# Invariants of $2 \times 2$ matrices

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Studying invariants of a set of $2 \times 2$ matrices is important for the theory of Kleinian and Fuchsian groups. There are many papers treating this subject (see, e.g., [FK], [Si], [He1], [Ho], [Ra], [Pr], [Tr], [LM], [Le], [Re], [Vi2], [VHM], [He2] and references there). The present paper has the following aims:

1) to systematize the results on invariants of $2 \times 2$ matrices;
2) to present them in a convenient form;
3) to show that they are easy consequences of the classical description of orthogonal invariants of vectors and standard facts of geometric invariant theory.

Some minor observations may be considered new.

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## 1. Statement of the main theorems.

We consider the action of the group $G = SL_2(\mathbb{C})$ on $G^n := \underbrace{G \times \cdots \times G}_{n}$ by simultaneous conjugations:

$$g \circ (X_1, \ldots, X_n) = (gX_1g^{-1}, \ldots, gX_ng^{-1}).$$

We put $G^* = G \times C_2$ , where $C_2$ is a cyclic group of order $2$ with generating element $\iota$ . The action of $G$ is extended to an action of $G^*$ by

$$\iota \circ (X_1, \ldots, X_n) = (X_1^{-1}, \ldots, X_n^{-1}).$$

For any set

$$\mathbf{X} = (X_1, \ldots, X_n) \in G^n$$

we put

$$t_{ij}(\mathbf{X}) = t_{ij} = \operatorname{tr} X_i X_j^{-1} \ (i,j = 0,1,\dots,n),$$

where $X_0 = E$ . In particular, we have

$$t_{ii} = 2, \quad t_{ij} = t_{ji}. \tag{1}$$

The $(n+1)\times(n+1)$ matrix

$$T(\mathbf{X}) = T = (t_{ij})$$

is called *the trace matrix* of $\mathbf{X}$ .

**Theorem 1** *The algebra* $\mathbb{C}[G^n]^{G^*}$ *of* $G^*$*–invariant polynomial functions on* $G^n$ *is generated by* $t_{ij}$ *'s, with defining relations* (1) *and*

$$\operatorname{rk} T \leq 4. \tag{2}$$

(The latter relations mean that all the minors of order 5 of $T$ equal 0. In fact, since $t_{00} = 2$ , one may restrict oneself with the minors containing $t_{00}$ ).

Consider *the factorization morphism*

$$\pi : G^n \to S, \quad \mathbf{X} \longmapsto T(\mathbf{X}),$$

where $S$ denotes the variety of $(n+1)\times(n+1)$ matrices satisfying (1) and (2).

**Theorem 2** *The morphism* $\pi$ *is surjective. Its fiber* $\pi^{-1}(T)$ *over* $T \in S$ *is described as follows:*

1) *if* $\operatorname{rk} T = 4$ *, the fiber consists of a single 3–dimensional* $G^*$*–orbit, which decomposes into two* $G$*–orbits;*

2) *if* $\operatorname{rk} T = 3$ *, the fiber consists of a single 3–dimensional* $G^*$*–orbit, which is one* $G$*–orbit;*

3) *if* $\operatorname{rk} T = 2$ *, the fiber consists of a 2–dimensional* $G^*$*–orbit, which is one* $G$*–orbit, and an* $(n-2)$*–parameter family of 3–dimensionl* $G^*$*–orbits, each of them decomposes into two* $G$*–orbits (if* $n = 2$ *, the family consists of a single orbit; if* $n = 1$ *, it is empty);*

4) *if* $\operatorname{rk} T = 1$ *, the fiber consists of a fixed point of* $G^*$ *and an* $(n-1)$*–parameter family of 2–dimensional* $G^*$*–orbits, each of them is one* $G$*–orbit (if* $n = 1$ *, the family consists of a single orbit).*

We can consider matrices of $M_2(\mathbb{C})$ as linear operators in $\mathbb{C}^2$ . In cases 1 and 2 any set of matrices in the fiber is irreducible. In cases 3 and 4 any set of matrices in the fiber is reducible, i.e. can be sinultaneoulsy put into a triangular form. For more details see Section 5.

From now on, let $\mathbf{X} = (X_1, \ldots, X_n) \in G^n$ be an irreducible set.

