Cellular algebras:
inflations and Morita equivalences

by

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1 Introduction

Cellular algebras have been introduced recently by Graham and Lehrer [5, 6] as a convenient axiomatization of all of the following algebras, each of them containing information on certain classical algebraic or finite groups: group algebras of symmetric groups in any characteristic, Hecke algebras of type A or B (or more generally, Ariki Koike algebras), Brauer algebras, Temperley–Lieb algebras, (q–)Schur algebras, ... The problem to determine a parameter set for or even to construct bases of simple modules is in this way reduced (but of course not solved in general) to questions of linear algebra.

The present paper has two aims. First, we make explicit an inductive construction of cellular algebras which has as input data of linear algebra and which produces in fact all cellular algebras (but no other ones). This is what we call 'inflation'. This construction also exhibits close relations between several of the above algebras as can be seen from the computations in [6]. Among the consequences of the construction is a natural way to generalize Hochschild cohomology. Another consequence is a construction of certain idempotents which is used in the second part of the paper.

The second aim is to study Morita equivalences of cellular algebras. Since the input of many constructions of representation theory of finite dimensional algebras is a basic algebra, it is useful to know whether any finite dimensional cellular algebra is Morita equivalent to a basic one by a Morita equivalence preserving the cellular structure. It turns out that the answer is yes if the underlying field has characteristic different from two. But there are counterexamples in case of characteristic two, or more generally for any ring in which 2 is not invertible. This also tells us that the notion of 'cellular' cannot be defined in terms of the module category only. However, in any characteristic we find some useful Morita equivalences which are compatible with cellular structures.

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In more detail, the contents of this paper is as follows: In section two we recall two equivalent definitions of cellular. In section three we define inflations, and in section four we explain how to produce cellular algebras by inflations. Section five illustrates this construction by examples from [6]. In section six we look at Morita equivalences which are compatible with given involutory anti–automorphisms, and we prove some technical results about the existence of such Morita equivalences. In particular, proposition 6.10 gives a sufficient criterion for a Morita equivalence to send a cell ideal to a cell ideal. More convenient for practical purposes is another sufficient criterion, given in corollary 6.12. In sections seven and eight we answer the question of whether any cellular algebra is Morita equivalent to a basic one in a way which preserves the cellular structure. Section seven is negative: it contains our counterexample in case of characteristic two and also an example of a full matrix algebra which is not cellular with respect to a certain involution. Section eight is positive; here we give the main result in case of characteristic different from two (using the techniques developed in section six). Among the consequences of the main result is that over fields of characteristic different from two any endomorphism ring of a projective module over a cellular algebra inherits the cellular structure of the algebra. Moreover, the maximal semisimple quotient of a cellular algebra always inherits the cellular structure (in any characteristic).

2 Definitions

Here we recall the original definition (due to Graham and Lehrer) and our equivalent definition [8] which we are going to use later on.

In the following by an A–module we mean a finitely generated A–module. By $A - mod$ or $mod - A$ we denote the category of finitely generated left or right A–modules, respectively.

The definition by Graham and Lehrer is as follows.

Definition 2.1 *(Graham and Lehrer, [6]) Let R be a noetherian commutative integral domain. An associative R–algebra A is called a* **cellular algebra** *with cell datum (I, M, C, i) if the following conditions are satisfied:*

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R–basis $C_{S,T}^{\lambda}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R–linear anti–automorphism of A which satisfies $i^2 = id$ and which sends $C_{S,T}^{\lambda}$ to $C_{T,S}^{\lambda}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC_{S,T}^{\lambda}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S) C_{U,T}^{\lambda}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in R$ do not depend on T.

In the following, we call the basis $\{C_{S,T}^{\lambda}\}$ a **cellular basis** for the cellular algebra A, and an R–linear anti–automorphism i of A with $i^2 = id$ an **involution**. By $J_{\leq\lambda}$ (or

$J_{<\lambda}$, respectively) we denote the ideal of A which has as R–basis all $C^{\mu}_{S,T}$ with $\mu \leq \lambda$ (with $\mu < \lambda$).

An equivalent definition which does not use bases is as follows:

Definition 2.2 *([8]) Let A be an algebra over a noetherian commutative integral domain R. Assume there is an involution i on A. A two–sided ideal J in A is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and there is an isomorphism of A–bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i–image of Δ) making the following diagram commutative:*

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta) \end{tikzcd}$$

The algebra A (with the involution i) is called **cellular** *if and only if there is a finite chain of two sided ideals $0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ (called* **cell chain***), each of them fixed by i, such that for each j the quotient J_j/J_{j-1} is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.*

Clearly, the left ideal Δ in the definition can be replaced by any left module Δ. Then $i(\Delta)$ has to be understood as the right module corresponding to Δ under the antiequivalence $A - mod \rightarrow mod - A$ defined by i.

3 Inflations

In this section we define inflations. In the next section we will show that this construction, if applied inductively to finitely many copies of the ground ring R, produces precisely all cellular algebras over R. For some examples discussed in [6] our construction (but not our result) is already implicit in the computations in [6] as we will make clear later on by discussing some of these examples. Throughout, the ground ring R is a noetherian commutative integral domain.

The inductive construction will have two ingredients. Firstly, inflating an R–algebra (which for the construction is just R itself) along a free R–module of finite rank produces another R–algebra (possibly without unit). Secondly, inflating a cellular algebra along another one, which has been produced in step one, gives another cellular algebra (larger than the one we started with).

3.1 Inflating algebras along free R–modules

We note that in the special case of Brauer's algebras, the following construction of the 'sections' of a cellular algebra has already been discovered by Hanlon and Wales in [7] (before cellular algebras were defined).

Given an R–algebra B, a finitely generated free R–module V, and a bilinear form $\varphi : V \otimes_R V \to B$ with values in B, we define an associative algebra (possibly without unit) $A(B, V, \varphi)$ as follows: As an R–module, A equals $V \otimes_R V \otimes_R B$. The multiplication is defined on basis elements as follows: $(a \otimes b \otimes x) \cdot (c \otimes d \otimes y) := a \otimes d \otimes x\varphi(b, c)y$.

We need an additional property, namely an involution on A: Assume there is an involution i on B. Assume moreover that φ satisfies $i(\varphi(v, w)) = \varphi(w, v)$. Then we can define an involution j on A by putting $j(a \otimes b \otimes x) = b \otimes a \otimes i(x)$.

Proposition 3.1 *This definition makes A into an associative R–algebra (possibly without unit), and j is an involutory anti–automorphism of A.*

Proof. By the associativity of multiplication in B,

$$((a\otimes b\otimes x)(c\otimes d\otimes y))(e\otimes f\otimes z) = (a\otimes d\otimes x\varphi(b,c)y)(e\otimes f\otimes z) = a\otimes f\otimes x\varphi(b,c)y\varphi(d,e)z$$

equals

$$(a\otimes b\otimes x)((c\otimes d\otimes y)(e\otimes f\otimes z)) = (a\otimes b\otimes x)(c\otimes f\otimes y\varphi(d,e)z) = a\otimes f\otimes x\varphi(b,c)y\varphi(d,e)z).$$

Hence the multiplication we defined on A is associative. Clearly, the left and the right R–action coincide. Similar computations show that j is an involutory anti–automorphism. ∎

If V has rank one and the image of φ contains the unit element of B, then A is clearly isomorphic to B. Otherwise, A need not have a unit element, but it may contain idempotents (see below).

We will use this construction in particular in the case that B is just R itself.

3.2 Inflating an algebra along another one

Suppose we are given an algebra B (maybe without unit) and an algebra C (with unit). We want to define an algebra structure on $A := B \oplus C$ which extends the given structures and which makes B into a two–sided ideal such that A/B becomes isomorphic to C. Multiplication is defined by fixing the eight summands of a multiplication map $(B \oplus C) \otimes_R (B \oplus C) \to (B \oplus C)$. In order to make B into an ideal we put the summands $B \otimes_R B \to C$, $C \otimes_R B \to C$ and $B \otimes_R C \to C$ all zero. The summands $C \otimes_R C \to C$ and $B \otimes_R B \to B$ are defined to be the given multiplications on C and B, respectively. Thus we have to choose three bilinear maps $\delta : C \otimes_R C \to B$, $\beta : B \otimes_R C \to B$ and $\gamma : C \otimes_R B \to B$. Then multiplication in A is defined by $(b_1 + c_1)(b_2 + c_2) = b_1b_2 + \beta(b_1, c_2) + \gamma(c_1, b_2) + \delta(c_1, c_2) + c_1c_2$.

