On the Zariski closure of the linear part of a properly discontinuous group of affine transformations

by

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1. Introduction

A well known classical theorem due to Bieberbach says that every discrete group Γ of isometries of the n—dimensional Euclidean space $\mathbb{R}^n$ with compact quotient $\Gamma \setminus \mathbb{R}^n$ contains a subgroup of finite index consisting of translations. Hence such a group Γ is virtually abelian, i.e. Γ contains an abelian subgroup of finite index.

Let us now look at the group of affine transformations instead of the group of isometries of $\mathbb{R}^n$. More precisely, let $G_n = \mathrm{Aff}(\mathbb{R}^n)$ denote the group of affine transformations of $\mathbb{R}^n$. The group G_n is the semidirect product $GL_n(\mathbb{R}) \ltimes \mathbb{R}^n$ where $\mathbb{R}^n$ is identified with its group of translations. A subgroup Γ of G_n is said to act *properly discontinuously* on $\mathbb{R}^n$ if for every compact subset K of $\mathbb{R}^n$ the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ of recurrencies is finite. If a discrete group Γ consists of isometries then Γ acts properly on $\mathbb{R}^n$. But this is not true for all discrete subgroups of G_n , e.g. for an infinite discrete subgroup of $GL_n(\mathbb{R})$.

A subgroup Γ of G_n will be called *crystallographic* if Γ acts properly discontinuously on $\mathbb{R}^n$ and the orbit space $\Gamma \setminus \mathbb{R}^n$ is compact. In [A] Auslander conjectured that every crystallographic subgroup Γ of G_n is virtually solvable, i.e. contains a solvable subgroup of finite index. In [Mi] Milnor asked whether Auslander's conjecture is true without the assumption that the orbit space $\Gamma \setminus \mathbb{R}^n$ be compact.

Milnor's question has a positive answer for $n \leq 2$ (easy for $n = 2$, trivial for $n = 1$). But for $n = 3$ the answer to Milnor's question is negative. In fact, the second named author proved that there is a nonabelian free subgroup Γ of $\mathrm{Aff}(\mathbb{R}^3)$ acting properly discontinuously on $\mathbb{R}^3$ ([Ma 1,2]). On the other hand, D. Fried and W. Goldman [FG] proved Auslander's conjecture for $n = 3$ using cohomological arguments. For higher dimensions the existing results confirming the Auslander conjecture are proved under the assumption that the linear part $l(\Gamma)$ of Γ belongs to some special subgroup of $GL_n(\mathbb{R})$ where $l : G_n \to GL_n(\mathbb{R})$ denotes the natural homomorphism ([GK, GrM, S, To]). The most general result in this direction was obtained by Soifer and Tomanov ([S, To]), as follows.

Theorem 1 *Let* Γ *be a crystallographic subgroup of* G_n. *Suppose* Γ *is a group of generalized Lorentz motions, i.e.* $l(\Gamma)$ *belongs to an algebraic subgroup* H *of* $GL_n(\mathbb{R})$ *such that the semisimple quotient* $S = H^\circ/R$ *of the connected component* H° *by its solvable radical* R *is of the form* $S = S_1 \ldots S_r$ *where the* S_i *are the simple factors of* S *and* $\mathrm{rank}_{\mathbb{R}} S_i \leq 1$ *for* $1 \leq i \leq r$. *Then* Γ *is virtually solvable.*

The two main results of the present paper are

Theorem 2 *For* n *even there is no properly discontinuous subgroup* Γ *of* G_{2n+1} *with linear part* $l(\Gamma)$ *Zariski dense in* $SO(n+1, n)$.

Theorem 3 *For* n *odd there are properly discontinuous free subgroups* Γ *of* G_{2n+1} *with linear part* $l(\Gamma)$ *Zariski dense in* $SO(n+1, n)$.

The Auslander conjecture in dimensions $n \leq 5$ follows from theorem 2, theorem 1 and other results of [To] and results announced in [Ma 3].

Theorem 2, its proof and the statement of theorem 3 were contained in the preprint [Ma 3] of Margulis circulated in 1991. The results of the present paper and more were announced in [AMS 2]

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2. A sufficient condition for non proper discontinuity

In this section we prove a general criterion for certain pairs of affine transformations to generate a subgroup of the affine group which is not properly discontinuous. Recall that $G_n = \mathrm{Aff}(\mathbb{R}^n)$ and that $l : G_n \longrightarrow GL_n(\mathbb{R})$ is the natural epimorphism. For any $g \in G_n$ let us consider the decomposition of $\mathbb{R}^n$ into a direct sum of the $l(g)$–invariant linear subspaces

$$\mathbb{R}^n = A^-(g) \oplus A^0(g) \oplus A^+(g) \ , \tag{2.1}$$

where $A^-(g), A^0(g)$ and $A^+(g)$ are determined by the condition that their sum is $\mathbb{R}^n$ and all eigenvalues of the restriction $l(g) \mid A^-(g)$ (resp. $l(g) \mid A^0(g)$, $l(g) \mid A^+(g)$) are of modulus less than 1 (resp. equal to 1, greater than 1) . We set $D^-(g) = A^-(g) \oplus A^0(g)$ and $D^+(g) = A^+(g) \oplus A^0(g)$. It is clear that $A^0(g) = D^-(g) \cap D^+(g)$. Let us call an element $g \in G_n$ *pseudohyperbolic* if $\dim A^\circ(g) = 1$ and the only eigenvalue of $l(g) \mid A^\circ(g)$ is 1 (not -1) . Let Ω be the set of pseudohyperbolic elements of G_n and let $\Omega_0 = \{g \in \Omega \mid g\,x \neq x \text{ for every } x \in \mathbb{R}^n\}$ be the subset of pseudohyperbolic fixed–point–free elements of G .

For $g \in \Omega$ there is exactly one g–invariant line C_g and C_g is parallel to $A^\circ(g)$. We call C_g the *axis* of g . The restriction of g to C_g is a parallel translation by a vector $\tau(g) \in A^\circ(g)$. The vector $\tau(g)$ is the $A^\circ(g)$–component with respect to the decomposition 2.1 of $g\,x - x$ for any $x \in \mathbb{R}^n$. We call $\tau(g)$ the *translational part* of g . We have $g \in \Omega_0$ if and only if $\tau(g) \neq 0$ .It is easy to see that

$$C_{g^m} = C_g \quad \text{ and } \quad \tau\left(g^m\right) = m\,\tau(g) \ \text{ for } m \neq 0 \ .$$

If $g \in \Omega_0$ then every $x \in D^+(g)$ has a unique decomposition $x = \lambda(x)\tau(g) + a(x)$ where $\lambda(x) \in \mathbb{R}$ and $a(x) \in A^+(g)$. We say that a vector $x \in D^+(g)$ is *positive with respect to* $g \in \Omega_0$ if $\lambda(x) > 0$.

Two elements $g_1, g_2 \in \Omega_0$ will be called *transversal* if

$$\mathbb{R}^n = A^+(g_1) \oplus D^+(g_2) = D^+(g_1) \oplus A^+(g_2)$$

or, equivalently, if $\dim(D^+(g_1) \cap D^+(g_2)) = 1$ and

$$\mathbb{R}^n = A^+(g_1) \oplus A^+(g_2) \oplus \left(D^+(g_1) \cap D^+(g_2)\right) .$$

For two transversal elements $g_1, g_2 \in \Omega_0$, we say that g_1 and g_2 *form a positive pair* if the semiline $\{x \in D^+(g_1) \cap D^+(g_2) \mid x \text{ is positive with respect to } g_1\}$ coincides with the semiline $\{x \in D^+(g_1) \cap D^+(g_2) \mid x \text{ is positive with respect to } g_2\}$. We shall denote by $E^-(g)$ and $E^+(g)$ the affine subspaces containing C_g and parallel to $D^-(g)$ and $D^+(g)$, respectively. Let us introduce an order on $E^+(g)$ by saying that $u \in E^+(g)$ is greater than $v \in E^+(g)$ if $u - v$ is positive with respect to g . The condition that g_1 and g_2 form a positive pair is equivalent to the condition that the orders on $E^+(g_1) \cap E^+(g_2)$ induced from $E^+(g_1)$ and $E^+(g_2)$ coincide.

