-known for distribution functions. For functions of bounded variation, it extends by linearity.

Let $m \in \mathbb{N}$. Assuming in addition that the random vector Y defined by (3.3) satisfies the condition $\mathcal{N}(p, (2\,s)^{-1}, s, Y)$, we shall prove that

$$\Delta_N \ll_s \frac{m}{p\,N} + \alpha + \frac{1}{p^6\,N}\,\big(\beta_4 + \beta_3^2 + \sigma^2 + \gamma_3 + \sigma^2\,\gamma_3 + \gamma_{2,2} + \sigma^2\,\gamma_{2,2}\big), \tag{4.15}$$

provided that $s \ge 13$, where

$$\alpha = \frac{\beta_3^2 + \sigma^2\,\gamma_{2,2}}{N}\,\Big(\frac{1}{|q_s|^s} + \frac{1}{|q_s|^6}\Big).$$

Furthermore, in the case that the condition (1.9) holds and $s \ge 9$, we shall prove

$$\Delta_N \ll_s \frac{m}{p\,N} + \alpha + \frac{1}{p^4\,N}\,\big(\beta_4 + \sigma^2 + \gamma_3 + \gamma_{2,2}\big). \tag{4.16}$$

The bounds (4.15) and (4.16) yield (4.2) and (4.3) respectively. Indeed, using Lemma 5.4 and estimating $\gamma_{2,3/2}^2 \le \sigma^2\,\gamma_{2,2}$ we see that

$$\mathcal{N}(p, (4\,s)^{-1}, s, G) \quad \text{implies} \quad \mathcal{N}(2^{-1}p, (2\,s)^{-1}, s, Y),$$

provided that

$$m \ge c_s\,p^{-2}\,\sigma^2\,\gamma_{2,2} + c_s\,p^{-1}\,\gamma_3. \tag{4.17}$$

Hence, choosing the minimal m in (4.17), we obtain (4.2) and (4.3).

In the proof of (4.15) and (4.16) we may assume that $pN > m$. Indeed, if $pN \le m$ then using the obvious bounds $|F| \le 1$, $|F_0| \le 1$ as well as (4.14) and the bound (8.210) of Lemma 8.5 for $|F_1|$, we have

$$\Delta_N \ll_s 1 + \frac{(\beta_3^2 + \sigma^2\,\gamma_{2,2})^{1/2}}{N^{1/2}\,|q_s|^3} \le \frac{m}{p\,N} + \frac{(\beta_3^2 + \sigma^2\,\gamma_{2,2})^{1/2}}{N^{1/2}\,|q_s|^3}\,\Big(\frac{m}{p\,N}\Big)^{1/2}$$
$$\ll \frac{m}{p\,N} + \frac{\beta_3^2 + \sigma^2\,\gamma_{2,2}}{N\,|q_s|^6},$$

and there is nothing to prove.

Choose $t_* = m^{-1}(pN/m)^{-1/2}$. Applying Lemma 4.2 to the distribution functions F and F_0, Lemma 8.5 with $\lambda = N\,t_*$ and (4.14) to the Edgeworth correction F_1, we obtain

$$\Delta_N \ll_s J + R, \qquad J \overset{\text{def}}{=} \int_{-N\,t_*}^{N\,t_*} \big|\,\widehat{F}(t) - \widehat{F}_0(t) - \widehat{F}_1(t)\,\big|\,\frac{dt}{|t|}, \tag{4.18}$$

where

$$|R| \ll_s \frac{m}{p\,N} + N^{-1/2}\,(\beta_3^2 + \sigma^2\,\gamma_{2,2})^{1/2}\,|q_s|^{-s/2}\,(N\,t_*)^{3-s/2}.$$

We have $Nt_* = \dfrac{N}{m}\Big(\dfrac{p\,N}{m}\Big)^{-1/2}$. Therefore $a\,b \ll a^2 + b^2$ combined with $p\,N \ge m$, $0 < p \le 1$ and $s \ge 9$ yields

$$|R| \ll_s \frac{m}{p\,N} + \frac{\beta_3^2 + \sigma^2\,\gamma_{2,2}}{|q_s|^s}. \tag{4.19}$$

Below we shall prove that

$$I_r \overset{\text{def}}{=} \int_{-N\,t_*}^{N\,t_*} |t|^r\,\varkappa(t)\,\frac{dt}{|t|} \ll_s p^{-r}, \qquad 1 \le r < s/2. \tag{4.20}$$

Using (4.20) and $p \le 1$, integrating the bounds for $\widehat{\Delta}_N(t) = \big|\,\widehat{F}(t) - \widehat{F}_0(t) - \widehat{F}_1(t)\,\big|$ of Lemma 8.1, we obtain

$$J \ll_s \frac{1}{p^6\,N}\,\Big(\beta_4 + \beta_3^2 + \sigma^2 + \gamma_3 + \sigma^2\,\gamma_3 + \gamma_{2,2} + \sigma^2\,\gamma_{2,2}\Big), \tag{4.21}$$

provided that $s \ge 13$, and

$$J \ll_s \frac{1}{p^4\,N}\,(\beta_4 + \sigma^2 + \gamma_3 + \gamma_{2,2}) \tag{4.22}$$

if (1.9) is fulfilled and $s \ge 9$. Collecting the bounds (4.18)–(4.22) we obtain (4.15) and (4.16).

It remains to prove (4.20). To estimate $\varkappa(t)$ we shall apply Theorem 6.2. The statistic in the definition (4.7) of $\varkappa_1$ is a statistic of type (1.13) with N replaced by

$l = [(N-2)/20]$. There M can be chosen arbitrary such that $1 \le M \le l$; for example, let $M = [l/2]$. Thus, using (2.14), we have $\varkappa_1(t) \ll_s \min\,\big\{1;\, \mathcal{M}^s(tm/N; pN/m)\big\}$. A similar bound holds for $\varkappa_2(t)$. Therefore, applying (2.13), we obtain

$$\begin{aligned} I_r &\ll_s \int_{-N\,t_*}^{N\,t_*} |t|^r \,\min\,\big\{1;\, \mathcal{M}^s(tm/N;\, pN/m)\big\}\,\frac{dt}{|t|} \\ &= 2\,N^r \int_0^{t_*} t^r\,\min\,\big\{1;\, \mathcal{M}^s(tm;\, pN/m)\big\}\,\frac{dt}{t} \\ &\ll N^r \int_0^{1/(pN)} t^r\,\frac{dt}{t} + N^r \int_{1/(pN)}^{t_*} t^r\,\frac{1}{(tpN)^{s/2}}\,\frac{dt}{t} \ll p^{-r}. \qquad \square \end{aligned}$$

5. Auxiliary results

Estimates of the convergence rate for expectations of smooth functions. Recall that X, X_j and G, G_j, $j \ge 1$, denote sequences of i.i.d. mean zero random vectors taking values in $\mathbb{R}^\infty$. We assume that X and G have equal covariances, that is, that (2.7) holds, and that G is Gaussian. Using the linearization (2.6) we can assume that $\phi(x,y)$ is linear in its arguments $x, y \in \mathbb{R}^\infty$. For $S_m = (X_1 + \dots + X_m)/\sqrt{m}$, we may write

$$\phi(S_m, \overline{S}_m) = m^{-1} \sum_{j=1}^m \sum_{k=1}^m \phi(X_j, \overline{X}_k). \tag{5.1}$$

Lemma 5.1. *For any three times continuously differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$, we have*

$$\Big|\, \mathbf{E}\,\varphi\big(\phi(S_m, \overline{S}_m)\big) - \mathbf{E}\,\varphi\big(\phi(G, \overline{G})\big) \Big| \ll m^{-1/2}\,|\varphi'''|_\infty\,(\gamma_{2,3/2} + m^{-1/2}\,\gamma_3), \tag{5.2}$$

where $|\varphi|_\infty = \sup\limits_{x\in\mathbb{R}} \big|\,\varphi(x)\,\big|$.

Proof. Notice that

$$\Big|\, \mathbf{E}\,\varphi\big(\phi(S_m, \overline{S}_m)\big) - \mathbf{E}\,\varphi\big(\phi(G, \overline{G})\big) \Big| \le \mathbf{E}\,J_1 + \mathbf{E}\,J_2,$$

where

$$\begin{aligned} J_1 &= \Big|\, \mathbf{E}_{S_m}\,\varphi\big(\phi(S_m, \overline{S}_m)\big) - \mathbf{E}_G\,\varphi\big(\phi(G, \overline{S}_m)\big) \Big|, \\ J_2 &= \Big|\, \mathbf{E}_{\overline{S}_m}\,\varphi\big(\phi(G, \overline{S}_m)\big) - \mathbf{E}_{\overline{G}}\,\varphi\big(\phi(G, \overline{G})\big) \Big|. \end{aligned}$$

In order to prove the lemma it is sufficient to show that both $\mathbf{E}\,J_1$ and $\mathbf{E}\,J_2$ are bounded by the right hand side of (5.2). For example, we shall bound $\mathbf{E}\,J_1$. Let us fix the value $z = \overline{S}_m$. Split

$$\begin{aligned} \phi(S_m, \overline{S}_m) &= \phi(S_m, z) = \xi_1 + \dots + \xi_m, &\qquad \text{where } \xi_j = \phi(X_j, z)/\sqrt{m}, \\ \phi(G, \overline{S}_m) &= \phi(G, z) \overset{\mathcal{D}}{=} \zeta_1 + \dots + \zeta_m, &\qquad \text{where } \zeta_j = \phi(G_j, z)/\sqrt{m}. \end{aligned}$$

Notice that random variables $\xi_1, \dots, \xi_m$ (resp. $\zeta_1, \dots, \zeta_m$) are i.i.d. Writing

$$\delta_j = \Big| \mathbf{E}\, \varphi\big(W_j + \xi_j\big) - \mathbf{E}\, \varphi\big(W_j + \zeta_j\big) \Big|, \qquad W_j = \zeta_1 + \dots + \zeta_{j-1} + \xi_{j+1} + \dots + \xi_m,$$

we obtain

$$J_1 \le \delta_1 + \dots + \delta_m. \tag{5.3}$$

Expanding in powers of ξ_j and of ζ_j and using (5.1), we get

$$\delta_j \le c \Big(\mathbf{E}\, |\xi_j|^3 + \mathbf{E}\, |\zeta_j|^3 \Big)\, |\varphi'''|_\infty. \tag{5.4}$$

The random variable $\zeta_j = \phi(G_j, z)/\sqrt{m}$ is centered and Gaussian. Therefore

$$\mathbf{E}\, |\zeta_j|^3 \ll \Big(\mathbf{E}\, \zeta_j^2 \Big)^{3/2} = \Big(\mathbf{E}\, \xi_j^2 \Big)^{3/2} \le \mathbf{E}\, |\xi_j|^3. \tag{5.5}$$

Collecting (5.3)–(5.5), we obtain $J_1 \ll m\, |\varphi'''|_\infty\, \mathbf{E}\, |\xi_1|^3$, whence, using $z = \overline{S}_m$, we have

$$\mathbf{E}\, J_1 \ll m^{-1/2}\, |\varphi'''|_\infty\, \mathbf{E}\, \big| \phi(X, \overline{S}_m) \big|^3. \tag{5.6}$$

Consequently, $\mathbf{E}\, J_1$ is bounded by the right-hand side of (5.2) since Rosenthal's inequality implies

$$\mathbf{E}\, \big| \phi(x, \overline{S}_m) \big|^3 = m^{-3/2}\, \mathbf{E} \Big| \sum_{j=1}^m \phi(x, \overline{X}_j) \Big|^3 \ll m^{-1/2}\, \mathbf{E}\, \big| \phi(x, \overline{X}) \big|^3 + \Big(\mathbf{E}\, \phi^2(x, \overline{X}) \Big)^{3/2},$$

for any $x \in \mathbb{R}^k$. $\square$

Let $Y_1, \dots, Y_s$ and $\overline{Y}_1, \dots, \overline{Y}_s$ be independent copies of S_m. For a sufficiently smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, write

$$D = \Big| \mathbf{E} \prod_{1 \le i,j \le s} \varphi\big(\phi(Y_i, \overline{Y}_j) - \delta_{ij} \big) - \mathbf{E} \prod_{1 \le i,j \le s} \varphi\big(\phi(G_i, \overline{G}_j) - \delta_{ij} \big) \Big|, \tag{5.7}$$

where δ_{ij} is the Kronecker symbol.

Lemma 5.2. *For D defined by (5.7) we have*

$$D \le C\, (m^{-1/2} \gamma_{2,3/2} + m^{-1} \gamma_3) \tag{5.8}$$

with

$$C \ll_s |\varphi|_\infty^{s^2-3} \Big(|\varphi|_\infty^2\, |\varphi'''|_\infty + |\varphi|_\infty\, |\varphi'|_\infty\, |\varphi''|_\infty + |\varphi'|_\infty^3 \Big).$$

Proof. The proof consists of $2s$ steps. In each step we shall replace one of the random vectors Y_j (or $\overline{Y}_j$) by a corresponding Gaussian random vector G_j (respectively, $\overline{G}_j$), for $1 \le j \le s$. We shall show that the errors of these replacements are bounded from above by the right-hand side of (5.8), thus proving the lemma.

For example, let us replace Y_1 by G_1 in

$$\mathbf{E} \prod_{1 \le i,j \le s} \varphi\big(\phi(Y_i, \overline{Y}_j) - \delta_{ij}\big). \tag{5.9}$$

Similarly as in the proof of Lemma 5.1, fix

$$y_i = Y_i,\ \bar{y}_j = \overline{Y}_j \quad \text{and} \quad z_i = G_i,\ \bar{z}_j = \overline{G}_j, \qquad \text{for}\ \ 2 \le i \le s,\ 1 \le j \le s. \tag{5.10}$$

Conditioning on the random vectors in (5.10), we see that it is sufficient to bound the difference

$$I = \Big| \mathbf{E} \prod_{1 \le j \le s} \varphi\big(\phi(Y_1, \bar{y}_j) - \delta_{1j}\big) - \mathbf{E} \prod_{1 \le j \le s} \varphi\big(\phi(G_1, \bar{y}_j) - \delta_{1j}\big) \Big|. \tag{5.11}$$

For $x = (x_1, \dots, x_s) \in \mathbb{R}^s$, define the function

$$\psi(x) = \varphi(x_1 - 1)\,\varphi(x_2) \dots \varphi(x_s). \tag{5.12}$$

This function is Frechet differentiable and

$$\|\psi'''\|_\infty \ll_s |\varphi|_\infty^{s-3} \big(|\varphi|_\infty^2 \, |\varphi'''|_\infty + |\varphi|_\infty \, |\varphi'|_\infty \, |\varphi''|_\infty + |\varphi'|_\infty^3 \big), \tag{5.13}$$

where $\|\psi'''\|_\infty = \sup_x \|\psi'''(x)\|$, and $\|\psi'''(x)\| = \sup\big\{ |\psi'''(x) h^3| \,:\, h \in \mathbb{R}^s,\, |h| = 1 \big\}$ denotes the norm of the Frechet derivative $\psi'''(x)$.

Introduce the random vectors

$$\mathbf{Y} = \big(\phi(Y_1, \bar{y}_1), \dots, \phi(Y_1, \bar{y}_s)\big), \quad \mathbf{G} = \big(\phi(G_1, \bar{y}_1), \dots, \phi(G_1, \bar{y}_s)\big) \tag{5.14}$$

which take values in $\mathbb{R}^s$. Using the notation (5.12) and (5.14), we can rewrite I defined by (5.11) as $I = \big| \mathbf{E}\, \psi(\mathbf{Y}) - \mathbf{E}\, \psi(\mathbf{G}) \big|$. Recall that $Y_1 \overset{\mathcal{D}}{=} (X_1 + \dots + X_m)/\sqrt{m}$. A similar representation is valid for G_1. Thus

$$\mathbf{Y} \overset{\mathcal{D}}{=} \xi_1 + \dots + \xi_m, \qquad \text{where}\ \ \xi_j = \big(\phi(X_j, \bar{y}_l)/\sqrt{m}\big)_{1 \le l \le s},$$

$$\mathbf{G} \overset{\mathcal{D}}{=} \zeta_1 + \dots + \zeta_m, \qquad \text{where}\ \ \zeta_j = \big(\phi(G_j, \bar{y}_l)/\sqrt{m}\big)_{1 \le l \le s}.$$

Notice that the random vectors $\xi_1, \dots, \xi_m$ (resp. $\zeta_1, \dots, \zeta_m$) take values in $\mathbb{R}^s$ and are i.i.d. Moreover, the random vectors ζ_j are Gaussian, have mean zero and their covariances are equal to those of ξ_j. Writing

$$\delta_j = \big| \mathbf{E}\, \psi\big(\mathbf{W}_j + \xi_j\big) - \mathbf{E}\, \psi\big(\mathbf{W}_j + \zeta_j\big) \big|, \qquad \mathbf{W}_j = \zeta_1 + \dots + \zeta_{j-1} + \xi_{j+1} + \dots + \xi_m,$$

we obtain

$$I = \big|\, \mathbf{E}\, \psi(\mathbf{Y}) - \mathbf{E}\, \psi(\mathbf{G}) \big| \le \delta_1 + \dots + \delta_m.$$

Expanding into a Taylor series in powers of ξ_j and of ζ_j and using the equality of the covariances of ξ_j and ζ_j, we get

$$\delta_j \ll \Big(\mathbf{E}\, |\xi_j|^3 + \mathbf{E}\, |\zeta_j|^3 \Big)\, \|\psi'''\|_\infty.$$

Here

$$|\xi_j|^3 = m^{-3/2} \Big(\sum_{l=1}^{s} \phi^2(X_j, \overline{y}_l) \Big)^{3/2} \ll_s m^{-3/2} \sum_{l=1}^{s} \big| \phi(X_j, \overline{y}_l) \big|^3,$$

and $\|\psi'''\|_\infty$ was estimated in (5.13). A similar bound holds for $|\zeta_j|^3$. Averaging and proceeding similarly as in the proof of Lemma 5.1, we see that Y_1 in (5.9) can be replaced by G_1 with an error bounded from above by the right hand side of (5.8). $\square$

The condition $\mathcal{N}(p, \delta, s, X)$.

Recall that the random $s \times s$ matrix $\mathbb{A}(X)$ has entries $a_{ij} = \phi(X_i, \overline{X}_j)$, where X_j and $\overline{X}_j$ denote independent copies of X. The matrix $\mathbb{A}(G)$ is defined similarly using independent copies of G. Furthermore, we write $\sigma^2 = q_1^2 + q_2^2 + \dots$ and $|\mathbb{A}|_\infty = \max \big\{ |a_{ij}| : \ 1 \le i, j \le s \big\}$.

Lemma 5.3. *Let $q_s \ne 0$, for some $s \ge 1$. Then, for any $\delta > 0$ and $\varepsilon > 0$, the condition $\mathcal{N}(p, \delta, s, \varepsilon\, G)$ holds with some $p > 0$, that is,*

$$\nu \overset{\text{def}}{=} \mathbf{P}\big\{ |\mathbb{A}(\varepsilon\, G) - \mathbb{I}|_\infty < \delta \big\} \ge p > 0. \tag{5.15}$$

Moreover, there exists a positive constant c_s such that the condition $\mathcal{N}(p, \delta, s, \varepsilon\, G)$ is fulfilled with $p = \exp\{-c_s/(\varepsilon^2\, |q_s|)\}$, that is,

$$\nu \ge \exp\{-c_s/(\varepsilon^2\, |q_s|)\} \tag{5.16}$$

provided that $\delta \le 1$ and $4\, s\, \sigma\, \varepsilon^2 \le \delta$.

Proof. Notice that $\phi(x, y) = \sum_{j \ge 1} q_j\, x_j\, y_j$, cf (2.5)–(2.6). Therefore in the proof of the lemma we can assume that $G = (0, \eta_1, \eta_2, \dots)$. Let us split the identity operator $\mathbb{J} : \mathbb{R}^\infty \to \mathbb{R}^\infty$ into the sum $\mathbb{J} = \mathbb{P}_s + \mathbb{P}_s^\perp$ of the projector $\mathbb{P}_s x = (0, x_1, \dots, x_s, 0, 0, \dots)$, $x = (x_0, x_1, \dots)$, and its orthogonal complement $\mathbb{P}_s^\perp$. Then $x = \mathbb{P}_s x + \mathbb{P}_s^\perp x$ and we have $\phi(\mathbb{P}_s x, \mathbb{P}_s^\perp x) = \big\langle \mathbb{Q}\mathbb{P}_s\, x, \mathbb{P}_s^\perp x \big\rangle \equiv 0$, for all $x \in \mathbb{R}^\infty$. Thus the Gaussian vectors $\mathbb{P}_s \varepsilon\, G$ and $\mathbb{P}_s^\perp \varepsilon\, G$ are orthogonal and independent since $\varepsilon\, G$ has independent coordinates. Therefore we can write

$$\mathbb{A}(\varepsilon\, G) = \mathbb{A}(\mathbb{P}_s\, \varepsilon\, G) + \mathbb{A}(\mathbb{P}_s^\perp\, \varepsilon\, G) \overset{\text{def}}{=} \mathbb{U} + \mathbb{V} \tag{5.17}$$

with independent random matrices $\mathbb{U} = \mathbb{A}(\mathbb{P}_s\, \varepsilon\, G)$ and $\mathbb{V} = \mathbb{A}(\mathbb{P}_s^\perp\, \varepsilon\, G)$.

Let v_{ij} denote entries of the matrix $\mathbb{V}$. The random variables v_{ij} are identically distributed and we have

$$\mathbf{P}\big\{\big|\mathbb{V}\big|_\infty \geq b\big\} \leq \frac{1}{b^2}\,\mathbf{E}\max_{1\leq i,j\leq s} v_{ij}^2 \leq \frac{s^2}{b^2}\,\mathbf{E}\,v_{11}^2 \leq \frac{s^2\,\varepsilon^4\,\sigma^2}{b^2}, \qquad b>0. \tag{5.18}$$

Let us prove (5.15). Replacing q_j by $\varepsilon^{1/2}\,q_j$, for all j, we can assume that $\varepsilon=1$. Choose $b=2\,s^2\,\sigma$. Then (5.18) yields $\mathbf{P}\big\{|\mathbb{V}|_\infty<b\big\}\geq 1/2$. Using (5.17), the independence of $\mathbb{U}$ and $\mathbb{V}$, and conditioning on $\mathbb{U}$, we have

$$\begin{aligned}\nu &\geq \mathbf{E}\,\mathbf{1}\big\{|\mathbb{V}|_\infty<b\big\}\,\mathbf{E}_{\mathbb{U}}\,\mathbf{1}\big\{|\mathbb{U}+\mathbb{V}-\mathbb{I}|_\infty<\delta\big\}\\ &\geq \frac{1}{2}\,\inf\,\Big\{\mathbf{P}\big\{|\mathbb{U}-\mathbb{C}|_\infty<\delta\big\}:\,|\mathbb{C}|_\infty\leq b+1\Big\}.\end{aligned} \tag{5.19}$$

Introduce the standard basis vectors $e_j=(\delta_{0j},\delta_{1j},\dots)$ in $\mathbb{R}^\infty$, where δ_{ij} is the Kronecker symbol. Let $\lambda>0$ be arbitrary. Write $\rho=\delta/(2\,|q_1|\,\lambda)$ and consider the events

$$A=\big\{|\mathbb{P}_s\overline{G}_j-e_j|\leq\rho,\ \text{for } 1\leq j\leq s\big\}, \qquad B=\big\{|\mathbb{P}_sG_i|\leq\lambda,\ \text{for } 1\leq i\leq s\big\}.$$

The events A and B are independent. Furthermore,

$$\mathbf{P}\,\{A\}=\prod_{j=1}^{s}\mathbf{P}\,\big\{|\mathbb{P}_s\overline{G}_j-e_j|\leq\rho\big\}>0, \qquad \text{for any } \rho>0.$$

Notice that the event $A\cap B$ implies that $\big|u_{ij}-\big\langle\mathbb{Q}\mathbb{P}_sG_i,e_j\big\rangle\big|\leq|q_1|\,\rho\,\lambda\leq\delta/2$ since $u_{ij}=\big\langle\mathbb{Q}\mathbb{P}_sG_i,P_s\,\overline{G}_j\big\rangle$ and $\rho=\delta/(2\,|q_1|\,\lambda)$. Therefore, denoting by c_{ij} the entries of the matrix $\mathbb{C}$, we obtain

$$\begin{aligned}\mathbf{P}\big\{|\mathbb{U}-\mathbb{C}|_\infty<\delta\big\} &\geq \mathbf{P}\big\{A\cap B\cap\big\{|\mathbb{U}-\mathbb{C}|_\infty<\delta\big\}\big\}\\ &\geq \mathbf{P}\big\{A\cap B\cap\big\{\max_{1\leq i,j\leq s}\big|\big\langle\mathbb{Q}\mathbb{P}_sG_i,e_j\big\rangle-c_{ij}\big|<\delta/2\big\}\big\}.\end{aligned} \tag{5.20}$$

The random vectors G, G_i and $\overline{G}_j$ are i.i.d. Thus (5.19) and (5.20) together imply

$$\nu\geq\frac{1}{2}\,\mathbf{P}\,\big\{A\big\}\inf_{|c|\leq b+1}\Big(\mathbf{P}\Big\{|\mathbb{P}_sG|\leq\lambda\ \text{ and }\ \max_{1\leq j\leq s}\big|\big\langle\mathbb{Q}\mathbb{P}_sG,e_j\big\rangle-c\big|<\frac{\delta}{2}\Big\}\Big)^s. \tag{5.21}$$

The Gaussian random vector G has independent standard normal coordinates and the eigenvalues q_j of $\mathbb{Q}$ are non-zero, for $1\leq j\leq s$. Therefore we can choose a sufficiently large λ depending on s, $\mathbb{Q}$ and δ such that the right hand side of (5.21) is positive, completing the proof of (5.15).

Let us prove (5.16). The condition $4\,s\,\varepsilon^2\,\sigma\leq\delta$ and the estimate (5.18) with $b=\delta/2$ yield $\mathbf{P}\,\big\{|\mathbb{V}|_\infty\leq\delta/2\big\}\geq1/2$. Therefore, applying (5.17) and $|\mathbb{A}+\mathbb{B}|_\infty\leq|\mathbb{A}|_\infty+|\mathbb{B}|_\infty$, using the independence of $\mathbb{U}$ and $\mathbb{V}$, we have

$$\nu\geq\mathbf{P}\Big\{|\mathbb{V}|_\infty<\frac{\delta}{2}\Big\}\,\nu_1\geq\frac{1}{2}\,\nu_1, \qquad \text{where } \nu_1=\mathbf{P}\Big\{|\mathbb{U}-\mathbb{I}|_\infty<\frac{\delta}{2}\Big\}. \tag{5.22}$$

Let henceforth Z_i, $\overline{Z}_j$ denote independent copies of the standard normal random vector $Z=(\eta_1,\dots,\eta_s)$ in $\mathbb{R}^s$. Then we can represent the entries u_{ij} of the matrix $\mathbb{U}$ as

$$u_{ij} \overset{\mathcal{D}}{=} \varepsilon^2 \big\langle \mathbb{Q} Z_j, \overline{Z}_k \big\rangle, \qquad \mathbb{Q}x = (q_1 x_1,\dots,q_s x_s) \in \mathbb{R}^s.$$

Let $e_1,\dots,e_s$ be the standard orthonormal basis in $\mathbb{R}^s$, and $B=\big\{x\in\mathbb{R}^s:\ |x|\le 1\big\}$ denote the unit ball. For $\rho>0$, consider the subsets

$$D_i = \frac{1}{\varepsilon\sqrt{|q_i|}}\,(e_i+\rho B), \qquad \overline{D}_j = \frac{\operatorname{sign} q_j}{\varepsilon\sqrt{|q_j|}}\,(e_j+\rho B), \qquad 1\le i,j\le s,$$

of $\mathbb{R}^s$. It is easy to verify that

$$\varepsilon^2\big\langle \mathbb{Q}x,y\big\rangle = \delta_{ij}+r_{ij}, \qquad |r_{ij}|\le (2\sigma\rho+\sigma\rho^2)/|q_s|, \qquad \text{for } x\in D_i \text{ and } y\in\overline{D}_j, \tag{5.23}$$

where δ_{ij} is the Kronecker symbol. Choose $\rho=\dfrac{\delta\,|q_s|}{6\sigma}$. Then $\rho\le 1$ since $\delta\le 1$. Thus (5.23) shows that the independent events

$$A_i=\big\{Z_i\in D_i\big\}, \qquad \overline{A}_j=\big\{\overline{Z}_j\in\overline{D}_j\big\}, \qquad 1\le i,j\le s,$$

satisfy

$$\nu_1 \ge \prod_{j=1}^s \mathbf{P}\{A_j\}\,\mathbf{P}\{\overline{A}_j\} = \prod_{j=1}^s \big(\mathbf{P}\{A_j\}\big)^2, \tag{5.24}$$

since Z_i, $\overline{Z}_j$ are i.i.d. and symmetric.

The random vector Z is centered and Gaussian. Let $e\in\mathbb{R}^s$. According to a result of Borell (1974), we have $\mathbf{P}\big\{Z\in a\,e+\rho B\big\}\ge \mathbf{P}\big\{Z\in b\,e+\rho B\big\}$ provided that $|a|\le|b|$. Thus, using $\sigma\ge|q_j|\ge|q_s|$ and shrinking the radius of the balls, we get

$$\mathbf{P}\{A_j\}\ge \mathbf{P}\big\{Z\in B_s\big\}, \qquad B_s \overset{\text{def}}{=} \frac{e_s}{\varepsilon\sqrt{|q_s|}} + \frac{\rho B}{\varepsilon\sqrt{\sigma}}.$$

The distribution of Z has the density $p(x)=(2\pi)^{-s/2}\exp\big\{-|x|^2/2\big\}$. Estimating $p(x)\ge\inf\big\{p(x):\ x\in B_s\big\}$ and writing $t=\rho/\big(\varepsilon\sqrt{\sigma}\big)$, we obtain

$$\mathbf{P}\{A_j\}\ge c_1(s)t^s\exp\Big\{-\frac12\Big(\frac{1}{\varepsilon\sqrt{|q_s|}}+t\Big)^2\Big\}\ge c_1(s)t^s\exp\Big\{-\frac{2}{\varepsilon^2|q_s|}\Big\} \tag{5.25}$$

since the assumption $\delta\le 1$ and the definitions of t and ρ imply that $\dfrac{1}{\varepsilon\sqrt{|q_s|}}\ge t$. Using the assumptions $4s\varepsilon^2\sigma\le\delta$ and $\delta\le 1$ it is easy to verify that $t\ge\varepsilon^2 q_s/2$ and that $\varepsilon^2|q_s|\le 1$. Therefore the inequality $z\ge\exp\{-1/z\}$ with $z=\varepsilon^2|q_s|\le 1$ combined with (5.24) and (5.25) implies

$$\nu_1\ge c_2(s)\exp\Big\{-\frac{c_3(s)}{\varepsilon^2|q_s|}\Big\},$$

whence (5.16) follows by (5.22) and since $\varepsilon^2|q_s|\le 1$. $\square$

Lemma 5.4. *Let $\delta > 0$. Then there exists a positive constant c_s such that the condition $\mathcal{N}(2p, \delta, s, G)$ implies condition $\mathcal{N}(p, 2\delta, s, S_m)$, provided that*

$$m \geq c_s\, \delta^{-3}\, p^{-1} \big(\delta^{-3} p^{-1} \gamma_{2,3/2}^2 + \gamma_3 \big).$$

Proof. It suffices to show that

$$2\,\mathbf{P}\big\{ |\mathbb{A}(S_m) - \mathbb{I}|_\infty \leq 2\,\delta \big\} \geq \mathbf{P}\big\{ |\mathbb{A}(G) - \mathbb{I}|_\infty \leq \delta \big\} \overset{\text{def}}{=} 2p, \tag{5.26}$$

for m specified in the condition of the lemma. We shall apply Lemma 5.2. Consider an infinitely differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ with bounded derivatives such that

$$0 \leq \varphi \leq 1, \qquad \varphi(x) = 1, \ \text{ for } |x| \leq 1, \qquad \varphi(x) = 0, \quad \text{for } \ |x| \geq 2.$$

Let $Y_1, \dots, Y_s$ and $\overline{Y}_1, \dots, \overline{Y}_s$ denote independent copies of S_m. Applying Lemma 5.2 we have

$$\begin{aligned} 2\,\mathbf{P}\big\{ |\mathbb{A}(S_m) - \mathbb{I}|_\infty \leq 2\,\delta \big\} &\geq 2\ \mathbf{E} \prod_{1 \leq i,j \leq s} \varphi\Big(\frac{\phi(Y_i, \overline{Y}_j) - \delta_{ij}}{\delta} \Big) \\ &\geq 2\ \mathbf{E} \prod_{1 \leq i,j \leq s} \varphi\Big(\frac{\phi(G_i, \overline{G}_j) - \delta_{ij}}{\delta} \Big) - c(s) \big(m^{-1/2} \gamma_{2,3/2} + m^{-1} \gamma_3 \big)\, \delta^{-3} \\ &\geq 2\,\mathbf{P}\big\{ |\mathbb{A}(G) - \mathbb{I}|_\infty \leq \delta \big\} - 2\,p = 2\,p \end{aligned}$$

provided m satisfies the condition of the lemma. □

A symmetrization lemma.