*The trace field* of $\mathbf{X}$ is the field $L$ generated over $\mathbb{Q}$ by all the traces $\operatorname{tr} X_{i_1} X_{i_2} \ldots X_{i_s}$ . The $L$–algebra generated by $X_1, \ldots, X_n$ is an $L$–form of $M_2(\mathbb{C})$ and either splits, i.e. isomorphic (and then conjugate) to $M_2(L)$ , or is a generalized quaternionic division algebra over $L$ .

Let us denote by $\Delta_{i_1 \ldots i_r}$ the principal minor of order $r$ of the trace matrix $T$ contained in the rows with numbers $i_1, \ldots, i_r$ .

**Theorem 3** *The trace field* $L$ *is at most quadratic extension of the field* $L_0$ *generated over* $\mathbb{Q}$ *by* $t_{ij}$ *'s. More precisely, if* $\operatorname{rk} T = 4$ *, then* $L = L_0\left(\sqrt{\Delta_{0ijk}}\right)$ *, where* $\Delta_{0ijk}$ *is any non–zero principal minor of order 4 of* $T$ *containing* $t_{00}$ *; if* $\operatorname{rk} T = 3$ *, then* $L = L_0$ *.*

**Theorem 4** *Let the rows of* $T$ *with numbers* $0$ *,* $i$ *,* $j$ *be linearly independent. The* $L$*–algebra generated by* $X_1, \ldots, X_n$ *, splits if and only if the quadratic form defined by the minor* $\Delta_{0ij}$ *is isotropic.*

*The invariant trace field* of $\mathbf{X}$ is the field $K$ generated over $\mathbb{Q}$ by all the traces $\operatorname{tr}\left(X_{i_1}^{\varepsilon_1} \ldots X_{i_s}^{\varepsilon_s}\right)^2$ , where $\varepsilon_p = \pm 1$ , or, equivalently, by $\left(\operatorname{tr} X_{i_1}^{\varepsilon_1} \ldots X_{i_r}^{\varepsilon_r}\right)^2$ .

If the subgroup generated by $X_1, \ldots, X_n$ is Zariski dense in $G$ (which means that it is infinite and is not contained in the normalizer of a maximal torus), then the invariant trace field is the least field, over which all the linear operators $\operatorname{Ad}(X_1), \ldots, \operatorname{Ad}(X_n)$ can be written simultaneously [Vi 1]. It does not change when replacing $X_1, \ldots, X_n$ by any matrices generating a commensurable subgroup [Vi 1].

The invariant trace field was also studied by A.W. Reid and W.D. Neumann [Re], [NR], who introduced the very name.

**Theorem 5** *The invariant trace field $K$ is at most quadratic extension of the field $K_0$ generated over $\mathbb{Q}$ by* the cyclic products

$$t_{i_1 i_2} t_{i_2 i_3} \dots t_{i_p i_1}. \tag{3}$$

*More precisely,*

$$K = K_0 \left( \sqrt{\Delta_{0ijk}} t_{ii_1} t_{i_1 i_2} \dots t_{i_p 0} t_{jj_1} t_{j_1 j_2} \dots t_{jqk} \right), \tag{4}$$

*where $i, j, k, i_1, \dots, i_p, j_1, \dots, j_q$ are chosen in any way, provided the product does not vanish; if such a choice is impossible, $K = K_0$ .*

Such a choice is possible if and only if the matrix $T$ is either indecomposable of rank 4, or a direct sum of two indecomposable matrices of rank 2.

If the subgroup generated by $X_1, \dots, X_n$ is a lattice in $SL_2(\mathbb{C})$ or $SL_2(\mathbb{R})$ , one may wonder, whether it is arithmetic. Finding the fields $L$ and $K$ plays the key role in solving this problem [Vi 2].

## 2. Some basic facts of invariant theory.

In this section we consider complex algebraic varieties and complex algebraic groups. The topological terms refer to the Zariski topology. A general reference for the facts of geometric invariant theory stated below is [PV].

Let us first recall that any orbit of an action of an algebraic group is open in its closure. It follows that the boundary of an orbit has a lesser dimension than the orbit itself. The dimension of an orbit equals the dimension of the group minus the dimension of the stabilizer of any point of the orbit.

Let now a reductive algebraic group $G$ act on an affine algebraic variety $X$ . We denote by $\mathbb{C}[X]$ the algebra of polynomial functions on $X$ and by $\mathbb{C}[X]^G$ the subalgebra of $G$ –invariants.