2.2 Lemma *Let $g_1, g_2 \in \Omega_0$ be two transversal elements which form a positive pair. Then there exist a compact set $K \subset \mathbb{R}^n$ and two sequences $\{m_i\}$ and $\{n_i\}$ of positive integers such that*

$$m_i \to \infty \quad \textit{and} \quad n_i \to \infty \quad \textit{as} \quad i \to \infty \quad \textit{and}$$

$$\left(g_1^{-m_i} g_2^{n_i} K\right) \cap K \neq \emptyset .$$

Proof For $x \in \mathbb{R}^n$ and $g \in \Omega_0$, let $B_g^+(x)$ denote the affine subspace containing x and parallel to $A^+(g)$. Let us define the projection $P_g : E^+(g) \to C_g$ by the equality

$$P_g(x) = C_g \cap B_g^+(x) \text{ for } x \in E^+(g) .$$

$C_g \cap B_g^+(x)$ is a point since $D^+(g) = A^+(g) \oplus A^0(g)$ and C_g is parallel to $A^0(g)$. The subspace $A^+(g)$ is $l(g)$–invariant. Therefore,

$$P_g(g\,x) = P_g(x) + \tau(g) \quad \text{ for } x \in E^+(g) . \tag{1}$$

Let d denote the Euclidean metric on $\mathbb{R}^n$. The absolute values of all eigenvalues of the restriction $l\left(g^{-1}\right) \mid A^+(g)$ are less than 1. This easily implies that there exist positive constants c and b such that for any $x \in E^+(g)$ and $n \in \mathbb{N}^+$

$$d\left(g^{-n}x, P_g\left(g^{-n}x\right)\right) \leq \mathrm{c}e^{-bn} d(x, P_g(x)) \ . \tag{2}$$

Let us fix a point $r(g) \in C_g$ and let $R(g)$ denote the interval $[r(g), r(g)+\tau_g) \subset C_g$. It follows from (1) that, for every $x \in E^+(g)$, there is a unique integer $k(x,g)$ such that

$$P_g\left(g^{k(x,g)}x\right) \in R(g) \ .$$

Note that if x is greater than $r(g)$ in $E^+(g)$ then $k(x,g) \leq 0$.

Since g_1 and g_2 form a positive pair one can find a vector $v \in D^+\left(g_1\right) \cap D^+\left(g_2\right)$ such that v is positive with respect to both g_1 and g_2 . Let us fix $x_0 \in E^+\left(g_1\right) \cap E^+\left(g_2\right)$ and set $x_i = x_0 + i\,v, i \in \mathbb{N}^+$. We have $x_i \in E^+\left(g_1\right) \cap E^+\left(g_2\right)$. Let

$$m_i = -k(x_i, g_1) \quad \text{and} \quad n_i = -k(x_i, g_2) \ .$$

Let $v_1 \in A^0\left(g_1\right)$ and $v_2 \in A^0\left(g_2\right)$ be such that $v \in v_1 + A^+\left(g_1\right)$ and $v \in v_2 + A^+\left(g_2\right)$. Since v is positive with respect to both g_1 and g_2 we have that $v_1 = \lambda_1 \tau\left(g_1\right)$ and $v_2 = \lambda_2 \tau\left(g_2\right)$ where $\lambda_1 > 0$ and $\lambda_2 > 0$. Then it is easy to see that

$$\lim_{i\to\infty} \frac{i}{m_i} = \lambda_1 \quad \text{and} \quad \lim_{i\to\infty} \frac{i}{n_i} = \lambda_2 \ . \tag{3}$$

It is clear that the growth of the functions $f_j(i) = d\left(x_i, P_{g_j}\left(x_i\right)\right), j = 1,2$, is asymptotically linear. Thus (1), (2) and (3) imply that

$$\lim_{i\to\infty} d\left(g_1^{-m_i}x_i, R\left(g_1\right)\right) = \lim_{i\to\infty} d\left(g_2^{-n_i}x_i, R\left(g_2\right)\right) = 0 \ . \tag{4}$$

We get from (4) that there exists a compact set $K \subset \mathbb{R}^n$ such that for all i

$$g_1^{-m_i}x_i \in K \quad \text{and} \quad g_2^{-n_i}x_i \in K \ . \tag{5}$$

We have $g_1^{-m_i}x_i = \left(g_1^{-m_i}g_2^{n_i}\right)g_2^{-n_i}x_i$. This and (5) imply that K is a compact set as claimed.

2.3 Corollary *Let* $g_1, g_2 \in \Omega_0$ *be two transversal elements which form a positive pair. Then the subgroup of* G_n *generated by* g_1 *and* g_2 *is not properly discontinuous.*

Proof Let $\{m_i\}$ and $\{n_i\}$ be as in Lemma 2.2. Then it is enough to check that the set $\left\{g_1^{-m_i}g_2^{n_i} \mid i \in \mathbb{N}^+\right\}$ is infinite. We may assume that $m_i < m_{i+1}$ and $n_i < n_{i+1}$ for every i since $m_i \to +\infty$ and $n_i \to +\infty$ as $i \to \infty$. Then for $i < j$ we have $g_1^{-m_i}g_2^{n_i} \neq g_1^{-m_j}g_2^{n_j}$ because otherwise $g_1^{m_j - m_i} = g_2^{n_j - n_i}$. But $g_1^m = g_2^n$ for positive m and n implies $A^+(g_1) = A^+(g_1^m) = A^+(g_2^n) = A^+(g_2)$ contradicting transversality.

3. Orientations on subspaces of $\mathbb{R}^{2n+1}$

In this section we introduce orientations on certain lines. In the next section we shall compare these orientations with the ones defined in the last section.

We fix a positive natural number n. Let $X = \mathbb{R}^{n+1}$ and $Y = \mathbb{R}^n$ and let B be the quadratic form on $X \times Y = \mathbb{R}^{2n+1}$ given by

$$B\left(x_1, \cdots, x_{n+1}, y_1, \cdots, y_n\right) = x_1^2 + \cdots + x_{n+1}^2 - y_1^2 - \cdots - y_n^2 ,$$

where $(x_1, \ldots, x_{n+1}) \in X$ and $(y_1, \ldots y_n) \in Y$. Consider the set Ψ of all maximal B–isotropic subspaces V of $\mathbb{R}^{2n+1}$. We have the two projections

$$p : \mathbb{R}^{2n+1} \longrightarrow X \quad \text{and} \quad q : \mathbb{R}^{2n+1} \longrightarrow Y .$$

The restriction of q to $V \in \Psi$ is a linear isomorphism $V \longrightarrow Y$. Hence if we fix an orientation on Y we have also fixed an orientation on each $V \in \Psi$. For $V \in \Psi$ let us denote the B–orthogonal complement of V by $V^\perp = \{z \in \mathbb{R}^{2n+1} \ ; \ B(z, V) = 0\}$. We have $V \subset V^\perp$ since V is B–isotropic. We also have

$$\dim V^\perp = 1 + \dim V = n + 1 .$$

The restriction of p to $V^\perp$ is a linear isomorphism $V^\perp \longrightarrow X$. Hence if we fix an orientation on X we have also fixed an orientation on $V^\perp$ for each $V \in \Psi$. Thus we have orientations on both V and $V^\perp$ and we can single out one of the two halfspaces of $V^\perp \smallsetminus V$ as positive by the following definition: if $(v_1, \ldots, v_n)$ is a positively oriented basis of V then a vector v in $V^\perp$ not in V is *positive* (with respect to V) if the basis $(v_1, \ldots, v_n, v)$ of $V^\perp$ is positively oriented. Accordingly, every affine line L in $V^{\perp}$ transversal to V is oriented: For two points x_0, x_1 of L we have $x_0 < x_1$ if $v = x_1 - x_0$ belongs to the positive half of $V^{\perp} \setminus V$.