Lemma 5.5. *Assume that the set $[1, N] = \cup_{j=1}^4 \Omega_j$ is a union of disjoint subsets. Let $\theta_1, \dots, \theta_N$ be arbitrary real-valued random variables such that the families of random variables*

$$\big\{ \theta_j, \ j \in \Omega_1 \big\}, \qquad \big\{ \theta_j, \ j \in \Omega_2 \big\}, \qquad \big\{ \theta_j, \ j \in \Omega_3 \big\}, \qquad \big\{ \theta_j, \ j \in \Omega_4 \big\}$$

are independent. Let a_{jk}, $1 \leq j, k \leq N$, be arbitrary real numbers. Assume that a statistic T can be represented as

$$T = \sum_{1 \leq j < k \leq N} \theta_j\, \theta_k\, a_{jk} + f_1 + f_2,$$

where f_1 (resp. f_2) is a function of θ_j, $j \in \Omega_1 \cup \Omega_4$ (resp. of θ_j, $j \in [1, N] \setminus \Omega_1$). Denote by $\bar\theta_1, \dots, \bar\theta_N$ an independent copy of $\theta_1, \dots, \theta_N$. Then

$$2 \,\Big| \,\mathbf{E}\ \mathrm{e}\big\{ T \big\} \Big|^2 \leq \mathbf{E}\ \mathrm{e}\big\{ T' \big\} + \mathbf{E}\ \mathrm{e}\big\{ T'' \big\},$$

where

$$T' = \sum_{j\in\Omega_1}\sum_{k\in\Omega_2} \widetilde\theta_j\,\widetilde\theta_k\,a_{jk}, \qquad T'' = \sum_{j\in\Omega_1}\sum_{k\in\Omega_3} \widetilde\theta_j\,\widetilde\theta_k\,a_{jk},$$

and $\widetilde\theta_j = \theta_j - \bar\theta_j$, *for* $1 \le j \le N$.

Proof. For a two-point set $A = \{j,k\} \subset [1,N]$ we shall denote $T_A = \theta_j\,\theta_k\,a_{jk}$. In this notation we can write

$$T = \sum_{|A|=2} T_A + f_1 + f_2.$$

Furthermore, excluding certain terms T_A from the sum and adding them to f_1 or f_1, we can represent T as follows:

$$T = R + R^* + f_1^* + f_2^*, \tag{5.27}$$

where

$$R = \sum_{A:\ |A\cap\Omega_1|=1,\ |A\cap\Omega_2|=1} T_A, \qquad R^* = \sum_{A:\ |A\cap\Omega_1|=1,\ |A\cap\Omega_3|=1} T_A,$$

and where f_1^* resp. f_2^* have the same structure as f_1 resp. f_2, which means that f_1^* depends on θ_j, $j \in \Omega_1\cup\Omega_4$ only resp. f_2^* is independent of θ_j, $j \in \Omega_1$. Indeed: if $A \subset \Omega_1\cup\Omega_4$, then we can include T_A in f_1; if $A \subset \Omega_2\cup\Omega_3\cup\Omega_4$, then we can add T_A to f_2; thus only terms T_A such that $|A\cap\Omega_1| = 1$ and $\bigl|A\cap(\Omega_2\cup\Omega_3)\bigr| = 1$ will remain; collecting them into sums R and R^*, we obtain (5.27).

Finally, for a subset $B \subset [1,N]$ and a statistic $f = f\bigl(\theta_j;\, j\in[1,N]\bigr)$, introduce the symmetrization with respect to the random variables θ_k, $k \in B$, as

$$\mathcal S_B f = f\bigl(\theta_j;\, j\in[1,N]\bigr) - f\bigl(\bar\theta_j, \theta_k;\, j\in B,\ k\in[1,N]\setminus B\bigr).$$

Using (5.27), conditioning on θ_j, $j \notin \Omega_1$, writing $\mathbf E_r$ for the partial integration with respect to the distributions of θ_j and $\bar\theta_j$, $j \in \Omega_r$, and applying Hölder's inequality we have

$$\begin{aligned}\Bigl|\,\mathbf E\ \mathrm e\bigl\{T\bigr\}\Bigr|^2 &\le \mathbf E\Bigl|\mathbf E_1\ \mathrm e\bigl\{R + R^* + f_1^*\bigr\}\Bigr|^2 = \mathbf E\ \mathrm e\bigl\{\mathcal S_{\Omega_1}R + \mathcal S_{\Omega_1}R^* + \mathcal S_{\Omega_1}f_1^*\bigr\}\\ &\le \mathbf E\Bigl|\mathbf E_2\,\mathbf E_3\ \mathrm e\bigl\{\mathcal S_{\Omega_1}R + \mathcal S_{\Omega_1}R^*\bigr\}\Bigr|.\end{aligned} \tag{5.28}$$

For given $\widetilde\theta_j$, $j \in \Omega_1$, the random variables $\mathcal S_{\Omega_1}R$ and $\mathcal S_{\Omega_1}R^*$ are independent. Thus (5.28) together with $2\,a\,b \le a^2 + b^2$ implies

$$\begin{aligned}2\Bigl|\,\mathbf E\ \mathrm e\bigl\{T\bigr\}\Bigr|^2 &\le 2\,\mathbf E\Bigl|\mathbf E_2\ \mathrm e\bigl\{\mathcal S_{\Omega_1}R\bigr\}\ \mathbf E_3\,\mathrm e\bigl\{\mathcal S_{\Omega_1}R^*\bigr\}\Bigr|\\ &\le \mathbf E\Bigl|\mathbf E_2\,\mathrm e\bigl\{\mathcal S_{\Omega_1}R\bigr\}\Bigr|^2 + \mathbf E\Bigl|\mathbf E_3\,\mathrm e\bigl\{\mathcal S_{\Omega_1}R^*\bigr\}\Bigr|^2\\ &= \mathbf E\ \mathrm e\bigl\{T'\bigr\} + \mathbf E\ \mathrm e\bigl\{T''\bigr\}.\end{aligned} \qquad \square$$

6. Bounds for characteristic functions of U-statistics

In this section we shall consider the statistic T_* defined by (1.13). The main result of the section is the following

Lemma 6.1. *Assume that the distribution of Gaussian random vector G satisfies the non-degeneracy condition $\mathcal{N}(p,(4\,s)^{-1},s,G)$. Write $m_0 = p^{-2}\gamma_{2,3/2}^2 + p^{-1}\gamma_3$. For any statistic T_* of type* (1.13) *we have*

$$\left|\,\mathbf{E}\ \mathrm{e}\bigl\{t\,T_*\bigr\}\right| \ll_s \mathcal{M}^s\bigl(t\,m_0;\,p\,M/m_0\bigr),$$

where the function $\mathcal{M}$ is defined by (2.13).

Lemma 6.1 is a corollary of the following Theorem 6.2, which is valid without any moment assumptions.

Theorem 6.2. *Let $m \in \mathbb{N}$. Assume that the sum $Y = (2\,m)^{-1/2}(\widetilde{X}_1 + \cdots + \widetilde{X}_m)$ satisfies the non-degeneracy condition $\mathcal{N}(p,\delta,(2\,s)^{-1},Y)$. Then, for any statistic T_* of type* (1.13) *we have*

$$\left|\,\mathbf{E}\ \mathrm{e}\bigl\{t\,T_*\bigr\}\right| \ll_s \mathcal{M}^s\bigl(t\,m;\,p\,M/m\bigr).$$

Proof of Lemma 6.1. Using Lemma 5.4, we have that

$$\mathcal{N}(p,(4\,s)^{-1},s,G) \quad \text{implies} \quad \mathcal{N}(2^{-1}p,(2\,s)^{-1},s,Y),$$

provided that

$$m \ge c_s\,m_0, \qquad\qquad m_0 = p^{-2}\gamma_{2,3/2}^2 + p^{-1}\gamma_3. \tag{6.1}$$

Thus, by Theorem 6.2, we obtain

$$\left|\,\mathbf{E}\ \mathrm{e}\bigl\{t\,T_*\bigr\}\right| \ll_s \mathcal{M}^s\bigl(c_1(s)\,t\,m;\,c_2(s)\,p\,M/m\bigr). \tag{6.2}$$

Using (2.16) and choosing the minimal $m \asymp c_s\,m_0$ such that (6.1) holds, Lemma 6.1 follows from (6.2). $\square$

To prove Theorem 6.2 we need the auxiliary Lemmas 6.3–6.6. Lemma 6.3 allows to replace the characteristic function of T_* by the characteristic function of a statistic of discrete random variables. Lemmas 6.4 and 6.5 are necessary steps for an application of the double large sieve type bound of Lemma 6.6. In Section 7 we extend the methods of this section for the proof of a multiplicative inequality.

Using randomization and conditioning techniques, we shall bound the characteristic function of T_* by the characteristic function of certain statistics of discrete random variables. To this end we need the following notation. Let $\varepsilon_1, \varepsilon_2, \ldots$ denote i.i.d. Rademacher random variables. As usually, we assume that all random variables and vectors are independent in aggregate, if the contrary is not clear from the context. In particular, the sequences introduced below are independent of $X_1, \dots, X_N$. Let m be an arbitrary natural number. Define the random variables $\vartheta_1 = \varepsilon_1, \dots, \vartheta_m = \varepsilon_1$. Similarly, the random variables ϑ_j, $j > m$, are defined block-wise, that is,

$$\vartheta_j = \varepsilon_l, \qquad \text{for } j \in I(l) \overset{\text{def}}{=} (m\,l - m,\, m\,l]. \tag{6.3}$$

Furthermore, introduce the sequence $\alpha_j = \vartheta_j/2+1/2$ of random variables taking values 0 and 1 with probability 1/2.

For natural numbers m and s, we introduce the non-negative integers

$$L = [M/(2\,m\,s)], \qquad K = s\,L \quad \text{and} \quad K_0 = m\,K = s\,m\,L \tag{6.4}$$

and the disjoint sets

$$\Omega_1 = [1,\, K_0], \quad \Omega_2 = (K_0, 2\,K_0], \quad \Omega_3 = (2\,K_0,\, 3\,K_0], \quad \Omega_4 = (3\,K_0, N]. \tag{6.5}$$

Lemma 6.3. *Assume that $m, s \in \mathbb{N}$. Then the statistic T_* defined by* (1.13) *satisfies*

$$\Bigl|\, \mathbf{E}\, \mathrm{e}\bigl\{ t T_* \bigr\} \Bigr| \le \mathbf{E} \Bigl| \mathbf{E}_\vartheta\, \mathrm{e}\bigl\{ t T^\vartheta \bigr\} \Bigr|, \tag{6.6}$$

where $\mathbf{E}_\vartheta = \mathbf{E}_{\{\vartheta_j,\, 1 \le j \le 3\, K_0\}}$,

$$T^\vartheta = \sum_{1 \le j < k \le 3\,K_0} \vartheta_j\, \vartheta_k\, a_{jk} + F_1 + F_2, \qquad a_{jk} \overset{\text{def}}{=} \frac{1}{4}\, \phi(\widetilde{X}_j, \widetilde{X}_k), \tag{6.7}$$

the number K_0 is defined in (6.4), *and where F_1 (resp. F_2) is a function of ϑ_j, $j \in [1,\, K_0]$, (resp. of ϑ_j, $j \in (K_0, 3\,K_0]$).*

Notice, that in (6.7) both statistics F_1 and F_2 may depend on $X_1, \dots, X_N$ and $\overline{X}_1, \dots, \overline{X}_N$.

Proof of Lemma 6.3. Define the disjoint sets $\Omega_1', \dots, \Omega_4' \subset [1, N]$ as follows:

$$\Omega_1' = [1,\, K_0], \quad \Omega_2' = (M, M + K_0], \quad \Omega_3' = (M + K_0, M + 2\,K_0], \tag{6.8}$$

and $\Omega_4' = [1, N] \setminus (\Omega_1' \cup \Omega_2' \cup \Omega_3')$. Without loss of generality we shall assume that

$$\Omega_1' = \Omega_1, \quad \Omega_2' = \Omega_2, \quad \Omega_3' = \Omega_3, \quad \Omega_4' = \Omega_4. \tag{6.9}$$

Indeed, we have to estimate $\Bigl|\, \mathbf{E}\, \mathrm{e}\bigl\{ t T_* \bigr\} \Bigr|$, and using the i.i.d. assumption we can reorder $X_1, \dots, X_N$ so that (6.9) holds.

Using (6.8), we can write

$$T_* = \sum_{\{j,k\} \subset \Omega_1 \cup \Omega_2 \cup \Omega_3} \phi(X_j, X_k) + f_1^* + f_2^*, \tag{6.10}$$

where f_1^* is a function of X_j with $j \in \Omega_1 \cup \Omega_4$, and f_2^* is a function of X_j with $j \in \Omega_2 \cup \Omega_3 \cup \Omega_4$. Indeed, the terms $\phi(X_j, X_k)$ with $\{j,k\} \cap \Omega_4 \neq \varnothing$ can be included in f_1^* or in f_2^*.

Introduce the statistic T^α replacing the random vectors X_j in (6.10) by the vectors $\alpha_j X_j + (1-\alpha_j)\overline{X}_j$, for $1 \le j \le 3K_0$. The statistics T_* and T^α have the same distribution since, for given $\alpha_1, \alpha_2, \dots$, the conditional distribution of T^α equals the distribution of T_*. Indeed, if $\alpha_j = 1$, then X_j remains unchanged; if $\alpha_j = 0$, then X_j are just exchanged with an independent copy $\overline{X}_j$. In any case this does not change the distribution of T_*. Therefore (6.10) implies

$$T_* \overset{\mathcal{D}}{=} T^\alpha = \sum_{\{j,k\} \subset \Omega_1 \cup \Omega_2 \cup \Omega_3} \phi\big(\alpha_j X_j + (1-\alpha_j)\overline{X}_j, \alpha_k X_k + (1-\alpha_k)\overline{X}_k\big) + f_1^{**} + f_2^{**}, \tag{6.11}$$

where f_1^{**} is a function of α_j with $j \in \Omega_1 \cup \Omega_4$, and f_2^{**} is a function of α_j with $j \in \Omega_2 \cup \Omega_3 \cup \Omega_4$. The equality $\alpha_j = 1/2 + \vartheta_j/2$ implies

$$\alpha_j X_j + (1-\alpha_j)\overline{X}_j = \frac{1}{2}\,\vartheta_j \widetilde{X}_j + \frac{1}{2}\,(X_j + \overline{X}_j), \qquad \widetilde{X}_j = X_j - \overline{X}_j. \tag{6.12}$$

Applying (6.12) to (6.11), using the linearity of $\phi(x,y)$ and rearranging the terms of the statistic, we obtain that $T_* \overset{\mathcal{D}}{=} T^\vartheta$ with T^ϑ defined by (6.7). Conditioning on all random variables and vectors except ϑ_j, $1 \le j \le 3K_0$, concludes the proof of (6.6). $\square$

Lemma 6.4. *Let $m \in \mathbb{N}$. Write*

$$Y = (2m)^{-1/2} \sum_{j=1}^{m} \widetilde{X}_j, \qquad \varLambda = \sum_{j=1}^{K} \widetilde{\varepsilon}_j Y_j, \tag{6.13}$$

where $Y_1, Y_2, \dots$ are independent copies of Y and K is given by (6.4). Then we have

$$\Big|\,\mathbf{E}\,\mathrm{e}\big\{tT_*\big\}\Big|^2 \le \mathbf{E}\,\mathrm{e}\big\{2^{-1} t m\, \phi(\varLambda, \overline{\varLambda})\big\}.$$

Proof. We shall apply the symmetrization Lemma 5.5 to the characteristic function $\mathbf{E}_\vartheta\,\mathrm{e}\big\{tT^\vartheta\big\}$ involved in the estimate of Lemma 6.3. Consider the sets Ω_l, $1 \le l \le 3$, defined by (6.5). Notice that the families of random variables

$$\big\{\vartheta_j,\ j \in \Omega_1\big\}, \qquad \big\{\vartheta_j,\ j \in \Omega_2\big\}, \qquad \big\{\vartheta_j,\ j \in \Omega_3\big\}$$

are independent.

Using the estimate of Lemma 6.3, conditioning on $X_1, \dots, X_M$ and $\overline{X}_1, \dots, \overline{X}_M$, writing $\mathbf{E}_\vartheta = \mathbf{E}_{\vartheta_1, \vartheta_2, \dots}$ and applying the symmetrization Lemma 5.5, we have

$$\begin{aligned} 2\Big|\,\mathbf{E}\,\mathrm{e}\big\{tT_*\big\}\Big|^2 &\le 2\,\mathbf{E}\Big|\mathbf{E}_\vartheta\,\mathrm{e}\big\{tT^\vartheta\big\}\Big|^2 \le \mathbf{E}\,\mathbf{E}_\vartheta\,\mathrm{e}\big\{tT'\big\} + \mathbf{E}\,\mathbf{E}_\vartheta\,\mathrm{e}\big\{tT''\big\} \\ &= \mathbf{E}\,\mathrm{e}\big\{tT'\big\} + \mathbf{E}\,\mathrm{e}\big\{tT''\big\} = 2\,\mathbf{E}\,\mathrm{e}\big\{tT'\big\}, \end{aligned} \tag{6.14}$$

since both statistics

$$T' = \sum_{j\in\Omega_1}\sum_{k\in\Omega_2} \widetilde{\vartheta}_j \widetilde{\vartheta}_k a_{jk}, \qquad \text{and} \qquad T'' = \sum_{j\in\Omega_1}\sum_{k\in\Omega_3} \widetilde{\vartheta}_j \widetilde{\vartheta}_k a_{jk} \tag{6.15}$$

(with a_{jk} defined by (6.7)) have the same distribution. Finally, notice that $\mathcal{L}(T') = \mathcal{L}\big(m\,\phi(\varLambda,\overline{\varLambda})/2\big)$, due to the definition (6.3) of ϑ_j and the linearity of $\phi(x,y)$. Indeed, we have

$$T' = \frac{1}{4}\,\phi\Big(\sum_{j\in\Omega_1} \widetilde{\vartheta}_j \widetilde{X}_j, \sum_{k\in\Omega_2} \widetilde{\vartheta}_k \widetilde{X}_k\Big) \tag{6.16}$$

and

$$\sum_{j\in\Omega_1} \widetilde{\vartheta}_j \widetilde{X}_j = \sum_{j=1}^{mK} \widetilde{\vartheta}_j \widetilde{X}_j = \sum_{l=1}^{K} \widetilde{\varepsilon}_l \sum_{j\in I(l)} \widetilde{X}_j = \sqrt{2m}\,\varLambda. \qquad \square$$

In the proof of the next Lemma 6.5 we shall combine random selections and the geometric-arithmetic mean inequality. This will reduce the problem to the case of sums of i.i.d. random vectors of a very special type.

By $\tau, \tau_1, \tau_2, \dots$ and τ_{kl} we shall denote independent copies of a symmetric random variable τ with non-negative characteristic function and such that

$$1 \le \mathbf{E}\,\tau^2 \le 2, \qquad\qquad \mathbf{P}\big\{|\tau| \le 2\big\} = 1. \tag{6.17}$$

An example of such τ is $\widetilde{\varepsilon}$. Define the random s-dimensional vector $W = (\tau_1, \dots, \tau_s)$. Let $W_1, \dots, W_q$ and $\overline{W}_1, \dots, \overline{W}_q$ denote independent copies of W. Introduce the sums

$$U = W_1 + \dots + W_q, \qquad\qquad V = \overline{W}_1 + \dots + \overline{W}_q. \tag{6.18}$$

Lemma 6.5. *Let $s \in \mathbb{N}$ and $L \in \mathbb{Z}_+$. Assume that a random vector Y takes values in $\mathbb{R}^\infty$ and satisfies the condition $\mathcal{N}(p,\delta,s,Y)$. Write*

$$\varLambda = \sum_{j=1}^{sL} \tau_j Y_j, \qquad \overline{\varLambda} = \sum_{j=1}^{sL} \overline{\tau}_j \overline{Y}_j, \qquad q = \big[p\,L/4\big], \tag{6.19}$$

where Y_j and $\overline{Y}_j$ are independent copies of Y. Then we have

$$\mathbf{E}\,\mathrm{e}\big\{t\,\phi(\varLambda,\overline{\varLambda})\big\} \le c_d(s)\,(p\,L)^{-d} + \sup_{\mathbb{A}}\,\mathbf{E}\,\mathrm{e}\big\{t\,\langle \mathbb{A}U, V\rangle\big\}, \qquad t\in\mathbb{R},\ d\ge 0, \tag{6.20}$$

where $\sup_{\mathbb{A}}$ denotes the supremum over all $s\times s$ (non-random) matrices $\mathbb{A}$ such that $|\mathbb{A} - \mathbb{I}|_\infty \le \delta$.

Proof. Introduce the sets $\mathcal{I}(j) = (j\,s - s, j\,s] \subset [1, s\,L]$ of natural numbers, for $1 \le j \le L$. Split the sum $\varLambda$ into a sum of blocks of length s, that is, write

$$\varLambda = Z(1) + \dots + Z(L), \qquad \text{where } Z(j) = \sum_{k\in\mathcal{I}(j)} \tau_k Y_k. \tag{6.21}$$

Similarly, write

$$\overline{A} = \overline{Z}(1) + \cdots + \overline{Z}(L), \qquad \text{where } \overline{Z}(j) = \sum_{k \in \mathcal{I}(j)} \overline{\tau}_k \overline{Y}_k. \tag{6.22}$$

Let us describe a random event, say D, which will play an essential role in our proof. For $1 \le j, k \le L$, define the $s \times s$-matrices

$$\mathbb{A}(j,k) = \big(\phi(Y_l, \overline{Y}_r)\big), \qquad l \in \mathcal{I}(j), \ \ r \in \mathcal{I}(k).$$

Introduce the random events and variables

$$B(j,k) = \big\{ |\mathbb{A}(j,k) - \mathbb{I}|_\infty \le \delta \big\}, \qquad \xi_{jk} = \mathbf{1}\{B(j,k)\}. \tag{6.23}$$

Condition $\mathcal{N}(p, \delta, s, Y)$ guarantees that $\mathbf{E}\, \xi_{jk} = \mathbf{P}\big\{B(j,k)\big\} \ge p$. Without loss of generality we shall assume that $\mathbf{E}\, \xi_{jk} = \mathbf{P}\big\{B(j,k)\big\} = p$. Furthermore, the events $B(j,k)$ occur with equal probabilities. Write

$$\eta_j = \sum_{k=1}^{L} \xi_{jk}, \qquad D = \Big\{ \sum_{j=1}^{L} \eta_j \ge \frac{p\,L^2}{2} \Big\}, \qquad D_j = \Big\{ \eta_j \ge \frac{p\,L}{4} \Big\}.$$

We shall use the following properties of D and D_j: the complement D^c of D satisfies

$$\mathbf{P}\big\{D^c\big\} = \mathbf{P}\Big\{ \sum_{j=1}^{L} \sum_{k=1}^{L} \xi_{jk} < \frac{p\,L^2}{2} \Big\} \ll_d (p\,L)^{-d}, \qquad \text{for } d \ge 0, \tag{6.24}$$

and

$$D \text{ occurs implies that at least } \frac{p\,L}{4} \text{ of the events } D_1, \dots, D_L \text{ occur.} \tag{6.25}$$

Let us prove (6.24). If $p\,L \le 1$ then (6.24) follows from $\mathbf{P}\{D^c\} \le 1$. Therefore we suppose without loss of generality that $p\,L \ge 1$. Assume as well that $d \ge 1$ since the desired result for $0 \le d \le 1$ follows from (6.24) with $d = 1$. Introducing a random vector ψ such that $\mathcal{L}(\psi) = \mathcal{L}(Y_1, \dots, Y_s)$, and let ψ_j and $\overline{\psi}_k$ denote its independent copies. Then we can represent ξ_{jk} as a function, say u, of ψ_j and $\overline{\psi}_k$, that is $\xi_{jk} = u(\psi_j, \overline{\psi}_k)$. Define the function $\rho(x) = \mathbf{E}\, u(x, \psi)$. Then we can write

$$-p\,L^2 + \sum_{j=1}^{L} \eta_j = -p\,L^2 + \sum_{j=1}^{L} \sum_{k=1}^{L} \xi_{jk} \stackrel{\text{def}}{=} R + R_1 + \overline{R}_1, \tag{6.26}$$

where

$$R = \sum_{j=1}^{L} \sum_{k=1}^{L} \big(\xi_{jk} - \rho(\psi_j) - \rho(\overline{\psi}_k) + p \big), \qquad R_1 = L \sum_{j=1}^{L} \big(\rho(\psi_j) - p \big),$$

and where $\overline{R}_1$ is defined similarly to R_1 with ψ_j replaced by $\overline{\psi}_j$. Using this notation and Chebyshev's inequality, we have

$$\mathbf{P}\big\{D^c\big\} \ll_d (pL^2)^{-2d}\Big(\mathbf{E}\,|R|^{2d} + \mathbf{E}\,|R_1|^{2d} + \mathbf{E}\,|\overline{R}_1|^{2d}\Big). \tag{6.27}$$

To bound $\mathbf{E}\,|R_1|^{2d} = \mathbf{E}\,|\overline{R}_1|^{2d}$ we use the inequality (2.17). We obtain

$$\mathbf{E}\,|R_1|^{2d} \ll_d pL + (pL)^d \ll_d (pL)^d, \quad \text{since } pL \ge 1,\ 0 \le \rho \le 1,\ \ \mathbf{E}\,\rho(\psi) = p.$$

Using (2.18) twice conditionally, we derive

$$\begin{aligned}
\mathbf{E}\,|R|^{2d} &= \mathbf{E}\,\mathbf{E}_\psi\,|R|^{2d} \ll_d L^d\,\mathbf{E}\,\mathbf{E}_\psi \Big|\sum_{k=1}^{L}\big(\xi_{1k} - \rho(\psi_1) - \rho(\overline{\psi}_k) + p\big)\Big|^{2d} \\
&= L^d\,\mathbf{E}\,\mathbf{E}_{\overline{\psi}} \Big|\sum_{k=1}^{L}\big(\xi_{1k} - \rho(\psi_1) - \rho(\overline{\psi}_k) + p\big)\Big|^{2d} \\
&\ll_d L^{2d}\,\mathbf{E}\,\big|\xi_{11} - \rho(\psi_1) - \rho(\overline{\psi}_1) + p\big|^{2d} \ll_d L^{2d}.
\end{aligned} \tag{6.28}$$

In the proof of (6.28) we used the i.i.d. assumption and the notation $\mathbf{E}_\psi = \mathbf{E}_{\psi_1,\dots,\psi_L}$. Collecting the bounds (6.27)–(6.28), we obtain (6.24).

Let us prove (6.25). Assume the contrary, that is, that D occurs and strictly less than $pL/4$ of events D_j occur. Then at least $L - pL/4$ of D_j^c occur, and we have

$$\frac{pL^2}{2} \le \Big(L - \frac{pL}{4}\Big)\frac{pL}{4} + \frac{pL}{4}\,L = \frac{pL^2}{2} - \frac{p^2L^2}{16},$$

which is impossible.

Let us return to the proof of (6.20). Define the random variable

$$J = \mathbf{E}_\tau\ \mathrm{e}\big\{t\,\phi(\Lambda, \overline{\Lambda})\big\}, \tag{6.29}$$

where $\mathbf{E}_\tau$ denotes the partial integration with respect to the distributions of the random variables which are independent copies of τ. Note that J is random since it depends on Y_j and $\overline{Y}_j$, $1 \le j \le sL$. Due to (6.24), the lemma follows provided that we can verify that

$$\mathbf{1}\{D\}\,J \le \sup_{\mathbb{A}}\ \mathbf{E}\ \mathrm{e}\big\{t\,\big\langle \mathbb{A}U, V\big\rangle\big\}. \tag{6.30}$$

So it remains to prove (6.30). If $\mathbf{1}\{D\} = 0$, then (6.30) is obviously fulfilled. Thus we can assume that $\mathbf{1}\{D\} = 1$ and hence (see (6.25)) that at least $pL/4$ of random variables $\eta_j = \sum_{k=1}^{L}\xi_{jk}$ satisfy $\eta_j \ge pL/4$. Thus there exist indices $j_1, \dots, j_q$ such that

$$\eta_{j_1} \ge pL/4,\ \dots,\ \eta_{j_q} \ge pL/4, \qquad\qquad \text{with } q = \big[pL/4\big]. \tag{6.31}$$

Given the random vectors Y_k, let $S(j)$ denote the sum of q independent copies of $Z(j)$, that is

$$S(j) = \sum_{l=1}^{q} \sum_{k\in\mathcal{I}(j)} \tau_{kl} Y_k, \tag{6.32}$$

where the random variables τ_{kl}, $1 \le l \le q$, $k \in \mathcal{I}(j)$, are i.i.d. Finally, let $\mathbf{E}_{\tau,Z(j)}$ (resp. $\mathbf{E}_{\tau,S(j)}$) denote the integration with respect to the distributions of the random variables τ_j (resp. τ_{kl}) occurring in the sum $Z(j)$ (resp., in $S(j)$).

Using the bilinearity of ϕ, conditioning, estimating

$$\mathbf{E}_{\tau,Z(j)}\ \mathrm{e}\big\{t\,\phi\big(Z(j),\overline{\varLambda}\big)\big\} \le 1, \qquad \text{for } j \notin \{j_1,\dots,j_q\},$$

and applying the geometric-arithmetic mean inequality, we have

$$\begin{aligned}
J &= \mathbf{E}_\tau \prod_{j=1}^{L} \mathbf{E}_{\tau,Z(j)}\ \mathrm{e}\big\{t\,\phi\big(Z(j),\overline{\varLambda}\big)\big\} \le \mathbf{E}_\tau \prod_{r=1}^{q} \mathbf{E}_{\tau,Z(j_r)}\ \mathrm{e}\big\{t\,\phi\big(Z(j_r),\overline{\varLambda}\big)\big\} \\
&\le \mathbf{E}_\tau \frac{1}{q}\sum_{r=1}^{q}\Big(\mathbf{E}_{\tau,Z(j_r)}\ \mathrm{e}\big\{t\,\phi\big(Z(j_r),\overline{\varLambda}\big)\big\}\Big)^q = \frac{1}{q}\sum_{r=1}^{q} \mathbf{E}_\tau\,\mathbf{E}_{\tau,S(j_r)}\ \mathrm{e}\big\{t\,\phi\big(S(j_r),\overline{\varLambda}\big)\big\} \\
&= \frac{1}{q}\sum_{r=1}^{q} \mathbf{E}_\tau \prod_{k=1}^{L} \mathbf{E}_{\tau,\overline{Z}(k)}\ \mathrm{e}\big\{t\,\phi\big(S(j_r),\overline{Z}(k)\big)\big\}.
\end{aligned} \tag{6.33}$$

Recall, that we assume that the event D occurs. Therefore (6.31) implies that $\eta_{j_r} \ge pL/4$, for any j_r. In other words, there exist (random) indices $k_1(j_r)$, …, $k_q(j_r)$ such that the events $B(j_r, k_l(j_r))$ occur (notice that this yields $|\mathbb{A}(j_r, k_l(j_r)) - \mathbb{I}|_\infty \le \delta$, for all $1 \le l \le q$). Therefore, using (6.33), (6.18) and arguing similarly as in the proof of (6.33), we have

$$\begin{aligned}
J &\le \frac{1}{q}\sum_{r=1}^{q} \mathbf{E}_\tau \prod_{l=1}^{q} \mathbf{E}_{\tau,\overline{Z}(k_l(j_r))}\ \mathrm{e}\big\{t\,\phi\big(S(j_r),\overline{Z}(k_l(j_r))\big)\big\} \\
&\le \frac{1}{q}\sum_{r=1}^{q} \mathbf{E}_\tau \frac{1}{q}\sum_{l=1}^{q}\Big(\mathbf{E}_{\tau,\overline{Z}(k_l(j_r))}\ \mathrm{e}\big\{t\,\phi\big(S(j_r),\overline{Z}(k_l(j_r))\big)\big\}\Big)^q \\
&= \frac{1}{q^2}\sum_{r=1}^{q}\sum_{l=1}^{q} \mathbf{E}_\tau\ \mathrm{e}\big\{t\,\phi\big(S(j_r),\overline{S}(k_l(j_r))\big)\big\} = \frac{1}{q^2}\sum_{r=1}^{q}\sum_{l=1}^{q} \mathbf{E}_\tau\ \mathrm{e}\big\{t\,\big\langle \mathbb{A}(j_r,k_l(j_r))U, V\big\rangle\big\} \\
&\le \frac{1}{q^2}\sum_{r=1}^{q}\sum_{l=1}^{q} \sup_{\mathbb{A}}\ \mathbf{E}\ \mathrm{e}\big\{t\,\big\langle \mathbb{A}U, V\big\rangle\big\} = \sup_{\mathbb{A}}\ \mathbf{E}\ \mathrm{e}\big\{t\,\big\langle \mathbb{A}U, V\big\rangle\big\},
\end{aligned} \tag{6.34}$$

thus proving (6.30) and the lemma. □

Lemma 6.6 (Bentkus and Götze 1996b, Lemma 4.7). *Let $\mathbb{A}$ be a $s \times s$ matrix. Let X denote a random vector taking values in $\mathbb{R}^s$ with covariance matrix $\mathbb{C}$. Assume that there exists a constant c_s such that*

$$\mathbf{P}\{|X| \le c_s\} = 1, \qquad |\mathbb{A}| \le c_s, \qquad |\mathbb{A}^{-1}| \le c_s, \qquad |\mathbb{C}^{-1}| \le c_s. \tag{6.35}$$

Let U and V denote independent random vectors which are sums of N independent copies of X. Then

$$\Big| \mathbf{E}\, \mathrm{e}\big\{ t \, \big\langle \mathbb{A} U, V \big\rangle \big\} \Big| \ll_s \mathcal{M}^{2s}(t; N), \qquad \text{for } t \in \mathbb{R}, \tag{6.36}$$

where the function $\mathcal{M}$ is defined by (2.13).