**Fact 1.** For any invariant closed subvariety $Y \subset X$ the restriction homomorphism $\mathbb{C}[X]^G \to \mathbb{C}[Y]^G$ is surjective.

According to the Hilbert–Nagata theorem, the algebra $\mathbb{C}[X]^G$ is finitely generated. Its spectrum is called *the categorical quotient* of $X$ with respect to $G$ and denoted by $X/\!\!/G$. The embedding $\mathbb{C}[X]^G \subset \mathbb{C}[X]$ induces a morphism

$$\pi_G : X \to X/\!\!/G,$$

which is called *the factorization morphism.*

If $\mathbb{C}[X]^G = \mathbb{C}[f_1, \dots, f_m]$ and $I$ is the ideal of algebraic relations between $f_1, \dots, f_m$, then $X/\!\!/G$ can be understood as the zero variety of $I$ in $\mathbb{C}^m$, and $\pi_G$ as the mapping

$$x \longmapsto (f_1(x), \dots, f_m(x)) \quad (x \in X).$$

**Fact 2.** $\pi_G$ is surjective.

This means that, for any point $(c_1, \dots, c_m) \in \mathbb{C}^m$ whose coordinates subject all the algebraic relations between $f_1, \dots, f_m$, there exists a point $x \in X$ such that $f_i(x) = c_i \, (i = 1, \dots, m)$.

Obviously, the fibers of $\pi_G$ consist of entire orbits. But in general a fiber may contain more than one orbit.

**Fact 3.** Any fiber of $\pi_G$ contains exactly one closed orbit.

This orbit is the orbit of the minimal dimension in the fiber.

We are going to apply this theory to the action of the group $GL_k(\mathbb{C})$ on

$$M_k(\mathbb{C})^n := \underbrace{M_k(\mathbb{C}) \oplus \cdots \oplus M_k(\mathbb{C})}$$

defined by

$$g \circ (X_1, \dots, X_n) = (gX_1g^{-1}, \dots, gX_ng^{-1}).$$

In this case, a description of closed orbits was obtained by M. Artin.

Let us consider matrices of $M_n(\mathbb{C})$ as linear operators in $\mathbb{C}^n$.

**Theorem** [Ar], [Ri]. The orbit of a set $\mathbf{X} = (X_1, \dots, X_n) \in M_k(\mathbb{C})^n$ is closed if and only if this set is completely reducible.

In the general case, the only closed orbit in the fiber of $\mathbf{X}$ (in fact, in the closure of the orbit of $\mathbf{X}$ ) is described as follows. Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{p-1} \subset V_p = \mathbb{C}^n$$

be a maximal chain of subspaces invariant under $X_1, \ldots, X_n$ . Let $U_j (j = 1, \ldots, p)$ be a complementary subspace of $V_{j-1}$ in $V_j$ , and $Y_{ij}$ be the linear operator induced by $X_i$ in $U_j$ . Put $Y_i = \bigoplus_{j=1}^{p} Y_{ij}$ . Then the set $\mathbf{Y} = (Y_1, \ldots, Y_n)$ belongs to the closed orbit contained in the closure of the orbit of $\mathbf{X}$ .

Note that for $k = 2$ there are only two possibilities: either the set $(X_1, \ldots, X_n)$ is irreducible, and then its orbit is closed, or it can be put into a triangular form, and then the sets in the closed orbit contained in its closure can be put into a diagonal form with the same diagonal entries.

## 3. Some formulas in $M_2(\mathbb{C})$ .

For $X \in M_2(\mathbb{C})$ denote by $X^*$ the algebraic adjoint matrix of $X$ defined by

$$X^* = (\operatorname{tr} X)E - X.$$

The mapping $X \longmapsto X^*$ is an involutary anti–automorphism of the algebra $M_2(\mathbb{C})$ . In view of the Hamilton–Cayley equation,

$$XX^* = X^*X = (\det X)E.$$

In particular, $X^* = X^{-1}$ for $X \in G = SL_2(\mathbb{C})$ .