If $V_1 \in \Psi$ and $V_2 \in \Psi$ are transversal then $V_1^{\perp} \cap V_2^{\perp}$ is a line which is transversal to both V_1 and V_2 . So there are two orientations ω_1 and ω_2 on L , where ω_i is defined if we consider L as a line in $V_i^{\perp}$. We have

3.1 Lemma *The orientations defined above on L are the same if n is even and are opposite if n is odd.*

Proof One can easily show that there exists a basis $(e_1, \ldots, e_{2n+1})$ in $\mathbb{R}^{2n+1}$ such that V_1 (resp. V_2) is the linear span of $(e_1, \ldots, e_n)$ (resp. of $(e_{n+1}, \ldots, e_{2n})$) and the form B with respect to this basis is given by the equation

$$B\,(x_1, \ldots, x_{2n+1}) = -x_1 x_{n+1} \cdots - x_n x_{2n} + x_{2n+1}^2 \;.$$

Let us note that $V_1^{\perp}$ (resp. $V_2^{\perp}$) is the linear span of $(e_1, \ldots, e_n, e_{2n+1})$ (resp. of $(e_{n+1}, \ldots, e_{2n+1})$) .

Let us denote by H_B the group of elements of $GL_{2n+1}(\mathbb{R})$ preserving the form B , and let H_B^0 denote the connected component (in the Euclidean topology) of the identity in H_B . Direct calculations show that, for every $j, 1 \le j \le n$, and every $t \in \mathbb{R}$, the transformation $h_j(t)$ defined by

$$e_i \to e_i, \;\; e_{n+i} \to e_{n+i} \;\text{ if }\; 1 \le i \le n, \; i \ne j$$

$$e_j \to \frac{e_j + e_{n+j}}{2} + \frac{\cos t\,(e_j - e_{n+j})}{2} + \frac{\sin t\, e_{2n+1}}{2}$$

$$e_{n+j} \to \frac{e_j + e_{n+j}}{2} - \frac{\cos t\,(e_j - e_{n+j})}{2} - \frac{\sin t\, e_{2n+1}}{2}$$

$$e_{2n+1} \to -\sin t\,(e_j - e_{n+j}) + \cos t\, e_{2n+1}$$

belongs to H_B . We have

$$h_j(\pi)e_i, = e_i, \;\; h_j(\pi)e_{n+i} = e_{n+i} \;\text{ if }\; 1 \le i \le n, \; i \ne j,$$

$$h_j(\pi)e_j = e_{n+j}, \;\; h_j(\pi)e_{n+j} = e_j,$$

$$h_j(\pi)e_{2n+1} = -e_{2n+1} \;.$$

Therefore, if we denote $h_1(\pi)\cdot\ldots\cdot h_n(\pi)$ by h_π , we have

$$h_\pi(e_i) = e_{n+i}, h_\pi(e_{n+i}) = e_i \quad \text{and} \quad h_\pi(e_{2n+1}) = (-1)^n e_{2n+1} \ . \tag{1}$$

Since $h_j(0) = \mathrm{Id}$ and $h_j(t) \in H_B$ depends continuously on t we have

$$h_\pi \in H_B^0 \ . \tag{2}$$

The orientations on V and $V^\perp$ defined above depend continuously on $V \in \Psi$. Hence these orientations are invariant under the action of the group H_B^0 . Now assuming that the basis $(e_1,\ldots,e_n)$ (resp. $(e_1,\ldots,e_n,e_{2n+1})$) is positively oriented in V_1 (resp. in $V_1^\perp$) we get from (1) and (2) that the basis $(e_{n+1},\ldots e_{2n})$ (resp. $(e_{n+1},\ldots,e_{2n},(-1)^n e_{2n+1})$) is positively oriented in V_2 (resp. in $V_2^\perp$) . This immediately implies the desired statement.

4. The sign of an affine transformation

In this section we define the notion of sign for certain affine transformations g . It tells us whether the translational part of g is positive or negative with respect to the orientation of the preceding section. We translate this into the orientations of the second section thus obtaining a criterion for (non–) proper discontinuity. The proof of theorem 2 will follow.

Let $H_B = SO(n+1,n)$ be the special orthogonal group of the quadratic form B on $\mathbb{R}^{2n+1}$ and let G_B be the subgroup of those elements of G_{2n+1} whose linear part is in H_B . Let g be a pseudohyperbolic element of G_B , i.e. $g \in G_B \cap \Omega$, then $\dim A^\circ(g) = 1$ and $\dim A^-(g) = \dim A^+(g) = n$. The subspaces $A^-(g)$ and $A^+(g)$ are isotropic, thus $A^-(g)$ and $A^+(g)$ are in Ψ . In the preceding section we fixed orientations on $A^+(g)$ and on $D^+(g) = A^+(g) \oplus A^\circ(g) = (A^+(g))^\perp$ and similarly on $A^-(g)$ and on $D^-(g) = A^-(g) \oplus A^\circ(g) = (A^-(g))^\perp$. We define the *sign* $\sigma(g)$ of $g \in G_B \cap \Omega$ as $+1$ or -1 or 0 according to whether the translational part $\tau(g)$ of g is a positive or a negative vector of $D^+(g) \setminus A^+(g)$

or is zero. Recall that $g \in \Omega$ is fixed point free, i.e. $g \in \Omega_0$, iff $\tau(g) = 0$. For $g \in G_B \cap \Omega$ and $m > 0$ we have

$$\sigma\left(g^m\right) = \sigma(g) \tag{4.1}$$

since $\tau\left(g^m\right) = m\,\tau(g)$ and $\overset{\pm}{A^{\circ}}(g) = \overset{\pm}{A^{\circ}}\left(g^m\right)$. For $g \in G_B \cap \Omega$ we have

$$\sigma\left(g^{-1}\right) = (-1)^{n+1}\sigma(g)\ . \tag{4.2}$$

This follows from $\tau\left(g^{-1}\right) = -\tau(g) \in A^{\circ}(g) = A^{\circ}\left(g^{-1}\right)$ and lemma 3.1 since $A^{+}(g)$ and $A^{-}(g) = A^{+}\left(g^{-1}\right)$ are transversal.

We shall need to compute $\sigma(g)$. Let $g \in G_B \cap \Omega$. Note that B restricts to a positive definite form on $A^{\circ}(g)$. Let $e^{\circ}(g)$ be the unique vector of $A^{\circ}(g)$ with $B\left(e^{\circ}(g), e^{\circ}(g)\right) = 1$ and positive with respect to the orientations of $A^{+}(g)$ and $D^{+}(g)$. Define $\alpha(g) \in \mathbb{R}$ by

$$\tau(g) = \alpha(g) e^{\circ}(g)\ . \tag{4.3}$$

Then

$$\sigma(g) = \operatorname{sgn}\,\alpha(g)\ . \tag{4.4}$$

We have

$$\alpha(g) = B\left(e^{\circ}(g), g\,x - x\right) \tag{4.5}$$

for every point x of the affine space. To see this recall the decomposition (2.1):

$$\mathbb{R}^n = A^{-}(g) \oplus A^{\circ}(g) \oplus A^{+}(g)\ . \tag{4.6}$$

Now (4.5) follows from (4.3) since the translational part $\tau(g)$ is the $A^{\circ}(g)$–component of $g\,x - x$ for every point x and $A^{\circ}(g)$ is orthogonal to both $A^{+}(g)$ and $A^{-}(g)$. Note that

$$\alpha\left(g^m\right) = m\,\alpha(g) \quad \text{for} \quad m > 0 \tag{4.7}$$

and

$$\alpha\left(g^{-1}\right) = (-1)^{n+1}\alpha(g) \qquad , \tag{4.8}$$

which is proved like (4.2) .