Lemma 6.7. *Let U and V denote independent random vectors in $\mathbb{R}^s$ which are sums of q independent copies of $W = (\tau_1, \dots, \tau_s)$ (see* (6.17)–(6.18) *for the definition). Assume that an $s \times s$ (non-random) matrix $\mathbb{A}$ satisfies $|\mathbb{A} - \mathbb{I}|_\infty \le 1/(2\,s)$. Then*

$$\mathbf{E}\, \mathrm{e}\big\{ t \, \big\langle \mathbb{A} U, V \big\rangle \big\} \ll_s \mathcal{M}^{2s}(t; q), \qquad \text{for } t \in \mathbb{R}. \tag{6.37}$$

Proof. The result follows by an application of Lemma 6.6 replacing N by q and X by W. We have to verify the condition (6.35). The covariance of W clearly satisfies (6.35). The matrix $\mathbb{A}$ satisfies (6.35) as well, since

$$2^{-1} |x| \le |\mathbb{A} x| \le 2\,|x|, \qquad \text{for all } x \in \mathbb{R}^s. \tag{6.38}$$

For example, let us verify the left-hand side inequality in (6.38). Using $|\mathbb{A} - \mathbb{I}|_\infty \le 1/(2\,s)$, writing $\mathbb{A} = \mathbb{I} + \mathbb{B}$ with some $\mathbb{B}$ such that $|\mathbb{B}|_\infty \le 1/(2\,s)$, and introducing the Hilbert–Schmidt norm $|\mathbb{B}|_2 \le 1/2$, we have

$$|\mathbb{A} x| \ge |x| - |\mathbb{B}|\,|x| \ge \big(1 - |\mathbb{B}|_2\big)\,|x| \ge |x|/2. \qquad \square$$

Proof of Theorem 6.2. We shall apply Lemmas 6.4, 6.5 and 6.7.

Without loss of generality we can assume that $p\,M > c_s\, m$ with a sufficiently large constant c_s. Indeed, otherwise $p\,M \le c_s\, m$ and the result follows from the trivial bound $\big| \mathbf{E}\, \mathrm{e}\big\{ t T_* \big\} \big| \le 1$ combined with (2.15). The assumption $p\,M > c_s\, m$ implies that the natural numbers L, K, K_0/m introduced in (6.4) are well defined. Moreover,

$$L \asymp_s K \asymp_s K_0/m \asymp_s M/m \asymp_s q/p, \qquad q = \big[p\,L/4 \big]. \tag{6.39}$$

Lemma 6.4 implies

$$\Big| \mathbf{E}\, \mathrm{e}\big\{ t T_* \big\} \Big|^2 \le \mathbf{E}\, \mathrm{e}\big\{ \beta\, \phi(\Lambda, \overline{\Lambda}) \big\}, \qquad \beta = t\,m/2. \tag{6.40}$$

Lemma 6.5 yields

$$\mathbf{E}\, \mathrm{e}\big\{ \beta\, \phi(\Lambda, \overline{\Lambda}) \big\} \le_{d,s} (p\,L)^{-d} + \sup_{\mathbb{A}} \, \mathbf{E}\, \mathrm{e}\big\{ \beta \, \big\langle \mathbb{A} U, V \big\rangle \big\}. \tag{6.41}$$

An application of Lemma 6.7 yields

$$\mathbf{E}\, \mathrm{e}\big\{ \beta \, \big\langle \mathbb{A} U, V \big\rangle \big\} \ll_s \mathcal{M}^{2s}(\beta; q). \tag{6.42}$$

Collecting the bounds (6.40)–(6.42), substituting $\beta = t\,m/2$, using (2.16) and (6.39), we obtain

$$\Big| \mathbf{E}\, \mathrm{e}\big\{ t T_* \big\} \Big| \ll_{d,s} (p\,M/m)^{-d/2} + \mathcal{M}^{s}(t\,m; p\,M/m). \tag{6.43}$$

By (2.15) we have $\inf_x \mathcal{M}^s(x; p\,M/m) = (p\,M/m)^{-s/4}$. Thus (6.43) implies the desired result provided that we choose $d = s/2$. $\square$

7. A multiplicative inequality for U-statistics

In this section we prove a multiplicative inequality for characteristic functions of the statistic T^ϑ defined by (6.7). This inequality yields the desired bound of order $\mathcal{O}(N^{-1})$ for an integral of the characteristic function of a U-statistic. Write

$$\psi(t) = \Big|\mathbf{E}_\vartheta \, \mathrm{e}\big\{t T^\vartheta\big\}\Big|. \tag{7.1}$$

Notice that $\psi(t)$ is random since it depends on $X_1, \dots, X_N$ and $\overline{X}_1, \dots, \overline{X}_N$.

Lemma 7.1. *Let $d \geq 0$ and $m, s \in \mathbb{N}$. Assume that $Y = (2m)^{-1/2} \sum_{k=1}^m \widetilde{X}_k$ satisfies the non-degeneracy condition $\mathcal{N}(p, (2s)^{-1}, s, Y)$. Then there exist constants $c_1(s,d)$ and $c_2(s,d)$ such that the event*

$$D = \Big\{\psi(t-\gamma)\,\psi(t+\gamma) \leq c_1(s,d)\,\mathcal{M}^s(\gamma m; pM/m), \quad \textit{for all } t, \gamma \in \mathbb{R}\Big\} \tag{7.2}$$

satisfies

$$\mathbf{P}\{D\} \geq 1 - c_2(s,d)\,(pM/m)^{-d}. \tag{7.3}$$

Proof. Recall that L, K, K_0 are defined by (6.4). Let $\rho_1, \rho_2, \dots$ be i.i.d. random variables such that $\mathbf{P}\big\{\rho_1 = 1\big\} = \mathbf{P}\big\{\rho_1 = 0\big\} = 1/2$. Write

$$Y_j = (2m)^{-1/2} \sum_{k \in I(j)} \widetilde{X}_k, \qquad I(j) = \big(mj - m,\, mj\big], \tag{7.4}$$

and (cf. (6.19))

$$\Lambda_r = \sum_{(r-1)\,s\,L < j \leq r\,s\,L} \rho_j\,\widetilde{\varepsilon}_j\,Y_j, \qquad \text{for } 1 \leq r \leq 3, \tag{7.5}$$

where $\widetilde{\varepsilon}_j$ denote symmetrizations of i.i.d. Rademacher random variables.

We shall derive the lemma from the following bound

$$2\,\psi^2(\tau-\gamma)\,\psi^2(\tau+\gamma) \leq \mathbf{E}_{\rho,\varepsilon}\,\mathrm{e}\big\{m\gamma\,\phi(\Lambda_1,\Lambda_2)\big\} + \mathbf{E}_{\rho,\varepsilon}\,\mathrm{e}\big\{m\gamma\,\phi(\Lambda_1,\Lambda_3)\big\}, \tag{7.6}$$

where $\mathbf{E}_{\rho,\varepsilon}$ denotes the expectation with respect to the random variables $\rho_1, \rho_2, \dots$ and $\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2, \dots$

In order to prove (7.6), let us start with a simple identity. Let $\zeta_1, \zeta_2, \dots$ denote an independent copy of the sequence $\vartheta_1, \vartheta_2, \dots$ introduced in (6.3). Then, for any function f and given $\vartheta_1, \vartheta_2, \dots$, we have

$$\mathbf{E}_\zeta\, f(\zeta_1\vartheta_1,\, \zeta_2\vartheta_2 \dots) = \mathbf{E}\, f(\zeta_1,\, \zeta_2, \dots), \tag{7.7}$$

where $\mathbf{E}_\zeta = \mathbf{E}_{\zeta_1,\zeta_2,\dots}$. For example, for any function g,

$$\mathbf{E}_\zeta\, g(\zeta_1\vartheta_1) = \tfrac{1}{2}\,g(\vartheta_1) + \tfrac{1}{2}\,g(-\vartheta_1) = \tfrac{1}{2}\,g(1) + \tfrac{1}{2}\,g(-1) = \mathbf{E}\; g(\zeta_1).$$

Define
$$Q^{\vartheta} = \sum_{1\le j<k\le 3\,K_0} \vartheta_j\,\vartheta_k\,a_{jk}, \tag{7.8}$$
and $T^{\zeta\vartheta}$ just as T^{ϑ} with ϑ_j replaced by $\vartheta_j\zeta_j$. Using (7.1), (7.7), (7.8), Fubini's theorem and Hölder's inequality, we get
$$\begin{aligned}\psi^2(\tau-\gamma)\,\psi^2(\tau+\gamma) &= \Big|\mathbf{E}_{\vartheta}\,\psi(\tau-\gamma)\ \mathrm{e}\big\{-(\tau+\gamma)\,T^{\vartheta}\big\}\Big|^2\\ &= \Big|\mathbf{E}_{\vartheta}\,\mathbf{E}_{\zeta}\ \mathrm{e}\big\{(\tau-\gamma)\,T^{\zeta\,\vartheta}-(\tau+\gamma)\,T^{\vartheta}\big\}\Big|^2\\ &\le \mathbf{E}_{\zeta}\,\Big|\mathbf{E}_{\vartheta}\,\mathrm{e}\big\{(\tau-\gamma)\,T^{\zeta\,\vartheta}-(\tau+\gamma)\,T^{\vartheta}\big\}\Big|^2\\ &= \mathbf{E}_{\zeta}\,J,\end{aligned} \tag{7.9}$$
where
$$J = \Big|\mathbf{E}_{\vartheta}\,\mathrm{e}\big\{\tau\big(Q^{\zeta\,\vartheta}-Q^{\vartheta}\big)-\gamma\big(Q^{\zeta\,\vartheta}+Q^{\vartheta}\big)+f_1+f_2\big\}\Big|^2,$$
with a statistic f_1 depending only on ϑ_j, $j\in[1,K_0]$, and a statistic f_2 independent of ϑ_j, $j\in[1,K_0]$; of course, f_1 and f_2 may depend on τ, γ, ζ_j, X_j and $\overline{X}_j$.

It is easy to see that
$$\begin{aligned}Q^{\zeta\,\vartheta}-Q^{\vartheta} &= \sum_{1\le j<k\le 3\,K_0}(\zeta_j\,\zeta_k-1)\,\vartheta_j\,\vartheta_k\,a_{jk},\\ Q^{\zeta\,\vartheta}+Q^{\vartheta} &= \sum_{1\le j<k\le 3\,K_0}(\zeta_j\,\zeta_k+1)\,\vartheta_j\,\vartheta_k\,a_{jk}.\end{aligned} \tag{7.10}$$
Notice that
$$\zeta_j\,\zeta_k-1=0 \quad\text{and}\quad \zeta_j\,\zeta_k+1=2 \qquad \text{whenever } \zeta_j=\zeta_k=1. \tag{7.11}$$
For given $\zeta_1,\zeta_2,\dots$, introduce the sets
$$\begin{aligned}\Omega_1 &= \big\{j\in[1,K_0]:\ \zeta_j=1\big\}, &\quad \Omega_2 &= \big\{j\in(K_0,2\,K_0]:\ \zeta_j=1\big\},\\ \Omega_3 &= \big\{j\in(2\,K_0,3\,K_0]:\ \zeta_j=1\big\}, &\quad \Omega_4 &= [1,\,3\,K_0]\setminus(\Omega_1\cup\Omega_2\cup\Omega_3).\end{aligned}$$
The sets Ω_j, $1\le j\le 4$, are random since they depend on $\zeta_1,\zeta_2,\dots$, but they are independent of $\vartheta_1,\vartheta_2,\dots$. Write $\mathbf{E}_1=\mathbf{E}_{\vartheta_j,\,j\in\Omega_1\cup\Omega_2\cup\Omega_3}$. Then, conditioning on ϑ_j, $j\notin\Omega_1\cup\Omega_2\cup\Omega_3$, regrouping the summands, and using (7.10) and (7.11), we obtain
$$J\le \mathbf{E}_{\vartheta}\,\Big|\mathbf{E}_1\ \mathrm{e}\big\{-2\,\gamma Q'+f_1^*+f_2^*\big\}\Big|^2, \qquad Q'=\sum_{\{j,k\}\subset\Omega_1\cup\Omega_2\cup\Omega_3}\vartheta_j\,\vartheta_k\,a_{jk}, \tag{7.12}$$
with a statistic f_1^* depending on ϑ_j only such that $j\in\Omega_1\cup\Omega_4$, and with a statistic f_2^* independent of ϑ_j such that $j\in\Omega_1$. In order to bound the expectation $\mathbf{E}_1\dots$ in (7.12) let us apply the symmetrization Lemma 5.5. Then we have
$$2\,J\le J_1+J_2, \tag{7.13}$$

where

$$J_1 = \mathbf{E}_\vartheta \, \mathbf{E}_1 \; \mathrm{e}\big\{-2\,\gamma\, Q_1\big\}, \qquad Q_1 = \sum_{j\in\Omega_1,\ k\in\Omega_2} a_{jk}\, \widetilde{\vartheta}_j\, \widetilde{\vartheta}_k, \tag{7.14}$$

$$J_2 = \mathbf{E}_\vartheta \, \mathbf{E}_1 \; \mathrm{e}\big\{-2\,\gamma\, Q_2\big\}, \qquad Q_2 = \sum_{j\in\Omega_1,\ k\in\Omega_3} a_{jk}\, \widetilde{\vartheta}_j\, \widetilde{\vartheta}_k. \tag{7.15}$$

Let us rewrite the conditional expectations in (7.14) and (7.15) as unconditional ones. To this end introduce the random variables $\sigma_j = (\zeta_j + 1)/2$. Notice that $\sigma_j = 1$, if $\zeta_j = 1$, and $\sigma_j = 0$, if $\zeta_j = -1$. Therefore we can write

$$Q_1 = \sum_{j=1}^{K_0} \sum_{k=K_0+1}^{2K_0} a_{jk}\, \sigma_j\, \sigma_k\, \widetilde{\vartheta}_j\, \widetilde{\vartheta}_k, \qquad Q_2 = \sum_{j=1}^{K_0} \sum_{k=2K_0+1}^{3K_0} a_{jk}\, \sigma_j\, \sigma_k\, \widetilde{\vartheta}_j\, \widetilde{\vartheta}_k,$$

and

$$J_1 = \mathbf{E}_\vartheta \; \mathrm{e}\big\{-2\,\gamma\, Q_1\big\}, \qquad J_2 = \mathbf{E}_\vartheta \; \mathrm{e}\big\{-2\,\gamma\, Q_2\big\}. \tag{7.16}$$

By the definition (see (6.3)), we have $\vartheta_j = \varepsilon_l$, for $j \in I(l)$. Since $\zeta_1, \zeta_2, \dots$ is an independent copy of $\vartheta_1, \vartheta_2, \dots$, we can introduce the random variables $\rho_l = \sigma_j$, for $j \in I(l)$. Due to the definition of σ_j as a function of ζ_j, the random variables $\rho_1, \rho_2, \dots$ are i.i.d. and assume only values 0 and 1, each with probability 1/2. Thus, using $4\, a_{jk} = \phi(\widetilde{X}_j, \widetilde{X}_k)$, $K_0 = m\, s\, L$ and the linearity of $\phi(x, y)$ in its arguments, we can write

$$Q_1 = \frac{1}{4}\, \phi\Big(\sum_{j=1}^{K_0} \sigma_j\, \widetilde{\vartheta}_j\, \widetilde{X}_j,\ \sum_{k=K_0+1}^{2\,K_0} \sigma_k\, \widetilde{\vartheta}_k\, \widetilde{X}_k \Big) = \frac{m}{2}\, \phi(\Lambda_1,\, \Lambda_2) \tag{7.17}$$

since, for example,

$$\sum_{j=1}^{K_0} \sigma_j\, \widetilde{\vartheta}_j\, \widetilde{X}_j = \sum_{j=1}^{s\,L} \rho_l\, \widetilde{\varepsilon}_l \sum_{j\in I(l)} \widetilde{X}_j = \sqrt{2\,m}\, \Lambda_1.$$

Similarly, $Q_2 = \dfrac{m}{2}\, \phi(\Lambda_1, \Lambda_3)$. Using the symmetry of the random variables $\widetilde{\varepsilon}_j$, writing $\mathbf{E}_{\zeta,\vartheta} = \mathbf{E}_{\rho,\varepsilon}$ and combining (7.9), (7.13), (7.16) and (7.17), we obtain (7.6).

Let us return to the proof of the lemma. Define the $s \times s$-matrices

$$\mathbb{A}(j,k) = \big(\phi(Y_l, Y_r)\big), \qquad l \in (j\,s - s,\, j\,s], \quad r \in (k\,s - s,\, k\,s].$$

Introduce the random events and variables

$$B(j,k) = \big\{ |\mathbb{A}(j,k) - \mathbb{I}|_\infty \le (2\,s)^{-1} \big\}, \qquad \xi_{jk} = \mathbf{1}\{B(j,k)\}.$$

The condition $\mathcal{N}\big(p, (2\,s)^{-1}, s, Y\big)$ guarantees that $\mathbf{E}\, \xi_{jk} = \mathbf{P}\,\big\{B(j,k)\big\} \ge p$, for $j \ne k$. Furthermore, the events $B(j,k)$ occur with equal probabilities, provided that $j \ne k$. Introduce the events

$$D_1 = \Big\{ \sum_{j=1}^{L} \sum_{k=L+1}^{2\,L} \xi_{jk} \ge \frac{p\,L^2}{2} \Big\}, \qquad D_2 = \Big\{ \sum_{j=1}^{L} \sum_{k=2\,L+1}^{3\,L} \xi_{jk} \ge \frac{p\,L^2}{2} \Big\}.$$

and $D = D_1 \cap D_2$. Then

$$\mathbf{P}\{D\} \geq 1 - c_d(pL)^{-d} \geq 1 - c_d(s)(pK)^{-d}, \tag{7.18}$$

for any $d \geq 0$, since by (6.24) each of events D_1 and D_2 satisfies an inequality similar to (7.18). Due to (7.18), it remains to show that the multiplicative inequality

$$\psi(t-\gamma)\,\psi(t+\gamma) \ll_{s,d} \mathcal{M}^s(\gamma m; pM/m)$$

is fulfilled whenever the event D occurs. The estimate (7.6) shows that it is sufficient to verify that

$$\mathbf{1}\{D_1\}\,\mathbf{E}\,\mathrm{e}\big\{m\gamma\,\phi(\varLambda_1,\varLambda_2)\big\} \leq \sup_{\mathbb{A}}\;\mathbf{E}\,\mathrm{e}\big\{m\gamma\,\langle \mathbb{A}U, V\rangle\big\} \tag{7.19}$$

$$\mathbf{1}\{D_2\}\,\mathbf{E}\,\mathrm{e}\big\{m\gamma\,\phi(\varLambda_1,\varLambda_3)\big\} \leq \sup_{\mathbb{A}}\;\mathbf{E}\,\mathrm{e}\big\{m\gamma\,\langle \mathbb{A}U, V\rangle\big\}, \tag{7.20}$$

$$\sup_{\mathbb{A}}\;\mathbf{E}\,\mathrm{e}\big\{m\gamma\,\langle \mathbb{A}U, V\rangle\big\} \ll_{s,d} \mathcal{M}^{2s}(m\gamma; pM/m). \tag{7.21}$$

The inequality (7.21) follows from Lemma 6.7, using $pL \asymp pM/m$. Inequalities (7.19) and (7.20) differ from (6.30) in notation only. For example, we get (7.19) by replacing in (6.30) $\varLambda$, $\overline{\varLambda}$ and τ_j by $\varLambda_1$, $\varLambda_2$ and $\rho_j\,\widetilde{\varepsilon}_j$ respectively. $\square$

8. Expansions of characteristic functions

In this section we shall obtain expansions for the difference of the characteristic functions of the statistics T and T_0 defined by (1.1) and (1.6). Recall that

$$\widehat{F}(t) = \mathbf{E}\,\mathrm{e}\big\{tT\big\}, \qquad \widehat{F}_0(t) = \mathbf{E}\,\mathrm{e}\big\{tT_0\big\},$$

where F and F_0 are distribution functions of T and T_0. The Edgeworth correction F_1 is defined by (4.1). The bound $\varkappa = \varkappa(t)$ is given by (4.6) and (4.7). The moments β_s, γ_s and $\gamma_{s,r}$ are defined by (1.3). Write

$$\widehat{\Delta}_N = \Big|\widehat{F}(t) - \widehat{F}_0(t) - \widehat{F}_1(t)\Big|.$$

The result of this section is the following lemma.

Lemma 8.1. *We have*

$$\widehat{\Delta}_N \ll \varkappa N^{-1}\Big(t^4\beta_4 + t^6\beta_3^2 + t^2\gamma_2 + |t|^3\gamma_3 + |t|^5\gamma_2\gamma_3 + t^4\gamma_{2,2} + t^6\gamma_2\gamma_{2,2}\Big). \tag{8.1}$$

If, in addition, the condition (1.9) *is fulfilled, then* $\widehat{\Delta}_N = \Big|\widehat{F}(t) - \widehat{F}_0(t)\Big|$, *and*

$$\widehat{\Delta}_N \ll \varkappa N^{-1}\big(t^4\beta_4 + t^2\gamma_2 + |t|^3\gamma_3 + t^4\gamma_{2,2}\big). \tag{8.2}$$

Proof of Lemma 8.1. The result follows from Lemmas 8.2–8.4 by an obvious application of (8.9)–(8.17). $\square$

This section is organized as follows. First we shall introduce Bergström type identities, which are valid for arbitrary statistics of arbitrary samples. Then we shall consider a specialized version of these identities for symmetric statistics of i.i.d. samples. Combined with Taylor expansions, these identities provide the Edgeworth expansions of Lemmas 8.3 and 8.4.

Bergström type identities. Consider two arbitrary samples

$$X_1, \dots, X_N, \qquad \text{and} \qquad G_1, \dots, G_N. \tag{8.3}$$

Let

$$S = S(X_1, \dots, X_N)$$

be an (eventually, non-symmetric) statistic based on the sample $X_1, \dots, X_N$. For a subset $A \subset [1, N]$, introduce the operation $\mathcal{R}_A$ as replacement of the random variables X_j, $j \in A$, by G_j, $j \in A$. For example,

$$\mathcal{R}_{[1,j]} S = S(G_1, \dots, G_j, X_{j+1}, \dots, X_N). \tag{8.4}$$

Using this notation, we have the obvious identity

$$S = \mathcal{R}_{[1,N]} S + \sum_{j=1}^{N} \Big\{ \mathcal{R}_{[1,j-1]} S - \mathcal{R}_{[1,j]} S \Big\}. \tag{8.5}$$

Repeating the procedure with the statistic in braces $\{\dots\}$ in (8.5), we can replace $X_{j+1}, \dots, X_N$ by $G_{j+1}, \dots, G_N$, and obtain

$$\begin{aligned} S - \mathcal{R}_{[1,N]} S &- \sum_{j=1}^{N} \Big\{ \mathcal{R}_{[1,N]\setminus\{j\}} S - \mathcal{R}_{[1,N]} S \Big\} \\ &= \sum_{j=1}^{N} \sum_{k=j+1}^{N} \Big\{ \mathcal{R}_{[1,k]\setminus\{j,k\}} S - \mathcal{R}_{[1,k]\setminus\{k\}} S - \mathcal{R}_{[1,k]\setminus\{j\}} S + \mathcal{R}_{[1,k]} S \Big\}. \end{aligned} \tag{8.6}$$

It is easy to extend (8.5) and (8.6) to higher order differences. The identities (8.5) and (8.6) will be called Bergström expansions (cf. with the classical Bergström expansions for sums of independent random variables).

In particular, if the statistic S is a *symmetric* function of its arguments and each of the (independent) samples (8.3) is *i.i.d.*, then (8.5) implies

$$\mathbf{E}\, S = \mathbf{E}\, \mathcal{R}_{[1,N]} S + \sum_{j=1}^{N} \Big\{ \mathbf{E}\, \mathcal{R}_{[2,j]} S - \mathbf{E}\, \mathcal{R}_{[1,j]} S \Big\}. \tag{8.7}$$

Similarly, changing the order of summation and collecting equal terms, we see that under the same conditions (8.6) turns into

$$\begin{aligned} \mathbf{E}\, S - \mathbf{E}\, \mathcal{R}_{[1,N]} S &- N \Big\{ \mathbf{E}\, \mathcal{R}_{[2,N]} S - \mathbf{E}\, \mathcal{R}_{[1,N]} S \Big\} \\ &= \sum_{k=2}^{N} (k-1) \Big\{ \mathbf{E}\, \mathcal{R}_{[3,k]} S - \mathbf{E}\, \mathcal{R}_{[2,k]} S - \mathbf{E}\, \mathcal{R}_{[1,k]\setminus\{2\}} S + \mathbf{E}\, \mathcal{R}_{[1,k]} S \Big\}. \end{aligned} \tag{8.8}$$

In order to simplify the extensive calculations we introduce the abbreviations

$$f = t\,\phi_1/\sqrt{N}, \qquad h = t\,\phi/N, \qquad h_0 = t\,\phi_0/(2N). \tag{8.9}$$

Then, for the statistic T defined by (1.1), we can write

$$t\,T = t\,T(X_1,\dots,X_N) = \sum_{1\le j<k\le N} h(X_j,X_k) + \sum_{1\le j\le N} f(X_j). \tag{8.10}$$

Using (1.6), (2.9) and (2.12), we may assume that

$$t\,T_0 = t\,T(G_1,\dots,G_N) + \sum_{1\le j\le N} h_0(G_j,G_j), \tag{8.11}$$

$$t\,\mathcal{R}_{[1,N]}T = t\,T(G_1,\dots,G_N) = \sum_{1\le j<k\le N} h(G_j,G_k) + \sum_{1\le j\le N} f(G_j). \tag{8.12}$$

$$\widehat{F}_1(t) = \frac{N}{6}\,\mathbf{E}\,\bigl(i\,L + i\,h(X,G_1)\bigr)^3\,\mathrm{e}\bigl\{t\,T_0\bigr\}, \qquad L = f(X) + \sum_{2\le j\le N} h(X,G_j). \tag{8.13}$$

Define as well the following modification of $\widehat{F}_1$:

$$\widehat{F}_1^*(t) = \frac{N}{6}\,\mathbf{E}\,\bigl(i\,L\bigr)^3\,\mathrm{e}\bigl\{U\bigr\}, \qquad U = \sum_{2\le j<k\le N} h(G_j,G_k) + \sum_{2\le j\le N} f(G_j). \tag{8.14}$$

Introducing

$$\overline{\beta}_s = \mathbf{E}\,\bigl|\,f(X)\,\bigr|^s, \qquad \overline{\gamma}_s = \mathbf{E}\,\bigl|\,h(X,\overline{X})\,\bigr|^s, \qquad \overline{\gamma}_{s,q} = \mathbf{E}\,\Bigl(\mathbf{E}_X\,\bigl|\,h(X,\overline{X})\,\bigr|^s\Bigr)^q, \tag{8.15}$$

we have

$$\overline{\beta}_s = |t|^s\,N^{-s/2}\,\beta_s, \qquad \overline{\gamma}_s = |t|^s\,N^{-s}\,\gamma_s, \qquad \overline{\gamma}_{s,q} = |t|^{sq}\,N^{-sq}\,\gamma_s. \tag{8.16}$$

Furthermore, we can rewrite the definition of $\varkappa = \varkappa_1 + \varkappa_2$ as

$$\begin{aligned} \varkappa_1(t) &= \sup\nolimits_A \Bigl|\,\mathbf{E}\ \mathrm{e}\Bigl\{\sum_{1\le j<k\le l} h(X_j,X_k) + A(X_1,\dots,X_l)\Bigr\}\Bigr|, \\ \varkappa_2(t) &= \sup\nolimits_A \Bigl|\,\mathbf{E}\ \mathrm{e}\Bigl\{\sum_{1\le j<k\le l} h(G_j,G_k) + A(G_1,\dots,G_l)\Bigr\}\Bigr|, \end{aligned} \tag{8.17}$$

where $\sup_A$ is taken over all linear statistics A, i.e. such that $A(x_1,\dots,x_l) = \sum_{j=1}^{l} f_j(x_j)$ with some functions $f_1,\dots,f_l$, and $l = \bigl[(N-2)/20\bigr]$. For the sake of brevity, in this section we shall write $\varkappa$ instead of $\varkappa(t)$.

Furthermore, we shall use the inequalities

$$\mathbf{E}\,\big|\,h_0(G,G)\,\big|^p \ll_p \overline{\gamma}_2^{\,p/2} \le \overline{\gamma}_{2,p/2} \le \overline{\gamma}_p, \qquad p \ge 2. \tag{8.18}$$

It suffices to prove (8.18) assuming that p is even. Using the representation (2.10) of $\phi_0(G,G)$ and applying the well-known Zygmund–Marcinkievicz inequality, we have

$$\mathbf{E}\,\big|\,\phi_0(G,G)\,\big|^p \ll_p \mathbf{E}\,\Big|\sum_{j\ge 1} q_j^2\,(\eta_j^4+1)\Big|^{p/2} \ll_p \sum_{j_1,\dots,j_{p/2}\ge 1} q_{j_1}^2 \dots q_{j_{p/2}}^2 \ll_p \gamma_2^{p/2},$$

whence (8.18) follows.

The proofs of the next Lemmas 8.2 and 8.3 are simpler than the proof of the main Lemma 8.4. Therefore in these proofs we demonstrate in more detail certain technicalities, which are widely used in the proof of Lemma 8.4. The proof of Lemma 8.2 is based on the already quite common splitting and conditioning techniques. It is possible to simplify considerably the proofs of Lemmas 8.3 and 8.4 allowing to depend the bounds on $\overline{\gamma}_6$ instead of $\overline{\gamma}_{2,2}$. In order to prove bounds in terms of $\overline{\gamma}_{2,2}$ we have to make extensive use of identities of type (8.44) combined with the inequalities (8.45). A rough scheme of the proof of these lemmas is to expand the characteristic function in Taylor's series with a remainder which contains only squares, say $h^2(X_1,X_2)$, of the kernel h as the highest power of h. In case that the order of the remainder term is still not of order $\mathcal{O}(N^{-1})$, we have to apply Taylor's expansions again involving $h(X_i,X_j)$ such that, say X_i, is independent of at least one of X_1 or X_2.