The decomposition of $M_2(\mathbb{C})$ into the sum of eigenspaces of $*$ looks as follows:

$$M_2(\mathbb{C}) = \mathbb{C}E \oplus M_2^0(\mathbb{C}),$$

where

$$\begin{aligned} M_2^0(\mathbb{C}) &= \{X \in M_2(\mathbb{C}) : X^* = -X\} = \\ &= \{X \in M_2(\mathbb{C}) : \operatorname{tr} X = 0\}. \end{aligned}$$

For any $X \in M_2(\mathbb{C})$ we put

$$\overset{\circ}{X} = X - \frac{1}{2}(\operatorname{tr} X)E \in M_2^0(\mathbb{C}).$$

Introduce in $M_2(\mathbb{C})$ a scalar product by

$$(X, Y) = \operatorname{tr} XY^* = -\operatorname{tr} XY + (\operatorname{tr} X)(\operatorname{tr} Y) \tag{5}$$

Obviously, it is invariant under $*$ .

Note, that

$$(X, E) = \operatorname{tr} X,$$
$$(X, X) = \operatorname{tr} XX^* = 2 \det X.$$

For $U, X, Y, Z \in M_2(\mathbb{C})$ we define $\det(U, X, Y, Z)$ as the determinant of the matrix of the coordinates of $U, X, Y, Z$ with respect to some fixed basis of $M_2(\mathbb{C})$ . Since the determinant of $*$ equals $-1$ , we have

$$\det(U^*, X^*, Y^*, Z^*) = -\det(U, X, Y, Z). \tag{6}$$

The product

$$\det(U, X, Y, Z) \cdot \det(U', X', Y', Z')$$

is equal to the determinant of the mutual Gram matrix of the systems $(U, X, Y, Z)$ and $(U', X', Y', Z')$ multiplied by a factor $c$ depending on the choice of the basis, with respect to which the determinants are defined. For the basis consisting of the matrix units, a straightforward calculation with concrete matrices shows that $c = 1$ .

The determinant $\det(U, X, Y, Z)$ can be expressed in terms of traces. Namely, for $X, Y \in M_2^0(\mathbb{C})$ we have

$$(XY + YX)^* = Y^*X^* + X^*Y^* = YX + XY,$$

whence

$$XY + YX \in \mathbb{C}E.$$

It follows that the trilinear form $\operatorname{tr} XYZ$ is skew–symmetric on $M_2^0(\mathbb{C})$ , so for any $X, Y, Z \in M_2(\mathbb{C})$ we have

$$\det(E, X, Y, Z) = \det(E, \overset{\circ}{X}, \overset{\circ}{Y}, \overset{\circ}{Z}) = a \operatorname{tr} \overset{\circ}{X}\overset{\circ}{Y}\overset{\circ}{Z} \quad (a \in \mathbb{C}).$$

Under the above choice of the basis in $M_2(\mathbb{C})$ , one can show that $a = -2$ . The general case reduces to this one by means of multiplying $U, X, Y, Z$ by $U^{-1}$ .

Conversely, making use of the above formula, we obtain for any $X, Y, Z \in M_2(\mathbb{C})$ :

$$\operatorname{tr} XYZ = \operatorname{tr}(\overset{\circ}{X} + \frac{1}{2}(\operatorname{tr} X)E)(\overset{\circ}{Y} + \frac{1}{2}(\operatorname{tr} Y)E)(\overset{\circ}{Z} + \frac{1}{2}(\operatorname{tr} Z)E) =$$
$$= -\frac{1}{2}\det(E, X, Y, Z) + R(\operatorname{tr} YZ, \operatorname{tr} ZX, \operatorname{tr} XY, \operatorname{tr} X, \operatorname{tr} Y, \operatorname{tr} Z), \qquad (7)$$

where $R$ is a polynomial with rational coefficients.

## 4. Proof of Theorem 1.

Consider the action of the group $G^*$ introduced in Section 1 on the space $M_2(\mathbb{C})$ defined by

$$g \circ X = g\, X\, g^{-1}, \quad \iota \circ X = X^*.$$

Obviously, the scalar product (5) is $G^*$–invariant.

Since $X^* = X^{-1}$ for $X \in G$ , the action of $G^*$ on $G$ coincides with that defined in Section 1.

The group $G^*$ fixes $E$ , while the action of $G^*$ (resp. $G$ ) on $M_2^0(\mathbb{C})$ reduces to the action of $O_3(\mathbb{C})$ (resp. $SO_3(\mathbb{C})$ ).