The following proposition is basic for our approach.

4.9 Proposition *Let* g_1, g_2 *be transversal pseudohyperbolic elements of* G_B *. If* $\sigma(g_1)\sigma\left(g_2^{-1}\right) < 0$ *then the group generated by* g_1 *and* g_2 *is not properly discontinuous.*

Proof Note that $g_1, g_2 \in G_B \cap \Omega_0$ since their signs are not zero. The proposition now follows from corollary 2.3 and the following lemma.

4.10 Lemma *Let* g_1, g_2 *be transversal elements of* $G_B \cap \Omega_0$ *. If* $\sigma(g_1)\sigma\left(g_2^{-1}\right) < 0$ *then* g_1 *and* g_2 *form a positive pair.*

Proof Given $g_i \in G_B \cap \Omega_0$ we have defined orientations on $V_i = A^+(g_i)$ and $V_i^\perp = D^+(g_i)$ and thus can talk about the positive half space of $V_i^\perp \setminus V_i$. The vector $e^0(g_i)$ defined before 4.3 is positive with respect to V_i . Let L be the line $V_1^\perp \cap V_2^\perp$. An element $\ell \neq 0$ in L is positive with respect to V_1 iff $(-1)^n \ell$ is positive with respect to V_2 , by lemma 3.1.. Write

$$\begin{aligned} \ell &= v_1 + \beta_1 \tau(g_1) \\ &= v_2 + \beta_2 \tau(g_2), \end{aligned}$$

$v_i \in V_i, \beta_i \in \mathbb{R}$. Then g_1, g_2 form a positive pair iff $\beta_1\beta_2 > 0$. But

$$\tau(g_i) = \alpha(g_i) e^0(g_i).$$

Thus, ℓ is positive for V_1 iff $\beta_1\alpha(g_1) > 0$ iff $(-1)^n\ell = (-1)^n v_2 + (-1)^n \beta_2 \alpha(g_2) e^0(g_2)$ is positive for V_2 iff $\beta_2(-1)^n\alpha(g_2) = -\beta_2\alpha(g_2^{-1}) > 0$, by (4.8). Thus g_1, g_2 form a positive pair iff $\alpha(g_1)\alpha(g_2^{-1}) < 0$, which proves our claim in view of (4.4).

4.11 Proof of theorem 2 Let Γ be a subgroup of G_n and let $H(\Gamma)$ be the Zariski closure of its linear part $\ell(\Gamma)$. Suppose we have $H(\Gamma)^\circ = H_B^\circ$ for the connected components with respect to the Euclidean topology. It then follows from the main algebraic result of [GM] that Γ contains an element γ_0 such that $\ell(\gamma)$ is pseudohyperbolic. To see this look a the natural representation ρ of H_B on $\bigwedge^n \mathbb{R}^{2n+1}$. The representation ρ restricted to H_B° is irreducible. Then $\rho(h)$ is proximal in the sense of [AMS 1] iff there is only one eigenvalue of h of modulus 1 , counting multiplicities, see [AMS 1, 5.1]. The main algebraic result of [GM] implies that $\rho(\ell(\gamma))$ contains a proximal element since $\rho(H_B\circ)$ does and the representation $\rho \mid H_B^\circ$ is irreducible, see [AMS 1, Theorem 4.1].

We have $\gamma_0 \in \Omega_0$ because otherwise all the eigenvalues of $\ell(\gamma_0)$ are $\neq 1$ and thus γ_0 has a fixed point and hence the cyclic subgroup of Γ generated by γ_0 does not act properly discontinuously on $\mathbb{R}^{2n+1}$. We can find an element $\gamma \in \Gamma$ such that

$$(A^+(\gamma_0) \cup A^-(\gamma_0) \cap \gamma(A^+(\gamma_0) \cup A^-(\gamma_0)) = 0 \tag{4.12}$$

since $H(\Gamma)^0 = H_B^\circ$. Put $\gamma_1 = \gamma\gamma_0\gamma^{-1}$. Then $A^+(\gamma_1) = \gamma A^+(\gamma_0)$ and $A^-(\gamma_1) = \gamma A^-(\gamma_0)$. Then 4.12 implies that γ_1 is transversal to both γ_0 and γ_0^{-1} . On the other hand we see from (4.2) for n even that the signs of γ_0 and γ_0^{-1} are different. It now remains to apply proposition 4.9 either to the pair $(g_1 = \gamma_0, g_2 = \gamma_1)$ or to the pair $(g_1 = \gamma_0^{-1}, g_2 = \gamma_1)$.

5. Pseudohyperbolicity of products

We are now heading for a proof of theorem 3. The method of proof is essentially the same as in [Ma 1, 2]. In this section we give a necessary condition for when the product of two linear maps is pseudohyperbolic. In the next section we prove a generalization of the basic additivity lemma. The rest of the proof is then exactly as in [Ma 1,2].

We need the following concept of pseudohyperbolicity which is both more general and more precise. Let V be a real vector space of dimension m . Fix a norm $\|\cdot\|$ on V and let d be the corresponding metric. Let $S(V)$ be the sphere $\{x \in V; \|x\| = 1\}$. Let $\Omega(p, s, \varepsilon)$, $p \in \{1, \ldots, m-1\}$, $0 < s < 1$, $\varepsilon > 0$, be the set of elements $g \in GL(V)$ with the following properties: There is a p–dimensional linear subspace V^+ of V and a complementary linear subspace V^- of V, both g–invariant, such that

$$\|g|V^-\| \leq s \cdot \|g^{-1}|V^+\|^{-1} \tag{5.1}$$

and $d(v^-, v^+) \geq \varepsilon$ for any two vectors $v^- \in S(V^-)$, $v^+ \in S(V^+)$, where, of course, $S(V^-) = S(V) \cap V^-$ etc. Note that

$$\|g^{-1}|V^+\|^{-1} = \min\{\|gv\|; v \in S(V^+)\}. \tag{5.2}$$

We have for every eigenvalue λ^- of g on V^- and λ^+ of g on V^+ :

$$|\lambda^-| \leq \|g|V^-\| \leq s\|g^{-1}|V^+\|^{-1} \leq s \cdot |\lambda^+|, \tag{5.3}$$

since the modulus of every eigenvalue is bounded above by the norm. Hence the spaces V^- and V^+ are uniquely determined by g and p, since $s < 1$, namely V^+ is the sum of the weight spaces of the p eigenvalues of largest modulus, counting multiplicities, and similarly V^- for the $n-p$ eigenvalues of smallest modulus.

5.4 Note that $g \in \Omega(p, s, \varepsilon)$ iff $g^{-1} \in \Omega(n-p, s, \varepsilon)$ with $V^-(g^{-1}) = V^+(g)$ and $V^+(g^{-1}) = V^-(g)$.

An element $g \in \bigcup_{s<1} \Omega(p, s, \varepsilon)$ will be called (p, ε)–pseudohyperbolic and we put for such g

$$s(g) := \|g|V^-\| \cdot \|g^{-1}|V^+\|. \tag{5.5}$$

So $s(g)$ is the smallest s such that $g \in \Omega(p, s, \varepsilon)$ for some $\varepsilon > 0$. Note that g is (r, ε)–proximal in the terminology of [AMS 1] iff g is $(1, \varepsilon)$–pseudohyperbolic with $s(g) \leq r^{-1}$. Two (p, ε)–pseudohyperbolic elements g, h are called (p, ε)–*transversal* if $d(v, w) \geq \varepsilon$ whenever $v \in S(V^-(g)) \cup S(V^-(h))$ and $w \in S(V^+(g)) \cup S(V^+(h))$.