Lemma 8.2. *The statistics* (8.11) *and* (8.12) *satisfy*

$$\Big|\,\mathbf{E}\ \mathrm{e}\big\{\mathcal{R}_{[1,N]}tT\big\} - \mathbf{E}\ \mathrm{e}\big\{tT_0\big\}\Big| \ll \varkappa\,\big(N\overline{\beta}_4 + N\,\overline{\gamma}_2 + N^2\,\overline{\gamma}_3\big). \tag{8.19}$$

For the Edgeworth corrections (8.13) *and* (8.14) *we have*

$$\big|\,\widehat{F}_1(t) - \widehat{F}_1^*(t)\,\big| \ll \varkappa\,\big(N\overline{\beta}_4 + N\,\overline{\gamma}_2 + N^2\,\overline{\gamma}_3 + N^3\,\overline{\gamma}_{2,2}\big). \tag{8.20}$$

Proof. Let us prove (8.19). Using (8.11) and (8.12) we can write

$$tT_0 = \mathcal{R}_{[1,N]}tT + d, \qquad d \stackrel{\mathrm{def}}{=} \sum_{j=1}^{N} h_0(G_j,G_j). \tag{8.21}$$

Expanding in powers of d in a Taylor series with remainder $\mathcal{O}(d^2)$ (that is, using (2.19) with $r=1$), we have

$$\Big|\,\mathbf{E}\ \mathrm{e}\big\{\mathcal{R}_{[1,N]}tT\big\} - \mathbf{E}\ \mathrm{e}\big\{tT_0\big\}\Big| \ll J_1 + \mathbf{E}\,J_2, \tag{8.22}$$

with

$$J_1 = \Big| \mathbf{E}\, d\, \mathrm{e}\big\{ \mathcal{R}_{[1,N]} tT \big\} \Big|, \qquad J_2 = \Big| \mathbf{E}^{\tau}\, d^2\, \mathrm{e}\big\{ \mathcal{R}_{[1,N]} tT + \tau\, d \big\} \Big|,$$

where τ is a random variable uniformly distributed on $[0,\,1]$ and independent of all other random variables. Recall that $\mathbf{E}^{\tau}$ denotes the conditional expectation given τ, and $\mathbf{E}_{\tau}$ stands for the conditional expectation given all random variables except τ. In order to prove (8.19), it suffices to verify that J_1 and J_2 in (8.22) satisfy

$$|J_1| \ll \varkappa (N\,\overline{\beta}_4 + N\,\overline{\gamma}_2 + N^2\,\overline{\gamma}_3), \qquad |J_2| \ll \varkappa N\,\overline{\gamma}_2. \tag{8.23}$$

Let us prove (8.23) for J_2. Split $d = d_1 + d_2 + d_3$, where

$$d_1 = \sum_{1 \le j \le N/3} h_0(G_j, G_j), \qquad d_2 = \sum_{N/3 < j \le 2\,N/3} h_0(G_j, G_j),$$

and $d_3 = d - d_1 - d_2$. Then

$$|J_2| \le \sum_{1 \le j,k \le 3} \Big| \mathbf{E}^{\tau}\, d_j\, d_k\, \mathrm{e}\big\{ \mathcal{R}_{[1,N]} tT + \tau\, d \big\} \Big|. \tag{8.24}$$

Any of the products $d_j\,d_k$ does not contain at least one of the terms d_1, d_2, d_3. For example, let us consider the case of $d_1\,d_2$. Conditioning on d_1 and d_2, using the definition of $\varkappa$, the i.i.d. assumption and the inequality $a\,b \le a^2 + b^2$, we have

$$\begin{aligned} \Big| \mathbf{E}^{\tau}\, d_1\, d_2\, \mathrm{e}\big\{ \mathcal{R}_{[1,N]} tT + \tau\, d \big\} \Big| &\le \mathbf{E}^{\tau}\, |d_1|\,|d_2|\, \Big| \mathbf{E}_{d_3}\, \mathrm{e}\big\{ \mathcal{R}_{[1,N]} tT + \tau\, d \big\} \Big| \\ &\le (\,\mathbf{E}\, d_1^2 + \mathbf{E}\, d_2^2)\, \varkappa \ll N\, \mathbf{E}\, (h_0(G,G))^2\, \varkappa \ll N\,\overline{\gamma}_2\, \varkappa, \end{aligned} \tag{8.25}$$

since $\mathbf{E}\, h_0(G,G) = 0$ and, by (8.18), we have $\mathbf{E}\,\big(h_0(G,G)\big)^2 \ll \overline{\gamma}_2$. Combining (8.24) and (8.25), we obtain (8.23) for $|J_2|$.

Let us verify (8.23) for J_1. Using the i.i.d. assumption and (8.21), we have

$$J_1 = N \Big| \mathbf{E}\, h_0(G_1, G_1)\, \mathrm{e}\big\{ \mathcal{R}_{[1,N]} tT \big\} \Big|. \tag{8.26}$$

Split

$$R_{[1,N]} tT = K + U, \qquad \text{where } K = f(G_1) + \sum_{j=2}^{N} h(G_1, G_j), \tag{8.27}$$

and where U is defined by (8.14) and is independent of G_1. Notice that

$$\mathbf{E}\, h_0(G_1, G_1) = 0, \qquad \text{and} \qquad \mathbf{E}_{G_1} h_0(G_1, G_1)\, K = 0, \tag{8.28}$$

due to the symmetry of G_1 (that is, $\mathcal{L}(G_1) = \mathcal{L}(-G_1)$). Using (8.26), (8.28) and expanding in powers of K, we obtain

$$|J_1| \ll N \Big| \mathbf{E}^{\tau} h_0(G_1, G_1)\, K^2\, \mathrm{e}\big\{ U + \tau\, K \big\} \Big|. \tag{8.29}$$

In order to bound the expectation in (8.29), let us repeat the arguments leading to (8.25): split the sum K into a sum of $f(G_1)$ and of three parts with approximately equal number of summands; condition on the part which is absent in the product, and so forth. We get

$$|J_1| \ll N \varkappa \, \mathbf{E} \, \big| h_0(G,G) \big| \, f^2(G) + N \varkappa \max_{1 \le m \le N} \mathbf{E} \, \big| h_0(G,G) \big| \Big(\sum_{j=2}^{m} h(G,G_j) \Big)^2 \tag{8.30}$$

since $\mathcal{L}(G_1) = \mathcal{L}(G)$. Using $a\,b \le a^2 + b^2$, applying (8.18) and replacing $\mathbf{E}\, f^4(G)$ by $\overline{\beta}_4$ by an application of (2.8), we have

$$\mathbf{E} \, \big| h_0(G,G) \big| \, f^2(G) \le \mathbf{E} \, \big(h_0(G,G) \big)^2 + \mathbf{E} \, f^4(G) \ll \overline{\gamma}_2 + \overline{\beta}_4. \tag{8.31}$$

Furthermore, using the i.i.d. assumption, Hölder's inequality, relations (2.8), (8.15), (8.18) and $\mathcal{L}(G_j) = \mathcal{L}(\overline{G})$, we obtain

$$\max_{1 \le m \le N} \mathbf{E} \, \big| h_0(G,G) \big| \Big(\sum_{j=1}^{m} h(G,G_j) \Big)^2 \le N \, \mathbf{E} \, \big| h_0(G,G) \big| \, h^2(G,\overline{G}) \ll N \, \overline{\gamma}_3. \tag{8.32}$$

Collecting (8.30)–(8.32), we obtain (8.23) for J_1, which concludes the proof of (8.19).

Let us prove (8.20). Using the notation (8.14), (8.13), (8.27) and (8.21), we can write

$$\widehat{F}_1^*(t) = \frac{N}{6} \, \mathbf{E} \, \big(i\,L \big)^3 \, \mathrm{e}\big\{ U \big\}, \qquad \widehat{F}_1(t) = \frac{N}{6} \, \mathbf{E} \, \big(i\,L + i\,h(X,G_1) \big)^3 \, \mathrm{e}\big\{ U + K + d \big\}.$$

Introducing

$$\widehat{F}_2(t) = \frac{N}{6} \, \mathbf{E} \, \big(i\,L \big)^3 \, \mathrm{e}\big\{ U + K + d \big\},$$

we reduce the proof of (8.20) to showing

$$\big| \widehat{F}_1(t) - \widehat{F}_2(t) \big| \ll \varkappa (N \, \overline{\beta}_4 + N \, \overline{\gamma}_2 + N^2 \, \overline{\gamma}_3), \tag{8.33}$$

$$\big| \widehat{F}_2(t) - \widehat{F}_1^*(t) \big| \ll \varkappa (N \, \overline{\beta}_4 + N^3 \, \overline{\gamma}_{2,2}). \tag{8.34}$$

Decompose $[1,N] = \cup_{s=1}^{5} \Omega_s$ into 5 disjoint sets with approximately equal cardinalities. Furthermore, split each of the sums L, K and d into the parts

$$L = \sum_{s=1}^{6} L_s, \qquad K = \sum_{s=1}^{6} K_s, \qquad d = \sum_{s=1}^{6} d_s$$

where, for example,

$$L_6 = f(X), \qquad K_6 = f(G_1), \qquad d_6 = 0, \qquad K_s = \sum_{j \in \Omega_s \setminus \{1\}} h(G_1, G_j), \qquad 1 \le s \le 5.$$

Using the splitting and conditioning arguments, applying $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ and several times $a^p b^q \ll a + b$, for $p, q \geq 0$ such that $p + q = 1$, we have

$$\begin{aligned} \bigl|\widehat{F}_1(t) - \widehat{F}_2(t)\bigr| &\ll \varkappa N \max_{1\leq s\leq 5} \mathbf{E}\bigl|h(X, G_1)\bigr| \bigl(h^2(X, G_1) + f^2(X) + L_s^2\bigr) \\ &\ll \varkappa N (\overline{\beta}_4 + \overline{\gamma}_2) + \varkappa N^2\, \mathbf{E}\bigl|h(X, G_1)\bigr|^3 + \varkappa N^{1/2} \max_{1\leq s\leq 5} \mathbf{E}\,|L_s|^3. \end{aligned} \tag{8.35}$$

The inequality (8.35) implies (8.33) since, by an application of (2.8) and (2.18), the last two expectations in (8.35) are bounded from above by $c\overline{\gamma}_3$ and $cN^{3/2}\overline{\gamma}_3$ respectively.

Let us prove (8.34). Expanding in powers of $K+d$ with remainder $\mathcal{O}(K+d)$, splitting and conditioning again, we have

$$\begin{aligned} \bigl|\widehat{F}_2(t) - \widehat{F}_1^*(t)\bigr| &\ll \varkappa N \max_{1\leq s_1, s_2, s_3\leq 6} \mathbf{E}\bigl|L_{s_1}\bigr|^3 \bigl(|K_{s_2}| + |d_{s_3}|\bigr) \\ &\ll \varkappa N \max_{1\leq s_1, s_2, s_3\leq 6} \mathbf{E}\bigl(L_{s_1}^4 + K_{s_2}^4 + d_{s_3}^4\bigr) \\ &\ll \varkappa N \overline{\beta}_4 + \varkappa N \Bigl(\max_{1\leq s\leq 5} \mathbf{E}\, L_s^4 + \max_{1\leq s\leq 5} \mathbf{E}\, K_s^4 + \max_{1\leq s\leq 5} \mathbf{E}\, d_s^4\Bigr). \end{aligned} \tag{8.36}$$

An application of (2.18), (2.8) and (8.18) shows that all expectations on the right hand side of (8.36) are bounded from above by $cN^2\overline{\gamma}_{2,2}$, proving (8.34). $\square$

In the proofs of Lemmas 8.3 and 8.4 we shall use the following notation and facts. Write

$$\varphi(x) = \mathbf{E}\, h^2(X, x), \qquad \varphi_3(x) = \mathbf{E}\bigl|h(X, x)\bigr|^3. \tag{8.37}$$

Then, for $0 \leq \alpha \leq 1$, Hölder's inequality implies

$$\mathbf{E}\bigl|h(X, x)\bigr|^{\alpha+2} \leq \varphi^{1-\alpha}(x)\, \varphi_3^{\alpha}(x), \qquad \mathbf{E}\,\varphi^{\alpha+1}(X) \leq \overline{\gamma}_2^{\,1-\alpha}\, \overline{\gamma}_{2,2}^{\,\alpha}, \tag{8.38}$$

$$\begin{aligned} \mathbf{E}\bigl|h(z, X)\bigr|^{1/2} h^2(y, X) &\leq \Bigl(\mathbf{E}\bigl|h(z, X)\bigr| h^2(y, X)\Bigr)^{1/2} \varphi^{1/2}(y) \\ &\leq \varphi_3^{1/6}(z)\, \varphi_3^{1/3}(y)\, \varphi^{1/2}(y). \end{aligned} \tag{8.39}$$

For a random vector Z, write

$$\widehat{Z} = f(Z) + \sum_{k=1}^{N-1} h(Z, X_k). \tag{8.40}$$

The sum $\widehat{Z}$ without summands $h(Z, X_i)$, $h(Z, X_j), \dots$ we shall denote as $\widehat{Z}(i, j, \dots)$. For example, $\widehat{Z}(k) = \widehat{Z} - h(Z, X_k)$, for $k \geq 1$. We write as well $\widehat{Z}(0) = \widehat{Z} - f(Z)$ and $\widehat{Z}(0, 1) = \widehat{Z} - f(Z) - h(Z, X_1)$. Introducing the statistic

$$S = \sum_{1\leq j<k\leq N-1} h(X_j, X_k) + \sum_{j=1}^{N-1} f(X_j), \tag{8.41}$$

we can write

$$tT = \mathcal{R}_{[2,1]} tT \overset{\mathcal{D}}{=} \widehat{X} + S \qquad \text{and} \qquad \mathcal{R}_{[1,1]} tT \overset{\mathcal{D}}{=} \widehat{G} + S. \tag{8.42}$$

To obtain (8.42), it suffices to change the numeration of random variables replacing X_1, G_1 and $X_2, \ldots, X_N$ by X, G, and $X_1, \ldots, X_{N-1}$ respectively. This is possible due to the i.i.d. assumption.

Finally, split $[N/2,\, N-2] \cap \mathbb{N} = \cup_{s=1}^{7} \Omega_s$ into a union of 7 disjoint subsets such that their cardinalities are approximately equal and bounded from below by $N/20$. Write

$$d_s(Z) = \sum_{j \in \Omega_s} h(Z, X_j), \quad 1 \le s \le 7, \qquad d_0(Z) = \sum_{j \notin [N/2,\, N-2]} h(Z, X_j). \tag{8.43}$$

We shall often use the following type relations

$$\mathbf{E}\, d_s(Z)\, \mathrm{e}\big\{ S + \widehat{Z} \big\} = \operatorname{card} \Omega_s\, \mathbf{E}\, h(Z, X_1)\, \mathrm{e}\big\{ S + \widehat{Z} \big\}, \tag{8.44}$$

and

$$\begin{aligned} \big| \,\mathbf{E}\, \widehat{Z}(0)\, \mathrm{e}\big\{ S + \widehat{Z} \big\} \big| &= (N-1) \big| \,\mathbf{E}\, h(Z, X_1)\, \mathrm{e}\big\{ S + \widehat{Z} \big\} \big| \\ &\ll \big| \,\mathbf{E}\, d_s(Z)\, \mathrm{e}\big\{ S + \widehat{Z} \big\} \big| \ll \big| \,\mathbf{E}\, \widehat{Z}(0)\, \mathrm{e}\big\{ S + \widehat{Z} \big\} \big|. \end{aligned} \tag{8.45}$$

Both (8.44) and (8.45) are consequences of the i.i.d. assumption. We have as well

$$\mathbf{E}^Z \big| d_1(Z) \big| \le \big(\mathbf{E}^Z d_1^2(Z) \big)^{1/2} \ll N^{1/2} \varphi^{1/2}(Z). \tag{8.46}$$

Lemma 8.3. *Assume that the third order moments of X and G coincide, that is, that* (1.9) *holds. Then we have*

$$\big| \,\mathbf{E}\ \mathrm{e}\{tT\} - \mathbf{E}\ \mathrm{e}\{\mathcal{R}_{[1,N]} tT\} \big| \ll \varkappa (N \overline{\beta}_4 + N^2\, \overline{\gamma}_3 + N^3\, \overline{\gamma}_{2,2}).$$

Proof. Let us apply the Bergström type identity (8.7) replacing S by $\mathrm{e}\{tT\}$. Then we have

$$\big| \,\mathbf{E}\ \mathrm{e}\{tT\} - \mathbf{E}\ \mathrm{e}\{\mathcal{R}_{[1,N]} tT\} \big| \le \sum_{j=1}^{N} \delta_j, \qquad \delta_j \overset{\text{def}}{=} \big| \,\mathbf{E}\ \mathrm{e}\big\{ \mathcal{R}_{[2,j]} tT \big\} - \mathbf{E}\ \mathrm{e}\big\{ \mathcal{R}_{[1,j]} tT \big\} \big|. \tag{8.47}$$

Due to (8.47), in order to prove the lemma it suffices to verify that

$$\delta_j \ll \varkappa (\overline{\beta}_4 + N\, \overline{\gamma}_3 + N^2\, \overline{\gamma}_{2,2}). \tag{8.48}$$

To simplify notation we shall estimate δ_1 only. An estimation of δ_j, $j > 1$, is similar since here $X_2, \dots, X_j$ are just replaced by the Gaussian random vectors $G_2, \dots, G_j$. Nevertheless we can repeat the proof using inequalities of type (2.8) for the estimation of moments involving Gaussian random vectors by moments involving X_k. Another unessential complication is that formulas similar to (8.52) can include factors of type $h^i(Z, G_j)$ which again leads to unessential although tedious additional technicalities. Using (8.42) we can represent δ_1 as

$$\delta_1 = \Big| \mathbf{E}\, \mathrm{e}\big\{ \widehat{X} + S \big\} - \mathbf{E}\, \mathrm{e}\big\{ \widehat{G} + S \big\} \Big|. \tag{8.49}$$

Expanding in (8.49) in a Taylor series with remainders $\mathcal{O}(\widehat{X}^3)$ and $\mathcal{O}(\widehat{G}^3)$, we get

$$\delta_1 \ll \mathbf{E} \big| \mathbf{E}^\tau a(X) - \mathbf{E}^\tau a(G) \big|, \qquad a(Z) \overset{\mathrm{def}}{=} \widehat{Z}^3\, \mathrm{e}\big\{ S + \tau \widehat{Z} \big\}. \tag{8.50}$$

Here τ denotes a random variable uniformly distributed on $[0, 1]$ independent of all other random variables, and $\mathbf{E}^\tau$ stands for the conditional expectation given τ. The terms of the expansion depending on the lower order derivatives cancel since by (2.7) moments of X and G up to the second order coincide.

Writing $\widehat{X}^3$ and $\widehat{G}^3$ in (8.50) as triple sums, using the i.i.d. assumption and collecting identical summands, we obtain

$$\mathbf{E}\, \delta_1 \ll \sum \mathbf{E}\, I(i_2), \qquad I(i_2) \overset{\mathrm{def}}{=} \big| \mathbf{E}^\tau b(X)\, \mathrm{e}\big\{ S + \widehat{X} \big\} - \mathbf{E}^\tau b(G)\, \mathrm{e}\big\{ S + \widehat{G} \big\} \big|, \tag{8.51}$$

with

$$b(Z) = N^{\theta(i_2) + \theta(i_3) + \theta(i_4)} f^{i_1}(Z)\, h^{i_2}(Z, X_1)\, h^{i_3}(Z, X_2)\, h^{i_4}(Z, X_3). \tag{8.52}$$

Here the sum in (8.51) extends over all non-negative integer indices i_1, i_2, i_3, i_4 such that $i_1 + i_2 + i_3 + i_4 = 3$, and $i_2 \ge i_3 \ge i_4$. Furthermore, we write $\theta(x) = 1$, for $x > 0$, and $\theta(x) = 0$, for $x \le 0$. It is clear that $I(i_2)$ depends on i_1, i_3 and i_4 as well. However, the further estimation of this quantity depends on the value of i_2 only.

Due to (8.51), in order to prove (8.48) it suffices to show that

$$I(i_2) \ll \varkappa (\overline{\beta}_4 + N\, \overline{\gamma}_3 + N^2\, \overline{\gamma}_{2,2}), \tag{8.53}$$

for all allowable values of the indices i_1, i_2, i_3, i_4.

We shall write, for example,

$$I(i_2) \sim \big\{ \widehat{Z}(0);\, b'(Z) \big\} \qquad \text{if} \;\; I(i_2) \ll I'(i_2) + \varkappa (\overline{\beta}_4 + N\, \overline{\gamma}_3 + N^2\, \overline{\gamma}_{2,2}),$$

where $I'(i_2)$ is defined similarly to $I(i_2)$ in (8.51) just replacing $\widehat{Z}$ and b by $\widehat{Z}(0)$ and b' respectively. In order to verify (8.53), we have to prove that $I(i_2) \sim \big\{ \dots;\, 0 \big\}$.

Let us prove (8.53) for $I(3)$. Then $i_1 = i_3 = i_4 = 0$ and using (8.15), (8.17), we have

$$I(3) \ll \max_{Z = X,\, Z = G} N \Big| \mathbf{E}^\tau h^3(Z, X_1)\, \mathrm{e}\big\{ S + \tau \widehat{Z} \big\} \Big| \ll \varkappa N \overline{\gamma}_3.$$

Let us prove (8.53) for $I(2)$. Then $i_4 = 0$ and we have

$$I(2) \sim \big\{ \widehat{Z};\, b(Z) \big\}, \qquad b(Z) = N^{1+\theta(i_3)}\, f^{i_1}(Z)\, h^2(Z, X_1)\, h^{i_3}(Z, X_2). \tag{8.54}$$

The indices i_1 and i_3 in (8.54) satisfy $i_1 + i_3 = 1$. Hence, there are only two possibilities: either $i_1 = 1$, $i_3 = 0$ or $i_1 = 0$, $i_3 = 1$. Using an identity of type (8.44), we can rewrite (8.54) as

$$I(2) \sim \big\{ \widehat{Z};\, b_1(Z) \big\}, \qquad b_1(Z) = N\, f^{i_1}(Z)\, h^2(Z, X_1)\, d_1^{i_3}(Z) \tag{8.55}$$

(we write $0^0 = 1$). Let us show that in (8.55) we can remove $f(Z)$ and $h(Z, X_1)$ from the corresponding exponents, that is, that

$$I(2) \sim \big\{ \widehat{Z}(0,1);\, b_1(Z) \big\}. \tag{8.56}$$

We derive (8.56) by expanding the exponents in Taylor series with remainders $\mathcal{O}\big(f(Z)\big)$ and $\mathcal{O}\Big(\sqrt{\big|h(Z, X_1)\big|}\Big)$. The errors are bounded from above by

$$\rho_1(i_1) \overset{\text{def}}{=} \varkappa N\, \mathbf{E}\, \big| f^{i_1+1}(Z)\, h^2(Z, X_1)\, d_1^{i_3}(Z) \big|$$

and

$$\rho_2(i_1) \overset{\text{def}}{=} \varkappa N\, \mathbf{E}\, \big| f^{i_1}(Z) \big| \big| h(Z, X_1) \big|^{5/2} \big| d_1^{i_3}(Z) \big|$$

respectively. The estimation of $\rho_j(i_1)$ will depend on i_1 only. We have to show that $\rho_1(i_1)$ and $\rho_2(i_1)$ are bounded from above by the right hand side of (8.53). Assume first that $i_1 = 1$. Then $i_3 = 0$. Using (8.15), (8.37), Hölder's inequality and $a\,b \ll a^2 + b^2$, we have

$$\begin{aligned} \rho_1(1) &\ll \varkappa N\, \mathbf{E}\, f^2(Z) h^2(Z, X_1) = \varkappa N\, \mathbf{E}\, f^2(Z)\, \varphi(Z) \\ &\le \varkappa N\, \overline{\beta}_4^{\,1/2}\, \overline{\gamma}_{2,2}^{\,1/2} \ll \varkappa \overline{\beta}_4 + \varkappa N^2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.57}$$

Furthermore, using (8.15), (8.38) and Hölder's inequality, we obtain

$$\begin{aligned} \rho_2(1) &\ll \varkappa N\, \mathbf{E}\, \big| f(Z) \big| \mathbf{E}_{X_1} \big| h(Z, X_1) \big|^{5/2} \ll \varkappa N\, \mathbf{E}\, \big| f(Z) \big|\, \varphi^{1/2}(Z)\, \varphi_3^{1/2}(Z) \\ &\ll \varkappa N \big(\mathbf{E}\, f^2(Z)\, \varphi(Z) \big)^{1/2}\, \overline{\gamma}_3^{\,1/2} \ll \varkappa N\, \overline{\beta}_4^{\,1/4}\, \overline{\gamma}_{2,2}^{\,1/4}\, \overline{\gamma}_3^{\,1/2} \\ &\ll \varkappa \overline{\beta}_4 + \varkappa N\, \overline{\gamma}_3 + \varkappa N^2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.58}$$

In the case $i_1 = 0$ we can proceed similarly as in the proof of (8.57) and (8.58). Using (8.15), (8.37), (8.38), (8.46) we have

$$\begin{aligned} \rho_1(0) &\ll \varkappa N^{3/2}\, \mathbf{E}\, \big| f(Z) \big| h^2(Z, X_1)\, \varphi^{1/2}(Z) = \varkappa N^{3/2}\, \mathbf{E}\, \big| f(Z) \big| \varphi^{3/2}(Z) \\ &\le \varkappa N^{3/2}\, \overline{\beta}_4^{\,1/4}\, \overline{\gamma}_{2,2}^{\,3/4} \ll \varkappa \overline{\beta}_4 + \varkappa N^2 \overline{\gamma}_{2,2}, \\ \rho_2(0) &\ll \varkappa N^{3/2}\, \mathbf{E}\, \mathbf{E}_{X_1} \big| h(Z, X_1) \big|^{5/2} \varphi^{1/2}(Z) \ll \varkappa N^{3/2}\, \mathbf{E}\, \varphi(Z)\, \varphi_3^{1/2}(Z) \\ &\ll \varkappa N^{3/2} \overline{\gamma}_3^{\,1/2}\, \overline{\gamma}_{2,2}^{\,1/2} \ll \varkappa N\, \overline{\gamma}_3 + \varkappa N^2\, \overline{\gamma}_{2,2}, \end{aligned}$$

proving (8.56).

Let us show that (8.56) implies $I(2) \sim \big\{ \widehat{Z}(0,1);\, 0 \big\}$. To this end it suffices to expand the exponents in powers of $\widehat{Z}(0,1)$ with remainder $\mathcal{O}(\widehat{Z}(0,1))$. Lower order terms cancel since by the assumption (1.9) the third order moments of X and G are equal, that is,

$$\mathbf{E}\, f^3(X) = \mathbf{E}\, f^2(X)\, h(X,x) = \mathbf{E}\, f(X)\, h^2(X,x) = \mathbf{E}\, h(X,x)\, h(X,y)\, h(X,z) = 0,$$

for all $x, y, z \in \mathcal{X}$. Using splitting of the sum $\widehat{Z}(0,1)$ of type (8.43) such that the sum $d_0(Z)$ is independent of X_1, we get a remainder term which is bounded from above by

$$\begin{aligned} \rho(i_1) &\stackrel{\text{def}}{=} \max_{0\le s\le 7} \varkappa N\, \mathbf{E} \,\big|\, f^{i_1}(Z)\, h^2(Z,X_1)\, d_1^{i_3}(Z)\, d_s(Z) \,\big| \\ &\ll \max_{0\le s\le 7} \varkappa N\, \mathbf{E} \,\big|\, f^{i_1}(Z) \varphi(Z)\, d_1^{i_3}(Z)\, d_s(Z) \,\big|, \end{aligned}$$

by an application of (8.44). We have

$$\rho(1) \ll \max_{0\le s\le 7} \varkappa N\, \mathbf{E} \,\big|\, f(Z) \varphi(Z)\, d_s(Z) \,\big| \ll \varkappa N^{3/2}\, \mathbf{E} \,\big|\, f(Z) \,\big|\, \varphi^{3/2}(Z) \ll \varkappa \overline{\beta}_4 + \varkappa N^2\, \overline{\gamma}_{2,2}$$

and

$$\begin{aligned} \rho(0) &= \max_{0\le s\le 7} \varkappa N\, \mathbf{E}\, h^2(Z,X_1) \,\big|\, d_1(Z)\, d_s(Z) \,\big| \\ &\ll \max_{0\le s\le 7} \varkappa N\, \mathbf{E}\, \varphi(Z) \left(\mathbf{E}^Z\, d_1^2(Z)\, \mathbf{E}^Z\, d_s^2(Z) \right)^{1/2} \ll \varkappa N^2\, \mathbf{E}\, \varphi^2(Z) = \varkappa N^2\, \overline{\gamma}_{2,2}, \end{aligned}$$

which completes the estimation of $I(2)$.

Let us prove (8.53) for $I(1)$. In this case

$$b(Z) = N^{1+\theta(i_3)+\theta(i_4)}\, f^{i_1}(Z)\, h(Z,X_1)\, h^{i_3}(Z,X_2)\, h^{i_4}(Z,X_3). \tag{8.59}$$

The relation (8.59) and identities of type (8.44) imply

$$I(1) \sim \big\{ \widehat{Z};\, b(Z) \big\}, \qquad b(Z) = f^{i_1}(Z)\, d_1(Z)\, d_2^{i_3}(Z)\, d_3^{i_4}(Z).$$

Expanding the exponents with remainder $\mathcal{O}(\widehat{Z})$, we obtain $I(1) \sim \big\{ \widehat{Z};\, 0 \big\}$. Lower order terms cancel again. Using splitting of the sum $\widehat{Z}(0,1)$ of type (8.43), we get a remainder term which is bounded from above by

$$\rho(i_1) \stackrel{\text{def}}{=} \max_{0\le s\le 7} \varkappa\, \mathbf{E} \,\big|\, f^{i_1}(Z) d_1(Z)\, d_2^{i_3}(Z)\, d_3^{i_4}(Z)\; d_s(Z) \,\big|.$$

Using (8.43), (8.46) and Hölder's inequality, we have

$$\begin{aligned} \rho(0) &= \max_{0\le s\le 7} \varkappa\, \mathbf{E}\, \mathbf{E}^Z \,\big|\, d_1(Z)\, d_2(Z)\; d_3(Z)\, d_s(Z) \,\big| \\ &\le \max_{0\le s\le 7} \varkappa\, \mathbf{E} \left(\mathbf{E}^Z\, d_1^2(Z)\, \mathbf{E}^Z\, d_2^2(Z)\, \mathbf{E}^Z\, d_3^2(Z)\, \mathbf{E}^Z\, d_s^2(Z) \right)^{1/2} \\ &\ll \varkappa N^2\, \mathbf{E}\, \varphi^2(Z) \big) \ll \varkappa N^2\, \overline{\gamma}_{2,2}. \end{aligned}$$

The estimation of

$$\rho(1) = \max_{0\le s\le 7} \varkappa\, \mathbf{E}\, \big| f(Z)\, d_1(Z)\, d_2(Z)\, d_s(Z) \big|, \quad \rho(2) = \max_{0\le s\le 7} \varkappa\, \mathbf{E}\, f^2(Z) \big| d_1(Z)\, d_s(Z) \big|$$

is similar to that of $\rho(0)$.

It remains to prove (8.53) for $I(0)$. In this case $b(Z) = f^3(Z)$. Similarly as in the case of $I(1)$, we can expand exponents with a remainder $\mathcal{O}(\widehat{Z})$. We obtain $I(0) \sim \big\{ \widehat{Z};\, 0 \big\}$. The remainder is bounded from above by $\max_{0\le s\le 7} \varkappa\, \mathbf{E}\, \big| f^3(Z)\, d_s(Z) \big|$, which can be estimated similarly as in (8.79). $\square$

Lemma 8.4. *We have*

$$\begin{aligned} &\Big| \mathbf{E}\, \mathrm{e}\{tT\} - \mathbf{E}\, \mathrm{e}\{\mathcal{R}_{[1,N]} tT\} - \widehat{F}_1^*(t) \Big| \\ &\qquad \ll \varkappa \big(N \overline{\beta}_4 + N^2 \overline{\beta}_3^{\,2} + N^2\, \overline{\gamma}_3 + N^4\, \overline{\gamma}_2\, \overline{\gamma}_3 + N^3\, \overline{\gamma}_{2,2} + N^5\, \overline{\gamma}_2\, \overline{\gamma}_{2,2} \big), \end{aligned} \tag{8.60}$$

where the Edgeworth correction $\widehat{F}_1^*$ is defined by (8.14).