Proposition 3.2 *This multiplication is associative if and only the following conditions are satisfied:*

(I) β is a homomorphism of left B–modules,

(II) γ is a homomorphism of right B–modules,

(III) for all b in B and c_1, c_2 in C there is an equality: $\beta(\beta(b, c_1), c_2) = b\delta(c_1, c_2) + \beta(b, c_1c_2)$,

(IV) for all b in B and c_1, c_2 in C there is an equality: $\gamma(c_1, \gamma(c_2, b)) = \gamma(c_1c_2, b) + \delta(c_1, c_2)b$,

(V) for all c_1, c_2, c_3 in C there is an equality: $\delta(c_1c_2, c_3) + \beta(\delta(c_1, c_2), c_3) = \delta(c_1, c_2c_3) + \gamma(c_1, \delta(c_2, c_3))$,

(VI) for all b_1, b_2 in B and c in C there is an equality: $\beta(b_1, c)b_2 = b_1\gamma(c, b_2)$,

(VII) for all c_1, c_2 in C and b in B there is an equality: $\beta(\gamma(c_1, b_1), c_2) = \gamma(c_1, \beta(b, c_2))$.

The **proof** is, of course, by direct computation. ∎

We need another condition in order to make sure that A has a unit element which is mapped to the unit element $1(C)$ of C by the quotient homomorphism.

(U) There exists an element b in B such that $b + 1(C)$ is a unit element in A.

Proposition 3.3 *An element b in B satisfies (U) if and only if it satisfies the following two equations:*

(U1) for all c in C there is an equality: $\delta(1, c) + \beta(b, c) = 0 = \delta(c, 1) + \gamma(c, b)$,

(U2) for all c in B there are equalities: $(b - 1)d = \gamma(1, d)$ and $d(b - 1) = \beta(d, 1)$.

Proof. Of course, $b + 1(C)$ is a unit element in A if and only if for all $d + c$ with d in B and c in C there are equalities: $(*)(b + 1(C))(d + c) = d + c = (d + c)(b + 1(C))$.

Plugging in $d = 0$ gives $\beta(b, c) + \delta(1, c) = 0$ and also $\delta(c, 1) + \gamma(c, b) = 0$, which is (U1). Using (U1), the equalities $(*)$ imply (U2).

Conversely, (U1) and (U2) imply that $b + 1(C)$ is a (hence the) unit element of A. ∎

3.3 Inflations

In order to produce cellular algebras we need additional assumptions which we introduce now:

Let C be any algebra and B be one of the form $V \otimes V \otimes R$ produced in step one of the construction. Let $A = C \oplus B$ be as in step two. In order to make B into a cell ideal we require the following two conditions:

(C) for any $c \in C$ and $a \otimes b \otimes r \in B$, the product $\gamma(c, a \otimes b \otimes r)$ lies in $V \otimes b \otimes R$, that is, is a linear combination of basis elements of the form $a' \otimes b \otimes 1$ (with fixed b at the second place); dually for any $c \in C$ and $a \otimes b \otimes r \in B$, the product $\beta(a \otimes b \otimes r, c)$ lies in $a \otimes V \otimes R$.

If this condition is satisfied, then we call A an **inflation** of C along B. Moreover, the result of iterated application of this construction also will be called an inflation (or, more precisely, an iterated inflation).

Proposition 3.4 *An inflation of a cellular algebra is cellular again. In particular, an iterated inflation of n copies of R is cellular with a cell chain of length n.*

More precisely, the second statement means: Start with C a full matrix ring over R and B an inflation of R along a free R–module. Form an inflation A. Then choose another B which is R inflated along some free R–module, and form a new A which is an inflation of the old A along the new B, and so on. Then after n steps we have produced a cellular algebra A with a cell chain of length n.

The proposition remains valid, if, more generally, one chooses for B any cellular algebra such that a modified condition (C) is satisfied. Here the multiplicative structure of C must be assumed to preserve the given cell chain of B. More precisely, the ideals in the given cell chain must be ideals in C as well and the sections of the chain must satisfy the defining condition of a cell ideal. This is by the modified version of condition (C), whose formulation we skip. In this case, we will call C an inflation of B. Iterating this construction (always we produce iterated inflations of cellular algebras, and such an iterated inflation then is cellular again.

4 Cellular algebras are inflations

In this section we first show that the class of cellular R–algebras is precisely the closure of the set $\{R\}$ under inflation, that is, an R–algebra is cellular if and only if it can be produced by iterated inflations of copies of R. Then we look at the two special cases of finite dimensional algebras and of orders and reprove the result in a more explicit way which will be used for applications later on. In the second subsection we discuss a generalization of Hochschild cohomology which appears in this construction.

4.1 The main result

Theorem 4.1 *Any cellular algebra over R is the iterated inflation of finitely many copies of R. Conversely, any iterated inflation of finitely many copies of R is cellular.*

Proof. First we note that a cell ideal J, regarded as an algebra (possibly without unit) always is an inflation of the ground ring R. In fact, the structural definition of cell ideal implies commutativity of the following diagram:

$$\begin{tikzcd} J \otimes_R J \arrow[r, "\alpha \otimes \alpha"] \arrow[d, "x \otimes y \mapsto xy"'] & \Delta \otimes_R i(\Delta) \otimes_R \Delta \otimes_R i(\Delta) \arrow[d, "u \otimes v \otimes w \otimes z \mapsto uvwz"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta) \end{tikzcd}$$

Since α is an isomorphism of bimodules and multiplication in J can be seen both as a left and as a right homomorphism, the product $uvwz$ must be a scalar multiple of $u \otimes z$. That is, there is a bilinear form $\varphi : \Delta \otimes i(\Delta) \to R$ such that $uvwz$ equals $\varphi(v,w)u \otimes z$. Hence identifying J with $V \otimes V \otimes R$ for a free R–module V having the same R–rank as Δ and as $i(\Delta)$ we can write J as an inflation.

We note that this statement is already implicit in [6] where the form φ is denoted by Φ.

To prove the first statement we proceed by induction on the length of the cell chain. By proposition 3.4 of [8], a cellular R–algebra A which is a cell ideal in itself is just a full matrix ring over R, say of size $n \times n$. Choose V to be a free R–module of rank n which we identify with $\Delta = Ae$ (where e is any primitive idempotent). We identify the second copy of V with $i(\Delta) = i(e)A$, and we identify $i(e)Ae$ with R. Using the above observation we can rewrite matrix multiplication $A \otimes_R A \to A$ as $A \otimes_R A \simeq (Ae \otimes_R i(e)A) \otimes_R (Ae \otimes_R i(e)A) \to Ae \otimes_R i(e)Ae \otimes_R i(e)A \simeq Ae \otimes_R i(e)A \simeq A$ where all maps are bimodule homomorphisms. As observed above, this provides us with a bilinear form $\varphi : V \otimes_R V \to R$ and shows how to write A as inflation of R along V. Note that the isomorphisms occurring in our rewriting of matrix multiplication go into the definition of φ.

Now we assume that A is cellular and has a cell chain of length greater than one. We fix a cell ideal J. By induction, the quotient algebra $B := A/J$ is an iterated inflation of copies of R. We have to show the following two assertions: (1) the cell ideal J (with multiplication as in A) is an inflation of R if considered as an algebra (without unit), and (2) the algebra A is an inflation of B along J.

Let us prove (1) first. We write J as $\Delta \otimes_R i(\Delta)$. Choose $V = \Delta$ as an R–module. Since $(u \otimes v) \times (x \otimes y)$ is a scalar multiple of $u \otimes y$, multiplication inside J is governed by a bilinear form $\varphi : i(\Delta) \otimes \Delta \to R$ (this is the form Φ of [6]), hence J can be written as inflation.

In order to prove (2) we use the isomorphism $J \simeq \Delta \otimes_R i(\Delta)$ and the fact that J is an ideal.

This finishes the proof of the first statement. The second statement is proposition 3.4. ■

Now we reprove the theorem in the two most important special cases, namely finite dimensional algebras over a field, and orders. This gives us more explicit information (to be used later on) about the process of inflation. Let Λ be either a finite dimensional algebra over a field k or an order over a commutative Noetherian integral domain k. Then Λ is cellular with a cell chain of length n if and only if it is the cellular inflation of n copies of k.