It follows that the algebra of $G^*$–invariants (resp. $G$–invariants) of a set of matrices $X_1, \ldots, X_n \in M_2(\mathbb{C})$ is the tensor product of the polynomial algebra $\mathbb{C}[\operatorname{tr} X_1, \ldots, \operatorname{tr} X_n]$ and the algebra of invariants of the set of vectors $\overset{\circ}{X}_1, \ldots, \overset{\circ}{X}_n \in M_2^0(\mathbb{C})$ with respect to $O_3(\mathbb{C})$ (resp. $SO_3(\mathbb{C})$ ).

It is a classical result (see, e.g., [We]) that the algebra of orthogonal invariants of a set of vectors $x_1, \ldots, x_n \in \mathbb{C}^3$ is generated by the scalar products $(x_i, x_j)$ , with defining relations

$$\operatorname{rk}((x_i, x_j)) \leq 3, \qquad (8)$$

which means that all the minors of order $4$ of the matrix $((x_i, x_j))$ equal $0$ .

If the group $O_3(\mathbb{C})$ is replaced by $SO_3(\mathbb{C})$ , one should add generators $\det(x_i, x_j, x_k)$ (the determinant of the matrix of the coordinates of $x_i, x_j, x_k$ with respect to a fixed basis) and defining relations of two types, one of which is

$$\det(x_i, x_j, x_k)\det(x_{i'}, x_{j'}, x_{k'}) = c\begin{vmatrix} (x_i, x_{i'}) & (x_i, x_{j'}) & (x_i, x_{k'}) \\ (x_j, x_{i'}) & (x_j, x_{j'}) & (x_j, x_{k'}) \\ (x_k, x_{i'}) & (x_k, x_{j'}) & (x_k, x_{k'}) \end{vmatrix},$$

where the factor $c$ depends on the fixed basis.

Going back to matrices, we should substitute $\overset{\circ}{X}_i$ 's instead of $x_i$ 's. We have

$$(\overset{\circ}{X}_i, \overset{\circ}{X}_j) = (X_i, X_j) - \frac{1}{2}(\operatorname{tr} X_i)(\operatorname{tr} X_j) =$$
$$= (X_i, X_j) - \frac{1}{2}(X_i, E)(X_j, E).$$

Hence the algebra of $G^*$–invariants of a set of matrices $X_1, \ldots, X_n \in M_2(\mathbb{C})$ is generated by the scalar products

$$(X_i, X_j)\ (i, j = 0, 1, \ldots, n), \tag{9}$$

where $X_0 = E$ . Obviously,

$$\operatorname{rk}((\overset{\circ}{X}_i, \overset{\circ}{X}_j))_{i,j=1}^n = \operatorname{rk}((X_i, X_j))_{i,j=0}^n - 1,$$

so the defining relations (8) can be represented in the form

$$\operatorname{rk}((X_i, X_j))_{i,j=0}^n \leq 4. \tag{10}$$

According to Fact 1 of Section 2, any invariant of a set of matrices of $G$ can be extended to an invariant of a set of matrices of $M_2(\mathbb{C})$ . Since for $X, Y \in G$ we have

$$(X, Y) = \operatorname{tr} XY^{-1},$$

Theorem 1 immediately follows.

To obtain $G$–invariants we should add to the scalar products (9) the determinants

$$\det(\overset{\circ}{X}_i, \overset{\circ}{X}_j, \overset{\circ}{X}_k) = \det(E, X_i, X_j, X_k). \tag{11}$$

Their pairwise products are minors of order $4$ of the Gram matrix $((X_i, X_j))_{i,j=0}^n$. In particular

$$(\det(E, X_i, X_j, X_k))^2 = \Delta_{0ijk}, \tag{12}$$

where $\Delta_{0ijk}$ is the principal minor of order $4$ of the Gram matrix contained in the rows with numbers $0, i, j, k$ .

It follows from the above consideration that the determinants (11) are defined by the scalar products (9) up to simultaneous multiplication by $-1$ . In view of (6) such a multiplication is realized when acting by $\iota$ .

Consequently, each fiber of $\pi_{G^*} = \pi$ decomposes into at most two fibers of $\pi_G$ , and it really decomposes into two fibers of $\pi_G$ if and only if not all of the determinants (11) vanish.

## 5. Proof of Theorem 2.

If a set $\mathbf{X} = (X_1, \ldots, X_n) \in G^n$ is reducible, then it can be put into a triangular form. Since the scalar product (5) is degenerate on the space of triangular matrices, we have $\operatorname{rk} T \leq 2$ in this case.