Proof. We shall use the Bergström type expansion (8.8). Substituting $S = \mathrm{e}\{T\}$ in (8.8) reduces the proof of (8.60) to a verification of

$$\Big| N\, \mathbf{E}\, \mathrm{e}\{\mathcal{R}_{[2,N]} tT\} - N\, \mathbf{E}\, \mathrm{e}\{\mathcal{R}_{[1,N]} tT\} - \widehat{F}_1^*(t) \Big| \ll \varkappa \big(N \overline{\beta}_4 + N^3\, \overline{\gamma}_{2,2} \big) \tag{8.61}$$

and

$$\max_{2\le k\le N} \big| J_1 - J_2 - J_3 + J_4 \big| \ll \varkappa \big(\overline{\beta}_3^{\,2} + \overline{\gamma}_3 + N^2\, \overline{\gamma}_2\, \overline{\gamma}_3 + N\, \overline{\gamma}_{2,2} + N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2} \big) \tag{8.62}$$

with

$$\begin{aligned} J_1 &= \mathbf{E}\, \mathrm{e}\{\mathcal{R}_{[3,k]} tT\}, & J_2 &= \mathbf{E}\, \mathrm{e}\{\mathcal{R}_{[2,k]} tT\}, \\ J_3 &= \mathbf{E}\, \mathrm{e}\{\mathcal{R}_{[1,k]\setminus\{2\}} tT\}, & J_4 &= \mathbf{E}\, \mathrm{e}\{\mathcal{R}_{[1,k]} tT\}. \end{aligned} \tag{8.63}$$

Let us prove (8.61). Setting $L(Z) = f(Z) + \sum_{j=2}^N h(Z, G_j)$ and using the notation (8.14), we can rewrite (8.61) as

$$\Big| 6\, \mathbf{E}\, \mathrm{e}\{U + L(X)\} - 6\, \mathbf{E}\, \mathrm{e}\{U + L(G)\} - \mathbf{E}\, (i\, L(X))^3\, \mathrm{e}\{U\} \Big| \ll \varkappa \overline{\beta}_4 + \varkappa N^2\, \overline{\gamma}_{2,2}. \tag{8.64}$$

The proof of (8.64) is simple—it suffices to expand in powers of $L(Z)$ with $Z = X, G$ and remainders $\mathcal{O}(Z^4)$, and to apply splitting and conditioning arguments like in the proof of Lemma 8.2. More precisely, split $L(Z) = \sum_{s=1}^6 L_j$ into approximately equal parts, for $1 \le j \le 5$, and put $L_6 = f(Z)$. Expanding and conditioning we reduce the proof of (8.64) to the bound $\mathbf{E}\, L_s^4 \ll \overline{\beta}_4 + N^2\, \overline{\gamma}_{2,2}$. But using (2.8) we have $\mathbf{E}\, L_6^4 \le$

$\mathbf{E}\, f^4(Z) \ll \overline{\beta}_4$. For $1 \le s \le 5$, an application of (2.18) yields $\mathbf{E}\, L_s^4 \ll N^2\, \mathbf{E}\, h^4(Z,G)$. Using (2.8) again, we have $\mathbf{E}\, h^4(Z,G) \ll \mathbf{E}\, \varphi^2(Z) \ll \overline{\gamma}_{2,2}$. Collecting these estimates, we obtain (8.64), proving (8.61).

To complete the proof of the lemma, we have to prove (8.62). To simplify the notation we shall assume henceforth that $k = 2$. That will not restrict the generality since in the case $k > 2$ more Gaussian random vectors are present in statistics $\mathcal{R}_{[\dots]}tT$ in (8.63). However, while estimating moments we can use (2.8) and replace them by random vectors distributed as X. Introduce the notation

$$\widehat{Z} = \sum_{1 \le j \le N-2} h(Z, X_j), \qquad W = \sum_{1 \le l < m \le N-2} h(X_l, X_m) + \sum_{1 \le l \le N-2} f(X_l). \tag{8.65}$$

Denote $\widehat{Z}(i,j,\dots)$ the sum $\widehat{Z}$ without summands with indices $i, j, \dots$ For example, $\widehat{Z}(1) = \widehat{Z} - h(Z, X_1)$. Notice that the notation (8.65) slightly differs from that used earlier in (8.40). Using (8.65) and replacing

$$X_1, X_2, G_1, G_2, X_3, \dots, X_N, G_3, \dots, G_N$$

by

$$X, \overline{X}, G, \overline{G}, X_1, \dots, X_{N-2}, G_1, \dots, G_{N-2}$$

respectively, we can rewrite J_s, $1 \le s \le 4$, in (8.62) as follows:

$$\begin{aligned}
J_1 &= \mathbf{E}\ \mathrm{e}\big\{ W + f(X) + f(\overline{X}) + \widehat{X} + \widehat{\overline{X}} + h(X, \overline{X}) \big\}, \\
J_2 &= \mathbf{E}\ \mathrm{e}\big\{ W + f(G) + f(\overline{X}) + \widehat{G} + \widehat{\overline{X}} + h(G, \overline{X}) \big\}, \\
J_3 &= \mathbf{E}\ \mathrm{e}\big\{ W + f(X) + f(\overline{G}) + \widehat{X} + \widehat{\overline{G}} + h(X, \overline{G}) \big\}, \\
J_4 &= \mathbf{E}\ \mathrm{e}\big\{ W + f(G) + f(\overline{G}) + \widehat{G} + \widehat{\overline{G}} + h(G, \overline{G}) \big\}.
\end{aligned}$$

Expanding in powers of $h(\cdot,\cdot)$ and conditioning, we obtain

$$\Big| J_s - J_{s0} - J_{s1} - \frac{1}{2}\, J_{s2} \Big| \ll \varkappa \overline{\gamma}_3, \qquad 1 \le s \le 4, \tag{8.66}$$

where, for example,

$$J_{1r} = \mathbf{E}\, \big(i\, h(X, \overline{X}) \big)^r\ \mathrm{e}\big\{ W + f(X) + f(\overline{X}) + \widehat{X} + \widehat{\overline{X}} \big\},$$

for $0 \le r \le 2$. Writing

$$I_r = J_{1r} - J_{2r} - J_{3r} + J_{4r}, \qquad 0 \le r \le 2,$$

and using (8.66), we have

$$\Big| J_1 - J_2 - J_3 + J_4 \Big| \ll |I_0| + |I_1| + |I_2| + \varkappa\,\overline{\gamma}_3. \tag{8.67}$$

Due to (8.67), instead of (8.62) it suffices to verify that

$$|I_r| \ll \varkappa\big(\overline{\beta}_3^{\,2} + \overline{\gamma}_3 + N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + N\,\overline{\gamma}_{2,2} + N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}\big), \qquad \text{for } r = 0, 1, 2. \tag{8.68}$$

It is easy to see that

$$|I_r| \ll \Big| \mathbf{E}\ \mathrm{e}\{W\}\big(a_r(X, \overline{X}) - a_r(G, \overline{X}) - a_r(X, \overline{G}) + a_r(G, \overline{G})\big)\Big|, \qquad r = 0, 1, 2, \tag{8.69}$$

with

$$a_r(Z, Y) = h^r(Z, Y)\ \mathrm{e}\big\{ f(Z) + \widehat{Z} \big\}\ \mathrm{e}\big\{ f(Y) + \widehat{Y} \big\}. \tag{8.70}$$

Here and below Z denotes one of X or G, and Y stands for $\overline{X}$ or $\overline{G}$.

Let us introduce the notation used throughout the rest of the proof. We shall write

$$\Gamma \sim \big\{ \widehat{Z}, \widehat{Y};\, b(Z, Y) \big\} \tag{8.71}$$

if

$$\Gamma \ll \Big| \mathbf{E}\ \mathrm{e}\{W\}\big(a(X, \overline{X}) - a(G, \overline{X}) - a(X, \overline{G}) + a(G, \overline{G})\big)\Big| + R$$

with some R bounded (up to an absolute constant) by the right hand side of (8.68), and some function a such that

$$a(Z, Y) = b(Z, Y)\ \mathrm{e}\big\{ \tau\,\widehat{Z} \big\}\ \mathrm{e}\big\{ \overline{\tau}\,\widehat{Y} \big\},$$

for some independent $0 \le \tau, \overline{\tau} \le 1$ which are independent of all other random vectors. Furthermore, we write

$$\Gamma \sim \Gamma_1 \cup \Gamma_2 \qquad \text{if } \Gamma \ll \Gamma_1 + \Gamma_2 + R. \tag{8.72}$$

In the notation (8.71) the bound (8.68) means that we have to prove that $I_r \sim \big\{ \dots;\, 0 \big\}$.

Let τ and $\overline{\tau}$ be independent random variables uniformly distributed on [0, 1]. Using expansions with remainders $\mathcal{O}(f^3(Z))$ and $\mathcal{O}(f^3(Y))$, from (8.69) and (8.70) we derive that

$$|I_0| \sim \bigcup_{0 \le k, l \le 3} I_{0,k,l}, \qquad I_{0,k,l} \sim \big\{ \delta_{k,3}\,\tau f(Z) + \widehat{Z},\, \delta_{l,3}\,\overline{\tau} f(Y) + \widehat{Y};\, f^k(Z)\, f^l(Y) \big\}, \tag{8.73}$$

where $\delta_{i,j}$ denotes the Kronecker symbol. Similarly to (8.73), expanding with remainders $\mathcal{O}(f^{3-r}(Z))$ and $\mathcal{O}(f^{3-r}(Y))$ (resp. $\mathcal{O}(f(Z))$ and $\mathcal{O}(f(Y))$), we obtain

$$|I_r| \sim \bigcup_{0 \le k, l \le 3-r} I_{r,k,l}, \qquad r = 1, 2, \tag{8.74}$$

where

$$I_{r,k,l} \sim \big\{ \delta_{k,3-r}\, \tau f(Z) + \widehat{Z},\, \delta_{l,3-r}\, \overline{\tau} f(Y) + \widehat{Y};\; f^k(Z)\, f^l(Y)\, h^r(Z,Y) \big\}. \tag{8.75}$$

Relations (8.68)–(8.75) reduce the proof of the lemma to showing that

$$I_{r,k,l} \sim \big\{ \dots;\, 0 \big\}, \tag{8.76}$$

for all allowable values of the indices r, k and l.

Proof of (8.76) *for* $I_{0,0,0}$. Expanding in powers of $\widehat{Z}$ and $\widehat{Y}$ with $Z = X, G$ and $Y = \overline{X}, \overline{G}$ with remainders $\mathcal{O}(\widehat{Z}^2)$ and $\mathcal{O}(\widehat{Y}^2)$, we obtain

$$I_{0,0,0} \sim \big\{ \widehat{Z}, \widehat{Y};\; \widehat{Z}^2 \widehat{Y}^2 \big\}, \tag{8.77}$$

Write $\widehat{Z}^2 \widehat{Y}^2$ as a fourfold sum. Using (8.77), the i.i.d. assumption of X_j, identities of type (8.44) and classifying summands according to multiplicities of the indices, we obtain

$$I_{0,0,0} \sim \bigcup_{s=1}^{9} \Gamma_s \tag{8.78}$$

with (see (8.43) for the definition of $d_s(Z)$)

$$\Gamma_1 \sim \big\{ \widehat{Z}, \widehat{Y};\; N\, h^2(Z, X_1)\, h^2(Y, X_1) \big\}, \tag{8.79}$$

$$\begin{aligned} \Gamma_2 &\sim \big\{ \widehat{Z}, \widehat{Y};\; N^2\, h^2(Z, X_1)\, h(Y, X_1)\, h(Y, X_2) \big\} \\ &\sim \big\{ \widehat{Z}, \widehat{Y};\; N\, h^2(Z, X_1)\, h(Y, X_1)\, d_1(Y) \big\}, \end{aligned} \tag{8.80}$$

$$\begin{aligned} \Gamma_3 &\sim \big\{ \widehat{Z}, \widehat{Y};\; N^2\, h^2(Y, X_1)\, h(Z, X_1)\, h(Z, X_2) \big\} \\ &\sim \big\{ \widehat{Z}, \widehat{Y};\; N\, h^2(Y, X_1)\, h(Z, X_1)\, d_1(Z) \big\}, \end{aligned} \tag{8.81}$$

$$\Gamma_4 \sim \big\{ \widehat{Z}, \widehat{Y};\; N^2\, h^2(Z, X_1)\, h^2(Y, X_2) \big\}, \tag{8.82}$$

$$\Gamma_5 \sim \big\{ \widehat{Z}, \widehat{Y};\; N^2\, h(Z, X_1)\, h(Z, X_2)\, h(Y, X_1)\, h(Y, X_2) \big\}, \tag{8.83}$$

$$\begin{aligned} \Gamma_6 &\sim \big\{ \widehat{Z}, \widehat{Y};\; N^3\, h^2(Z, X_1)\, h(Y, X_2)\, h(Y, X_3) \big\} \\ &\sim \big\{ \widehat{Z}, \widehat{Y};\; N\, h^2(Z, X_1)\, d_1(Y)\, d_2(Y) \big\}, \end{aligned} \tag{8.84}$$

$$\begin{aligned} \Gamma_7 &\sim \big\{ \widehat{Z}, \widehat{Y};\; N^3\, h(Z, X_2)\, h(Z, X_3)\, h^2(Y, X_1) \big\} \\ &\sim \big\{ \widehat{Z}, \widehat{Y};\; N\, d_1(Z)\, d_2(Z)\, h^2(Y, X_1) \big\}, \end{aligned} \tag{8.85}$$

$$\begin{aligned} \Gamma_8 &\sim \big\{ \widehat{Z}, \widehat{Y};\; N^3\, h(Z, X_1)\, h(Z, X_2)\, h(Y, X_1)\, h(Y, X_3) \big\} \\ &\sim \big\{ \widehat{Z}, \widehat{Y};\; N\, h(Z, X_1)\, d_1(Z)\, h(Y, X_1)\, d_2(Y) \big\}, \end{aligned} \tag{8.86}$$

$$\begin{aligned} \Gamma_9 &\sim \big\{ \widehat{Z}, \widehat{Y};\; N^4\, h(Z, X_1)\, h(Z, X_2)\, h(Y, X_3)\, h(Y, X_4) \big\} \\ &\sim \big\{ \widehat{Z}, \widehat{Y};\; d_1(Z)\, d_2(Z)\, d_3(Y)\, d_4(Y) \big\}. \end{aligned} \tag{8.87}$$

Let us prove that $\Gamma_1 \sim \big\{ \dots ; 0 \big\}$. Using (2.8) and (8.37), we have $\Gamma_1 \sim \big\{ \dots ; 0 \big\}$ with an error bounded from above by

$$\varkappa N \, \mathbf{E} \, h^2(Z, X_1) \, h^2(Y, X_1) \ll \varkappa N \, \mathbf{E} \, \varphi^2(X_1) = \varkappa N \, \overline{\gamma}_{2,2}.$$

Below we shall use systematically without refering (8.15), (8.37), (8.38) and (8.46).

Let us prove that $\Gamma_2 \sim \big\{ \dots ; 0 \big\}$. Using a Taylor expansion with remainder $\mathcal{O}\big(\big| h(Y, X_1) \big|^{1/2} \big)$ and Hölder's inequality, we obtain

$$\Gamma_2 \sim \big\{ \widehat{Z}, \widehat{Y}(1); \; N \, h^2(Z, X_1) \, h(Y, X_1) \, d_1(Y) \big\} \tag{8.88}$$

with an error bounded from above by

$$\begin{aligned} \varkappa N \, \mathbf{E} \, h^2(Z, X_1) \, \big| h(Y, X_1) \big|^{3/2} \big| d_1(Y) \big| &\ll \varkappa N^{3/2} \, \mathbf{E} \, \varphi(X_1) \, \varphi^{1/2}(Y) \big| h(Y, X_1) \big|^{3/2} \\ &\ll \varkappa N^{3/2} \, \big(\overline{\gamma}_2 \, \overline{\gamma}_3 \, \overline{\gamma}_{2,2} \big)^{1/2} \le \varkappa \overline{\gamma}_3 + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.89}$$

Expanding in powers of $\widehat{Y}(1)$, using (8.88) and a splitting of $\widehat{Y}(1)$ of type (8.43) with d_0 independent of X_1, conditioning and applying $a\,b \le a^2 + b^2$, we obtain $\Gamma_2 \sim \big\{ \dots ; 0 \big\}$ (notice, that lower order terms cancel since the covariances of X and G resp. of $\overline{X}$ and $\overline{G}$ are equal; here the decisive role plays a second order type difference, cf. (8.69) and the notation (8.71)) with a remainder bounded from above by

$$\begin{aligned} \varkappa N \max_{0 \le s \le 7} \, \mathbf{E} \, h^2(Z, X_1) \big| h(Y, X_1) \, d_1(Y) \, d_s(Y) \big| &\ll \varkappa N^2 \, \mathbf{E} \, \varphi(X_1) \big| h(Y, X_1) \big| \varphi(Y) \\ &\ll \varkappa N^2 \, \overline{\gamma}_{2,2} \, \overline{\gamma}_2^{\,1/2} \ll \varkappa N \overline{\gamma}_{2,2} + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.90}$$

Notice that Γ_3 differs from Γ_2 only in that Z and Y are exchanged. Thus, we can apply to Γ_3 the bound for Γ_2.

Let us estimate Γ_4. We shall proceed similarly to the case of Γ_2. Expanding with remainders $\mathcal{O}\big(h(Z, X_1) \big)$, $\mathcal{O}\big(h(Z, X_2) \big)$, $\mathcal{O}\big(h(Y, X_1) \big)$ and $\mathcal{O}\big(h(Y, X_2) \big)$, we obtain

$$\Gamma_4 \sim \big\{ \widehat{Z}(1,2), \widehat{Y}(1,2); \; N^2 \, h^2(Z, X_1) \, h^2(Y, X_2) \big\}. \tag{8.91}$$

with errors bounded from above by

$$\varkappa N^2 \, \mathbf{E} \, \big| h(Z, X_1) \big|^3 \, h^2(Y, X_2) \ll \varkappa N^2 \, \overline{\gamma}_2 \, \overline{\gamma}_3, \tag{8.92}$$

$$\begin{aligned} \varkappa N^2 \, \mathbf{E} \, h^2(Z, X_1) \big| h(Z, X_2) \big| h^2(Y, X_2) &\ll \varkappa N^2 \, \mathbf{E} \, \varphi(Z) \, \varphi(X_2) \big| h(Z, X_2) \big| \\ &\ll \varkappa N^2 \, \overline{\gamma}_2^{\,1/2} \, \overline{\gamma}_{2,2} \ll \varkappa N \, \overline{\gamma}_{2,2} + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}, \end{aligned} \tag{8.93}$$

$$\varkappa N^2 \, \mathbf{E} \, h^2(Z, X_1) \big| h(Y, X_1) \big| h^2(Y, X_2) \ll \varkappa N \, \overline{\gamma}_{2,2} + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}, \tag{8.94}$$

$$\varkappa N^2 \, \mathbf{E} \, h^2(Z, X_1) \big| h(Y, X_2) \big|^3 \ll \varkappa N^2 \, \overline{\gamma}_2 \, \overline{\gamma}_3. \tag{8.95}$$

The relation (8.91) implies $\Gamma_4 \sim \big\{ \dots ; 0 \big\}$. Indeed, expanding in powers of $\widehat{Z}(1,2)$ and $\widehat{Y}(1,2)$, splitting and conditioning we obtain $\Gamma_4 \sim \big\{ \dots ; 0 \big\}$ with a remainder bounded from above by

$$\begin{aligned} &\varkappa N^2 \max_{0 \le s, r \le 7} \mathbf{E}\, h^2(Z, X_1)\, h^2(Y, X_2) \, \big| \, d_s(Z)\, d_r(Y) \, \big| \\ &\qquad \ll \varkappa N^3 \, \mathbf{E}\, \varphi^{3/2}(Z)\, \varphi^{3/2}(Y) \ll \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.96}$$

Let us estimate Γ_5 proceeding similarly to the case of Γ_4. Removing terms depending on X_1 and X_2 from $\widehat{Z}$ and $\widehat{Y}$, we get

$$\Gamma_5 \sim \big\{ \widehat{Z}(1,2), \widehat{Y}(1,2);\; N^2\, h(Z, X_1)\, h(Z, X_2)\, h(Y, X_1)\, h(Y, X_2) \big\}. \tag{8.97}$$

with errors bounded from above by expectations of the following type

$$\begin{aligned} &\varkappa N^2 \, \mathbf{E}\, h^2(Z, X_1) \, \big| \, h(Z, X_2)\, h(Y, X_1)\, h(Y, X_2) \, \big| \\ &\qquad \ll \varkappa N^2 \, \mathbf{E}\, h^2(Z, X_1) \, \big| \, h(Z, X_2) \, \big| \, \varphi^{1/2}(X_1)\, \varphi^{1/2}(X_2) \\ &\qquad \ll \varkappa N^2 \, \overline{\gamma}_2^{\,1/2} \, \mathbf{E}\, h^2(Z, X_1)\, \varphi^{1/2}(X_1)\, \varphi^{1/2}(Z) \\ &\qquad \ll \varkappa N^2 \, \overline{\gamma}_2^{\,1/2} \, \overline{\gamma}_3^{\,2/3} \big(\, \mathbf{E}\, \varphi^{3/2}(X_1) \, \mathbf{E}\, \varphi^{3/2}(Z) \big)^{1/3} \\ &\qquad \ll \varkappa N^2 \, \overline{\gamma}_2^{\,1/2} \, \overline{\gamma}_3^{\,2/3} \, \overline{\gamma}_2^{\,1/3} \, \overline{\gamma}_{2,2}^{\,1/3} \ll \varkappa \, \overline{\gamma}_3 + \varkappa N^2 \, \overline{\gamma}_2 \, \overline{\gamma}_3 + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.98}$$

The relation (8.97) implies $\Gamma_5 \sim \big\{ \dots ; 0 \big\}$ since we can expand with remainders $\mathcal{O}\big(\widehat{Z}(1,2) \big)$ and $\mathcal{O}\big(\widehat{Y}(1,2) \big)$. The error is bounded by

$$\begin{aligned} &\varkappa N^2 \max_{0 \le s, r \le 7} \mathbf{E} \, \big| \, h(Z, X_1)\, h(Z, X_2)\, h(Y, X_1)\, h(Y, X_2)\, d_s(Z)\, d_r(Y) \, \big| \\ &\qquad \ll \varkappa N^3 \, \mathbf{E}\, \varphi(Z)\, \varphi(Y) \, \big(\varphi(Z) + \varphi(Y) \big) \ll \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.99}$$

Let us estimate Γ_6. Expanding with remainders $\mathcal{O}\big(h(Z, X_1) \big)$ and $\mathcal{O}\big(h(Y, X_1) \big)$ we obtain

$$\Gamma_6 \sim \big\{ \widehat{Z}(1), \widehat{Y}(1);\; N\, h^2(Z, X_1)\, d_1(Y)\, d_2(Y) \big\} \tag{8.100}$$

with errors bounded by

$$\begin{aligned} &\varkappa N \, \mathbf{E}\, h^2(Z, X_1) \, \big| \, h(Z, X_1)\, d_1(Y)\, d_2(Y) \, \big| \\ &\qquad \ll \varkappa N^2 \, \mathbf{E} \, \big| \, h(Z, X_1) \, \big|^3 \, \varphi(Y) \ll \varkappa N^2 \, \overline{\gamma}_2 \, \overline{\gamma}_3, \\ &\varkappa N \, \mathbf{E}\, h^2(Z, X_1) \, \big| \, h(Y, X_1)\, d_1(Y)\, d_2(Y) \, \big| \\ &\qquad \ll \varkappa N^2 \, \mathbf{E}\, \varphi(X_1) \, \big| \, h(Y, X_1) \, \big| \, \varphi(Y) \ll \varkappa N \overline{\gamma}_{2,2} + \varkappa N^3 \; \overline{\gamma}_2 \, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.101}$$

The i.i.d. assumption and (8.100) yield

$$\Gamma_6 \sim \bigl\{ \widehat{Z}(1), \widehat{Y}(1);\ N^3\, h^2(Z, X_1)\, h(Y, X_2)\, h(Y, X_3) \bigr\}. \tag{8.102}$$

Expanding with remainder $\mathcal{O}(\widehat{Z}(1))$, we get

$$\Gamma_6 \sim \bigl\{ \widehat{Z}(1), \widehat{Y}(1);\ N^3\, h^2(Z, X_1)\, h(Y, X_2)\, h(Y, X_3)\, \widehat{Z}(1) \bigr\} \sim \Gamma_{6,1} \cup \Gamma_{6,2}, \tag{8.103}$$

where

$$\begin{aligned}
\Gamma_{6,1} &\sim \bigl\{ \widehat{Z}(1), \widehat{Y}(1);\ N^3\, h^2(Z, X_1)\, h(Y, X_2)\, h(Z, X_2)\, h(Y, X_3) \bigr\} \\
&\sim \bigl\{ \widehat{Z}(1), \widehat{Y}(1);\ N^2\, h^2(Z, X_1)\, h(Y, X_2)\, h(Z, X_2)\, d_1(Y) \bigr\}, && (8.104) \\
\Gamma_{6,2} &\sim \bigl\{ \widehat{Z}(1), \widehat{Y}(1);\ N^4\, h^2(Z, X_1)\, h(Y, X_2)\, h(Y, X_3)\, h(Z, X_4) \bigr\} \\
&\sim \bigl\{ \widehat{Z}(1), \widehat{Y}(1);\ N\, h^2(Z, X_1)\, d_1(Y)\, d_2(Y)\, d_3(Z) \bigr\}, && (8.105)
\end{aligned}$$

Expanding with remainder $\mathcal{O}\Bigl(\bigl| h(Y, X_2) \bigr|^{1/2} \Bigr)$, we have

$$\Gamma_{6,1} \sim \bigl\{ \widehat{Z}(1), \widehat{Y}(1,2);\ N^2\, h^2(Z, X_1)\, h(Y, X_2)\, h(Z, X_2)\, d_1(Y) \bigr\}, \tag{8.106}$$

with an error bounded by

$$\begin{aligned}
&\varkappa N^2\, \mathbf{E}\, h^2(Z, X_1) \bigl| h(Z, X_2) \bigr| \bigl| h(Y, X_2) \bigr|^{3/2} d_1(Y) \\
&\qquad \ll \varkappa N^{5/2}\, \mathbf{E}\, \varphi(Z) \bigl| h(Y, X_2) \bigr|^{3/2} \bigl| h(Z, X_2) \bigr|\, \varphi^{1/2}(Y) \\
&\qquad \ll \varkappa N^{5/2}\, \overline{\gamma}_{2,2}^{\,1/2}\, \overline{\gamma}_3^{\,1/2}\, \overline{\gamma}_2 \ll \varkappa N^2\, \overline{\gamma}_2\, \overline{\gamma}_3 + \varkappa N^3 \overline{\gamma}_2\, \overline{\gamma}_{2,2}. && (8.107)
\end{aligned}$$

Using (8.106) and expanding with remainder $\mathcal{O}\bigl(\widehat{Y}(1,2)\bigr)$, we have $\Gamma_{6,1} \sim \bigl\{ \dots ; 0 \bigr\}$ with a remainder bounded by

$$\begin{aligned}
&\varkappa N^2\, \mathbf{E}\, h^2(Z, X_1) \bigl| h(Z, X_2)\, h(Y, X_2)\, d_1(Y)\, d_s(Y) \bigr| \\
&\qquad \ll \varkappa N^3\, \mathbf{E}\, \varphi(Z) \bigl| h(Y, X_2) \bigr| \bigl| h(Z, X_2) \bigr| \varphi(Y) \ll \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}. && (8.108)
\end{aligned}$$

Let us estimate $\Gamma_{6,2}$ defined by (8.105). Expanding with remainder $\mathcal{O}\bigl(\widehat{Z}(1)\bigr)$, we obtain $\Gamma_{6,2} \sim \bigl\{ \dots ; 0 \bigr\}$ with an error bounded by

$$\varkappa N\, \mathbf{E}\, h^2(Z, X_1) \bigl| d_s(Z)\, d_1(Y)\, d_2(Y)\, d_3(Y) \bigr| \ll \varkappa N^3\, \mathbf{E}\, \varphi^{3/2}(Z)\, \varphi^{3/2}(Y) \ll \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}.$$

The estimation of Γ_7 is similar to that of Γ_6 since Z and Y are just exchanged.

Let us estimate Γ_8. Expanding in powers of $h(Z, X_1)$ and $h(Y, X_1)$ we obtain

$$\Gamma_8 \sim \bigl\{ \widehat{Z}(1), \widehat{Y}(1);\ N\, h(Z, X_1)\, d_1(Z)\, h(Y, X_1)\, d_2(Y) \bigr\} \tag{8.109}$$

with an error bounded by

$$\begin{aligned}\varkappa N\,\mathbf{E}\,h^2(Z,X_1)\,\bigl|\,d_1(Z)\,h(Y,X_1)\,d_2(Y)\,\bigr| \\ &\ll \varkappa N^2\,\mathbf{E}\,\varphi^{1/2}(Z)\,h^2(Z,X_1)\,\mathbf{E}_Y\,\varphi^{1/2}(Y)\,\bigl|\,h(Y,X_1)\,\bigr| \\ &\ll \varkappa N^2\,\overline{\gamma}_2^{\,1/2}\,\mathbf{E}\,\varphi^{1/2}(Z)\,\varphi^{1/2}(X_1)\,h^2(Z,X_1) \ll \varkappa\,\overline{\gamma}_3 + \varkappa N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2},\end{aligned}$$

using (8.98). Relation (8.109) combined with an expansion in powers of $\widehat{Z}(1)$ yields (cf. (8.102)–(8.105))

$$\Gamma_8 \ll \Gamma_{8,1} \cup \Gamma_{8,2} \cup \Gamma_{8,3}$$

with

$$\Gamma_{8,1} \sim \bigl\{\,\widehat{Z}(1),\,\widehat{Y}(1);\; N^2\,h(Z,X_1)\,h^2(Z,X_2)\,h(Y,X_1)\,d_1(Y)\,\bigr\}, \tag{8.110}$$

$$\Gamma_{8,2} \sim \bigl\{\,\widehat{Z}(1),\,\widehat{Y}(1);\; N^2\,h(Z,X_1)\,d_1(Z)\,h(Y,X_1)h(Y,X_3)h(Z,X_3)\,\bigr\}, \tag{8.111}$$

$$\Gamma_{8,3} \sim \bigl\{\,\widehat{Z}(1),\,\widehat{Y}(1);\; N\,h(Z,X_1)\,d_1(Z)\,h(Y,X_1)d_2(Y)d_3(Z)\,\bigr\}. \tag{8.112}$$

Now we can expand in powers of $\widehat{Y}(1)$. Using splittings and Hölder's inequality, we obtain $\Gamma_{8,1} \sim \bigl\{\dots;\,0\bigr\}$ with an error bounded by

$$\begin{aligned}\varkappa N^2\,\mathbf{E}\,\bigl|\,h(Z,X_1)\,\bigr|\,h^2(Z,X_2)\,\bigl|\,h(Y,X_1)\,d_1(Y)\,d_s(Y)\,\bigr| \\ &\ll \varkappa N^3\,\mathbf{E}\,\bigl|\,h(Z,X_1)\,h(Y,X_1)\,\bigr|\,h^2(Z,X_2)\,\varphi(Y) \\ &\ll \varkappa N^3\,\mathbf{E}\,\varphi^{1/2}(Z)\,\varphi^{1/2}(Y)\,\varphi(Z)\,\varphi(Y) \\ &\ll \varkappa N^3\,\bigl(\,\mathbf{E}\,\varphi^{3/2}(X)\,\bigr)^2 \ll \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned} \tag{8.113}$$

An estimation similar to (8.113) shows that $\Gamma_{8,s} \sim \bigl\{\dots;\,0\bigr\}$, for $s = 2, 3$, with errors bounded by the right hand side of (8.113). Thus $\Gamma_8 \sim \bigl\{\dots;\,0\bigr\}$.