The first case we deal with is that of k being a field and Λ being a **finite dimensional cellular k–algebra**. Fix a cell chain $0 = J_0 \subset J = J_1 \subset \ldots \subset J_n = \Lambda$. We proceed by induction on the length n of the cell chain. If n is one, then Λ is a full matrix ring itself, and we copy the above argument. Thus we may assume that n is bigger than one. Hence, by induction, Λ/J is the cellular inflation of $n-1$ copies of k. We put $C := \Lambda/J$. By definition, the cell ideal $J \simeq \Delta \otimes_k i(\Delta)$ has quadratic dimension, say m^2. Thus we can try to put $B := M_m(k)$, a full matrix ring. This clearly describes the additive structure of J. But we have to change the multiplicative structure. For this, we have to distinguish two subcases. Since k is a field, proposition 4.1 of [8] tells us that either J^2 equals zero, or J is a heredity ideal.

Suppose first that J^2 equals zero. Then we put $V = k^m$ and write B as an inflation $V \otimes_k V \otimes k$ of the field k along the vector space V, where the bilinear form $\varphi : V \otimes_k V \to k$ is just zero. This gives the correct multiplicative structure on $B = J$. Since J^2 equals zero, the left and right Λ–module structure on J factor via the quotient algebra $C = \Lambda/J$. Thus we can define the inflation of C along B in the following way: For β and γ we choose the right and the left C–module structures on B. For α we choose the Hochschild two–cocycle defining the extension $J \subset \Lambda \to C$. It remains to verify that all the above axioms are satisfied, which is easy.

Now we consider the second subcase: J is a heredity ideal. This means that there is a primitive idempotent e in Λ such that J is generated by e as a two–sided ideal. In particular, J^2 equals J. Moreover, the endomorphism ring $e\Lambda e$ equals k, and multiplication in Λ provides an isomorphism of left and right Λ–modules $\Lambda e \otimes_k e\Lambda \simeq J$. Here, again we put $V = k^m$ and write B as inflation $V \otimes_k V \otimes k$. But now the bilinear form φ is obtained in the following way: We identify the first copy of V with Λe, the second one with $i(e)\Lambda$ and k with $i(e)\Lambda e$ (note that $i(e)$ is equivalent to e by [8], proposition 5.1 – hence $i(e)Ae$ has the same k–dimension as eAe which equals k). As in the case of a full matrix ring we rewrite multiplication (using the definition of a cell ideal) as $J \otimes_k J \simeq (Ae \otimes_k i(e)A) \otimes_k (Ae \otimes_k i(e)A) \to Ae \otimes_k i(e)Ae \otimes_k i(e)A \simeq Ae \otimes_k i(e)A \simeq J$. As before this defines the bilinear form φ and shows how to write J as an inflation of k along V.

In order to define α, β and γ, we write Λ as a k–direct sum $J \oplus (\Lambda/J)$. The choices of the bilinear maps will depend on the choice of this direct sum decomposition. In fact, multiplication in Λ now can be written uniquely as $(b_1 + c_1)(b_2 + c_2) = b_1 b_2 + b_1 c_2 + c_1 b_2 + c_1 c_2$. The first term on the right hand side is the multiplication inside J (as controlled by φ). The second term defines β, the third term defines γ, and the difference $c_1 c_2 - c_1 \circ c_2$, where $\circ$ is the product taken in the quotient algebra C, defines α. Again it is easy to check the axioms. We note that in this case, α is not a Hochschild cocycle, and β and γ are not module actions.

Now we come to the second case of Λ being an **order** over R. Put $K = frac(R)$. Here this means that the K–algebra $A = K \otimes_R \Lambda$ is a direct product of finitely many full matrix rings over K. Moreover, Λ contains a K–basis of A. Fix a cell chain $J_0 = 0 \subset J = J_1 \subset \ldots \subset J_n = \Lambda$ of Λ. First we notice that extending scalars we get a cell chain $0 \subset K \otimes_R J_1 \subset \ldots \subset K \otimes_R \Lambda = A$ of the algebra A. But a semisimple algebra does not contain any nilpotent ideal, hence the cell chain of A must be a heredity chain. Thus, each ideal J_i contains an element e_i in the K–span of which there is an idempotent in A which becomes primitive after factoring out J_{i-1}.

If n equals one, then proposition 3.4 of [8] implies that Λ is isomorphic to a full matrix ring over R. Thus we may suppose that n is bigger than one, and by induction we know already that Λ/J is an inflation of full matrix rings. As we have seen above, we can write $K \otimes_R J$ (which is a heredity ideal in A) as an inflation $V \otimes_K V \otimes_K K$, where the first copy of V is Ae, the second one is $i(e)A$, and K is the endomorphism ring eAe which again can be identified with $i(e)Ae$. This provides us with a bilinear map φ describing the multiplication inside $K \otimes_R J$. Restricting to Λ we get the multiplicative structure of J as an inflation, since $Ae \cap \Lambda$ and $i(e)A \cap \Lambda$ both are R–free of the same rank and since the isomorphism $(Ae \cap \Lambda) \otimes_R (i(e)A \cap \Lambda) \to J$ is the restriction of the isomorphism $Ae \otimes_K i(e)A \to K \otimes_R J$. In order to get α, β and γ, we again use the above description of A as inflation and then restrict to Λ. It is clear that the axioms are satisfied.

4.2 A generalization of Hochschild cohomology

Let A be a finite dimensional cellular algebra. As we have seen above, there are two kinds of ideals occurring in any cell chain of A, distinguished by the bilinear form φ being zero or non–zero. In the first case the data defining A as extension are just the usual data of second Hochschild cohomology as we will show now. In the second case, however, the situation is rather different. This means that inflation can be seen as a proper generalization of Hochschild extension.

Proposition 4.2 *Suppose that β and γ are module actions, that is, $J^2 = 0$. Then α is a Hochschild 2–cocycle, and conversely, any such cocycle which is i–stable can be used for defining α.*

Proof. One condition on α is in axiom (V) (in proposition 3.2). In case of β and γ being right and left module actions (respectively) of C on B, this condition is equivalent to: for all c_1, c_2, c_3 in C, there is an equality $\alpha(c_1c_2, c_3) + \alpha(c_1, c_2)c_3 = \alpha(c_1, c_2c_3) + c_1\alpha(c_2, c_3)$: which is precisely the condition for α being a Hochschild cocycle in $H^2(C, B)$. Thus it remains to show that the other two conditions (III) and (IV) which involve α become empty in case of β and γ being module actions. It is enough to look at (III): If β is a module action, this reduces to $(bc_1)c_2 = b\alpha(c_1, c_2) + b(c_1c_2)$ which is satisfied because of $J^2 = 0$ (which implies $b\alpha(c_1, c_2) = 0$). ∎

If A is finite dimensional and each ideal in a cell chain is generated by an idempotent, then A is in fact a quasi–hereditary algebra. Conversely, any quasi–hereditary algebra admitting an involution which fixes the ideals in a heredity chain is cellular. In this way, inflation provides us with a new inductive construction of these quasi–hereditary algebras which is different from the two previously known (more general) constructions in [10] and in [4].

5 Examples

The inflation construction does not require copies of the ground ring R as input. In fact, we have seen, that inflating any cellular algebra one gets another cellular algebra.

Some of the examples discussed in [6] are indeed related in this way. As we will see now, Brauer algebras are inflations of group algebras of symmetric groups. Temperley–Lieb algebras and Jones' annular algebras are inflations of local cellular algebras. These statements are already implicit in [6].