5.6 Lemma *For any* $\varepsilon > 0$ *there is a constant* $a(\varepsilon) > 1$ *such that if* g *and* h *are* (p, ε)*–transversal,* $s(g) < a(\varepsilon)^{-1}$ *and* $s(h) < a(\varepsilon)^{-1}$ *then*

1) gh *is* $(p, \frac{\varepsilon}{2}$*–pseudohyperbolic*

2) $a(\varepsilon)^{-1}s(g)s(h) < s(gh) < a(\varepsilon)s(g)s(h)$

3) $d(V^+(gh), V^+(g)) < a(\varepsilon)s(g)$

4) $d(V^-(gh), V^-(h)) < a(\varepsilon)s(h)$.

Here d is the Hausdorff distance associated with $\|\cdot\|$: For any two compact subsets A, B of V put

$$d(A,B) = \left\{ \max_{b\in B} \min_{A\in A} d(a,b), \max_{a\in A} \min_{b\in B} d(a,b) \right\}$$

So $d(A,B)$ is the minimum of the numbers r such that $A \subset B_r(B)$ and $B \subset B_r(A)$, where $B_r(X) = \bigcup\limits_{x\in X} B_r(x)$ and $B_r(x) = \{y \in V; d(x,y) \le r\}$ is the ball of radius r with center x . For two linear subspaces V_1, V_2 of V put

$$d(V_1, V_2) = d(S(V_1), S(V_2)).$$

Proof We start with the case $p = 1$. We only show the claims 1), 3) and the second inequality of 2). The remaining inequality of 2) and also 4) will be obtained later by passing to inverses, using 5.4, but for a different dimension. If g is $(1,\varepsilon)$–pseudohyperbolic then $V^+(g)$ is a g–invariant line corresponding to a real eigenvalue of g which we call $\lambda(g)$, so $gv = \lambda(g)v$ for $v \in V^+(g)$.

5.7 Lemma *For every $\varepsilon > 0$ there is a constant $a(\varepsilon) > 1$ with the following property. For any two $(1,\varepsilon)$–transversal elements g, h in $GL(V)$ with $s(g) < a(\varepsilon)^{-1}$, $s(h) < a(\varepsilon)^{-1}$, $\lambda(g) > 0$ and $\lambda(h) > 0$ we have*

1) gh is $(1, \frac{\varepsilon}{2})$–proximal.

2) $s(gh) < a(\varepsilon)s(g)s(h)$

3) $a(\varepsilon)^{-1}\lambda(g)\lambda(h) < \lambda(gh) < a(\varepsilon)\lambda(g)\lambda(h)$

4) $d(V^+(gh), V^+(g)) < a(\varepsilon)s(g)$

Proof Let $B(x,r) = \{y \in V; d(x,y) \le r\}$ be the ball of radius r with center x . For any $(1,\varepsilon)$–pseudohyperbolic element g fix one vector $x^+(g)$ in $S(V^+(g))$. Put

$$g_1(x) = \frac{a(x)}{\|g(x)\|} \quad \text{for } x \neq 0$$

and

$$U(g) = B(x^+(g), \frac{\varepsilon}{2}).$$

It is easy to see that there is a constant $a(\varepsilon) > 0$ for every $\varepsilon > 0$ such that if g and h are $(1, \varepsilon)$–transversal with $s(g) \leq a(\varepsilon)^{-1}$ and $s(h) \leq a(\varepsilon)^{-1}$ and $\lambda(g)$ and $\lambda(h)$ are positive then

$$d(x^+(h), h_1 U(g)) < \min(a(\varepsilon)s(h), \frac{\varepsilon}{2}) \tag{5.8}$$

$$a(\varepsilon)^{-\frac{1}{2}} \lambda(h) \|x\| < \|h(x)\| < a(\varepsilon)^{\frac{1}{2}} \lambda(h) \|x\| \quad \text{for } x \in U(g) \tag{5.9}$$

$$\|h_1(x) - h_1(y)\| < a(\varepsilon)^{\frac{1}{2}} s(h) \|x - y\| \quad \text{for } x, y \in U(g). \tag{5.10}$$

The same holds with g und h interchanged. So we have by (5.9)

$$h_1 U(g) \subset \overset{\circ}{U}(h) \tag{5.11}$$

and

$$g_1 U(h) \subset \overset{\circ}{U}(g). \tag{5.12}$$

Thus $g_1 \circ h_1 : U(g) \to \overset{\circ}{U}(g)$ and

$$\|g_1 h_1(x) - g_1 h_1(y)\| < a(\varepsilon) s(g) s(h) \|x - y\| \tag{5.13}$$

for $x, y \in Y(g)$. It follows from Tits' lemma [AMS 1, 2.1] and thus basically from the Banach fixed point theorem that gh is proximal, the unique fixed point of $g_1 h_1$ in $U(g)$ is one of the two points in $S(V^+(gh))$ and $V^-(gh)$ does not intersect $\overset{\circ}{U}(g)$. This proves our claim 1), claim 4) follows from 5.8 and the corresponding inequality with g and h interchanged, and claim 3) follows from 5.9 for g and h . Finally 5.13 implies that $s(gh) < b(\varepsilon)^m s(g) s(h)$ for some constant $b(\varepsilon)$ depending only on ε and of course the given norm. We may assume that $b(\varepsilon) \geq a(\varepsilon)$ and thus replace $a(\varepsilon)$ by $b(\varepsilon)$ to retain everything and also obtain claim 2).

5.14 We shall constantly use the following notion. Fix p . Let $k \geq 1$ be a natural number and let two functions f_1, f_2 be given on some subset E of the direct product of k copies of $GL(V)$. We say that f_1 is *dominated* by f_2 and write $f_1 \ll f_2$ if for every $\varepsilon > 0$ there is a $c(\varepsilon) > 0$ such that $f_1(g_1, \ldots, g_k) \leq c(\varepsilon) f_2(g_1, \ldots, g_k)$ whenever $(g_1, \ldots, g_k) \in E$ and the elements $g_i, i = 1, \ldots, k$, are pairwise (p, ε)–transversal. If $f_1 \ll f_2$ and $f_2 \ll f_1$ then we write $f_1 \sim f_2$ and say that f_1 and f_2 are *equivalent.*

Proof of lemma 5.6 for arbitrary p Note that if $g \in \Omega(p,s,\varepsilon)$ then $\bigwedge^p g : \bigwedge^p V \longrightarrow \bigwedge^p V$ is proximal, i.e. has a unique eigenvalue of maximal modulus and this eigenvalue has multiplicity one. The corresponding line $V^+(\bigwedge^p g)$ is $\bigwedge^p V^+(g)$, the hyperplane $V^<(\bigwedge^p g)$ is $\{x \in \bigwedge^p V; x \wedge \bigwedge^{n-p} V^-(g) = 0\}$, the eigenvalue $\lambda^+(\bigwedge^p g)$ of maximal modulus is the product of the p eigenvalues of g of largest modulus and all the eigenvalues of $\bigwedge^p g$ on $V^<(\bigwedge^p g)$ are of modulus $\leq s \cdot |\lambda^+(\bigwedge^p g)|$.

In order to proceed we need the quantitative specifications of proximality as follows:

$$d(x^+(\bigwedge^p g), S(V^<(\bigwedge^p g))) \quad \sim \quad 1 \tag{5.15}$$

$$\frac{\|g \mid V^<(\wedge^p g)\|}{|\lambda^+(\bigwedge^p g)|} \quad \sim \quad s(g) \tag{5.16}$$

To see this we may assume that our norm comes from a positive definite quadratic form Q on V . Now change the form Q to the form Q_g with the following properties: V^- is orthogonal to V^+ with respect to Q_g , $Q_g|V^+ = Q|V^+$, $Q_g|V^- = Q|V^-$. Then $\sup\limits_{Q(x)=1} Q_g(x)$ and $\inf\limits_{Q(x)=1} Q_g(x)$ are ~ 1 as functions of g , i.e. there is a compact set $K = K(\varepsilon)$ of quadratic forms, depending on ε , such that $Q_g \in K$ for every (p,ε)–pseudohyperbolic g . For the induced form on $\bigwedge^p V$, also denoted Q_g , and which also belongs to a compact set $K'(\varepsilon)$ of quadratic forms, we then have $V^+(\wedge^p g) \perp V^<(\wedge^p g)$, which implies 5.15.