Let us show that $\Gamma_9 \sim \bigl\{\dots;\,0\bigr\}$. Expanding in powers of $\widehat{Z}$ with remainder $\mathcal{O}(\widehat{Z})$ and using the i.i.d. assumption, we have (cf. (8.102)–(8.105))

$$\Gamma_9 \ll \Gamma_{9,1} \cup \Gamma_{9,2} \cup \Gamma_{9,3} \tag{8.114}$$

with

$$\Gamma_{9,1} \sim \bigl\{\,\widehat{Z},\,\widehat{Y};\; d_1(Z)\,d_2(Z)\,d_3(Y)\,d_4(Y)\,d_5(Z)\,\bigr\}, \tag{8.115}$$

$$\Gamma_{9,2} \sim \bigl\{\,\widehat{Z},\,\widehat{Y};\; N\,h^2(Z,X_1)\,d_1(Z)\,d_2(Y)\,d_3(Y)\,\bigr\}, \tag{8.116}$$

$$\Gamma_{9,3} \sim \bigl\{\,\widehat{Z},\,\widehat{Y};\; N\,h(Y,X_1)\,h(Z,X_1)\,d_1(Z)\,d_2(Z)\,d_3(Y)\,\bigr\}. \tag{8.117}$$

Expanding in powers of $\widehat{Y}$ with remainder $\mathcal{O}(\widehat{Y})$ we get $\Gamma_{9,1} \sim \bigl\{\dots;\,0\bigr\}$ with an error bounded by

$$\varkappa\,\mathbf{E}\,\bigl|\,d_1(Z)\,d_2(Z)\,d_3(Y)\,d_4(Y)\,d_5(Z)\,d_s(Y)\,\bigr| \ll \varkappa N^3\,\mathbf{E}\,\varphi^{3/2}(Z)\,\varphi^{3/2}(Y) \ll \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.$$

In order to estimate $\Gamma_{9,s}$, for $s = 2, 3$, we first remove $h(Y, X_1)$ from the exponents. Expansions with remainders $\mathcal{O}\big(\bigl| h(Y, X_1)\bigr|^{1/2}\big)$ produce errors bounded by

$$\begin{aligned}
&\varkappa N\, \mathbf{E}\, h^2(Z, X_1) \bigl| h(Y, X_1)\bigr|^{1/2} \bigl| d_1(Z)\, d_2(Y)\, d_3(Y)\bigr| \\
&\qquad \ll \varkappa N^{5/2}\, \mathbf{E}\, \varphi(Y)\, \mathbf{E}^Y \bigl| h(Z, X_1)\bigr|^{3/2} \Bigl(\bigl| h(Z, X_1)\, h(Y, X_1)\bigr|\, \varphi(Z)\Bigr)^{1/2} \\
&\qquad \ll \varkappa N^{5/2}\, \mathbf{E}\, \varphi(Y)\, \overline{\gamma}_3^{\,1/2} \Bigl(\mathbf{E}^Y \bigl| h(Z, X_1)\, h(Y, X_1)\bigr|\, \varphi(Z)\Bigr)^{1/2} \\
&\qquad \ll \varkappa N^{5/2}\, \mathbf{E}\, \varphi(Y)\, \overline{\gamma}_3^{\,1/2} \Bigl(\mathbf{E}^Y\ \varphi^{1/2}(Y)\, \varphi^{3/2}(Z)\Bigr)^{1/2} \\
&\qquad = \varkappa N^{5/2}\, \mathbf{E}\, \varphi^{5/4}(Y)\, \overline{\gamma}_3^{\,1/2} \Bigl(\mathbf{E}\, \varphi^{3/2}(Z)\Bigr)^{1/2} \\
&\qquad \ll \varkappa N^{5/2}\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}^{\,1/2}\, \overline{\gamma}_3^{\,1/2} \\
&\qquad \ll \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2} + \varkappa N^2\, \overline{\gamma}_2\, \overline{\gamma}_3
\end{aligned} \tag{8.118}$$

and

$$\begin{aligned}
&\varkappa N\, \mathbf{E} \bigl| h(Z, X_1)\bigr| \bigl| h(Y, X_1)\bigr|^{3/2} \bigl| d_1(Z)\, d_2(Z)\, d_3(Y)\bigr| \\
&\qquad \ll \varkappa N^{5/2}\, \mathbf{E}\, \varphi^{1/2}(Z) \Bigl(\mathbf{E}^Y \bigl| h(Y, X_1)\bigr|^3\Bigr)^{1/2} \varphi(Z)\, \varphi^{1/2}(Y) \\
&\qquad \ll \varkappa N^{5/2}\, \mathbf{E}\, \varphi^{3/2}(Z)\, \overline{\gamma}_2^{\,1/2}\, \overline{\gamma}_3^{\,1/2} \ll \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2} + \varkappa N^2 \overline{\gamma}_2 \overline{\gamma}_3.
\end{aligned} \tag{8.119}$$

Once terms depending on Y and X_1 are removed from the exponents, we can expand in powers of $\widehat{Y}(1)$. Using splittings and analyzing various cases, we obtain $\Gamma_{9,s} \sim \bigl\{ \dots ; 0 \bigr\}$ with remainders bounded by

$$\varkappa N^3\, \mathbf{E}\, \varphi^{3/2}(Z)\, \varphi^{3/2}(Y) \ll \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}, \tag{8.120}$$

completing the estimation of $I_{0,0,0}$.

Proof of (8.76) *for* $I_{0,0,1}$ *and* $I_{0,1,0}$. We shall proceed similarly to the case of $I_{0,0,0}$. Here we shall estimate $I_{0,0,1}$ only. Expanding in powers of $\widehat{Y}$ and $\widehat{Z}$ with remainders $\mathcal{O}(\widehat{Y})$ and $\mathcal{O}(\widehat{Z}^2)$, writing $\widehat{Y}\,\widehat{Z}^2$ as a triple sum, using the i.i.d. assumption, classifying the diagonals and applying (8.44), we obtain (cf. (8.77)–(8.87))

$$I_{0,0,1} \sim \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \tag{8.121}$$

with

$$\Gamma_1 \sim \bigl\{ \widehat{Z}, \widehat{Y};\ N\, f(Y)\, h(Y, X_1)\, h^2(Z, X_1) \bigr\}, \tag{8.122}$$

$$\Gamma_2 \sim \bigl\{ \widehat{Z}, \widehat{Y};\ N\, f(Y)\, d_1(Y)\, h^2(Z, X_1) \bigr\}, \tag{8.123}$$

$$\Gamma_3 \sim \bigl\{ \widehat{Z}, \widehat{Y};\ N\, f(Y)\, h(Y, X_1)\, h(Z, X_1)\, d_1(Z) \bigr\}, \tag{8.124}$$

$$\Gamma_4 \sim \bigl\{ \widehat{Z}, \widehat{Y};\ f(Y)\, d_1(Y)\, d_2(Z)\, d_3(Z) \bigr\}. \tag{8.125}$$

Let us estimate Γ_1. Expanding with remainder $\mathcal{O}\big(\sqrt{|h(Y,X_1)|}\big)$, we have

$$\Gamma_1 \sim \big\{\widehat{Z},\widehat{Y}(1);\; N\,f(Y)\,h(Y,X_1)\,h^2(Z,X_1)\big\}, \tag{8.126}$$

with a remainder bounded by

$$\begin{aligned}
\varkappa N\;\mathbf{E}\,\big|\,f(Y)\big|\big|\,h(Y,X_1)\big|^{3/2}\,h^2(Z,X_1) &\ll \varkappa N\,\overline{\beta}_3^{\,1/3}\;\mathbf{E}\,\Big(\mathbf{E}_Y\big|\,h(Y,X_1)\big|^{9/4}\Big)^{2/3}\varphi(X_1)\\
&\ll \varkappa N\,\overline{\beta}_3^{\,1/3}\,\mathbf{E}\,\varphi^{1/2}(X_1)\,\varphi_3^{1/6}(X_1)\,\varphi(X_1) \ll \varkappa N\,\overline{\beta}_3^{\,1/3}\,\overline{\gamma}_3^{\,1/6}\,\big(\,\mathbf{E}\,\varphi^{9/5}(X_1)\big)^{5/6}\\
&\ll \varkappa N\,\overline{\beta}_3^{\,1/3}\,\overline{\gamma}_3^{\,1/6}\,\overline{\gamma}_2^{\,1/6}\,\overline{\gamma}_{2,2}^{\,2/3} \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^{6/5}\,\overline{\gamma}_3^{\,1/5}\,\overline{\gamma}_2^{\,1/5}\,\overline{\gamma}_{2,2}^{\,4/5}\\
&\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa\overline{\gamma}_3 + \varkappa N^{3/2}\,\overline{\gamma}_2^{\,1/4}\,\overline{\gamma}_{2,2} \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa\overline{\gamma}_3 + \varkappa N\,\overline{\gamma}_{2,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
\end{aligned} \tag{8.127}$$

In (8.127) we used Hölder's inequality, (8.38) and several times $a^p\,b^q \ll a+b$, for $p,q\geq 0$ such that $p+q=1$.

The relation (8.126) implies $\Gamma_1 \sim \big\{\dots;0\big\}$. Indeed, expanding with remainder $\mathcal{O}(\widehat{Y}(1))$, we obtain $\Gamma_1 \sim \big\{\dots;0\big\}$ with an error bounded by

$$\begin{aligned}
\varkappa N\;\mathbf{E}\,\big|\,f(Y)\,h(Y,X_1)\,h^2(Z,X_1)\,d_s(Y)\big| &\ll \varkappa N^{3/2}\;\mathbf{E}\,\big|\,f(Y)\,h(Y,X_1)\big|\,\varphi(X_1)\,\varphi^{1/2}(Y)\\
&\ll \varkappa N^{3/2}\;\mathbf{E}\,\big|\,f(Y)\big|\,\varphi(Y)\,\overline{\gamma}_{2,2}^{\,1/2} \ll \varkappa N^{3/2}\,\overline{\beta}_3^{\,1/3}\big(\,\mathbf{E}\,\varphi^{3/2}(Y)\big)^{2/3}\,\overline{\gamma}_{2,2}^{\,1/2}\\
&\ll \varkappa N^{3/2}\,\overline{\beta}_3^{\,1/3}\,\overline{\gamma}_2^{\,1/3}\,\overline{\gamma}_{2,2}^{\,5/6} \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^{9/5}\,\overline{\gamma}_2^{\,2/5}\,\overline{\gamma}_{2,2}\\
&\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N\,\overline{\gamma}_{2,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
\end{aligned} \tag{8.128}$$

Let us estimate Γ_2. Using (8.123) and expanding with remainder $\mathcal{O}\big(h(Y,X_1)\big)$, we obtain

$$\Gamma_2 \sim \big\{\widehat{Z},\widehat{Y}(1);\; N\,f(Y)\,d_1(Y)\,h^2(Z,X_1)\big\} \tag{8.129}$$

with an error bounded by (cf. (8.128))

$$\begin{aligned}
\varkappa N\;\mathbf{E}\,\big|\,f(Y)\,d_1(Y)\,h^2(Z,X_1)\,h(Y,X_1)\big| &\ll \varkappa N^{3/2}\;\mathbf{E}\,\big|\,f(Y)\,\varphi^{1/2}(Y)\,\varphi(X_1)\,h(Y,X_1)\big|\\
&\ll \varkappa N^{3/2}\;\mathbf{E}\,\big|\,f(Y)\,\varphi(Y)\big|\,\overline{\gamma}_{2,2}^{\,1/2} \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N\,\overline{\gamma}_{2,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
\end{aligned} \tag{8.130}$$

Using (8.129), expanding with remainder $\mathcal{O}\big(\widehat{Y}(1)\big)$ and using the i.i.d. assumption, we obtain

$$\Gamma_2 \sim \big\{\widehat{Z},\widehat{Y}(1);\; N\,f(Y)\,d_1(Y)\,h^2(Z,X_1)\,\widehat{Y}(1)\big\} \sim \Gamma_{2,1}\cup\Gamma_{2,2}, \tag{8.131}$$

where

$$\Gamma_{2,1} \sim \big\{\widehat{Z},\widehat{Y}(1);\; N\,f(Y)\,d_1(Y)\,d_2(Y)\,h^2(Z,X_1)\big\}, \tag{8.132}$$

$$\Gamma_{2,2} \sim \big\{\widehat{Z},\widehat{Y}(1);\; N^2\,f(Y)\,h^2(Z,X_1)\,h^2(Y,X_2)\big\}. \tag{8.133}$$

Using an expansion with remainder $\mathcal{O}\big(\sqrt{|\,h(Z,X_1)\,|}\big)$, we have

$$\Gamma_{2,1} \sim \big\{\,\widehat{Z}(1),\,\widehat{Y}(1);\;N\,f(Y)\,d_1(Y)\,d_2(Y)\,h^2(Z,X_1)\,\big\}, \tag{8.134}$$

with an error bounded by (cf. (8.128))

$$\begin{aligned}\varkappa N\,\mathbf{E}\,\big|\,f(Y)\,d_1(Y)\,d_2(Y)\,\big|\,\big|\,h(Z,X_1)\,\big|^{5/2} &\ll \varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,\varphi(Y)\,\big|\,\varphi^{1/2}(Z)\,\varphi_3^{1/2}(Z)\\ \ll \varkappa N^2\,\overline{\gamma}_2^{\,1/2}\,\overline{\gamma}_3^{\,1/2}\,\mathbf{E}\,\big|\,f(Y)\,\varphi(Y)\,\big| &\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned} \tag{8.135}$$

The relation (8.134) yields $\Gamma_{2,1} \sim \big\{\dots;0\big\}$. Indeed, we can expand with remainder $\mathcal{O}\big(\widehat{Z}(1)\big)$, and obtain $\Gamma_{2,1} \sim \big\{\dots;0\big\}$ with an error bounded by (cf. (8.128))

$$\begin{aligned}\varkappa N\,\mathbf{E}\,\big|\,f(Y)\,d_1(Y)\,d_2(Y)\,d_s(Z)\,\big|\,h^2(Z,X_1) &\ll \varkappa N^{5/2}\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\varphi(Y)\,\varphi^{3/2}(Z)\\ \ll \varkappa N^{5/2}\,\overline{\gamma}_2^{\,1/2}\,\overline{\gamma}_{2,2}^{\,1/2}\,\mathbf{E}\,\big|\,f(Y)\,\varphi(Y)\,\big| &\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned} \tag{8.136}$$

In order to prove $\Gamma_{2,2} \sim \big\{\dots;0\big\}$ it suffices to use Taylor expansions with remainders $\mathcal{O}\big(\sqrt{|\,h(Z,X_1)\,|}\big)$, $\mathcal{O}\big(\sqrt{|\,h(Z,X_2)\,|}\big)$ and $\mathcal{O}\big(\widehat{Z}(1,2)\big)$. The errors are bounded by (cf. (8.135) and (8.136))

$$\begin{aligned}\varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\big|\,h(Z,X_1)\,\big|^{5/2}\,h^2(Y,X_2) &\ll \varkappa N^2\,\overline{\gamma}_2^{\,1/2}\,\overline{\gamma}_3^{\,1/2}\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\varphi(Y)\\ &\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2},\end{aligned} \tag{8.137}$$

$$\begin{aligned}&\varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,\big|\,h^2(Z,X_1)\,\big|\,h(Z,X_2)\,\big|^{1/2}\,h^2(Y,X_2)\\ &\quad\ll \varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\varphi(Z)\,\varphi_3^{1/6}(Z)\,\varphi_3^{1/3}(Y)\,\varphi^{1/2}(Y)\\ &\quad\ll \varkappa N^2\,\overline{\beta}_3^{\,1/3}\,\big(\,\mathbf{E}\,\varphi_3^{1/2}(Y)\,\varphi^{3/4}(Y)\big)^{2/3}\,\overline{\gamma}_3^{\,1/6}\big(\,\mathbf{E}\,\varphi^{6/5}(Z)\big)^{5/6}\\ &\quad\ll \varkappa N^2\,\overline{\beta}_3^{\,1/3}\,\big(\,\mathbf{E}\,\varphi^{3/2}(Y)\big)^{1/3}\,\overline{\gamma}_3^{\,1/2}\,\overline{\gamma}_2^{\,2/3}\,\overline{\gamma}_{2,2}^{\,1/6}\\ &\quad\ll \varkappa N^2\,\overline{\beta}_3^{\,1/3}\;\overline{\gamma}_3^{\,1/2}\,\overline{\gamma}_2^{\,5/6}\,\overline{\gamma}_{2,2}^{\,1/3}\\ &\quad\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2},\end{aligned} \tag{8.138}$$

$$\begin{aligned}\varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,\big|\,h^2(Z,X_1)\,h^2(Y,X_2)\,\big|\,d_s(Z)\,\big| &\ll \varkappa N^{5/2}\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\varphi^{3/2}(Z)\,\varphi(Y)\\ \ll \varkappa N^{5/2}\,\overline{\gamma}_2^{\,1/2}\,\overline{\gamma}_{2,2}^{\,1/2}\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\varphi(Y) &\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2},\end{aligned} \tag{8.139}$$

In the proof of (8.138) we used (8.39).

Let us estimate Γ_3. Expanding with remainder $\mathcal{O}\big(h(Y,X_1)\big)$, we obtain

$$\Gamma_3 \sim \big\{\, \widehat{Z},\, \widehat{Y}(1);\; N\, f(Y)\, h(Y,X_1)\, h(Z,X_1)\, d_1(Z)\,\big\}, \tag{8.140}$$

with an error bounded by

$$\begin{aligned} &\varkappa N^{3/2}\, \mathbf{E}\,\big|\, f(Y)\, h^2(Y,X_1)\, h(Z,X_1)\,\big|\, \varphi^{1/2}(Z) \\ &\qquad \ll \varkappa N^{3/2}\, \overline{\gamma}_2^{\,1/2}\, \mathbf{E}\,\big|\, f(Y)\,\big|\, h^2(Y,X_1)\; \varphi^{1/2}(X_1) \\ &\qquad \ll \varkappa N^{3/2}\, \overline{\gamma}_2^{\,1/2} \overline{\beta}_3^{\,1/3}\, \mathbf{E}\, \varphi^{1/2}(X_1)\, \varphi_3^{2/3}(X_1) \\ &\qquad \ll \varkappa N^{3/2}\, \overline{\beta}_3^{\,1/3}\, \overline{\gamma}_2^{\,2/3}\, \overline{\gamma}_3^{\,2/3}\, \overline{\gamma}_{2,2}^{\,1/6} \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^2\, \overline{\gamma}_2\, \overline{\gamma}_3 + \varkappa N\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.141}$$

Using (8.140) and expanding with remainder $\mathcal{O}\big(\widehat{Y}(1)\big)$, we obtain

$$\Gamma_3 \sim \Gamma_{3,1} \cup \Gamma_{3,2} \tag{8.142}$$

with

$$\begin{aligned} \Gamma_{3,1} &\sim \big\{\, \widehat{Z},\, \widehat{Y}(1);\; N\, f(Y)\, h(Y,X_1)\, h(Z,X_1)\, d_1(Z)\, d_2(Y)\,\big\}, \\ \Gamma_{3,2} &\sim \big\{\, \widehat{Z},\, \widehat{Y}(1);\; N^2\, f(Y)\, h(Y,X_1)\, h(Z,X_1)\, h(Z,X_2)\, h(Y,X_2)\,\big\}. \end{aligned} \tag{8.143}$$

Let us estimate $\Gamma_{3,1}$. Expanding with remainder $\mathcal{O}\Big(\sqrt{|\,h(Z,X_1)\,|}\Big)$, we have

$$\Gamma_{3,1} \sim \big\{\, \widehat{Z}(1),\, \widehat{Y}(1);\; N\, f(Y)\, h(Y,X_1)\, h(Z,X_1)\, d_1(Z)\, d_2(Y)\,\big\} \tag{8.144}$$

with an error bounded by (see (8.135))

$$\begin{aligned} &\varkappa N^2\, \mathbf{E}\,\big|\, f(Y)\, h(Y,X_1)\,\big|\,\big|\, h(Z,X_1)\,\big|^{3/2}\, \varphi^{1/2}(Z)\, \varphi^{1/2}(Y) \\ &\qquad \ll \varkappa N^2\, \mathbf{E}\,\big|\, f(Y)\,\big|\, \varphi(Y)\; \varphi^{1/2}(Z)\; \varphi_3^{1/2}(Z) \\ &\qquad \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^2\, \overline{\gamma}_2\, \overline{\gamma}_3 + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.145}$$

The relation (8.144) yields $\Gamma_{3,1} \sim \big\{ \dots ; 0 \big\}$ since expanding with remainder $\mathcal{O}(\widehat{Z}(1))$ we have an error bounded by (see (8.136))

$$\begin{aligned} &\varkappa N\, \mathbf{E}\,\big|\, f(Y)\, h(Y,X_1)\, h(Z,X_1)\, d_1(Z)\, d_2(Y)\, d_s(Z)\,\big| \\ &\qquad \ll \varkappa N^{5/2}\, \mathbf{E}\,\big|\, f(Y) h(Y,X_1)\, h(Z,X_1)\,\big|\, \varphi(Z)\, \varphi^{1/2}(Y) \\ &\qquad \ll \varkappa N^{5/2}\, \mathbf{E}\,\big|\, f(Y)\,\big|\, \varphi^{3/2}(Z)\, \varphi(Y) \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.146}$$

Let us estimate $\Gamma_{3,2}$. The relation (8.143) combined with Taylor expansions with remainders $\mathcal{O}\big(\,|\,h(Z,X_1)\,|^{1/2}\big)$ and $\mathcal{O}\big(\,|\,h(Z,X_2)\,|^{1/2}\big)$ yields

$$\Gamma_{3,2} \sim \big\{\,\widehat{Z}(1,2),\,\widehat{Y}(1);\; N^2 f(Y)\,h(Y,X_1)\,h(Z,X_1)\,h(Z,X_2)\,h(Y,X_2)\,\big\} \tag{8.147}$$

with errors bounded by (cf. (8.135))

$$\begin{aligned}&\varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,h(Y,X_1)\,h(Z,X_1)\,h(Y,X_2)\,\big|\,\big|\,h(Z,X_2)\,\big|^{3/2}\\ &\ll \varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\varphi(Y)\,\varphi^{1/2}(Z)\,\varphi_3^{1/2}(Z) \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned} \tag{8.148}$$

The relation (8.147) yields $\Gamma_{3,2} \sim \big\{\ldots;0\big\}$ since expanding the exponents with remainder $\mathcal{O}(\widehat{Z}(1,2))$ we have an error bounded by (cf. (8.136))

$$\begin{aligned}&\varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,h(Y,X_1)\,h(Z,X_1)\,h(Y,X_2)\,h(Z,X_2)\,d_s(Z)\,\big|\\ &\qquad \ll \varkappa N^{5/2}\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\varphi^{3/2}(Z)\,\varphi(Y) \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned} \tag{8.149}$$

Let us estimate Γ_4. Expanding with remainder $\mathcal{O}(\widehat{Z})$, using the i.i.d. assumption and analyzing various cases, we get

$$\Gamma_4 \sim \Gamma_{4,1} \cup \Gamma_{4,2} \cup \Gamma_{4,3} \tag{8.150}$$

with

$$\begin{aligned}\Gamma_{4,1} &\sim \big\{\,\widehat{Z},\,\widehat{Y};\; N\,f(Y)\,h(Y,X_1)\,h(Z,X_1)\,d_1(Z)\,d_2(Z)\,\big\},\\ \Gamma_{4,2} &\sim \big\{\,\widehat{Z},\,\widehat{Y};\; N\,f(Y)\,h^2(Z,X_1)\,d_1(Y)\,d_2(Z)\,\big\},\\ \Gamma_{4,3} &\sim \big\{\,\widehat{Z},\,\widehat{Y};\; f(Y)\,d_1(Y)\,d_2(Z)\,d_3(Z)\,d_4(Z)\,\big\}.\end{aligned} \tag{8.151}$$

Let us estimate $\Gamma_{4,1}$. Expanding with remainder $\mathcal{O}\big(\,|\,h(Y,X_1)\,|^{1/2}\big)$ we obtain

$$\Gamma_{4,1} \sim \big\{\,\widehat{Z},\,\widehat{Y}(1);\; N\,f(Y)\,h(Y,X_1)\,h(Z,X_1)\,d_1(Z)\,d_2(Z)\,\big\} \tag{8.152}$$

with an error bounded by

$$\begin{aligned}&\varkappa N\,\mathbf{E}\,\big|\,f(Y)\,\big|\,\big|\,h(Y,X_1)\,\big|^{3/2}\,\big|\,h(Z,X_1)\,d_1(Z)\,d_2(Z)\,\big|\\ &\qquad\ll \varkappa N^2\,\overline{\beta}_3^{\,1/3}\,\mathbf{E}\,\Big(\mathbf{E}_Y\big|\,h(Y,X_1)\,\big|^{9/4}\Big)^{2/3}\,\big|\,h(Z,X_1)\,\big|\,\varphi(Z)\\ &\qquad\ll \varkappa N^2\overline{\beta}_3^{\,1/3}\,\mathbf{E}\,\varphi^{1/2}(X_1)\,\varphi_3^{1/6}(X_1)\,\varphi^{1/2}(X_1)\,\overline{\gamma}_{2,2}^{1/2}\\ &\qquad\ll \varkappa N^2\overline{\beta}_3^{\,1/3}\,\Big(\mathbf{E}\,\varphi^{6/5}(X_1)\Big)^{5/6}\,\overline{\gamma}_3^{\,1/6}\,\overline{\gamma}_{2,2}^{\,1/2} \ll \varkappa N^2\overline{\beta}_3^{\,1/3}\,\overline{\gamma}_2^{\,2/3}\,\overline{\gamma}_3^{\,1/6}\,\overline{\gamma}_{2,2}^{\,2/3}\\ &\qquad\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned} \tag{8.153}$$

Using (8.152) and expanding with remainder $\mathcal{O}(\widehat{Y}(1))$, we have $\Gamma_{4,1} \sim \big\{ \dots ; 0 \big\}$ with an error bounded by (cf. (8.136))

$$\begin{aligned} &\varkappa N\, \mathbf{E} \,\big|\, f(Y) \,\big|\, \big|\, h(Y, X_1)\, h(Z, X_1)\, d_1(Z)\, d_2(Z)\, d_s(Y) \,\big| \\ &\qquad \ll \varkappa N^{5/2}\, \mathbf{E} \,\big|\, f(Y) \,\big|\, \varphi^{3/2}(Z)\, \varphi(Y) \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.154}$$

Let us estimate $\Gamma_{4,2}$. Expanding with remainder $\mathcal{O}\Big(\big|\, h(Y, X_1) \,\big|^{1/2} \Big)$ we have

$$\Gamma_{4,2} \sim \big\{ \widehat{Z},\, \widehat{Y}(1);\; N\, f(Y)\, h^2(Z, X_1)\, d_1(Y)\, d_2(Z) \,\big\} \tag{8.155}$$

with an error bounded by

$$\begin{aligned} &\varkappa N\, \mathbf{E} \,\big|\, f(Y) \,\big|\, h^2(Z, X_1) \,\big|\, h(Y, X_1) \,\big|^{1/2} \,\big|\, d_1(Y)\, d_2(Z) \,\big| \\ &\qquad \ll \varkappa N^2\, \mathbf{E} \,\big|\, f(Y) \,\big|\, \varphi_3^{1/6}(Y)\, \varphi_3^{1/3}(Z)\, \varphi^{1/2}(Z)\, \varphi^{1/2}(Y)\, \varphi^{1/2}(Z) \\ &\qquad \ll \varkappa N^2\, \overline{\beta}_3^{\,1/3} \big(\mathbf{E}\, \varphi_3^{1/4}(Y)\, \varphi^{3/4}(Y) \big)^{2/3}\, \mathbf{E}\, \varphi_3^{1/3}(Z)\, \varphi(Z) \\ &\qquad \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^2\, \overline{\gamma}_2\, \overline{\gamma}_3 + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.156}$$

The relation (8.155) yields $\Gamma_{4,2} \sim \big\{ \dots ; 0 \big\}$ since expanding with remainder $\mathcal{O}(\widehat{Y}(1))$ we have an error bounded by

$$\begin{aligned} &\varkappa N\, \mathbf{E} \,\big|\, f(Y) \,\big|\, h^2(Z, X_1)\, d_1(Y)\, d_2(Z)\, d_s(Y) \,\big| \\ &\qquad \ll \varkappa N^{5/2}\, \mathbf{E} \,\big|\, f(Y) \,\big|\, \varphi^{3/2}(Z)\, \varphi(Y) \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.157}$$

Let us estimate $\Gamma_{4,3}$. Expanding with remainder $\mathcal{O}(\widehat{Y})$ we have $\Gamma_{4,3} \sim \big\{ \dots ; 0 \big\}$ with an error bounded by

$$\begin{aligned} &\varkappa\, \mathbf{E} \,\big|\, f(Y) \,\big|\, d_1(Y)\, d_2(Z)\, d_3(Z)\, d_4(Z)\, d_s(Y) \,\big| \\ &\qquad \ll \varkappa N^{5/2}\, \mathbf{E} \,\big|\, f(Y) \,\big|\, \varphi^{3/2}(Z)\, \varphi(Y) \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.158}$$

Proof of (8.76) *for* $I_{0,0,2}$ *and* $I_{0,2,0}$. We shall estimate $I_{0,0,2}$ only. Expanding in powers of $\widehat{Y}$ and $\widehat{Z}$ with remainders $\mathcal{O}(\widehat{Y})$ and $\mathcal{O}(\widehat{Z}^2)$, writing $\widehat{Y}\,\widehat{Z}^2$ as a triple sum, using the i.i.d. assumption, classifying the diagonals and applying (8.44), we obtain (cf. (8.77)–(8.87))

$$I_{0,0,2} \sim \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \tag{8.159}$$

with

$$\Gamma_1 \sim \big\{ \widehat{Z},\, \widehat{Y};\; N\, f^2(Y)\, h(Y, X_1)\, h^2(Z, X_1) \,\big\}, \tag{8.160}$$

$$\Gamma_2 \sim \big\{ \widehat{Z},\, \widehat{Y};\; N\, f^2(Y)\, d_1(Y)\, h^2(Z, X_1) \,\big\}, \tag{8.161}$$

$$\Gamma_3 \sim \big\{ \widehat{Z},\, \widehat{Y};\; N\, f^2(Y)\, h(Y, X_1)\, h(Z, X_1)\, d_1(Z) \,\big\}, \tag{8.162}$$

$$\Gamma_4 \sim \big\{ \widehat{Z},\, \widehat{Y};\; f^2(Y)\, d_1(Y)\, d_2(Z)\, d_3(Z) \,\big\}. \tag{8.163}$$

Let us estimate Γ_1. We have $\Gamma_1 \sim \big\{ \dots ; 0 \big\}$ with an error bounded by

$$\begin{aligned}
\varkappa N \, \mathbf{E} \, f^2(Y) \big| h(Y, X_1) \big| h^2(Z, X_1) &\ll \varkappa N \, \overline{\beta}_3^{\,2/3} \, \mathbf{E} \, \varphi_3^{1/3}(Y) \, \varphi(X_1) \\
&\ll \varkappa N \, \overline{\beta}_3^{\,2/3} \, \overline{\gamma}_3^{\,1/3} \big(\mathbf{E} \, \varphi^{3/2}(X_1) \big)^{2/3} \ll \varkappa N \, \overline{\beta}_3^{\,2/3} \, \overline{\gamma}_3^{\,1/3} \, \overline{\gamma}_2^{\,1/3} \, \overline{\gamma}_{2,2}^{\,1/3} \\
&\ll \varkappa \overline{\beta}_3^{\,2} + \varkappa \, \overline{\gamma}_3 + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}.
\end{aligned} \tag{8.164}$$

Let us estimate Γ_2. Expanding with remainder $\mathcal{O}\Big(\sqrt{\big| h(Z, X_1) \big|} \Big)$, we have

$$\Gamma_2 \sim \big\{ \widehat{Z}(1), \widehat{Y}; \, N f^2(Y) \, d_1(Y) \, h^2(Z, X_1) \big\}, \tag{8.165}$$

with an error bounded by

$$\begin{aligned}
\varkappa N \, \mathbf{E} \, f^2(Y) \big| d_1(Y) \big| \big| h(Z, X_1) \big|^{5/2} &\ll \varkappa N^{3/2} \, \mathbf{E} \, f^2(Y) \, \varphi^{1/2}(Y) \, \varphi^{1/2}(Z) \, \varphi_3^{1/2}(Z) \\
&\ll \varkappa N^{3/2} \, \overline{\beta}_3^{\,2/3} \overline{\gamma}_2^{\,1/2} \, \overline{\gamma}_3^{\,1/2} \big(\mathbf{E} \, \varphi^{3/2}(Y) \big)^{1/3} \ll \varkappa N^{3/2} \, \overline{\beta}_3^{\,2/3} \overline{\gamma}_2^{\,2/3} \, \overline{\gamma}_3^{\,1/2} \, \overline{\gamma}_{2,2}^{\,1/6} \\
&\ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^2 \, \overline{\gamma}_2 \, \overline{\gamma}_3 + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}.
\end{aligned} \tag{8.166}$$