The main examples are Brauer algebras. We closely follow the notation used in [6]. We do not repeat the results of [6]. Instead we just give a list of references to statements in [6] which altogether show how to write Brauer algebras as iterated inflations. Recall that the Brauer algebra (for a fixed n) has a basis consisting of diagrams. Each diagram D consists of n strings. Some of these strings, say $t(D)$ of them, connect the top of the diagram with the bottom, whereas the others go from top to top or from bottom to bottom, that is, they define two involutions (called 'annular involutions'), say $S_1(D)$ and $S_2(D)$, on the top and on the bottom line, respectively. The natural number $t(D)$ here runs through $n, n-2, n-4, ...$

Corollary 5.1 *The Brauer algebra A is an iterated inflation of the group algebras (over the same field) of the symmetric groups $\Sigma_n, \Sigma_{n-2}, ...$ (with Σ_n as top ingredient of the construction). More precisely, A admits a chain of two–sided ideals $0 \subset \ldots \subset J_{n-2} \subset J_n = A$ such that: Each subquotient J_l/J_{l-2} is a (one step) inflation of the group algebra of Σ_l (where in case of n being even, the bottom section J_0 is an inflation of the ground field). And this chain of two–sided ideals can be refined to a cell chain of A by taking preimages of cell chains in the inflations of the sections.*

We indicate how to prove this corollary by suitable quotations from [6]. First, 4.4 states that the Brauer algebra A has a basis of the form $a \otimes b \otimes c$ where c runs through bases of the above group algebras of symmetric groups (where permutations are realized by strings from top to bottom) and a and b correspond to the above elements $S_1(D)$ and $S_2(D)$, that is, they run through a basis of a vector space which depends on $t(D)$ only. In fact, the basis of this vector space are the 'annular involutions', that is, its dimension is the number of possibilities to connect $n - t(D)$ out of n vertices (say in the top line) by $(n - t(D))/2$ edges. A multiplication rule for such basis elements is given in 4.7. If both factors in a product $(a \otimes b \otimes c) \times (d \otimes e \otimes f)$ are in the same section, then the argument following 4.10.1 (together with a dual argument) shows that the product is a scalar multiple of a basis element $g \otimes h \otimes i$ in the same section, and the scalar is given explicitly as the value of a certain bilinear form. Since the group algebra of the symmetric group is cellular, hence an iterated inflation of copies of the ground field, we can refine this chain of ideals to a cell chain where each section is an inflation of the ground field along a finite dimensional vector space. (The multiplication rule 4.7 then ensures that this chain consists of two–sided ideals in the Brauer algebra.) A section of the refined chain is a section of the cell chain of the group algebras of symmetric groups tensored with one of the above vector spaces (given by 'annular involutions'). Therefore, the Brauer algebra is an iterated inflation. ■

In [6] it is also shown how to view the Temperley–Lieb algebra and Jones' annular algebra as subalgebras of the Brauer algebra. Both subalgebras have bases consisting of diagrams of a special shape. Thus restricting the above arguments shows that both of these algebras are inflations again. The inputs now are copies of the field for the Temperley–Lieb algebra, copies of the group algebra of the cyclic group of order two for the covering of the Temperley–Lieb algebra, and several group algebras of cyclic groups (of varying size) for the Jones' annular algebra. All details can be found in chapter six of [6].

6 Morita equivalences: technical tools

In this section we recall shortly the basic facts on Morita equivalence of two algebras, which can be found in the usual algebra textbooks (see for example [1, 2] or the original article [9].) Then we study Morita equivalences of cellular algebras and obtain several technical tools to be applied later on.

Let A and B be two algebras (necessarily with unit elements). Then A and B are said to be Morita equivalent if the module categories of A and B are equivalent: there are two functors $F : A - mod \to B - mod$ and $G : B - mod \to A - mod$ such that $FG \cong 1_{B-mod}$ and $GF \cong 1_{A-mod}$.

The following characterization of Morita equivalences is well–known.

Theorem 6.1 *Let A and B be two algebras. Then the following statements are equivalent:*

(1) A and B are Morita equivalent, and the equivalence is given by functors F and G as above.

(2) There are two bimodules ${}_AP_B$ and ${}_BQ_A$ and a pair of surjective bimodule homomorphisms

$$\theta : P \otimes_B Q \longrightarrow A \quad \text{and} \quad \phi : Q \otimes_A P \longrightarrow B$$

such that for $x, y \in P$ and $f, g \in Q$,

$$\theta(x, f)y = x\phi(f, y) \quad \text{and} \quad f\theta(x, g) = \phi(f, x)g.$$

In this case, there are isomorphisms
(i) $F \cong Q \otimes_A - \cong \mathrm{Hom}_A(P, -)$,
(ii) $G \cong P \otimes_B - \cong \mathrm{Hom}_B(Q, -)$,
(iii) $A \simeq \mathrm{End}(P_B)$ and $B \simeq \mathrm{End}({}_AP)$.

If two algebras are Morita equivalent, then the lattices of ideals are isomorphic:

Proposition 6.2 *Suppose A and B are Morita equivalent via the inverse equivalence $F : A\text{–mod} \to B\text{–mod}$. Then the lattice of all (two–sided) ideals in A and the lattice of ideals in B are isomorphic via $I \to \Phi(I)$, where $\Phi(I)$ is defined to be the left annihilator of $F(A/I)$ in B. Moreover, for each ideal I in A, the two algebras A/I and $B/\Phi(I)$ are Morita equivalent.*

Remark. The correspondence in the above proposition can be described by using the bimodules P and Q as follows: The ideal I in A corresponds to $\Phi(I)$ in B if and only if $IP = P\Phi(I)$ if and only if $QI = \Phi(I)Q$.

This shows also that $\Phi(I) = \phi(QI \otimes_A P)$ and $I = \theta(P\Phi(I) \otimes_B Q)$. We refer to the second chapter in [2] for a proof.

For a left A–module M, we denote by $l_A(M)$ the set $\{a \in A \mid am = 0 \,\text{for all}\, m \in M\}$, the left annihilator of M in A. If there is no confusion, we denote also by $l_A(M)$ the right annihilator of a right A–module M_A.

Lemma 6.3 *If M and N are two isomorphic modules, then $l_A(M) = l_A(N)$.*

Suppose A and B are Morita equivalent. Then, by theorem 6.1, we may identify B with $\mathrm{End}({}_AP)$. Furthermore, we have

Proposition 6.4 *For any A–module X, $l_B(FX) = l_B(F(A/l_A(X)))$.*

Proof. First we show that $l_B(FX) \subseteq l_B(F(A/l_A(X)))$. Pick an element $b \in B = \mathrm{End}({}_AP)$ with $bf = 0$ for all $f \in \mathrm{Hom}_A(P, X) = FX$. Since the A-module X is a faithful $A/l_A(X)$–module, there exists an injective $A/l_A(X)$–homomorphism $\mu : A/l_A(X) \to X^n$. (This is also an A–homomorphism). If $g : P \to A/l_A(X)$ is an A– homomorphism, then $bg\mu = 0$. Thus $bg = 0$ since μ is injective. This implies that $b \in l_B(F(A/l_A(X)))$.

Conversely, if $b \in l_B(F(A/l_A(X)))$ then $bf = 0$ for any $f : P \to A/l_A(X)$. Since X is also a module over $A/l_A(X)$, there is a surjective $A/l_A(X)$–homomorphism $\pi : (A/l_A(X))^n \to X$. This is also an A–homomorphism. Now we take an A–homomorphism $g : P \to X$. As ${}_AP$ by assumption is projective as an A–module, there exists a homomorphism $g' :_A P \to (A/l_A(X))^n$ of A–modules such that $g = g'\pi$. Thus $bg = bg'\pi = 0 \cdot \pi = 0$ and $b \in l_B(FX)$. This finishes the proof. ∎

Corollary 6.5 *Suppose X and Y are two A–modules. If $l_A(X) = l_A(Y)$, then $l_B(FX) = l_B(FY)$.* ∎

As a cellular algebra involves an involution (that is, a k–linear anti–automorphism), we also need to consider the behaviour of the annihilators of modules under the duality induced by an involution. For each module ${}_AX$, we denote by $i({}_AX)$ the right A–module which is induced by i, that is, $i(X) := \{\hat{x} \mid x \in X\}$ and $\hat{x}a = \widehat{i(a)x}$. In the following $\hat{x}$ will be denoted by $i(x)$ if no confusion can arise.

Lemma 6.6 *Let $i : A \to A$ be an involution of the algebra A and denote by i also the induced duality functor from A–mod to mod–A. Then $l_A(i(M)) = i(l_A(M))$ for each left A–module M.*

Proof. It is clear that $aM = 0$ if and only if $i(M) \cdot i(a) = 0$. But the latter is equivalent to saying $i(a) \in l_A(i(M))$. ■

Remark. If we consider the functor $F_1 := - \otimes_A P \cong Hom_A(Q_A, -)$ from mod–A to mod–B, then statements similar to 6.4, 6.5, 6.6 are also true.