For orthonormal vectors $x_1, \ldots, x_r$ in V^+ , $x_{r+1}, \ldots, x_p$ in V^- we have for the norm $\|\cdot\|$ corresponding to Q_g :

$$\|g(x_1 \wedge \cdots \wedge x_p)\| = \|g(x_1 \wedge \cdots \wedge x_r)\| \cdot \|g(x_{r+1} \wedge \cdots \wedge x_p)\|. \tag{5.17}$$

The second factor can be estimated by

$$\|gx_{r+1} \wedge \cdots \wedge gx_p\| \leq \|gx_{r+1}\| \cdot \|gx_{r+2}\| \cdot \ldots \cdot \|gx_p\| \leq \|g \mid V^-\|^{p-r}, \tag{5.18}$$

since

$$\|z_1 \wedge \cdots \wedge z_{t+1}\| = \|z_1 \wedge \cdots \wedge z_t\| \cdot \|\pi(z_{t+1})\|$$

where π is the orthogonal projection of z_{t+1} onto the span of $z_1, \ldots, z_t$.

As to the first factor in (5.17), take the Cartan decomposition of $g \mid V^+ = k_1 \cdot d \cdot k_2$ where k_1, k_2 are unitary with respect to $Q_g \mid V^+ = Q \mid V^+$ and d is diagonal with entries $d_1 \geq d_2 \geq \cdots \geq d_p > 0$ with respect to an orthonormal basis $y_1, \dots, y_p$ of V^+. Assuming that $x_1, \dots, x_r$ forms part of the basis $k_2^{-1} y_1, \dots, k_2^{-1} y_p$, as we may, we obtain

$$\|g(x_1 \wedge \cdots \wedge x_r)\| = \prod_i d_i \leq d_1 \dots d_r, \tag{5.19}$$

where the index i runs through an r–element subset of $\{1, \dots p\}$. We thus obtain

$$\|g \mid V^<(\bigwedge^p g)\| < C \cdot s \cdot |\lambda^+(\bigwedge^p (g))|$$

for some constant C, since we have a corresponding inequality for every element of an orthonormal basis of $V^<(\bigwedge^p g)$ by 5.17 – 5.19. Changing back to the original quadratic form Q gives the inequality $\ll$ in 5.16. The opposite inequality is easily proved using the Cartan decompositions of $g \mid V^+$ and $g \mid V^-$ with respect to Q_g.

It follows now from lemma 5.7, the case $p = 1$, that there is a constant $a_1(\varepsilon) > 1$ such that if g and h are (p, ε)-transversal and $s(g) < a_1(\varepsilon)^{-1}$, $s(h) < a_1(\varepsilon)^{-1}$ then $\bigwedge^p(gh)$ is proximal since (p, ε)–transversality of g and h implies $(1, c(\varepsilon))$–transversality of $\bigwedge^p g$ and $\bigwedge^p h$ for some $c(\varepsilon) > 0$. The proof of 5.7 and of the Banach fixed point theorem show furthermore that then $V^+(\bigwedge^p(gh)) = \lim\,(gh)^n\ V^+(\bigwedge^p g)$ which implies that the sequence $(gh)^n\ V^+(g)$ converges to a p–dimensional linear subspace of V which we call V^+. Similarly, using inverses and 5.4, the sequence $(gh)^{-n}\ V^-(h)$ converges to an $n-p$–subspace V^- of V. By 5.7 4 we have

$$d(V^+(g), V^+) < a_2(\varepsilon) s(g) \tag{5.20}$$

and

$$d(V^-(h), V^-) < a_2(\varepsilon) s(h). \tag{5.21}$$

Finally, for $x \in U(g) := B(S(V^+(g)), \varepsilon/2)$ we have

$$a_3(\varepsilon)^{-1} \|\pi_h^+(x)\| \leq \|hx\| \leq a_3(\varepsilon) \|\pi_h^+(x)\| \tag{5.22}$$

where π_h^+ and π_h^- are the projections in $V = V^+(h) \oplus V^-(h)$, and similarly with g and h interchanged. This implies

$$\|(gh)(x)\| \geq a_4(\varepsilon)^{-1} \cdot \|g^{-1} \mid V^+(g)\|^{-1} \cdot \|h^{-1} \mid V(h)\|^{-1} \cdot \|x\|$$

for $x \in V^+$ by 5.20 and that there is a point $x \in S(V^+)$ with

$$\|gh(x)\| \leq a_4(\varepsilon) \cdot \|g^{-1} \mid V^+(g)\|^{-1} \cdot \|h^{-1} \mid V^+(h)\|^{-1} \cdot \|x\|,$$

since $\pi_h^+ : V^+ \longrightarrow V^+(h)$ is surjective and $\inf\limits_{x \in S^+(V)} \|\pi_h^+(x)\| \geq a_5(\varepsilon)\|x\|$, and similarly for g . Thus

$$\begin{aligned} &a_4(\varepsilon)^{-1} \cdot \|g^{-1} \mid V^+(g)\|^{-1} \cdot \|h^{-1} \mid V^+(h)\|^{-1} \\ &\qquad \leq \|(gh)^{-1} \mid V^+\|^{-1} \leq a_4(\varepsilon) \cdot \|g^{-1} \mid V^+(g)\|^{-1} \cdot \|h^{-1} \mid V^+(h)\|^{-1}. \end{aligned}$$

Similarly for $(gh)^{-1}$:

$$a_4(\varepsilon)^{-1} \cdot \|g \mid V^-(g)\| \cdot \|h \mid V^-(h)\| \leq \|gh \mid V^-\| \leq a_4(\varepsilon) \cdot \|g \mid V^-(g)\| \cdot \|h \mid V^-(h)\|.$$

Hence for $a(\varepsilon) = \max(2, a_i(\varepsilon))$ and $s(g) \leq a(\varepsilon)^{-1}$, $s(h) \leq a(\varepsilon)^{-1}$, we obtain that gh is pseudohyperbolic with $V^+(gh) = V^+$, $V^-(gh) = V^-$,

$$s(gh) \sim s(g)s(h)$$

and $d(V^+(gh), V^+) \ll s(g)$, $d(V^-(gh)\ ,\ V^-(h)) \ll s(h)$.

6. The basic lemma

The main claim of the basic lemma states additivity of the function α of 4.3. The proof of theorem 3 then follows as in [Ma 1,2].

So in this section we return to affine maps. Recall that $H_B = SO(n+1, n)$ is the orthogonal group of the form B on $\mathbb{R}^{2n+1}$ and G_B is the subgroup of $G_{2n+1} = \mathrm{Aff}(\mathbb{R}^{2n+1})$ of those elements with linear part in H_B . Also, Ω was the set of those $g \in G_n$ for which $\dim A^\circ(g) = 1$ and the eigenvalue of $\ell(g) \mid A^\circ(g)$ is $+1$, see paragraphs following 2.1. These elements were called pseudohyperbolic in sections 2–4. Let us clarify how this notion of pseudohyperbolic is related to the one of section 5. If $\ell(g)$ is in $\Omega(n,s,\varepsilon)$ of section 5 for some $s<1$ and $\varepsilon>0$ and $g\in G_B^\circ$ then g is pseudohyperbolic. Conversely, if $g\in\Omega$ then some power of g is in $\Omega(n,s,\varepsilon)$ for some $s<1$ and $\varepsilon>0$. Cf. the corresponding discussion for proximality in [AMS 1, section 2].

We need some auxiliary lemmas before we can prove the basic lemma.