Combining (8.165) and an expansion with remainder $\mathcal{O}\big(\widehat{Z}(1) \big)$ we have $\Gamma_2 \sim \big\{ \dots ; 0 \big\}$ with an error bounded by

$$\begin{aligned}
\varkappa N \, \mathbf{E} \, f^2(Y) \big| d_1(Y) \, d_s(Z) \big| h^2(Z, X_1) &\ll \varkappa N^2 \, \mathbf{E} \, f^2(Y) \, \varphi^{1/2}(Y) \, \varphi^{3/2}(Z) \\
&\ll \varkappa N^2 \overline{\beta}_3^{\,2/3} \, \big(\mathbf{E} \, \varphi^{3/2}(X) \big)^{4/3} \ll \varkappa N^2 \overline{\beta}_3^{\,2/3} \, \overline{\gamma}_2^{\,2/3} \, \overline{\gamma}_{2,2}^{\,2/3} \\
&\ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}.
\end{aligned} \tag{8.167}$$

Let us estimate Γ_3. Expanding with remainder $\mathcal{O}\Big(\sqrt{\big| h(Z, X_1) \big|} \Big)$, we have

$$\Gamma_3 \sim \big\{ \widehat{Z}(1), \widehat{Y}; \, N f^2(Y) \, h(Y, X_1) \, h(Z, X_1) \, d_1(Z) \big\} \tag{8.168}$$

with an error bounded by (cf. (8.166))

$$\begin{aligned}
&\varkappa N^{3/2} \, \mathbf{E} \, f^2(Y) \big| h(Y, X_1) \big| \big| h(Z, X_1) \big|^{3/2} \varphi^{1/2}(Z) \\
&\qquad \ll \varkappa N^{3/2} \, \mathbf{E} \, f^2(Y) \, \varphi^{1/2}(Y) \, \varphi^{1/2}(Z) \, \varphi_3^{1/2}(Z) \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^2 \, \overline{\gamma}_2 \, \overline{\gamma}_3 + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}.
\end{aligned}$$

Now (8.168) yields $\Gamma_3 \sim \big\{ \dots ; 0 \big\}$ since, expanding with remainder $\mathcal{O}\big(\widehat{Z}(1) \big)$ we have an error bounded by (cf. (8.167))

$$\begin{aligned}
&\varkappa N \, \mathbf{E} \, f^2(Y) \big| h(Y, X_1) \, h(Z, X_1) \, d_1(Z) \, d_s(Z) \big| \\
&\qquad \ll \varkappa N^2 \, \mathbf{E} \, f^2(Y) \, \varphi^{3/2}(Z) \, \varphi^{1/2}(Y) \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}.
\end{aligned} \tag{8.169}$$

In order to prove $\Gamma_4 \sim \big\{ \dots ; 0 \big\}$ it suffices to expand with remainder $\mathcal{O}(\widehat{Z})$. An error is bounded by (cf. (8.167))

$$\begin{aligned} &\varkappa \mathbf{E} f^2(Y) \, \big| d_1(Y) \, d_2(Z) \, d_3(Z) \, d_s(Z) \big| \\ &\qquad \ll \varkappa N^2 \, \mathbf{E} f^2(Y) \, \varphi^{3/2}(Z) \, \varphi^{1/2}(Y) \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N^3 \, \overline{\gamma}_2 \, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.170}$$

Proof of (8.76) *for* $I_{0,1,1}$. Expanding in powers of $\widehat{Y}$ and $\widehat{Z}$ with remainders $\mathcal{O}(\widehat{Y})$ and $\mathcal{O}(\widehat{Z})$, writing $\widehat{Y} \, \widehat{Z}$ as a double sum, we obtain

$$I_{0,1,1} \sim \Gamma_1 \cup \Gamma_2 \tag{8.171}$$

with

$$\Gamma_1 \sim \big\{ \widehat{Z}, \widehat{Y}; \, N f(Y) \, f(Z) \, h(Y, X_1) \, h(Z, X_1) \, \big\}, \tag{8.172}$$

$$\Gamma_2 \sim \big\{ \widehat{Z}, \widehat{Y}; \; f(Y) \, f(Z) \, d_1(Y) \, d_2(Z) \, \big\}. \tag{8.173}$$

Let us estimate Γ_1. Introduce the function $p(t) = \mathrm{e}\{t\} - 1$, for $t \in \mathbb{R}$. Then, writing $\widehat{Z} = h(Z, X_1) + \widehat{Z}(1)$ and $\widehat{Y} = h(Y, X_1) + \widehat{Y}(1)$, replacing $\mathrm{e}\{\tau \, h(Z, X_1)\}$ and $\mathrm{e}\{\tau \, h(Y, X_1)\}$ by $p\big(\tau \, h(Z, X_1)\big) + 1$ and $p\big(\tau \, h(Y, X_1)\big) + 1$ respectively, we obtain

$$\Gamma_1 \sim \Gamma_{1,1} \cup \Gamma_{1,2} \cup \Gamma_{1,3} \cup \Gamma_{1,4} \tag{8.174}$$

where: $\Gamma_{1,1}$ is defined as Γ_1 just replacing both $\mathrm{e}\{\tau h(Z, X_1)\}$ and $\mathrm{e}\{\tau h(Y, X_1)\}$ by $p\big(\tau h(Z, X_1)\big)$ and $p\big(\tau h(Y, X_1)\big)$ respectively; $\Gamma_{1,2}$ and $\Gamma_{1,3}$ are defined as Γ_1 replacing one of the exponents by the corresponding function $p(\dots)$ and other exponent by 1; $\Gamma_{1,4}$ is defined as Γ_1 replacing both exponents by 1.

In order to estimate $\Gamma_{1,1}$ we use $|p(t)| \ll \sqrt{|t|}$ and conditioning. We obtain $\Gamma_{1,1} \sim \big\{ \dots ; 0 \big\}$ with an error bounded by (cf. (8.164))

$$\begin{aligned} &\varkappa N \, \mathbf{E} \, \big| f(Y) \, f(Z) \big| \big| h(Y, X_1) \, h(Z, X_1) \big|^{3/2} \\ &\qquad \ll \varkappa N \overline{\beta}_3^{\,2/3} \, \mathbf{E} \, \Big(\mathbf{E}_Y \, \big| h(Y, X_1) \big|^{9/4} \mathbf{E}_Z \, \big| h(Z, X_1) \big|^{9/4} \Big)^{2/3} \\ &\qquad \ll \varkappa N \overline{\beta}_3^{\,2/3} \, \mathbf{E} \, \varphi(X_1) \, \varphi_3^{1/3}(X_1) \ll \varkappa N \, \overline{\beta}_3^{\,2/3} \, \overline{\gamma}_2^{\,1/3} \, \overline{\gamma}_{2,2}^{\,1/3} \, \overline{\gamma}_3^{\,1/3} \\ &\qquad \ll \varkappa \overline{\beta}_3^{\,2/3} + \varkappa \overline{\gamma}_3 + \varkappa N^3 \overline{\gamma}_2 \, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.175}$$

Notice that

$$\Gamma_{1,4} \sim \big\{ \widehat{Z}(1), \widehat{Y}(1); \, N f(Y) \, f(Z) \, h(Y, X_1) \, h(Z, X_1) \, \big\}. \tag{8.176}$$

Expansions with remainders $\mathcal{O}\big(\widehat{Z}(1)\big)$ and $\mathcal{O}\big(\widehat{Y}(1)\big)$ show that $\Gamma_{1,4} \sim \big\{\dots;0\big\}$ with an error bounded by

$$\begin{aligned}\varkappa N\,\mathbf{E}\,\big|\,f(Y)\,f(Z)\,h(Y,X_1)\,h(Z,X_1)\,d_r(Y)\,d_s(Z)\big| &\ll \varkappa N^2\,\mathbf{E}\,\big|\,f(Y)\,f(Z)\,\big|\,\varphi(Y)\,\varphi(Z)\\ \ll \varkappa N^2\overline{\beta}_3^{\,2/3}\,\big(\,\mathbf{E}\,\varphi^{3/2}(X)\big)^{4/3} \ll \varkappa N^2\overline{\beta}_3^{\,2/3}\,\overline{\gamma}_2^{\,2/3}\,\overline{\gamma}_{2,2}^{\,2/3} &\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}. \end{aligned} \tag{8.177}$$

Both $\Gamma_{1,2}$ and $\Gamma_{1,3}$ can be estimated similarly. Let us estimate $\Gamma_{1,2}$ only. Since $\mathrm{e}\big\{\tau\,h(Y,X_1)\big\}$ is replaced by 1, we can expand with remainder $\mathcal{O}\big(\widehat{Y}(1)\big)$. Using $a\,b \ll a^2+b^2$, we obtain $\Gamma_{1,2} \sim \big\{\dots;0\big\}$ with an error bounded by (cf. (8.175) and (8.177))

$$\begin{aligned}&\varkappa N\,\mathbf{E}\,\big|\,f(Y)\,f(Z)\,h(Y,X_1)\big|\,\big|\,h(Z,X_1)\,\big|^{3/2}\,\big|\,d_s(Y)\,\big|\\ &\quad\ll \varkappa N^{3/2}\,\mathbf{E}\,\mathbf{E}_Y\big|\,f(Y)\,h(Y,X_1)\,\big|\,\varphi^{1/2}(Y)\,\mathbf{E}_Z\big|\,f(Z)\,\big|\,\big|\,h(Z,X_1)\,\big|^{3/2}\\ &\quad\ll \varkappa N^2\,\mathbf{E}\,\Big(\mathbf{E}_Y\big|\,f(Y)\,h(Y,X_1)\,\big|\,\varphi^{1/2}(Y)\Big)^2 + \varkappa N\,\mathbf{E}\,\Big(\mathbf{E}_Z\big|\,f(Z)\,\big|\,\big|\,h(Z,X_1)\,\big|^{3/2}\Big)^2\\ &\quad\ll \varkappa\overline{\beta}_3^{\,2} + \varkappa\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}. \end{aligned} \tag{8.178}$$

Let us estimate Γ_2. Expanding with remainder $\mathcal{O}\big(\widehat{Z}\big)$, we obtain

$$\Gamma_2 \sim \Gamma_{2,1} \cup \Gamma_{2,2} \cup \Gamma_{2,3} \tag{8.179}$$

with

$$\Gamma_{2,1} \sim \big\{\widehat{Z},\,\widehat{Y};\;N\,f(Y)\,f(Z)\,h(Y,X_1)\,h(Z,X_1)\,d_1(Z)\,\big\}, \tag{8.180}$$

$$\Gamma_{2,2} \sim \big\{\widehat{Z},\,\widehat{Y};\;N\,f(Y)\,f(Z)\,d_1(Y)\,h^2(Z,X_1)\,\big\}, \tag{8.181}$$

$$\Gamma_{2,3} \sim \big\{\widehat{Z},\,\widehat{Y};\;f(Y)\,f(Z)\,d_1(Y)\,d_2(Z)\,d_3(Z)\,\big\}. \tag{8.182}$$

In order to show $\Gamma_{2,1} \sim \big\{\dots;0\big\}$ we use expansions with remainders $\mathcal{O}\Big(\sqrt{|\,h(Y,X_1)\,|}\Big)$ and $\mathcal{O}\big(\widehat{Y}(1)\big)$. The errors are bounded by

$$\varkappa N\,\mathbf{E}\,\big|\,f(Y)\,f(Z)\,\big|\,\big|\,h(Y,X_1)\,\big|^{3/2}\,\big|\,h(Z,X_1)\,d_1(Z)\,\big| \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2} \tag{8.183}$$

(exchanging Z with Y and using (8.178)),

$$\varkappa N\,\mathbf{E}\,\big|\,f(Y)\,f(Z)\,\big|\,\big|\,h(Y,X_1)h(Z,X_1)\,d_r(Z)\,d_s(Y)\,\big| \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2} \tag{8.184}$$

(using (8.177)).

In order to prove $\Gamma_{2,2} \sim \big\{\dots;0\big\}$ we again use expansions with remainder terms $\mathcal{O}\big(\sqrt{|h(Y,X_1)|}\big)$ and $\mathcal{O}\big(\widehat{Y}(1)\big)$. The errors are bounded by

$$
\begin{aligned}
\varkappa N\,\mathbf{E}\,\big|f(Y)\,f(Z)\big|\,\big|h(Y,X_1)\big|^{1/2}\,h^2(Z,X_1)\,\big|d_1(Y)\big| \\
&\ll \varkappa N^{3/2}\,\mathbf{E}\,\big|f(Y)\,f(Z)\big|\,\varphi^{1/2}(Y)\,\varphi_3^{1/6}(Y)\,\varphi_3^{1/3}(Z)\,\varphi^{1/2}(Z) \\
&\ll \varkappa N^{3/2}\,\overline{\beta}_3^{\,2/3}\Big(\mathbf{E}\,\varphi^{3/4}(Y)\,\varphi_3^{1/4}(Y)\,\mathbf{E}\,\varphi^{1/2}(Z)\,\varphi_3^{3/4}(Z)\Big)^{2/3} \\
&\ll \overline{\beta}_3^{\,2/3}\,\overline{\gamma}_3^{\,1/2}\,\overline{\gamma}_2^{\,2/3}\,\overline{\gamma}_{2,2}^{\,1/6} \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^2\,\overline{\gamma}_2\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}
\end{aligned}
\tag{8.185}
$$

(using (8.39)), and by

$$
\begin{aligned}
\varkappa N\,\mathbf{E}\,\big|f(Y)\,f(Z)\big|\,h^2(Z,X_1)\,\big|d_1(Y)\,d_s(Y)\big| \\
&\ll \varkappa N^2\,\mathbf{E}\,\big|f(Y)\,f(Z)\big|\,\varphi(Z)\,\varphi(Y) \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}
\end{aligned}
\tag{8.186}
$$

(using (8.177)).

In order to prove $\Gamma_{2,3} \sim \big\{\dots;0\big\}$ we expand with remainder $\mathcal{O}\big(\widehat{Y}\big)$. The error is bounded by (cf. (8.177))

$$
\begin{aligned}
\varkappa N\,\mathbf{E}\,\big|f(Y)\,f(Z)\,d_1(Y)\,d_s(Y)\,d_2(Z)\,d_3(Z)\big| \\
&\ll \varkappa N^2\,\mathbf{E}\,\big|f(Y)\,f(Z)\big|\,\varphi(Z)\,\varphi(Y) \ll \varkappa\overline{\beta}_3^{\,2/3} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
\end{aligned}
\tag{8.187}
$$

Proof of (8.76) *for* $I_{0,1,2}$, $I_{0,2,1}$, $I_{0,0,3}$ *and* $I_{0,3,0}$. We shall estimate $I_{0,0,3}$ only. Expanding in powers of $\widehat{Z}$ with remainder $\mathcal{O}(\widehat{Z}^2)$ and writing $\widehat{Z}^2$ as a double sum, we obtain

$$
I_{0,0,3} \sim \Gamma_1 \cup \Gamma_2
\tag{8.188}
$$

with

$$
\Gamma_1 \sim \big\{\widehat{Z},\widehat{Y};\; N\,f^3(Y)\,h^2(Z,X_1)\big\}, \qquad \Gamma_2 \sim \big\{\widehat{Z},\widehat{Y};\; f^3(Y)\,d_1(Z)\,d_2(Z)\big\}.
\tag{8.189}
$$

In order to show $\Gamma_1 \sim \big\{\dots;0\big\}$ we use expansions with remainders $\mathcal{O}\big(\sqrt{|h(Z,X_1)|}\big)$ and $\mathcal{O}\big(\widehat{Z}(1)\big)$. The errors are bounded by

$$
\begin{aligned}
\varkappa N\,\mathbf{E}\,\big|f^3(Y)\big|\,\big|h(Z,X_1)\big|^{5/2} &\ll \varkappa N\,\overline{\beta}_3\,\mathbf{E}\,\varphi^{1/2}(Z)\varphi_3^{1/2}(Z) \\
&\ll \varkappa N\,\overline{\beta}_3\,\overline{\gamma}_2^{\,1/2}\,\overline{\gamma}_3^{\,1/2} \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^2\overline{\gamma}_2\,\overline{\gamma}_3,
\end{aligned}
$$

$$
\varkappa N\,\mathbf{E}\,\big|f^3(Y)\big|\,h^2(Z,X_1)\,\big|d_1(Z)\big| \ll \varkappa N^{3/2}\,\overline{\beta}_3\,\mathbf{E}\,\varphi^{3/2}(Z) \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
$$

In order to prove $\Gamma_2 \sim \{\dots;0\}$ we expand with remainder $\mathcal{O}(\widehat{Z})$. The error is bounded by

$$\varkappa\,\mathbf{E}\,\bigl|f^3(Y)\bigr|\,\bigl|d_1(Z)\,d_2(Z)\,d_s(Z)\bigr| \ll \varkappa N^{3/2}\,\overline{\beta}_3\,\mathbf{E}\,\varphi^{3/2}(Z) \ll \varkappa\overline{\beta}_3^{\,2} + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.$$

Proof of (8.76) *for* $I_{0,3,1}$, $I_{0,2,2}$ *and* $I_{0,1,3}$. Let us estimate $I_{0,2,2}$ only. Expanding with remainders $\mathcal{O}(\widehat{Z})$ and $\mathcal{O}(\widehat{Y})$, we have $I_{0,2,2} \sim \{\dots;0\}$ with an error bounded by

$$\begin{aligned}\varkappa\,\mathbf{E}\,f^2(Y)\,f^2(Z)\,\bigl|d_s(Z)\,d_r(Y)\bigr| &\ll \varkappa N\,\mathbf{E}\,f^2(Y)\,f^2(Z)\,\varphi^{1/2}(Y)\varphi^{1/2}(Z)\\ &\ll \varkappa N\,\overline{\beta}_3^{\,4/3}\bigl(\mathbf{E}\,\varphi^{3/2}(X)\bigr)^{2/3} \ll \varkappa N\,\overline{\beta}_3^{\,4/3}\,\overline{\gamma}_2^{\,1/3}\,\overline{\gamma}_{2,2}^{\,1/3} \ll \varkappa\overline{\beta}_3^{\,2} + N^3\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned}$$

Proof of (8.76) *for* $I_{0,3,2}$ *and* $I_{0,2,3}$. In the case of $I_{0,2,3}$ we can expand with remainder $\mathcal{O}(\widehat{Z})$, and we have $I_{0,2,3} \sim \{\dots;0\}$ with an error bounded by

$$\begin{aligned}\varkappa N\,\mathbf{E}\,\bigl|f^3(Y)\bigr|\,f^2(Z)\,\bigl|d_1(Z)\bigr| &\ll \varkappa N^{1/2}\,\overline{\beta}_3^{\,5/3}\bigl(\mathbf{E}\,\varphi^{3/2}(X)\bigr)^{1/3}\\ &\ll \varkappa N^{1/2}\,\overline{\beta}_3^{\,5/3}\,\overline{\gamma}_2^{\,1/6}\,\overline{\gamma}_{2,2}^{\,1/6} \ll \varkappa\overline{\beta}_3^{\,2} + N^3\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned} \tag{8.190}$$

Proof of (8.76) *for* $I_{0,3,3}$. We have $I_{0,3,3} \sim \{\dots;0\}$ with an error bounded by $\varkappa\overline{\beta}_3^{\,2}$.

Proof of (8.76) *for* $I_{1,0,0}$. Expanding with remainders $\mathcal{O}(\widehat{Z}^2)$ and $\mathcal{O}(\widehat{Y})$, we obtain (cf. (8.77)–(8.87))

$$I_{1,0,0} \sim \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \tag{8.191}$$

where

$$\Gamma_1 \sim \bigl\{\widehat{Z},\widehat{Y};\; N\,h(Z,Y)\,h(Y,X_1)\,h^2(Z,X_1)\bigr\}, \tag{8.192}$$

$$\Gamma_2 \sim \bigl\{\widehat{Z},\widehat{Y};\; N\,h(Z,Y)\,d_1(Y)\,h^2(Z,X_1)\bigr\}, \tag{8.193}$$

$$\Gamma_3 \sim \bigl\{\widehat{Z},\widehat{Y};\; N\,h(Z,Y)\,h(Y,X_1)\,h(Z,X_1)\,d_1(Z)\bigr\}, \tag{8.194}$$

$$\Gamma_4 \sim \bigl\{\widehat{Z},\widehat{Y};\; h(Z,Y)\,d_1(Y)\,d_2(Z)\,d_3(Z)\bigr\}. \tag{8.195}$$

Let us estimate Γ_1. We have $\Gamma_1 \sim \{\dots;\,0\}$ with an error bounded by

$$\begin{aligned}\varkappa N\,\mathbf{E}\,\bigl|h(Z,Y)\,h(Y,X_1)\bigr|\,h^2(Z,X_1) &\ll \varkappa N\,\mathbf{E}\,h^2(Z,X_1)\,\varphi^{1/2}(Z)\varphi^{1/2}(X_1)\\ &\ll \varkappa N\,\overline{\gamma}_3^{\,2/3}\bigl(\mathbf{E}\,\varphi^{3/2}(X)\bigr)^{2/3} \ll \varkappa N\,\overline{\gamma}_3^{\,2/3}\,\overline{\gamma}_2^{\,1/3}\,\overline{\gamma}_{2,2}^{\,1/3} \ll \varkappa\,\overline{\gamma}_3 + \varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.\end{aligned}$$

Let us estimate Γ_2. Expanding with remainder $\mathcal{O}\bigl(\sqrt{|h(Y,X_1)|}\bigr)$, we have

$$\Gamma_2 \sim \bigl\{\widehat{Z},\widehat{Y}(1);\; N\,h(Z,Y)\,d_1(Y)\,h^2(Z,X_1)\bigr\} \tag{8.196}$$

with an error bounded by

$$\begin{aligned}
&\varkappa N\,\mathbf{E}\,\bigl|\,h(Z,Y)\,\bigr|\,\bigl|\,h(Y,X_1)\,\bigr|^{1/2}\,\bigl|\,d_1(Y)\,\bigr|\,h^2(Z,X_1)\\
&\qquad\ll \varkappa N^{3/2}\,\mathbf{E}\,\bigl|\,h(Z,Y)\,\bigr|\,\varphi^{1/2}(Y)\,\varphi_3^{1/6}(Y)\,\varphi^{1/2}(Z)\,\varphi_3^{1/3}(Z)\\
&\qquad\ll \varkappa N^{3/2}\,\mathbf{E}\,\varphi(Z)\,\varphi_3^{1/3}(Z)\bigl(\,\mathbf{E}\,\varphi(Y)\,\varphi_3^{1/3}(Y)\bigr)^{1/2}\\
&\qquad\ll \varkappa N^{3/2}\,\overline{\gamma}_2^{\,1/2}\,\overline{\gamma}_3^{\,1/2}\,\overline{\gamma}_{2,2}^{\,1/2}\ll \varkappa\overline{\gamma}_3+\varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2},
\end{aligned}$$

using (8.39) and (8.175).

Combining (8.196) and an expansion with remainder $\mathcal{O}\bigl(\widehat{Y}(1)\bigr)$ we have $\Gamma_2\sim\bigl\{\ldots;0\bigr\}$ with an error bounded by

$$\begin{aligned}
\varkappa N\,\mathbf{E}\,\bigl|\,h(Z,Y)\,\bigr|\,\bigl|\,d_1(Y)\,d_s(Y)\,\bigr|\,h^2(Z,X_1)&\ll \varkappa N^2\,\mathbf{E}\,\bigl|\,h(Z,Y)\,\bigr|\,\varphi(Y)\,\varphi(Z)\\
&\ll \varkappa N^2\,\overline{\gamma}_2^{\,1/2}\,\gamma_{2,2}\ll \varkappa N\,\overline{\gamma}_{2,2}+\varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
\end{aligned}\tag{8.197}$$

Let us estimate Γ_3. Expanding with remainder $\mathcal{O}\Bigl(\sqrt{\bigl|\,h(Y,X_1)\,\bigr|}\Bigr)$, we have

$$\Gamma_3\sim\bigl\{\,\widehat{Z},\,\widehat{Y}(1);\;N\,h(Z,Y)\,h(Y,X_1)\,h(Z,X_1)\,d_1(Z)\,\bigr\},\tag{8.198}$$

with an error bounded by

$$\begin{aligned}
&\varkappa N\,\mathbf{E}\,\bigl|\,h(Z,Y)\,\bigr|\,\bigl|\,h(Y,X_1)\,\bigr|^{3/2}\,\bigl|\,h(Z,X_1)\,\bigr|\,\bigl|\,d_1(Z)\,\bigr|\\
&\qquad\ll \varkappa N^{3/2}\,\mathbf{E}\,\bigl|\,h(Z,Y)\,\bigr|\,\varphi_3^{1/2}(Y)\,\varphi(Z)\ll \varkappa N^{3/2}\,\overline{\gamma}_{2,2}^{\,1/2}\,\mathbf{E}\,\varphi_3^{1/2}(Y)\,\varphi^{1/2}(Y)\\
&\qquad\ll \varkappa N^{3/2}\,\overline{\gamma}_2^{\,1/2}\,\overline{\gamma}_3^{\,1/2}\,\overline{\gamma}_{2,2}^{\,1/2}\ll \varkappa\overline{\gamma}_3+\varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
\end{aligned}$$

Now (8.198) yields $\Gamma_3\sim\bigl\{\ldots;0\bigr\}$ since expanding with remainder $\mathcal{O}(\widehat{Y}(1))$ we have an error bounded by (cf. (8.197))

$$\begin{aligned}
&\varkappa N\,\mathbf{E}\,\bigl|\,h(Z,Y)\,h(Y,X_1)\,h(Z,X_1)\,d_1(Z)\,d_s(Y)\,\bigr|\\
&\qquad\ll \varkappa N^2\,\mathbf{E}\,\bigl|\,h(Z,Y)\,\bigr|\,\varphi(Z)\,\varphi(Y)\ll \varkappa N\,\overline{\gamma}_{2,2}+\varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
\end{aligned}$$

In order to prove $\Gamma_4\sim\bigl\{\ldots;0\bigr\}$ it suffices to expand with remainder $\mathcal{O}(\widehat{Y})$. The error is bounded by

$$\begin{aligned}
&\varkappa\,\mathbf{E}\,\bigl|\,h(Z,Y)\,d_1(Y)\,d_2(Z)\,d_3(Z)\,d_s(Y)\,\bigr|\\
&\qquad\ll \varkappa N^2\,\mathbf{E}\,\bigl|\,h(Z,Y)\,\bigr|\,\varphi(Z)\,\varphi(Y)\ll \varkappa N\,\overline{\gamma}_{2,2}+\varkappa N^3\,\overline{\gamma}_2\,\overline{\gamma}_{2,2}.
\end{aligned}$$

Proof of (8.76) *for* $I_{1,1,0}$ *and* $I_{1,0,1}$. We shall estimate $I_{1,0,1}$ only. Expanding in powers of $\widehat{Z}$ with remainder $\mathcal{O}(\widehat{Z}^2)$ and writing $\widehat{Z}^2$ as a double sum, we obtain

$$I_{1,0,1}\ll\Gamma_1\cup\Gamma_2\tag{8.199}$$

with

$$\Gamma_1 \sim \big\{\, \widehat{Z}, \widehat{Y};\; N\, f(Y)\, h(Z,Y)\, h^2(Z,X_1)\,\big\}, \qquad \Gamma_2 \sim \big\{\, \widehat{Z}, \widehat{Y};\; f(Y)\, h(Z,Y)\, d_1(Z)\, d_2(Z)\,\big\}.$$

In order to show $\Gamma_1 \sim \big\{\dots;0\big\}$ we use expansions with remainders $\mathcal{O}\Big(\sqrt{|\,h(Y,X_1)\,|}\,\Big)$ and $\mathcal{O}\big(\widehat{Y}(1)\big)$. The errors are bounded by

$$\begin{aligned}
&\varkappa N\, \mathbf{E}\,\big|\, f(Y)\, h(Z,Y)\,\big|\,\big|\, h(Y,X_1)\,\big|^{1/2}\, h^2(Z,X_1)\\
&\qquad \ll \varkappa N\, \mathbf{E}\,\big|\, f(Y)\, h(Z,Y)\,\big|\, \varphi_3^{1/6}(Y)\, \varphi_3^{1/3}(Z)\, \varphi^{1/2}(Z)\\
&\qquad \ll \varkappa N\, \overline{\beta}_3^{\,1/3}\, \mathbf{E}\,\Big(\mathbf{E}_Y\big|\, h(Z,Y)\,\big|^{3/2}\, \varphi_3^{1/4}(Y)\Big)^{2/3} \varphi_3^{1/3}(Z)\, \varphi^{1/2}(Z)\\
&\qquad \ll \varkappa N\, \overline{\beta}_3^{\,1/3}\, \overline{\gamma}_3^{\,1/6}\, \mathbf{E}\, \varphi_3^{1/3}(Z)\, \varphi(Z)\\
&\qquad \ll \varkappa N\, \overline{\beta}_3^{\,1/3}\, \overline{\gamma}_2^{\,1/3}\, \overline{\gamma}_3^{\,1/2}\, \overline{\gamma}_{2,2}^{\,1/3} \ll \varkappa \overline{\beta}_3^{\,2} + \overline{\gamma}_3 + N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}
\end{aligned} \tag{8.200}$$

(using (8.39), (8.175)), and

$$\begin{aligned}
&\varkappa N\, \mathbf{E}\,\big|\, f(Y)\, h(Z,Y)\,\big|\, h^2(Z,X_1)\,\big|\, d_s(Y)\,\big| \ll \varkappa N^{3/2}\, \mathbf{E}\,\big|\, f(Y)\, h(Z,Y)\,\big|\, \varphi(Z)\, \varphi^{1/2}(Y)\\
&\qquad \ll \varkappa N^{3/2}\, \overline{\gamma}_{2,2}^{\,1/2}\, \mathbf{E}\,\big|\, f(Y)\,\big|\, \varphi(Y) \ll \varkappa N^{3/2}\, \overline{\beta}_3^{\,1/3}\, \overline{\gamma}_2^{\,1/3}\, \overline{\gamma}_{2,2}^{\,5/6}\\
&\qquad \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N\, \overline{\gamma}_{2,2} + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}.
\end{aligned} \tag{8.201}$$

In order to prove $\Gamma_2 \sim \big\{\dots;0\big\}$ we expand with remainder $\mathcal{O}\big(\widehat{Y}\big)$. The error is bounded by (cf. (8.201))

$$\begin{aligned}
&\varkappa\, \mathbf{E}\,\big|\, f(Y)\, h(Z,Y)\,\big|\,\big|\, d_1(Z)\, d_2(Z)\, d_s(Y)\,\big|\\
&\qquad \ll \varkappa N^{3/2}\, \mathbf{E}\,\big|\, f(Y)\, h(Z,Y)\,\big|\, \varphi(Z)\, \varphi^{1/2}(Y) \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa N\, \overline{\gamma}_{2,2} + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}.
\end{aligned}$$

Proof of (8.76) *for* $I_{1,1,1}$, $I_{1,2,0}$ *and* $I_{1,0,2}$. Let us estimate $I_{1,1,1}$ only. Expanding with remainders $\mathcal{O}(\widehat{Z})$ and $\mathcal{O}(\widehat{Y})$, we have $I_{1,1,1} \sim \big\{\dots;0\big\}$ with an error bounded by

$$\begin{aligned}
&\varkappa\, \mathbf{E}\,\big|\, f(Y)\, f(Z)\, h(Z,Y)\, d_s(Z)\, d_r(Y)\,\big| \ll \varkappa N\, \mathbf{E}\,\big|\, f(Y)\, f(Z)\, h(Z,Y)\,\big|\, \varphi^{1/2}(Y) \varphi^{1/2}(Z)\\
&\qquad \ll \varkappa N\, \overline{\gamma}_3^{\,1/3} \Big(\mathbf{E}\,\big|\, f(X)\,\big|^{3/2} \varphi^{3/4}(X)\Big)^{4/3} \ll \varkappa N\, \overline{\beta}_3^{\,2/3}\, \overline{\gamma}_2^{\,1/3}\, \overline{\gamma}_3^{\,1/3}\, \overline{\gamma}_{2,2}^{\,1/3}\\
&\qquad \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa \overline{\gamma}_{2,2} + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}.
\end{aligned} \tag{8.202}$$