Now we look at cellular algebras under Morita equivalences.

As we have seen, the definition of cellular algebras involves an involution. This involution is not only an additional datum, but also makes the other conditions in the definition more restrictive and more subtle (see [8]). Hence a natural question arises: How to impose involutions on the Morita context? One natural way to make them compatible may be the following hypothesis (see also the remarks at the end of this section).

Hypothesis. $(*)$ Let A be a cellular algebra with respect to an involution i and B an algebra with an involution i' which is Morita equivalent to A via the mutually inverse equivalences F and G as in 6.1.

Our hypothesis is that the following diagram of functors commutes up to natural isomorphism

$$\begin{tikzcd} A - mod \arrow[r, "{_B Q \otimes_A -}"] \arrow[d, "i"'] & B - mod \arrow[d, "i'"] \\ mod - A \arrow[r, "- \otimes_A P_B"] & mod - B \end{tikzcd}$$

i.e., $i'(Q \otimes_A X) \simeq i(X) \otimes_A P$ for all A–modules X, and this isomorphism is natural in X.

Since $P \otimes_B -$ and $- \otimes_B Q$ are the inverse functors of $F = Q \otimes_A -$ and $F_1 := - \otimes_A P$ respectively, we have also the following natural equivalence : $i(P \otimes_B Y) \simeq i'(Y) \otimes_B Q$ as right A–modules.

A stronger hypothesis (which may be easier to handle in practice) is:

Hypothesis. $(\dagger)$ Let A be a cellular algebra with respect to an involution i and B an algebra with an involution i' which is Morita equivalent to A via the inverse equivalences F and G as in 6.1.

The hypothesis means that the following diagram of functors commutes

$$\begin{tikzcd} A - mod \arrow[r, "{_B Q \otimes_A -}"] \arrow[d, "i"'] & B - mod \arrow[d, "i'"] \\ mod - A \arrow[r, "- \otimes_A P_B"] & mod - B \end{tikzcd}$$

i.e., $i'(Q \otimes_A X) = i(X) \otimes_A P$ for all A–modules X.

An important special case of Morita equivalences is defined by multiplication with an idempotent, say $e = e^2 \in A$. Then the hypothesis (†) tells us the following: Right multiplication by e on A goes to left multiplication by $i(e)$ under i. But the Morita equivalence sends right multiplication by e to the identity on $B = eAe$ which goes to itself under i'. Hence left multiplication by $i(e)$ must be the identity on B as well. Thus $i(e) \cdot e = e$. Hence $i(e) = i(i(e) \cdot e) = i(e) \cdot i^2(e) = e$. It is easy to check that conversely a Morita equivalence which is multiplication by $e = i(e)$ satisfies (†). We have shown:

Proposition 6.7 *A Morita equivalence which is multiplication by an idempotent e satisfies the hypothesis (†) if and only if e equals $i(e)$.*

This explains why (†) is handy for practical considerations. However, most of our results need only the weaker hypothesis $(*)$.

Remark. If we are interested in finite dimensional k–algebras over a field k, the hypothesis $(*)$ is equivalent to the commutativity up to natural isomorphism of the following diagram of functors (here, we denote by D the usual duality functor $\mathrm{Hom}_k(-, k)$ and by ϵ the duality Di from A–mod to itself):

$$\begin{tikzcd} A-mod \arrow[r, "{}_BQ \otimes_A -"] \arrow[d, "\epsilon"'] & B-mod \arrow[d, "\epsilon' = Di'"] \\ A-mod \arrow[r, "{}_BQ \otimes_A -"] & B-mod \end{tikzcd}$$

i.e., $Di'(Q \otimes_A X) \simeq Q \otimes_A Di(X)$ for all A–modules X, and this isomorphism is natural in X. Of course, a similar assertion is valid for the hypothesis (†).

Proof. First we observe that

$$D(i(X) \otimes_A P) \simeq Hom_k(i(X) \otimes_A P, k) \simeq Hom_A(P, Di(X)) \simeq Q \otimes_A Di(X).$$

Now $(*)$ implies that $Di'(Q \otimes_A X) \simeq D(i(X) \otimes_A P) \simeq Q \otimes_A Di(X)$. The converse is proved in a similar way. ■

From now on, we assume hypothesis $(*)$ to be satisfied. Later on, we will restrict to the more special but also more handy situation of Morita equivalences satisfying (†).

Notice that the involutions i and i' also give rise to a new bimodule structure on the $B - A$ bimodule ${}_BQ_A$: we denote this new module by $\bar{Q} := \{\bar{q} \mid q \in Q\}$ and then define $a \cdot \bar{q} \cdot b = \overline{i(b)qi(a)}$ for all $a \in A, b \in B$ and $q \in Q$. Similarly, we have a new $B - A$ bimodule $\bar{P}$.

As a consequence of $(*)$, we have

Corollary 6.8 *(1)* ${}_A\bar{Q}_B \simeq_A P_B$ *as* $A-B$ *bimodules.*
(2) $\bar{P} \simeq Q$ *as* $B-A$ *bimodules.*
(3) $\overline{Q \otimes_A M} \simeq \bar{M} \otimes_A P$ *as* $A-B$ *bimodules, where* M *is an* $A-A$ *bimodule.*

Thus we may identify P with $\bar{Q}$ and consider the homomorphism ϕ in 6.1 as a bimodule homomorphism from $Q \otimes_A \bar{Q}$ to B. Let us now compare the i–stable ideals in A with i'–stable ideals in B. Again we identify B with $\mathrm{End}({}_AP)$.

Proposition 6.9 *If* J *is an ideal in* A *with* $i(J) = J$*, then* $i'(\Phi(J)) = \Phi(J)$*.*

Proof. It follows from 6.6 that $l_A((A/J)_A) = J = i(J) = il_A(A/J) = l_A(i({}_A(A/J)))$. Note that $\Phi(J) = l_B(F_1((A/J)_A))$. Hence, again by 6.6 and its right–handed version, we have that

$$\Phi(J) = l_B(F_1(A/J)) = l_B(F_1 i({}_A(A/J)))$$

$$= l_B(i'F(A/J)) = i'(l_B(F(A/J))) = i'(\Phi(J)). \blacksquare$$

Now we can formulate a sufficient condition for a Morita equivalence to send a cell ideal to a cell ideal. Technically, this is our main result.

Proposition 6.10 *Suppose that* $\sum_j \phi(q_j, \bar{p}_j) = 1_B$ *for some* $q_j, p_j \in Q$ *implies that* $\sum_j \phi(p_j, \bar{q}_j) = 1_B$*. If* J *is a cell ideal in* A*, then* $\Phi(J)$ *is a cell ideal in* B*.*

Proof. We use the structural definition of cell ideal to prove this proposition. Since J is a cell ideal, there is an A–module Δ and a bimodule isomorphism α such that the following diagram is commutative:

$$\begin{tikzcd} {}_A\Delta \otimes_R i(\Delta)_A \arrow[r, "\alpha"] \arrow[d, "\beta_\Delta"'] & J \arrow[d, "i"] \\ {}_A\Delta \otimes_R i(\Delta)_A \arrow[r, "\alpha"] & J, \end{tikzcd}$$

where β_Δ denotes the map sending $x \otimes i(y)$ to $y \otimes i(x)$.

Put $\Gamma = Q \otimes_A \Delta$. Then $i'(\Gamma) \simeq i(\Delta) \otimes_A \bar{Q}$ by our hypothesis $(*)$ and by 6.8. In fact, this isomorphism sends $i'(q \otimes \delta)$ to $i(\delta) \otimes \bar{q}$ for all $q \in Q$ and $\delta \in \Delta$. Now we consider the map

$$\alpha' : (Q \otimes_A \Delta) \otimes_R i'(Q \otimes_A \Delta) \longrightarrow \Phi(J), \quad (q \otimes \delta) \otimes_R i'(p \otimes \sigma) \longmapsto \phi(q, \alpha(\delta, i(\sigma))\bar{p}),$$

and another map

$$\zeta : Q \otimes_A \bar{Q} \longrightarrow Q \otimes_A \bar{Q}$$

which maps $q \otimes \bar{p}$ to $p \otimes \bar{q}$ for $q, p \in Q$. We can verify that α' and ζ are well-defined. The condition $\sum_j \phi(q_j, \bar{p}_j) = 1_B = \sum_j \phi(p_j, \bar{q}_j)$ implies that the diagram

$$\begin{tikzcd}
Q \otimes_A \bar{Q} \arrow[r, "\phi"] \arrow[d, "\zeta"'] & B \arrow[d, "i'"] \\
Q \otimes_A \bar{Q} \arrow[r, "\phi"] & B
\end{tikzcd}$$

is commutative. In fact, we can write each element $b \in B$ in the form $b = \sum_j \phi(q_j, \bar{p}_j b) = \sum_j \phi(q_j, \overline{i'(b)p_j})$ and and we also can write $i'(b)$ in the similar form $i'(b) = i'(b) \sum_j \phi(p_j, \bar{q}_j) = \sum_j \phi(i'(b)p_j, \bar{q}_j)$.