Consequently, if $\operatorname{rk} T \geq 3$ , the set $\mathbf{X}$ is irreducible, so its $G$–orbit is closed and exhausts the fiber $\pi_G^{-1}(T)$ (see Section 2). Its $G^*$–orbit is also closed and exhausts the fiber $\pi^{-1}(T)$ . It decomposes into two $G$–orbits if and only if not all of the determinants (12) vanish (see the end of Section 4), i.e. if $\operatorname{rk} T = 4$ ; otherwise it coincides with the $G$–orbit of $\mathbf{X}$ .

In view of the Schur lemma the stabilizer of an irreducible set $\mathbf{X}$ in $G$ reduces to $\{\pm E\}$ . Hence, the $G$–orbit of such a set, as well as its $G^*$–orbit, is 3–dimensional.

If $\operatorname{rk} T = 2$ , but $\operatorname{rk}(E, X_1, \ldots, X_n) = 3$ , then the linear span $\langle E, X_1, \ldots, X_n \rangle$ contains a non–zero matrix $U$ , which is orthogonal to all $X_i$'s $(i = 0, 1, \ldots, n)$ . In particular,

$$\operatorname{tr} U = (U, E) = 0,$$

$$2 \det U = (U, U) = 0.$$

This means that $U$ is nilpotent. We may assume that $U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then all $X_i$'s are upper triangular, and the linear span $\langle E, X_1, \ldots, X_n \rangle$ coincides with the space of all upper triangular matrices. The stabilizer of $\mathbf{X}$ in $G$ still reduces to $\{\pm E\}$, so the orbit of $\mathbf{X}$ is 3–dimensional.

More precisely, let

$$X_i = \begin{pmatrix} \lambda_i & x_i \\ 0 & \lambda_i^{-1} \end{pmatrix} \quad (i = 1, \ldots, n),$$

not all of $\lambda_i$'s being equal to $\pm 1$. Put

$$Y_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix} \quad (i = 1, \ldots, n).$$

Then the set $\mathbf{Y} = (Y_1, \ldots, Y_n)$ belongs to the closed orbit contained in $\pi^{-1}(T)$ (see Section 2). The stabilizer of $\mathbf{Y}$ in $G$ is the subgroup of diagonal matrices, hence the orbit of $\mathbf{Y}$ is 2–dimensional. Obviously, the set $(Y_1^{-1}, \ldots, Y_n^{-1})$ is $G$–equivalent to $\mathbf{Y}$ (by means of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$), while the set $(X_1^{-1}, \ldots, X_n^{-1})$ is not $G$–equivalent to $\mathbf{X}$. It follows that the $G^*$–orbit of $\mathbf{Y}$ coincides with the $G$–orbit, while the $G^*$–orbit of $\mathbf{X}$ decomposes into two $G$–orbits.

Let $\mathbf{X}' = (X_1', \ldots, X_n')$, where

$$X_i' = \begin{pmatrix} \lambda_i & x_i' \\ 0 & \lambda_i^{-1} \end{pmatrix} \quad (i = 1, \ldots, n).$$

If $\mathbf{X}' = g \circ \mathbf{X} \, (g \in G)$, then $g$ is a triangular matrix, say, $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, and

$$x_i' = a^2 x_i + (\lambda_i - \lambda_i^{-1}) ab.$$

Hence the family of orbits with given $\lambda_1, \ldots, \lambda_n$ (not all of them being equal to $\pm 1$) depends on $n - 2$ parameters.

If $\operatorname{rk}(E, X_1, \ldots, X_n) = 2$, then $\langle E, X_1, \ldots, X_n \rangle = \langle E, Z \rangle$ with $Z \in M_2(\mathbb{C}), Z \neq \lambda E$. If $Z$ is diagonalizable, then $\operatorname{rk} T = 2$ and $\mathbf{X}$ is diagonalizable. The orbit of such $\mathbf{X}$ is closed and belongs to a fiber of the type considered above.