Proof of (8.76) *for* $I_{1,1,2}$ *and* $I_{1,2,1}$. In the case of $I_{1,1,2}$ we can expand with remainder $\mathcal{O}(\widehat{Z})$, and we have $I_{1,1,2} \sim \big\{\dots;0\big\}$ with an error bounded by

$$\begin{aligned}
&\varkappa\, \mathbf{E}\, f^2(Y)\,\big|\, f(Z)\, h(Z,Y)\, d_s(Z)\,\big| \ll \varkappa N^{1/2}\, \overline{\beta}_3^{\,1/3}\, \mathbf{E}\, f^2(Y) \Big(\mathbf{E}_Z\big|\, h(Z,Y)\,\big|^{3/2} \varphi^{3/4}(Z)\Big)^{2/3}\\
&\qquad \ll \varkappa N^{1/2}\, \overline{\beta}_3^{\,1/3}\, \mathbf{E}\, f^2(Y)\, \varphi_3^{1/3}(Y)\, \overline{\gamma}_2^{\,1/6}\, \overline{\gamma}_{2,2}^{\,1/6} \ll \varkappa N^{1/2}\, \overline{\beta}_3\, \overline{\gamma}_3^{\,1/3}\, \overline{\gamma}_2^{\,1/6}\, \overline{\gamma}_{2,2}^{\,1/6}\\
&\qquad \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa \overline{\gamma}_3 + \varkappa N^3 \overline{\gamma}_2\, \overline{\gamma}_{2,2}.
\end{aligned} \tag{8.203}$$

Proof of (8.76) *for* $I_{1,2,2}$. We have

$$I_{1,2,2} \ll \varkappa\, \mathbf{E}\, f^2(Z)\, f^2(Y)\, \big|\, h(Z,Y)\,\big| \ll \varkappa \overline{\beta}_3^{\,4/3} \overline{\gamma}_3^{\,1/3} \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa \overline{\gamma}_3. \tag{8.204}$$

Proof of (8.76) *for* $I_{2,0,0}$. Expanding with remainders $\mathcal{O}(\widehat{Z})$ and $\mathcal{O}(\widehat{Y})$, we obtain $I_{2,0,0} \sim \big\{\dots;0\big\}$ with an error bounded by

$$\begin{aligned} \varkappa\, \mathbf{E}\, h^2(Z,Y)\, \big|\, d_s(Z)\, d_r(Y)\,\big| &\ll \varkappa N\, \mathbf{E}\, h^2(Z,Y)\, \varphi^{1/2}(Y) \varphi^{1/2}(Z) \\ &\ll \varkappa N\, \overline{\gamma}_3^{\,2/3}\, \overline{\gamma}_2^{\,1/3}\, \overline{\gamma}_{2,2}^{\,1/3} \ll \varkappa \overline{\gamma}_3 + \varkappa N^3\, \overline{\gamma}_2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.205}$$

Proof of (8.76) *for* $I_{2,1,0}$ *and* $I_{2,0,1}$. In the case of $I_{2,0,1}$ we can expand with remainder $\mathcal{O}(\widehat{Z})$, and we have $I_{2,0,1} \sim \big\{\dots;0\big\}$ with an error bounded by

$$\begin{aligned} \varkappa\, \mathbf{E}\, \big|\, f(Y)\,\big|\, h^2(Z,Y)\, \big|\, d_s(Z)\,\big| &\ll \varkappa N^{1/2}\, \mathbf{E}\, \big|\, f(Y)\,\big|\, h^2(Z,Y)\, \varphi^{1/2}(Z) \\ &\ll \varkappa N^{1/2}\, \overline{\beta}_3^{\,1/3}\, \mathbf{E}\, \varphi_3^{2/3}(Z) \varphi^{1/2}(Z) \ll \varkappa N^{1/2}\, \overline{\beta}_3^{\,1/3}\, \overline{\gamma}_3^{\,2/3}\, \overline{\gamma}_2^{\,1/6}\, \overline{\gamma}_{2,2}^{\,1/6} \\ &\ll \varkappa \overline{\beta}_3^{\,2} + \varkappa \overline{\gamma}_3 + \varkappa N^3 \overline{\gamma}_2\, \overline{\gamma}_{2,2}. \end{aligned} \tag{8.206}$$

Proof of (8.76) *for* $I_{2,1,1}$. We have $I_{2,1,1} \sim \big\{\dots;0\big\}$ with an error bounded by

$$\varkappa\, \mathbf{E}\, \big|\, f(Z)\, f(Y)\,\big|\, h^2(Z,Y)\,\big| \ll \varkappa \overline{\beta}_3^{\,2/3} \overline{\gamma}_3^{\,2/3} \ll \varkappa \overline{\beta}_3^{\,2} + \varkappa \overline{\gamma}_3. \qquad \square \tag{8.207}$$

Lemma 8.5. *The Edgeworth correction* F_1 *satisfies*

$$\big|\, \widehat{F}_1(t)\,\big| \ll N^{-1/2}\, |t|^3\, (\beta_3^2 + \sigma^2\, \gamma_{2,2})^{1/2} \prod_{j\ge 1} (1 + 2\, t^2\, q_j^2/25)^{-1/4}. \tag{8.208}$$

Furthermore, for $s \ge 7$ *we have*

$$\int_{|t|\ge\lambda} \big|\, \widehat{F}_1(t)\,\big|\, \frac{dt}{|t|} \ll_s N^{-1/2}\, (\beta_3^2 + \sigma^2\, \gamma_{2,2})^{1/2}\, |q_s|^{-s/2}\, \lambda^{3-s/2}, \qquad \textit{for } \lambda > 0, \tag{8.209}$$

$$\int_{\mathbb{R}} \big|\, \widehat{F}_1(t)\,\big|\, \frac{dt}{|t|} \ll_s N^{-1/2}\, (\beta_3^2 + \sigma^2\, \gamma_{2,2})^{1/2}\, |q_s|^{-3}. \tag{8.210}$$

Proof. We can write $G \overset{\mathcal{D}}{=} (G_1 + \dots + G_5)/\sqrt{5}$. Using the linearity of ϕ_1, the relation (2.11) with $N = 5$ and (2.12), we have

$$T_0 \overset{\mathcal{D}}{=} \frac{1}{5} \sum_{1\le i<j\le 5} \phi(G_i, G_j) + \sum_{j=1}^{5} f_j(G_j) \tag{8.211}$$

with some statistics f_j. Using the explicit formula (4.1) for $\widehat{F}_1(t)$, sorting the terms of the statistic (8.211) and conditioning (cf., for example, the proof of (8.23)), we obtain

$$\big|\,\widehat{F}_1(t)\,\big| \ll N^{-1/2}\,|t|^3(\beta_3+\gamma_{2,3/2})\,\chi(t), \tag{8.212}$$

where

$$\chi(t) = \sup_{f_1,f_2}\Big|\,\mathbf{E}\ \mathrm{e}\big\{\,t\,\phi(G_1,G_2)/5+f_1(G_1)+f_2(G_2)\,\big\}\,\Big|. \tag{8.213}$$

Using $G_1-\overline{G}_1 \overset{\mathcal{D}}{=} \sqrt{2}\,G$ and $\mathbf{E}\ \mathrm{e}\big\{\,\phi(x,G_2)\,\big\}\ge 0$, we obtain

$$\begin{aligned}\chi^2(t) &\le \sup_{f_1,f_2}\ \mathbf{E}\,\Big|\mathbf{E}_{G_1}\ \mathrm{e}\big\{\,t\,\phi(G_1,G_2)/5+f_1(G_1)\,\big\}\,\Big|^2\\ &= \sup_{f_1}\ \mathbf{E}\ \mathrm{e}\big\{\,\sqrt{2}\,t\,\phi(G,G_2)/5+f_1(G_1)-f_1(\overline{G}_1)\,\big\}\\ &\le \mathbf{E}\,\Big|\mathbf{E}_{G_2}\ \mathrm{e}\big\{\,\sqrt{2}\,t\,\phi(G,G_2)/5\,\big\}\Big| = \mathbf{E}\ \mathrm{e}\big\{\,\sqrt{2}\,t\,\phi(G,\overline{G})/5\,\big\}.\end{aligned} \tag{8.214}$$

Recall that $\phi(x,y)=\sum_{j\ge 1} q_j\,x_j\,y_j$ (see (2.6)). Hence

$$\mathbf{E}\ \mathrm{e}\big\{\,\theta\,\phi(G,\overline{G})\,\big\} = \prod_{j\ge 1}(1+\theta^2\,q_j^2)^{-1/2}, \qquad \theta\in\mathbb{R}. \tag{8.215}$$

Estimating $\gamma_{2,3/2}^2 \ll \sigma^2\,\gamma_{2,2}$ and collecting (8.212)–(8.215), we obtain (8.208).

Using (8.208) and estimating $1+2t^2\,q_j^2/25\ge 1$, for $j>s$, and $q_s^2t^2 \ll 1+2t^2\,q_j^2/25$, for $j\le s$, we obtain (8.209). For the proof of (8.210) we use $1+q_s^2t^2 \ll 1+2t^2\,q_j^2/25$, for $j\le s$. $\square$

9. An extension of bounds to von Mises statistics. Applications

Assuming that the kernels ϕ and ϕ_1 are degenerate, consider the von Mises statistic

$$M = \frac{1}{2N}\sum_{1\le i,j\le N}\phi(X_i,X_j) + \frac{1}{\sqrt{N}}\sum_{1\le i\le N}\phi_1(X_i). \tag{9.1}$$

Introducing the function $\psi(x)=\big(\phi(x,x)-\nu\big)/2$ with $\nu=\mathbf{E}\,\phi(X,X)$, we can rewrite (9.1) as

$$M-\nu/2 = T+\frac{1}{N}\sum_{1\le i\le N}\psi(X_i) \tag{9.2}$$

with T defined by (1.1). In this section we shall extend the bounds to statistics of type (9.2) assuming that $\mathbf{E}\,\psi(X)=0$ and $\varrho \overset{\text{def}}{=} \mathbf{E}\,\psi^2(X)<\infty$.

Similarly to the case of T, we can represent the kernel ϕ (respectively ϕ_1 and ψ) as bilinear (respectively linear) function defined on $\mathbb{R}^\infty$. However in this case we have to assume that $\mathbb{R}^\infty$ has an additional coordinate since ψ can be linearly independent of ϕ_1 and of the eigenfunctions of $\mathbb{Q}$. To fix notation, we shall assume that $\mathbb{R}^\infty$ consists of vectors $x = (x_{-1}, x_0, x_1, \dots)$. Then all representations and results of Section 2 concerning ϕ and ϕ_1 still hold, and for ψ we have $\psi(x) = \langle b, x \rangle$ with some $b = (b_{-1}, b_0, b_1, \dots)$ such that $\sum_{j \geq -1} b_j^2 < \infty$. Write $\psi_0(x) = \sum_{j \geq 0} b_j x_j$.

Introduce the function F_* of bounded variation (provided that $q_3 \neq 0$) with the Fourier–Stieltjes transform

$$\widehat{F}_*(t) = \frac{it}{\sqrt{N}}\, \mathbf{E}\, \psi(G)\, \mathrm{e}\big\{ tT_0 \big\} = \frac{it}{\sqrt{N}}\, \mathbf{E}\, \psi_0(G)\, \mathrm{e}\big\{ tT_0 \big\}$$

and such that $F_*(-\infty) = 0$. Below we shall show that (see Lemma 9.3)

$$\widehat{F}_*(t) = \frac{(it)^2}{\sqrt{N}}\, \mathbf{E}\, \psi(X)\, (\phi_1(X) + \phi(X, G))\, \mathrm{e}\big\{ tT_0 \big\}. \tag{9.3}$$

Notice that $F_* = 0$ whenever $\phi_1 = 0$.

Write $H_1 = F_1 + F_*$, and let H denote the distribution function of $M - \nu/2$. Define

$$\delta_N = \sup_x \big| \delta_N(x) \big|, \qquad \delta_N(x) = H(x) - F_0(x) - H_1(x).$$

Theorem 9.1.

i) Assume that $q_{13} \neq 0$. Then we have

$$\delta_N \leq \frac{C}{N} \Big(\frac{\beta_4}{\sigma^4} + \frac{\beta_3^2}{\sigma^6} + \frac{\gamma_3}{\sigma^3} + \frac{\gamma_{2,2}}{\sigma^4} + \frac{\varrho}{\sigma^2} \Big), \qquad \text{where } C \leq \exp\Big\{ \frac{c\sigma}{|q_{13}|} \Big\}. \tag{9.4}$$

ii) Assume that (1.9) is fulfilled and $q_9 \neq 0$. Then (9.4) holds with $C \leq \exp\Big\{ \dfrac{c\sigma}{|q_9|} \Big\}$.

Proof. We shall use the following estimates. Write

$$\xi = \frac{t}{N} \sum_{1 \leq j \leq N} \psi(X_j), \qquad \zeta = \frac{t}{N} \sum_{1 \leq j \leq N} \psi(G_j). \tag{9.5}$$

Expanding with remainder $\mathcal{O}(\xi)$, splitting the sum ξ in parts and conditioning (cf. the proof of (8.23)), we have

$$\Big| \mathbf{E}\, \mathrm{e}\big\{ tT + \xi \big\} - \mathbf{E}\, \mathrm{e}\big\{ tT \big\} - i\, \mathbf{E}\, \xi\, \mathrm{e}\big\{ tT \big\} \Big| \ll \varkappa N^{-1} t^2 \varrho. \tag{9.6}$$

Proceeding similarly as in the proof of Lemma 8.2, we obtain

$$\Big| \widehat{F}_*(t) - i\, \mathbf{E}\, \zeta\, \mathrm{e}\big\{ \mathcal{R}_{[1,N]} T \big\} \Big| \ll \varkappa N^{-1} t^2 (\varrho + \sigma^2). \tag{9.7}$$

Applying the Bergström type identity (8.7) with $S = \xi\ \mathrm{e}\{tT\}$ and proceeding similarly to the proof of Lemma 8.3, we get (we omit cumbersome technical details)

$$\Big| \mathbf{E}\, \xi\, \mathrm{e}\big\{ tT \big\} - \mathbf{E}\, \zeta\, \mathrm{e}\big\{ \mathcal{R}_{[1,N]} T \big\} \Big| \ll \varkappa N^{-1} (t^2 + t^4) (\varrho + \beta_4 + \beta_3^2 + \gamma_3 + \gamma_{2,2} + \gamma_2\, \gamma_{2,2}). \tag{9.8}$$

Arguments similar to the proof of Lemma 8.5 allow to prove

$$\bigl|\,\widehat{F}_*(t)\,\bigr| \ll N^{-1/2}\,|t|\,\varrho^{1/2}\,\prod_{j\geq 1}(1+2t^2\,q_j^2/25)^{-1/4}, \tag{9.9}$$

and, for $s\geq 3$,

$$\int_{|t|\geq\lambda}\bigl|\,\widehat{F}_*(t)\,\bigr|\,\frac{dt}{|t|} \ll_s N^{-1/2}\,\varrho^{1/2}\,|q_s|^{-s/2}\,\lambda^{1-s/2}, \qquad \lambda>0, \tag{9.10}$$

$$\int_{\mathbb{R}}\bigl|\,\widehat{F}_*(t)\,\bigr|\,\frac{dt}{|t|} \ll_s N^{-1/2}\,\varrho^{1/2}\,|q_s|^{-1}. \tag{9.11}$$

The estimates (9.6)–(9.11) allow to proceed similarly as in the proof of Theorem 1.1, using a lemma similar to Lemma 4.1. Proving such a lemma, we have to apply Lemma 4.2 to the distribution function H. This is possible since the statistic $M-\nu/2$ is a statistic of type (4.8). The estimates (9.10)–(9.11) allow to apply the Fourier inversion (4.14) to the function F_*. As a result (cf. integral J in (4.18)) we arrive at

$$\int_{-N\,t_*}^{N\,t_*}\bigl|\,\widehat{H}(t)-\widehat{F}_0(t)-\widehat{H}_1(t)\,\bigr|\,\frac{dt}{|t|}.$$

Here, however, we have $\widehat{H}(t) = \mathbf{E}\,\mathrm{e}\bigl\{\,tT+\xi\,\bigr\}$, and

$$\begin{aligned}\bigl|\,\widehat{H}(t)-\widehat{F}_0(t)-\widehat{H}_1(t)\,\bigr| \leq{}& \bigl|\,\widehat{F}(t)-\widehat{F}_0(t)-\widehat{F}_1(t)\,\bigr| \\ &+\Bigl|\,\mathbf{E}\,\mathrm{e}\bigl\{\,tT+\xi\,\bigr\}-\mathbf{E}\,\mathrm{e}\bigl\{\,tT\,\bigr\}-i\,\mathbf{E}\,\xi\,\mathrm{e}\bigl\{\,tT\,\bigr\}\Bigr| \\ +\Bigl|\,\widehat{F}_*(t)-i\,\mathbf{E}\,\zeta\,\mathrm{e}\bigl\{\,\mathcal{R}_{[1,N]}T\,\bigr\}\Bigr| &+\Bigl|\,\mathbf{E}\,\xi\,\mathrm{e}\bigl\{\,tT\,\bigr\}-\mathbf{E}\,\zeta\,\mathrm{e}\bigl\{\,\mathcal{R}_{[1,N]}T\,\bigr\}\Bigr|. \end{aligned} \tag{9.12}$$

Therefore, using (9.6)–(9.8), we can proceed as in the proof of Lemma 4.1. As a final result we get bounds similar to those of Theorem 1.1, with the additional summand ϱ/σ^2. $\square$

Theorem 9.2. *Assume that $q_{d+1}=0$ and $q_d\neq 0$, for some $d\geq 9$, and that the weights a_j in* (1.6) *satisfy $a_0=0$ and $a_j=0$, for $j>d$. Then we have*

$$\Delta_N \ll_d \frac{C}{N}\Bigl(\,\frac{\beta_4}{\sigma^4}+\frac{\beta_3^2}{\sigma^6}+\frac{\gamma_{2,2}}{\sigma^4}+\frac{\beta_2^{3/2}\,\gamma_{2,2}}{\sigma^7}\,\Bigr), \qquad \text{where } C\leq \exp\Bigl\{\,\frac{c\sigma}{|q_d|}\,\Bigr\}. \tag{9.13}$$

Proof. The assumptions of the theorem allow to assume that the random vectors X and G take values in $\mathbb{R}^d$, have mean zero and identity covariances, and that

$$\phi(x,y) = \bigl\langle \mathbb{Q}x, y\bigr\rangle = \sum_{j=1}^{d} q_j\,x_j\,y_j, \qquad \phi_1(x) = \bigl\langle \mathbb{Q}x, a\bigr\rangle = \sum_{j=1}^{d} a_j\,x_j,$$

where $\mathbb{Q}:\mathbb{R}^d\to\mathbb{R}^d$ is linear diagonal operator with eigenvalues q_j. Introducing $S_N = N^{-1/2}(X_1+\cdots+X_N)$ we can write

$$T = T_* - \frac{1}{N}\sum_{j=1}^N \psi(X_j), \qquad 2T_* = \big\langle \mathbb{Q}(S_N-v),(S_N-v)\big\rangle - \big\langle \mathbb{Q}v,v\big\rangle - \nu,$$

where $2\,\psi(x) = \big\langle \mathbb{Q}x,x\big\rangle - \nu$, $v=\mathbb{Q}^{-1}\,a$ and $\nu = \sum_{j=1}^d q_j$. Similarly

$$2\,T_0 = \big\langle \mathbb{Q}(G-v),(G-v)\big\rangle - \big\langle \mathbb{Q}v,v\big\rangle - \nu.$$

We have $\Delta_N \le \Delta_{N,1}+\Delta_{N,2}$ where

$$\Delta_{N,1} = \sup_x \big|\,\mathbf{P}\big\{T_*\le x\big\} - \mathbf{P}\big\{T_0\le x\big\} - F_*(x) - F_1(x)\,\big|,$$
$$\Delta_{N,2} = \sup_x \big|\,\mathbf{P}\big\{T\le x\big\} - \mathbf{P}\big\{T_*\le x\big\} + F_*(x)\,\big|.$$

To estimate $\Delta_{N,1}$ we shall apply Theorem 1.5 in Bentkus and Götze (1996b). Let $\mathbb{C}$ be the diagonal operator with eigenvalues $|q_j|$. Then

$$\Delta_{N,1} \ll_d C\,N^{-1}\,\theta^{-4}\,\mathbf{E}\,|\mathbb{C}^{1/2}X|^4\big(1+\theta^{-3}|\mathbb{C}^{1/2}v|^3\big), \qquad C\le \exp\Big\{\frac{c\,\theta^2}{|q_d|}\Big\}, \tag{9.14}$$

where $\theta^2 = \sum_{j=1}^d |q_j|$. However, using

$$\theta^4 \asymp_d \sigma^2, \qquad |\mathbb{C}^{1/2}X|^2 \le |q_d|^{-1}\,\mathbf{E}_{\overline{X}}\,\phi^2(X,\overline{X}), \qquad |\mathbb{C}^{1/2}v|^2\le |q_d|^{-1}\beta_2$$

and (9.14), we see that $\Delta_{N,1}$ is bounded from above by the right hand side of (9.13).

The estimation of $\Delta_{N,2}$ is similar to the proof of Theorem 9.1. Using the notation (9.5) and writing $tT_* = tT+\xi$ we reduce the estimation of $\Delta_{N,2}$ to the estimation of $\widehat{\Delta}_{N,2} = \big|\,\mathbf{E}\ \mathrm{e}\{tT\} - \mathbf{E}\ \mathrm{e}\{tT+\xi\} + \widehat{F}_*(t)\,\big|$. It is easy to see that $\widehat{\Delta}_{N,2}$ is bounded from above by the right hand side of (9.12). Thus, using $\varrho \le q_d^{-2}\gamma_{2,2}\le C\,\gamma_{2,2}/\sigma^2$, we see that $\Delta_{N,2}$ can be estimated by the right hand side of (9.13). □

Lemma 9.3. *Let $f:\mathbb{R}\to\mathbb{C}$ be a bounded infinitely many times differentiable function with bounded derivatives. Then we have*

$$\mathbf{E}\,\psi(G)\,f(T_0) = \mathbf{E}\,\psi(X)\,\big(\phi_1(X)+\phi(X,G)\big)\,f'(T_0) \tag{9.15}$$

provided that $\beta_2<\infty$, $\varrho<\infty$ and $\gamma_{2,2}<\infty$.

Proof. Using $G \overset{\mathcal{D}}{=} N^{-1/2}(G_1+\cdots+G_N)$, the i.i.d. assumption, representations (2.12) and (2.9), the linearity of ψ and the symmetry of T_0, we obtain

$$\mathbf{E}\,\psi(G)\,f(T_0) = \sqrt{N}\,\mathbf{E}\,\psi(G_1)\,f\big((2\,N)^{-1}\,\phi_0(G_1)+L(G_1)+W\big), \tag{9.16}$$

where $L(Z) = N^{-1/2}\,\phi(Z) + N^{-1}\sum_{j=2}^{N}\phi(Z, G_j)$, and the statistic W is defined as the right hand side of (2.12) just omitting summands depending on G_1. Expanding in (9.16) with remainders $\mathcal{O}\big((2N)^{-1}\phi_0(G_1)\big)$ and $\mathcal{O}\big(L^2(G_1)\big)$, we get

$$\mathbf{E}\,\psi(G)\,f(T_0) = \sqrt{N}\,\mathbf{E}\,\psi(G_1)\,L(G_1)\,f'\big(W\big) + \mathcal{O}(N^{-1/2}) \tag{9.17}$$

since $\mathbf{E}\,\psi(G_1) = 0$. The equality of the covariances of G_1 and X yields

$$\mathbf{E}\,\psi(G_1)\,L(G_1)\,f'\big(W\big) = \mathbf{E}\,\psi(X)\,L(X)\,f'\big(W\big). \tag{9.18}$$

Writing now $W = T_0 - L(G_1) - (2N)^{-1}\phi_0(G_1)$, expanding with remainders $\mathcal{O}\big(L(G_1)\big)$ and $\mathcal{O}\big((2N)^{-1}\phi_0(G_1)\big)$ and using $L(X) \overset{\mathcal{D}}{=} N^{-1/2}\phi_1(X) + N^{-1}\sqrt{N-1}\,\phi(X, G)$, we obtain

$$\sqrt{N}\,\mathbf{E}\,\psi(X)\,L(X)\,f'\big(W\big) = \mathbf{E}\,\psi(X)\,\big(\phi_1(X) + \phi(X, G)\big)\,f'\big(T_0\big) + \mathcal{O}(N^{-1/2}). \tag{9.19}$$

Collecting (9.16)–(9.19) and passing to the limit as $N \to \infty$, we get (9.15). $\square$

Estimates of the convergence rate in the CLT for conic sections in $\mathbb{R}^d$ and Hilbert spaces. Let $\mathbb{R}^d$ denote the standard d-dimensional Euclidean space with scalar product $\langle x, y\rangle$ and norm $|x|^2 = \langle x, x\rangle$. In the case $d = \infty$ in this subsection we shall understand $\mathbb{R}^d$ as the Hilbert space l_2. Consider i.i.d. random vectors $Y, Y_1, \dots, Y_N$ taking values in $\mathbb{R}^d$ such that $\mathbf{E}\,Y = 0$. Write $S_N = N^{-1/2}(Y_1 + \dots + Y_N)$. Let $\mathbb{A} : \mathbb{R}^d \to \mathbb{R}^d$ be a symmetric bounded linear operator. Introduce the quadratic form $\mathbb{A}[x] = \langle \mathbb{A}x, x\rangle$. We are interested in approximations of the probability $\mathbf{P}\,\big\{\,\mathbb{A}[S_N - w] \le x\,\big\}$, $w \in \mathbb{R}^d$. Without loss of generality we can replace the aforementioned probability by $\mathbf{P}\,\big\{\,M - \nu/2 \le x\,\big\}$, where $M - \nu/2 = \mathbb{A}[S_N] - 2\langle \mathbb{A}S_N, w\rangle - \nu$ is the statistic (9.2) with

$$\phi(x, y) = 2\,\langle \mathbb{A}x, y\rangle, \quad \phi_1(x) = -2\,\langle \mathbb{A}x, w\rangle, \quad \psi(x) = \langle \mathbb{A}x, x\rangle - \nu, \quad \nu = \mathbf{E}\,\langle \mathbb{A}Y, Y\rangle.$$

We shall use the following fact: for any bounded linear operators $\mathbb{A}, \mathbb{B} : \mathbb{R}^d \to \mathbb{R}^d$ such that $\mathbb{A} \ge 0$, the collections of non-zero eigenvalues of $\mathbb{A}\mathbb{B}$, $\mathbb{B}\mathbb{A}$ and $\sqrt{\mathbb{A}}\,\mathbb{B}\,\sqrt{\mathbb{A}}$ coincide.

Introducing the covariance operator $\mathbb{C}$ of Y, we have $\nu = \operatorname{Tr}\,\mathbb{C}^{1/2}\,\mathbb{A}\mathbb{C}^{1/2} = \operatorname{Tr}\,\mathbb{A}\mathbb{C}$.

Let u_j, $j \ge 1$, be a complete orthonormal system of eigenvectors of the operator $2\,\mathbb{C}^{1/2}\,\mathbb{A}\mathbb{C}^{1/2}$ with the corresponding eigenvalues q_j. Consider the system of functions

$$e_j(x) = \langle \mathbb{C}^{-1/2}u_j, x\rangle = \frac{2}{q_j}\,\langle \mathbb{A}\mathbb{C}^{1/2}u_j, x\rangle, \qquad q_j \ne 0,$$

in $L^2(\mathbb{R}^d, \mathcal{L}(Y))$. It is easy to verify that this system is an orthonormal system of eigenfunctions of the operator $\mathbb{Q}$ associated with the kernel $\phi(x, y)$ in $L^2(\mathbb{R}^d, \mathcal{L}(Y))$, and that the corresponding eigenvalues are q_j. Moreover, we have $\sigma^2 = 4\,\mathrm{Tr}\,(\mathbb{A}\mathbb{C})^2 = \sum_{j\ge 1} q_j^2$.

Obviously,

$$\gamma_s \ll_s \mathbf{E}\,\big|\big\langle \mathbb{A}Y, \overline{Y}\big\rangle\big|^s, \qquad \gamma_{2,2} \ll \mathbf{E}\,\big\langle \mathbb{A}\mathbb{C}\mathbb{A}Y, Y\big\rangle^2,$$
$$\beta_s \ll_s \mathbf{E}\,\big|\big\langle \mathbb{A}Y, w\big\rangle\big|^s, \qquad \varrho \ll \mathbf{E}\,\big\langle \mathbb{A}Y, Y\big\rangle^2.$$

Thus we can apply to δ_N the bounds of Theorem 9.1. In particular, if $\mathbb{A} = \mathbb{I}$ then we have

$$\delta_N \ll \frac{C}{N}\Big(\frac{\mathbf{E}\,|Y|^4}{\sigma^2} + \frac{(\,\mathbf{E}\,|\langle Y, w\rangle|^3)^2}{\sigma^6} + \frac{\mathbf{E}\,\langle Y, w\rangle^4}{\sigma^4}\Big)$$

with C as in Theorem 9.1 and q_s being the eigenvalues of $2\mathbb{C}$. Comparing with the bounds in Bentkus and Götze (1996b), the bounds for δ_N implied by Theorem 9.1 show an improved dependence on w and moments.

Introduce a centered Gaussian random vector Z taking values in $\mathbb{R}^d$ and such that the covariances of Y and Z are equal. Then we can write

$$T_0 \overset{\mathcal{D}}{=} M_0 \overset{\mathrm{def}}{=} \big\langle \mathbb{A}Z, Z\big\rangle - \big\langle \mathbb{A}Z, w\big\rangle - \nu.$$

Furthermore, we have $H_1 = F_1 + F_*$, where (using (9.3))

$$\widehat{F}_1(t) = \frac{4\,(it)^3}{3\sqrt{N}}\,\mathbf{E}\,\big\langle \mathbb{A}Y, Z - w\big\rangle^3\, \mathrm{e}\big\{t\,M_0\big\},$$
$$\widehat{F}_*(t) = \frac{2\,(it)^2}{\sqrt{N}}\,\mathbf{E}\,\big\langle \mathbb{A}Y, Y\big\rangle\big\langle \mathbb{A}Y, Z - w\big\rangle\, \mathrm{e}\big\{tT_0\big\}.$$

Approximations of multiple Wiener–Itô integrals. Consider a probability distribution μ on a measurable space $(\mathcal{X}, \mathcal{B})$. Let $I_n(f)$ be the n-fold multiple Wiener–Itô integral of a symmetric function $f : \mathcal{X}^n \to \mathbf{R}$ with respect to an orthogonal Gaussian random measure with control measure μ (see Itô 1951, Major 1981). The Itô formula yields

$$\frac{1}{2}\,I_2(\phi) + I_1(\phi_1) \overset{\mathcal{D}}{=} \frac{1}{2}\sum_{j\ge 1} q_j\,(\eta_j^2 - 1) + \sum_{j\ge 0} a_j\,\eta_j = T_0.$$

Thus, assuming that the sample $X_1, \dots, X_N$ is taken from the distribution μ, i.e., $\mathcal{L}(X) = \mu$, we can use the statistic T defined by (1.1) to approximate the random variable $\dfrac{1}{2}\,I_2(\phi) + I_1(\phi_1)$ in distribution. In particular, Theorem 1.1 shows that approximating $I_2(\phi)$ by a corresponding statistic T of type (1.1), we get an error of the order $\mathcal{O}(N^{-1})$, provided that $\phi \in L^4$ and $q_{13} \ne 0$.

Example 9.4. Let X denote a random variable uniformly distributed on $\mathcal{X} = [0,1]$. For $0 \le x \le 1$ and $\alpha < 1$ consider the functions

$$f(x,y) = |x-y|^{-\alpha}, \qquad f_1(x) = \mathbf{E}\, f(x,X) = (1-\alpha)^{-1}\big(x^{1-\alpha} + (1-x)^{1-\alpha}\big).$$

Introduce the degenerate kernel $\phi(x,y) = f(x,y) - f_1(x) - f_1(y) + \mathbf{E}\, f(X,\overline{X})$. Since the function f_1 is bounded, we have

$$\gamma_s < \infty \Longleftrightarrow \int_0^1 \int_0^1 |x-y|^{-s\alpha}\, dx\, dy < \infty \Longleftrightarrow \alpha < \frac{1}{s},$$
$$\gamma_{2,2} < \infty \Longleftrightarrow \int_0^1 \Big(\int_0^1 |x-y|^{-2\alpha}\, dx\Big)^2 dy < \infty \Longleftrightarrow \alpha < \frac{1}{2}.$$

Consequently, the bounds of Theorem 1.1 are applicable when $\alpha < 1/3$, whereas the bounds (1.12) depending on γ_4 are applicable only when $\alpha < 1/4$.

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