Since Q_A is a projective right A–module, we can identify $Q \otimes_A J$ with QJ. Thus we have the following commutative diagram

$$\begin{tikzcd}
\Gamma \otimes_R i'(\Gamma) \arrow[r, "\bar{\alpha}"] \arrow[d, "\beta_\Gamma"'] & Q \otimes_A J \otimes_A \bar{Q} \arrow[r, "\phi"] \arrow[d, "\zeta"'] & \Phi(J) \arrow[d, "i'"] \\
\Gamma \otimes_R i'(\Gamma) \arrow[r, "\bar{\alpha}"] & Q \otimes_A J \otimes_A \bar{Q} \arrow[r, "\phi"] & \Phi(J)
\end{tikzcd}$$

where $\bar{\alpha}$ sends $q \otimes \delta \otimes i'(q_1 \otimes \delta_1)$ to $q \otimes \alpha(\delta, i(\delta_1)) \otimes \bar{q}_1$. Since α' is the composition of the $B - B$ bimodule isomorphisms $\bar{\alpha}$ and ϕ, we get the desired equality $\alpha' i' = \beta_\Gamma \alpha'$. Thus $\Phi(J)$ is a cell ideal in B. ■

Theorem 6.11 *If $\sum_j \phi(q_j, \bar{p}_j) = 1_B$ implies $\sum_j \phi(p_j, \bar{q}_j) = 1_B$, then B is a cellular algebra having as a cell chain the Φ–image of a cell chain of A.*

Proof. By 6.10, if J is a cell ideal in A, then $\Phi(J)$ is a cell ideal in B. We know from 6.2 that A/J is Morita equivalent to $B/\Phi(J)$. Moreover, this equivalence can be defined by

$$\begin{aligned}
\theta_1 : P/JP \otimes_{B/\Phi(J)} Q/QJ &\longrightarrow A/J, \\
(p + JP) \otimes (q + QJ) &\longmapsto \theta(p, q) + J, \\
\phi_1 : Q/QJ \otimes_{A/J} P/JP &\longrightarrow B/\Phi(J), \\
(q + QJ) \otimes (p + JP) &\longmapsto \phi(p, q) + \Phi(J).
\end{aligned}$$

(Note that $P/JP = P/P\Phi(J)$ is an $A/J - B/\Phi(J)$ bimodule by the remark following 6.2.) The condition in 6.10 is satisfied. Hence we can proceed by induction on the length of a cell chain to show that B is cellular. ■

Corollary 6.12 *Let B be an algebra with an involution i. Suppose that there is an idempotent element $e \in B$ such that $i(e) = e$ and $BeB = B$. If eBe is cellular (with respect to the restriction of i to eBe) then B is cellular with respect to i.*

Proof. Set $A = eBe, Q = Be$ and $P = eB$. Then the multiplication maps

$$\theta : eB \otimes_B Be \longrightarrow A \quad \text{and} \quad \phi : Be \otimes_A eB \longrightarrow B$$

define a Morita equivalence of A and B (see [2], p.68 and [1], exercise 22.7). It is clear that $\bar{Q}$ is isomorphic to P under the homomorphism $\overline{be} \longmapsto ei(b)$. More generally, for each ${}_AX$, the map $\eta : i(Q \otimes_A X) \longrightarrow i(X) \otimes_A \bar{Q}$ via $i(q \otimes x) \longmapsto i(x) \otimes \overline{i(q)}$ is a B–isomorphism and natural in X. If there are b_j and c_j in B such that $\sum_j b_j e c_j = 1$, then $\sum_j i(c_j) e i(b_j) = 1$. Therefore the condition in proposition 6.10 and the hypothesis $(*)$ are satisfied. Now the corollary follows immediately from theorem 6.11. ∎

As an application of 6.12 we can prove the following result.

Proposition 6.13 *Let A and B be Morita equivalent algebras with involutions i and i' respectively, such that the hypothesis $(*)$ is satisfied and that $\theta(\bar{p}, q) = i\theta(\bar{q}, p)$ and $\phi(q, \bar{p}) = i'\phi(p, \bar{q})$ hold true for all $p, q \in Q$. Then A is cellular if and only if B is cellular.*

Proof. First, we form the algebra

$$S = \begin{pmatrix} A & \bar{Q} \\ Q & B \end{pmatrix}$$

which will turn out to be Morita equivalent to either of the algebras A and B. Its additive structure is given by the usual addition of matrices. Multiplication is given by

$$\begin{pmatrix} a & \bar{q} \\ p & b \end{pmatrix} \begin{pmatrix} a_1 & \bar{q}_1 \\ p_1 & b_1 \end{pmatrix} = \begin{pmatrix} aa_1 + \theta(\bar{q}, p_1) & a\bar{q}_1 + \bar{q}b_1 \\ pa_1 + bp_1 & \phi(p, \bar{q}_1) + bb_1 \end{pmatrix}$$

and, as an involution j on S, we define j as follows:

$$j \begin{pmatrix} a & \bar{q} \\ p & b \end{pmatrix} = \begin{pmatrix} i(a) & \bar{p} \\ q & i'(b) \end{pmatrix}.$$

By the assumptions on θ and ϕ and by hypothesis $(*)$, we can verify that j is an involution of S. Let us consider the idempotents

$$e = \begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}.$$

It is easy to see that e and f are fixed by j. Clearly, $SeS = S = SfS, eSe = A$ and $fSf = B$. This implies that S is Morita equivalent to either of the algebras A and B. Since the involutions i and i' are the restrictions of j to eSe and fSf respectively, we know that if one of A and B is cellular, then S is a cellular algebra by 6.12, and therefore the other one is cellular by [8], Prop.4.3. ∎

As a special case of 6.12, we get the following fact.

Corollary 6.14 *Let A be an algebra with an involution i, which fixes a complete set of primitive orthogonal idempotents. Then A is cellular if and only if its basic algebra is cellular.*

Proof. Suppose $1 = \sum_{j=1}^{n} e_j$ with $\{e_j\}$ a complete set of orthogonal primitive idempotents such that $i(e_j) = e_j$ for all j. Then $A \simeq (Ae_{j_1})^{l_1} \oplus \ldots \oplus (Ae_{j_m})^{l_m}$ with $Ae_{j_t} \not\simeq Ae_{j_s}$ for $j_t \neq j_s$, where $l_1, ..., l_m$ are positive integers, and $\{j_1, ..., j_m\}$ is a subset of $\{1, 2, ..., n\}$. Put $e = \sum_t e_{j_t}$ and consider the basic algebra eAe. For two primitive idempotents e_1 and e_2, there is an equality of two–sided ideals $Ae_1A = Ae_2A$ if and only if there is an isomorphism of left ideals $Ae_1 \simeq Ae_2$. This yields that $AeA = A$ holds true. So, if eAe is cellular, A is cellular by 6.12. The converse statement follows from [8], 4.3. ■

As another application, we have

Example. If A is a cellular algebra with respect to an involution i, then the $n \times n$ matrix algebra $M_n(A)$ is cellular with respect to the involution j defined by $j(a_{kl}) = (b_{kl})$ with $b_{kl} = i(a_{lk})$.

Note that $M_n(A) = M_n(R) \otimes_R A$. In fact, more is true: If two algebras A and B are cellular, then their tensor product again is cellular. We skip the details of the proof of this fact.

Finally, let us make the following remarks concerning the hypothesis $(*)$.