If $Z$ is not diagonalizable, one may assume that $Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $\operatorname{rk} T = 1$ and

$$X_i = \begin{pmatrix} \varepsilon_i & x_i \\ 0 & \varepsilon_i \end{pmatrix} \quad (i = 1, \ldots, n),$$

where $\varepsilon_i = \pm 1$ and not all of $x_i$'s vanish. The stabilizer of $\mathbf{X}$ in $G$ is the subgroup of matrices of the form $\pm \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, so the orbit of $\mathbf{X}$ is 2–dimensional. The set $(X_1^{-1}, \dots, X_n^{-1})$ is $G$–equivalent to $\mathbf{X}$ (by means of $\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$), and hence the $G^*$–orbit of $\mathbf{X}$ coincides with the $G$–orbit.

In this case, the closed orbit contained in $\pi^{-1}(T)$ is the fixed point $\mathbf{Y} = (Y_1, \dots, Y_n)$, where

$$Y_i = \begin{pmatrix} \varepsilon_i & 0 \\ 0 & \varepsilon_i \end{pmatrix} \qquad (i = 1, \dots, n).$$

Let $\mathbf{X}' = (X_1', \dots, X_n')$, where

$$X_i' = \begin{pmatrix} \varepsilon_i & x_i' \\ 0 & \varepsilon_i \end{pmatrix} \qquad (i = 1, \dots, n).$$

If $\mathbf{X}' = g \circ \mathbf{X}\, (g \in G)$, then $g$ is triangular, say, $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, and

$$x_i' = a^2 x_i.$$

Hence the family of 2–dimensional orbits in $\pi^{-1}(T)$ depends on $n-1$ parameters (and represents the projective space $\mathbb{C}P^{n-1}$ ).

Finally, if $\operatorname{rk}(E, X_1, \dots, X_n) = 1$, then $X_i = \pm E \quad (i = 1, \dots, n)$, which is the case already considered.

## 6. Proof of Theorem 3.

As we proved in Section 4, the algebra $\mathbb{C}[G^n]^G$ is generated by the scalar products (9) and determinants (11). Let $\mathbb{Q}[G^n]$ denote the $\mathbb{Q}$–algebra of polynomials on $G^n$ with rational coefficients. Then

$$\mathbb{C}[G^n] = \mathbb{C} \otimes \mathbb{Q}[G^n]$$

and, since the action of $G$ is defined over $\mathbb{Q}$ ,

$$\mathbb{C}[G^n]^G = \mathbb{C} \otimes \mathbb{Q}[G^n]^G.$$

Note that the scalar products (9) and determinants (11) belong to $\mathbb{Q}[G^n]$ . Hence the $\mathbb{Q}$–algebra $\mathbb{Q}[G^n]^G$ is generated by them.

The scalar products (9) and determinants (11) are expressed as polynomials with rational coefficients in the traces $\mathrm{tr} X_{i_1} X_{i_2} \dots X_{i_s}$ (see Section 3) and, conversely, these traces, being elements of $\mathbb{Q}[G^n]^G$ , are expressed as polynomials with rational coefficients in the scalar products (9) and determinants (11).

It follows that the trace field $L$ of a concrete set $\mathbf{X} = (X_1, \dots, X_n) \in G^n$ is generated over $\mathbb{Q}$ by the numbers

$$t_{ij} = (X_1, X_j) = \mathrm{tr}\, X_i X_j^{-1} \ (i, j = 0, 1, \dots, n)$$

and

$$d_{ijk} = \det (E, X_i, X_j, X_k) = \sqrt{\Delta_{0ijk}} \ (i, j, k = 1, \dots, n).$$

However the pairwise products of $d_{ijk}$ 's are expressed as polynomials with rational coefficients in $t_{ij}$ 's (see Section 4). This consideration immediately implies Theorem 3.

## 7. Proof of Theorem 4.

An $L$–form of the algebra $M_2(\mathbb{C})$ splits, i.e. is isomorphic to $M_2(L)$ , if and only if the quadratic form $(X, X) = 2 \det X$ is isotropic on it. Theorem 4 is implied by the following lemma, which slightly strengthens the above assertion.

**Lemma.** The quadratic form $(X, X) = 2 \det X$ is isotropic on any 3–dimensional subspace of $M_2(L)$ .

**Proof** Any 3–dimensional subspace $P \subset M_2(L)$ is the orthogonal subspace of some non–zero matrix $Z \in M_2(L)$ . We are to prove that $P$ contains a non–zero degenerate matrix. If $Z = \lambda E$ , then $P$ contains, for example, the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Otherwise, there exists a non–zero vector $e \in L^2$ that is not an eigen-vector of $Z$ . The vectors $e$ and $Ze$ constitute a basis of $L^2$ . Passing to this basis, we may assume that $Z$ has the form

$$Z = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}.$$

Then $P$ contains the matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , which is degenerate.