6.1 Lemma

a) *Let* $g\in G_B^\circ$, $\ell(g)\in\Omega(n,s,\varepsilon)$. *Then* $\ell(g)\in\Omega(n{+}1,s',b(\varepsilon))$ *for some constant* $b(\varepsilon)$ *and* $\frac{s'}{s}$ *and* $\frac{s}{s'}$ *are bounded, independently of* ε , *if* s *is sufficiently small.*

b) $d(v,w)\le c(\varepsilon)$ *for every* $g\in\Omega(n,s,\varepsilon), v\in S(A^\circ(g))$ *and every isotropic vector* w *with* $\|w\|=1$.

c) *For the given norm* $\|\cdot\|$ *and the norm* $\|\cdot\|_B$ *induced by* B *on* $A^\circ(g)$ *we have* $\|\cdot\|\sim\|\cdot\|_B$, *i.e.* $c_1(\varepsilon)>\sup\limits_{x\in A^\circ(g)}\frac{\|x\|}{\|x\|_B}>\inf\limits_{x\in A^\circ(g)}\frac{\|x\|}{\|x\|_B}>c_1(\varepsilon)^{-1}$ *for some* $c_1(\varepsilon)$ *and every* $g\in\Omega(n,s,\varepsilon)$.

d)
$$\|g\mid A^\circ(g)\|\sim 1$$
and
$$\|g^{-1}\mid A^\circ(g)\|\sim 1 .$$

e) $s(g)=\|g\mid D^-(g)\|\cdot\|g^{-1}\mid A^+(g)\|\sim\|g\mid A^-(g)\|\cdot\|g^{-1}\mid D^+(g)\|\sim s(g^{-1})$.

Proof a) Take a basis $e_1,\dots e_{2n+1}$ of V such that $B(e_i,e_j)=0$ for $i\ne j$ and $B(e_i,e_i)=1$ for $i\le n{+}1$, $B(e_i,e_i)=-1$ for $i>n{+}1$. Suppose our norm $\|\cdot\|$ is given by the Euclidean form Q with respect to this basis. Then for $a\in H_B$ we have $a^{-1}=Ba^tB^{-1}$. Write $a=k_1dk_2$ in Cartan form, so $k_1,k_2\in O(Q)$, $d=(d_1,\dots,d_{2n+1})$ is diagonal with $d_1\ge d_2\ge\cdots\ge d_{2n+1}>0$. Then a is (n,s,ε_1)–pseudohyperbolic iff $d_{n+1}\le sd_n$ iff a^t is (n,s,ε_2)–pseudohyperbolic iff $a^{-1}=Ba^tB^{-1}$ is (n,s,ε_3)–pseudohyperbolic, since $B\in O(Q)$ iff a is $(n+1,s,\varepsilon_4)$–pseudohyperbolic, by 5.4.

To see the claim concerning $b(\varepsilon)$, look at the map *orth* which associates to every subspace W of V its orthogonal space with respect to B . Regard it as a map from a subset of a Grassmannian to another Grassmannian. Then *orth* is continuous and the claim follows from $D^+(g)=A^+(g)^\perp$ and $D^-(g)^\perp=A^-(g)$.

b) Otherwise there are sequences g_i of (n, ε)–pseudohyperbolic elements such that $D^+(g_i)$ and $D^-(g_i)$ converge to spaces D^+, D^- containing an isotropic vector w . Then $A^+(g_i) = D^+(g_i)^\perp$ and $A^-(g_i) = D^-(g_i)^\perp$ converge to spaces $A^+ = D^{+\perp}$ and $A^- = D^{-\perp}$ containing w , contradicting (n, ε)–pseudohyperbolicity.

c) The set of pairs of spaces $(A^+(g), A^-(g))$, g (n, ε)–pseudohyperbolic, in the product of Grassmannians is compact, hence so is the set of $A^\circ(g)'$s , by part b) of the proof since $A^0(g) = D^+(g) \cap D^-(g)$.

d) follows from c) since g is a unitary operator on $A^\circ(g)$ with respect to $B \mid A^\circ(g)$.

e) is proved by the same method as a).

We also need the following inequality:

$$d(x, E^+(g)) \ll s(g) d(x, g^{-1}x) \tag{6.2}$$

for $s(g) \ll 1$.

To see this first note that $d(x, E^+(g)) \sim \|p^-(x - c)\|$ for arbitrary $c \in C(g)$ where p^- is the projection to $A^-(g)$ in the decomposition $V = A^-(g) \oplus A^\circ(g) \oplus A^+(g)$, by (n, s, ε)–pseudohyperbolicity. Thus

$$d(x, E^+(g)) \ll s(g) d(g^{-1}x, E^+(g))$$

by 6.1 c) and a). The triangle inequality gives

$$s(g)^{-1} d(x, E^+(g)) \ll d(g^{-1}x, x) + d(x, E^+(g))$$

which yields 6.2.

We are now ready for the basic lemma.

6.3 Basic lemma *Let* g, h *be* (n, ε)*–transversal elements of* G *with* $s(g) \ll 1$ *and* $s(h) \ll 1$ *. Then* gh *is* $(n, \frac{\varepsilon}{2})$*–pseudohyperbolic and for every point* q *we have*

$$d(q, C(gh)) \ll s(gh)(d(q, gh(q)) + d(q, (gh)^{-1}q)) \tag{a}$$

and

$$|\alpha(gh) - \alpha(g) - \alpha(h)| \ll d(C(g), C(h)) + s(h)|\alpha(h)| + s(g)|\alpha(g)| \tag{b}$$

Proof (Compare Drumm [D])

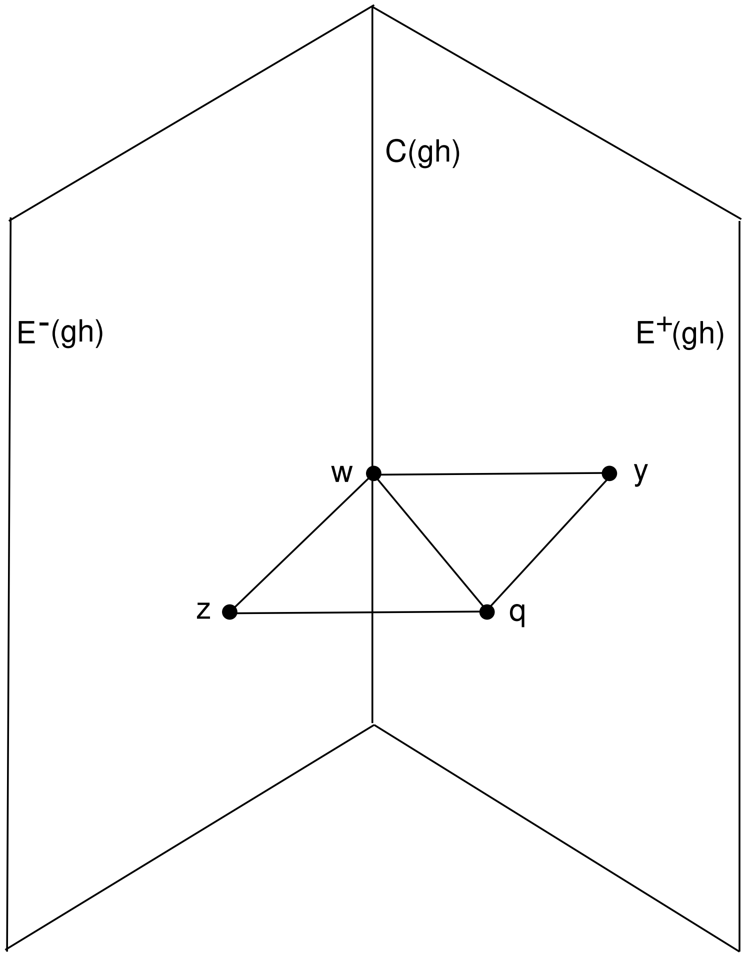

Figure 1

Let us choose $s(g)$ and $s(h)$ small enough so that gh is $(n, \frac{\varepsilon}{2})$–pseudohyperbolic and lemma 5.6 applies. Let again $\| \cdot \|$ be given by the Euclidean form Q with respect to a fixed basis of V . Let R be the hyperplane through q orthogonal to $C(gh)$. Let $w = R \cap C(gh)$, so $d(q, w) = d(q, C(gh))$. Let $y \in E^+(gh) \cap R$ be such that $d(q, y) = d(q, E^+(gh))$ and $z \in E^+(gh) \cap R$ such that $d(q, z) = d(q, E^-(gh))$. Note that $C(gh) = E^+(gh) \cap E^-(gh)$, see Figure 1.