Remarks. (1) If we only assume that A and B are Morita equivalent and that A is a cellular algebra with respect to an involution i, then there is an anti-automorphism $\sigma : B \longrightarrow B$ such that σ^2 is an inner automorphism of B and that the duality induced by σ satisfies hypothesis $(*)$ (see [3]). But in this case we cannot ensure that this σ is an involution of B.

(2) The hypothesis $(*)$ is necessary for our question. In fact, an algebra can be cellular with respect to one involution and not cellular with respect to another, although it is, of course, Morita equivalent to itself. We will see an easy example in the next section.

7 Morita equivalences: two examples

When dealing with Morita equivalences defined by certain idempotents we will need the following special case. This in particular yields an important counterexample in case of characteristic two. A variation of this example will provide us with a counterexample to some other questions.

Let k be a commutative ring (of any characteristic, for the moment) and let A be the k–algebra of two by two matrices. We first determine all primitive idempotents in this algebra. A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ equals its own square if and only if the following equations are satisfied:

$$a = a^2 + bc, b = b(a + d), c = c(a + d), d = d^2 + bc.$$

We distinguish between the following two cases:

If $a + d \neq 1$, then $b = c = 0$, hence $a = a^2$ and $d = d^2$, thus we get the zero matrix and the identity matrix.

If $a + d = 1$, then the first equation becomes $ad = bc$, that is, the matrix has determinant zero. The last equation becomes equivalent to the first one, and the second and the third equation become vacuous. Thus any matrix with $d = 1 - a$ and determinant zero is an idempotent (and even primitive, since the matrix is singular).

Now we consider the following involution i on A: it sends $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to $\begin{pmatrix} d & b \\ c & a \end{pmatrix}$. Thus, it interchanges the diagonal entries and leaves fixed the other entries. Clearly, i is k–linear and an involution. Moreover, it is an anti–automorphism, as is easily checked.

Now the following observation is the key for our study of Morita equivalences later on:

If k is a field of characteristic different from two, then i fixes some primitive idempotent, for example the one with all entries being 2^{-1}. But in case of characteristic two, there is no primitive idempotent fixed by i, since a cannot equal $d = 1 - a = 1 + a$ in that case.

However, the algebra A together with the involution i is cellular over any ring k. In fact, one can define Δ to be the left module consisting of matrices with $a = b$ and $c = d$. Then $i(\Delta)$ is the right module of matrices with $a = c$ and $b = d$. Then an isomorphism $\Delta \otimes_k i(\Delta) \to A$ which satisfies all the conditions required in the definition of a cell ideal is given by sending $\begin{pmatrix} a & a \\ b & b \end{pmatrix} \otimes \begin{pmatrix} c & d \\ c & d \end{pmatrix}$ to 'the product divided by two', $\begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$.

In case of a field of characteristic different from two, one could use ordinary matrix multiplication as well. Actually, in that case one can write A as AeA for some primitive idempotent e fixed by i, whereas in the other case one has to write A as $Ae \otimes_k i(e)A$ which is, of course, isomorphic to $Ae \otimes_k eA$, but the isomorphism to be used here is responsible for the 'division by two' in the above explicit isomorphism.

Let us recall here that by [8], proposition 3.4, a simple cellular algebra must be a full matrix ring, hence there are many examples of non–cellular simple algebras. Of course, there are also many examples of non–cellular products of full matrix rings, since an involution belonging to a cellular structure has to fix the isomorphism classes of primitive idempotents (see [8], 5.1). The above discussion now shows that although equivalence classes of primitive idempotents must be fixed by i, the same need not be true for their elements: it may happen that i does not fix any primitive idempotent. We have shown:

Proposition 7.1 *A cellular algebra (A, i) over a ring in which 2 is not invertible need not have a primitive idempotent fixed by i. Thus there are cellular structures on non–basic algebras which do not correspond to any cellular structure on a basic algebra.*

In the next section we will show that over a field this can happen in characteristic two only.

We remark that this example also shows that an algebra can be cellular with respect to one involution and not cellular with respect to another, as we mentioned already at the end of the previous section.

Now we give an example of a full matrix algebra A over a field together with an involution i such that (A, i) is not cellular.

Let k be any field of characteristic different from two, let A as above be the algebra of two by two matrices over k and let i be the map which sends a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ to the matrix $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. It is easy to check that i in fact is an antiautomorphism of A and an involution. Now we show that (A, i) is **not cellular**. Assume to the contrary it would be cellular. Since A is simple, it must be a cell ideal in itself. Hence A as a bimodule over itself is isomorphic to $\Delta \otimes_k i(\Delta)$ for some left ideal Δ. Moreover, the involution i acts on the tensor product in the following way: an element $x \otimes y$ is sent to $i(y) \otimes i(x)$. The ideal Δ has k–dimension two. Pick a basis, say u and v. Then the vectors $u \otimes i(u)$, $v \otimes i(v)$ and $(u+v) \otimes (i(u) + i(v))$ all are fixed by i. Thus the space of fixed points of i has k–dimension at least three. But a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ fixed by i must satisfy $b = -b$, $c = -c$ and $d = a$, hence the space of i–fixed points has k–dimension one, a contradiction.

We remark that this example may also be used to illustrate that the commutativity of the diagram in the definition of cell ideal is really needed.

8 Morita equivalences from cellular algebras to their basic algebras

Our general theory of Morita equivalences as developed above suggests considering Morita equivalences which are defined by idempotents fixed under i. In fact, if A admits such an idempotent, then, by our previous results (in particular 6.12 and [8],4.3), (A, i) is cellular if and only if (eAe, i) is cellular. Moreover, the Morita equivalence between these two algebras then preserves the cellular structure. Thus the question now is, whether such Morita equivalences exist. In the case of finite dimensional algebras over fields, it is clearly enough to look for structure preserving Morita equivalences between a given cellular algebra and its basic algebra. (Note that we can assume the algebra to be cellular by our previous results.) This question is also central because of practical reasons. If the answer is yes, then one may work with basic algebras (which are the standard input for most of the techniques of representation theory of finite dimensional algebras) without loss of generality. Moreover, a positive answer would imply that 'cellular' is a notion depending only on the module category, not on the algebra chosen.

The first example in the previous section shows that, however, the answer in general is negative in case of characteristic two. The best what we can hope for is a positive answer in all other cases. This is given in the next theorem. We remark that most of the computations in the proof of this theorem are valid in general. Thus given an explicit algebra over a field of any characteristic one may follow the proof of the theorem in order to construct an idempotent fixed by the involution. At a crucial point of the proof there will appear two different cases; one of them allows the construction of the desired idempotent (without using the assumption on k), the second case, however, needs characteristic different from two. Thus, at this point one can see whether the construction works for the algebra one is interested in.

However, this argument does not tell us that for certain algebras there is no Morita equivalence preserving a cellular structure. In fact, the construction in the proof will depend on the choice of the data defining the algebra involved as an iterated inflation; that is, the construction depends on the choice of a basis, and it may depend on this choice whether the idempotent resulting from the construction has the desired property.

Theorem 8.1 *Let (A, i) be a cellular algebra over a field k of characteristic different from two. Then there is an idempotent e which is a sum of primitive idempotents – one from each equivalence class – and which is fixed by i. Hence there is a Morita equivalence between A and its basic algebra which respects the cellular structure.*

Proof. We proceed by induction on the sum of the multiplicities of indecomposable projective modules in a decomposition of A as a left A–module. It is enough to show that inside each equivalence class of primitive idempotents there exists a primitive one, say e, which is fixed by i. If this equivalence class belongs to a projective module with multiplicity bigger than one, then we can use the idempotent $1 - e$ (which is fixed by i) to reduce the sum of multiplicities by one. In fact, in that case multiplication by $1 - e$ defines a Morita equivalence between A and $(1-e)A(1-e)$, and this Morita equivalence respects the cellular structure. Proceeding inductively we can in this way reduce all multiplicities to one, which brings us to the basic algebra of A.

Thus it is enough to prove the following proposition (which gives a stronger assertion than the theorem).

Proposition 8.2 *Let (A, i) be a cellular algebra over a field k of characteristic different from two. Then each equivalence class of idempotents contains an idempotent which is fixed by i.*

Proof. It is enough to deal with primitive idempotents.

We will use the language of 'inflations' which has been explained in sections three and four. That is, we write the cellular algebra A as an iterated inflation of copies of the field k. The equivalence classes of primitive idempotents are in bijection with the indices λ of inflation steps with non–zero bilinear form φ. We proceed by induction on λ.