## 8. Proof of Theorem 5.

Consider the action of the group $D = \{\pm 1\}^n$ on $G^n$ defined by

$$(\varepsilon_1, \ldots, \varepsilon_n) \circ (X_1, \ldots, X_n) = (\varepsilon_1 X_1, \ldots, \varepsilon_n X_n) \quad (\varepsilon_i = \pm 1).$$

It commutes with the action of $G$ and hence induces an action of $D$ on $\mathbb{Q}[G^n]^G$ . It is proved in [Vi 2] that the $\mathbb{Q}$–algebra generated by the polynomials $\operatorname{tr}(X_{i_1}^{\varepsilon_1} \ldots X_{i_s}^{\varepsilon_s})^2$ $(\varepsilon_p = \pm 1)$ coincides with the $\mathbb{Q}$–algebra

$$(\mathbb{Q}[G^n]^G)^D = \mathbb{Q}[G^n]^{G \times D}$$

and is generated by the polynomials of the following four types:

$$\operatorname{tr} X_{i_1} \cdot \operatorname{tr} X_{i_1} X_{i_2} \cdot \operatorname{tr} X_{i_2} X_{i_3} \cdot \ldots \cdot \operatorname{tr} X_{i_{p-1}} X_{i_p} \cdot \operatorname{tr} X_{i_p}, \tag{13}$$

$$\operatorname{tr} X_{i_1} X_{i_2} \cdot \operatorname{tr} X_{i_2} X_{i_3} \cdot \ldots \cdot \operatorname{tr} X_{i_p} X_{i_1}, \tag{14}$$

$$\begin{aligned}\operatorname{tr} X_i X_j X_k \cdot \operatorname{tr} X_i X_{i_1} \cdot \operatorname{tr} X_{i_1} X_{i_2} \cdot \ldots \cdot \operatorname{tr} X_{i_{p-1}} X_{i_p} \cdot \operatorname{tr} X_{i_p} \times \\ \times \operatorname{tr} X_j X_{j_1} \cdot \operatorname{tr} X_{j_1} X_{j_2} \cdot \ldots \cdot \operatorname{tr} X_{j_{r-1}} X_{j_r} \cdot \operatorname{tr} X_{j_r} \times \\ \times \operatorname{tr} X_k X_{k_1} \cdot \operatorname{tr} X_{k_1} X_{k_2} \cdot \ldots \cdot \operatorname{tr} X_{k_{s-1}} X_{k_s} \cdot \operatorname{tr} X_{k_s}, \end{aligned} \tag{15}$$

$$\begin{aligned}\operatorname{tr} X_i X_j X_k \cdot \operatorname{tr} X_i X_{i_1} \cdot \operatorname{tr} X_{i_1} X_{i_2} \cdot \ldots \cdot \operatorname{tr} X_{i_{p-1}} X_{i_p} \cdot \operatorname{tr} X_{i_p} \times \\ \times \operatorname{tr} X_j X_{j_1} \cdot \operatorname{tr} X_{j_1} X_{j_2} \cdot \ldots \cdot \operatorname{tr} X_{j_q} X_k. \end{aligned} \tag{16}$$

In view of (5) and (7) we may replace $\operatorname{tr} X_i X_j$ with $(X_i, X_j)$ and $\operatorname{tr} X_i X_j X_k$ with $\det(E, X_i, X_j, X_k)$. Moreover, if we introduce $X_0 = E$ and replace $\operatorname{tr} X_i$ with $(X_i, X_0)$, then (13) will become a partial case of (14), and (15) a partial case of (16).

It follows that the invariant trace field $K$ of a concrete set $\mathbf{X} = (X_1, \ldots, X_n) \in G^n$ is generated over $\mathbb{Q}$ by the cyclic products (3) and the numbers

$$d_{ijk} t_{ii_1} t_{i_1 i_2} \ldots t_{i_p 0} t_{jj_1} t_{j_1 j_2} \ldots t_{j_q k},$$

where

$$d_{ijk} = \det(E, X_i, X_j, X_k) = \sqrt{\Delta_{0ijk}}.$$

However the pairwise products of the latter numbers are expressed as polynomials with rational coefficients in the cyclic products. This implies Theorem 5.

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