The angle between $E^-(gh)$ and $E^+(gh)$ at w is bounded below since gh is $(n, \frac{\varepsilon}{2})$–pseudohyperbolic. Hence $d(q,w) \ll d(q,y)$ or $d(q,w) \ll d(q,z)$, thus

$$\begin{aligned} d(q, C(gh)) \ll d(q,y) + d(q,z) &= d(q, E^+(gh)) + d(q, E^-(gh)) \\ &\ll s(gh) d(q, (gh)^{-1} q) + s(gh) d(q, (gh)(q)) \end{aligned}$$

where we have used the inequality 6.2.

To show (b) first note that by bilinearity of B we have

$$B(\tau(g) + \tau(h), v) = B(\tau(g), v) + B(\tau(h), v).$$

So the left hand side of (b) is

$$|(B(\tau(g), v) - \alpha(g)) + B(\tau(h), v) - \alpha(h) + (\alpha(gh) - B(\tau(g) + \tau(h), v))|. \quad (6.4)$$

Note that $B(\tau(g), v) = 0$ for $v \in A^+(g) \oplus A^-(g)$ and

$$\alpha(g) = B(\tau(g), x_g^\circ + A^+(g) + A^-(g)).$$

Take $v = x_{gh}^\circ$. Then 6.4 is bounded above by

$$|B(\tau(g), v - (A_g^+ + x_g^\circ))| + |B(\tau(h), v - (A_h^- + x_h^\circ))| + |\alpha(gh) - B(\tau(g) + \tau(h), v)|. \quad (6.5)$$

The first term of 6.5 is bounded by $\|\tau(g)\| \cdot \|v - (A_g^+ + x_g^\circ)\|$ if we again take the Euclidean norm. We show that $\|v - (A_g^+ + x_g^\circ)\| \ll s(g)$. For $v = x_{gh}^\circ$ there is a point $w \in D^+(g) = \mathbb{R} x_g^\circ + A_g^+$ of distance $\ll s(g)$, by 5.6 3), say $w = \mu x_g^\circ + a$, $\mu \in \mathbb{R}$, $a \in A_g^+$. Then $|\mu| \sim 1$ since $A^+(g)$ is $\ll s(g)$–close to $A^+(gh)$ and $v = x_{gh}^\circ$ is $\varepsilon/2$–distant from $A^+(gh)$, hence x_{gh}° is $\varepsilon/4$–distant from $A^+(g)$. Finally, $\mu > 1$ by continuity of the orientations: $D^+(g)$ is $\ll s(g)$-close to $D^+(gh)$ and $A^+(g)$ is $\ll s(g)$-close to $A^+(gh)$.

Similarly for the second summand of 6.5

$$|B(\tau(h), x_{gh}^\circ - (A_h^- + x_h^\circ))| \ll \|\tau(h)\| \cdot s(h).$$

To estimate the third term of 6.5 note that

$$\alpha(gh) = B(gy - h^{-1} y, x_{gh}^\circ)$$

for any y . So the third term of 6.5 is

$$|B((gy-y)+(y-h^{-1}y)-\tau(g)-\tau(h),x^{\circ}_{gh})|. \tag{6.6}$$

Take now $u \in C(g)$ and $z \in C(h)$ such that $d(u,z)=d(C(g),C(h))$, $y=(A^{-}(g)+u)\cap E^{+}(h)$ and $w=(A^{+}(h)+y)\cap C(h)$, see Figure 2.

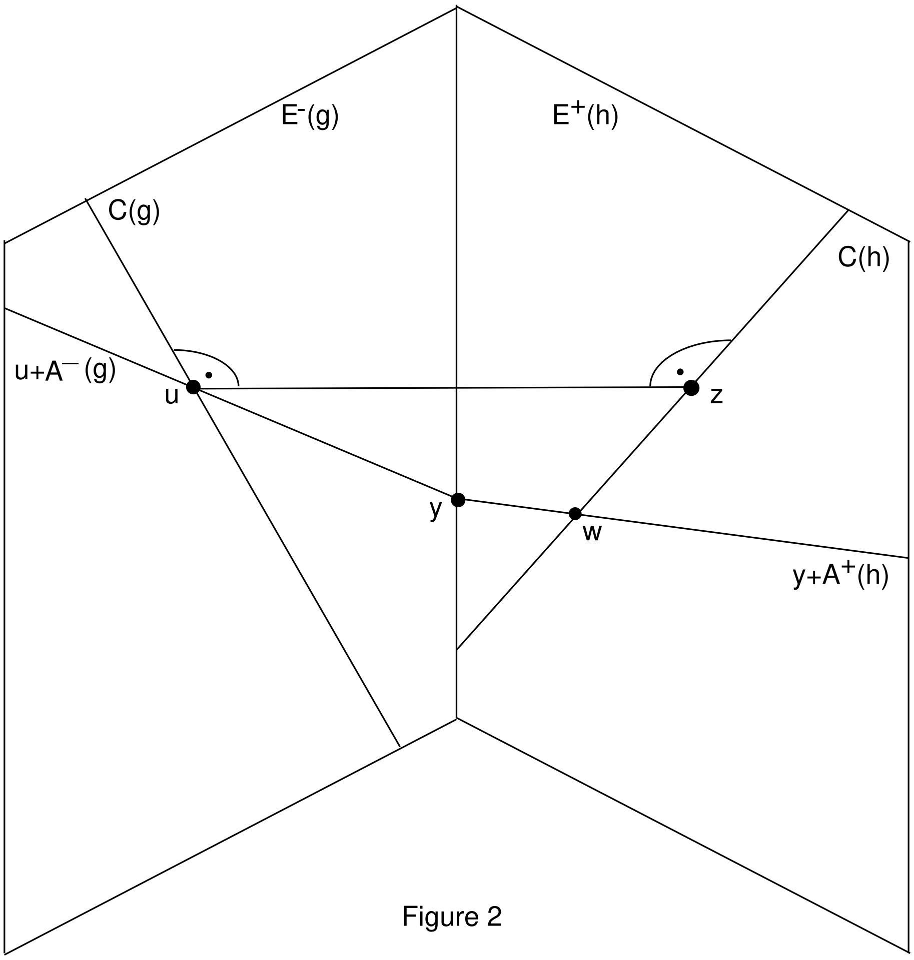

Figure 2

We can regard y as $u+r$ with $r\in A^{-}(g)$ and $\|r\|=d(u,y)$ or as $w+t$ with $t\in A^{+}(h)$ and $\|t\|=d(w,y)$. Thus $gy-y=(gu-u)+\ell(g)r=\tau(g)+\ell(g)r$ and similarly $(y-h^{-1}y)-\tau(h)=\ell(h^{-1})t$ and hence 6.6 can be estimated by $(d(u,y)+d(y,w))\cdot\|x^{\circ}_{gh}\|$. We have $d(u,y)\ll d(y,z)$, by the law of sines for the triangle u,y,z , since g and h are (n,ε)–transversal. We have $d(y,w)\ll d(y,z)$ by the law of sines since h is (n,ε)–transversal. And $d(y,z)\le d(u,z)+d(u,y)$ by the triangle inequality so that finally 6.6, the third term of 6.5 is bounded by $\ll d(C(g),C(h))$, thus completing the proof of 6.3 b).

The rest of the proof of theorem 3 now works exactly as in case $n = 1$, see [Ma 1,2], compare [D].

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