For the induction start we have to fix an index λ which is minimal among those indices having non–zero associated bilinear form φ. In other words, we fix an ideal J_λ in the cell chain such that J_λ^2 is not contained in $J_{<\lambda}$. Moreover, since λ is the minimal index such that J_λ contains a primitive idempotent, the ideal $J_{<\lambda}$ is nilpotent. The quotient $J_\lambda/J_{<\lambda}$ is an inflation, say $V \otimes V \otimes k$. Since φ is not zero, we can find elements $a, b \in V$ such that $\varphi(a,b) \neq 0$. In order to find a primitive idempotent, which is fixed by i, in the equivalence class associated with λ, we proceed as follows: We first define an element e, which is fixed by i, which is contained in J_λ, but not in $J_{<\lambda}$, and which is an idempotent modulo $J_{<\lambda}$. Having obtained that, we will argue that a linear combination of powers of e is the primitive idempotent which we want to construct.

For defining e we have to distinguish two cases. In case $a = b$ we put

$$e = \frac{1}{\varphi(a,a)}(a \otimes b \otimes 1).$$

In the other case we put

$$e = \frac{1}{2}(\frac{1}{\varphi(b,a)}(a \otimes b \otimes 1) + \frac{1}{\varphi(a,b)}(a \otimes a \otimes 1) + \frac{1}{\varphi(a,b)}(b \otimes a \otimes 1) + \frac{1}{\varphi(b,a)}(b \otimes b \otimes 1)).$$

(Note that $\varphi(b,a) = i(\varphi(a,b))$, hence all denominators are non–zero.) The involution i sends $a \otimes b \otimes 1$ to $b \otimes a \otimes 1$ and $\varphi(a,b)$ to $\varphi(b,a)$, hence it fixes e in both cases. An easy computation using the multiplication rule inside $J_\lambda/J_{<\lambda}$ shows that $e^2 = e + f$ where f is an element of $J_{<\lambda}$. Similary we can check that $e(J_\lambda/J_{<\lambda})e$ has k–dimension one. The element $f = e^2 - e$ is nilpotent, since it is contained in the nilpotent ideal J_λ. Thus a certain power of f is zero, which implies that some power of e, say e^l for some $l \in \mathbb{N}$, can be written as $e^l = e^{l+1}p(e)$ where $p(e)$ is some integral linear combination of powers of e. Now a well–known trick produces an idempotent. In fact,

$$e^l = e^{l+1}p(e) = e^l ep(e) = e^{l+1}p(e)ep(e) = e^{l+2}p(e)^2 = \ldots = e^{2l}p(e)^l$$

implies that $g := e^l p(e)^l$ is an idempotent element:

$$g^2 = e^{2l}p(e)^l p(e)^l = e^l p(e)^l = g.$$

As a linear combination of powers of e, the element g is fixed by i. Moreover, the space $g(J_\lambda/J_{<\lambda})g$ must have dimension one, since it contains the non–zero residue class of g and is contained in $e(J_\lambda/J_{<\lambda})e$. Thus g is a primitive idempotent.

This finishes the induction start. To complete the proof we show how to reduce the case of arbitrary μ (with non–zero φ) to the case of minimal λ. Fix μ not minimal. By λ we denote as before a minimal index. By induction we know already that there is a primitive idempotent g of class λ which is fixed by i. Hence the algebra $(1-g)A(1-g)$ is cellular with respect to the restriction of i. Applying the same argument again we pass to a cellular algebra $(1-g')A(1-g')$ which does not have any idempotents of class λ any more, in which, however, all the other equivalence classes of idempotents still are there. Continuing in the same way, we eventually arrive at a centralizer algebra $B \subset A$ which is cellular with respect to the restriction of i and for which μ is the minimal index associated with a non–zero bilinear form. Applying now the induction start to B we find a primitive idempotent of class μ there and this idempotent solves the problem for A as well, since it is in fact an element of A, and the effect of i on this idempotent is the same whether computed in A or in B. ∎

This finishes the proof of the proposition, hence also of the theorem.

The proof shows on the one hand that the assumption on the characteristic of k is needed (if one runs into the second case). On the other hand, in case J_λ is already a cell ideal, then under the assumption on k it is easy to find a primitive idempotent which is fixed under i: in fact, we may put e as above. Then it is easy to check that $e^2 = e$ since multiplication in a cell ideal is given by the multiplication rule for an inflation. Moreover, e is primitive since eAe has dimension one over k. Thus up to getting rid of terms in nilpotent cell ideals, the proof is genuinely constructive.

As a consequence we get a rich supply of cellular algebras:

Corollary 8.3 *Let (A, i) be a cellular algebra over a field of characteristic different from two. Let P a finitely generated projective A–module. Then $End_A(P)$ is a cellular algebra with respect to an involution induced by i.*

Proof. If P is a direct summand of A and each indecomposable direct summand of P occurs with multiplicity one, then P is isomorphic to Ae, where e is a sum of (pairwise orthogonal) primitive idempotents which by the theorem can be chosen to be fixed by i. Thus $End_A(P)$ is isomorphic to the cellular algebra eAe whose involution is the restriction of i.

In general, $End_A(P)$ has an algebra like the aforementioned one as basic algebra. Define an involution on $End_A(P)$ to correspond to the involution on the basic algebra via the Morita equivalence. Then $End_A(P)$ is cellular as well. ∎

Another consequence of the theorem is that a semisimple quotient of a cellular algebra is cellular again. Recall that in the previous section we gave an example of a full matrix algebra which is not cellular with respect to a given involution. Hence the assertion of the corollary is not trivial.

Corollary 8.4 *Let A be a finite dimensional algebra over a field k which is cellular with respect to an involution i. Then the maximal semisimple quotient algebra $A/rad(A)$ is cellular with respect to i as well.*

Proof. It is clear that i induces an anti–isomorphism of $A/rad(A)$. Since i preserves the equivalence classes of primitive idempotents in A, it sends each indecomposable ring direct summand of $A/rad(A)$ into itself by [8], 5.1. Let B be such a ring direct summand. Then B has precisely one isomorphism class of simple modules. We fix a representative, say L, which can be written as $L = \Delta(\lambda)/rad(\Delta(\lambda))$ for some standard module $\Delta(\lambda)$ occurring in a cell ideal, say J_λ, of A, with $J_\lambda/J_{<\lambda}$ a heredity ideal. Hence $\Delta(\lambda)$ has endomorphism ring k over A and it follows that L also has endomorphism ring k, both over A and over B. Consequently B is a full matrix ring over k. Now we distinguish two cases:

If k has characteristic different from two, then by the theorem A contains a primitive idempotent, say e, of equivalence class λ which is fixed by i. Denoting the image of e in B also by e we can write B as a cell ideal in the following way: $Be \otimes_k eB \simeq B$, where the map is multiplication and eB equals $i(e)B$ and eBe is isomorphic to k.

Now let k have characteristic two. We pick any primitive idempotent e in B. The element $v := i(e)e$ is fixed by i. If v is not zero, then the left ideal Bv is simple and its image under i is the right ideal vB. Now it is easy to check that the map

$$\alpha : Bv \otimes_k vB \to B,$$

$$av \otimes vb \mapsto avb$$

is a well–defined homomorphism of bimodules. Since the image of α is not zero, α is an isomorphism. The same argument works in case $u := ei(e)$ is not zero.

Finally, we are in the case of $v = i(e)e$ and $u = ei(e)$ both being zero, that is, e and $i(e)$ are orthogonal. The space $eBi(e)$ has dimension one and is fixed by i. We choose any non–zero element, say $x \in eBi(e)$, and get that $i(x) = sx$ for some scalar $s \in k$. Because of $i^2(x) = x$, the scalar s must satisfy $s^2 = 1$. Thus s equals 1 since we are in characteristic two. Hence x is fixed by i. Now we proceed as above to check that B can be written as a cell ideal via $Bx \otimes_k xB \to B, ax \otimes xb \mapsto axb$. ∎

We remark that the argument in the case of characteristic two proves that a full matrix ring over a field of characteristic two is cellular with respect to any involution. We need the theorem only to exclude examples like the one given at the end of the previous section, and such examples exist only over fields of characteristic different from two.

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