Revised and Enlarged Version

Irreducible Modules of Quantized Enveloping Algebras at Roots of 1, II

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0. Introduction

A central topic in representation theory is to understand the irreducible modules. In this paper we are concerned with finite dimensional irreducible modules of quantized hyperalgebras and of hyperalgebras. Recently, big progress has been made on the characters of the irreducible modules, namely the Lusztig's conjectural formula for the characters is true for quantized hyperalgebras and is true for hyperalgebras when the characteristic of the groud field is sufficiently large, see [AJS, KL2, KT, L6].

In [X2] we see that certain monomials in a quantized hyperalgebra U_ξ (resp. a hyperalgebra $\mathfrak{U}_\mathfrak{k}$) at a root of 1 are interesting in realizing the finite dimensional irreducible U_ξ-modules (resp. $\mathfrak{U}_\mathfrak{k}$-modules). The realization seems useful in understanding the irreducible modules, for example, based on the realization and assuming char$\mathfrak{k}$=2, Xu and Ye constructed explicitly certain bases for finite dimensional irreducible $\mathfrak{U}_\mathfrak{k}$-modules for type A_4, B_3, C_3, A_5, see [XY, Y1, Y2] and then get the irreducible characters. Until now it seems difficult to get the information by other methods.

One purpose of this paper is to give a unified proof and to romove a restriction for the results in [X2, §5]. Due to technical reasons, the proof in [X2] is lengthy and involves case by case analysis, what is worse, it doesnot work for type G_2. In this paper we will give a simple and unified proof for the main results in [X2, §5], the proof works for all types. To do this we work in the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand, see §2.

Another purpose is to give some applications of the main results in [X2], such as (a) the description of the socle of a quantized Frobenius kernel (see 3.3), (b) a realization of the tensor product of two irreducible modules of a quantized Frobenius kernel (see 3.5), (c) calculation of the dimensions of some irreducible modules for type A (see 4.5-6). In [X4-5] we show that the main results in [X2] can be used to find out some (quasi) primitive elements in a Verma modules. For type A_2, B_2 we can find all (quasi) primitive elements in a Verma module of a quantized Frobenius kernel and in a Weyl module, see [X4-5].

The contents of the paper are as follows. In section 1 we introduce the basic notation and recall some basic facts. In section 2 we give a unified proof for the main results of [X2,§5], this section is based on the preprint RIMS-1067. In section 3 we describe the socle of the regular representation of $\mathbf{u}$ (the quantized Frobenius kernel) and realize the tensor product of two irreducible $\mathbf{u}$-modules as a submodule of the regular module, this section is based on part of preprint RIMS-1087. In section 4 we consider some submodules and quotient modules of the Verma modules of $\mathbf{u}$. These modules seem interesting, for example, by means of them we can compute the dimensions of irreducible modules for type A.

All results in §§3-4 have their counterparts for hyperalgebras, we will not state them in the paper.

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1. Preliminaries

1.1. Let U be the quantized enveloping algebra over $\mathbf{Q}(v)$ (v an inderterminate) corresponding to a Cartan matrix (a_{ij}) of rank n. Then U is an associative $\mathbf{Q}(v)$-algebra with generators $E_i,\ F_i,\ K_i, K_i^{-1}$ $(i = 1, 2, ..., n)$ which satisfy the quantized Serre relations. The algebra U has a Hopf algebra structure. Let $U_{\mathcal{A}}$ be the $\mathcal{A} = \mathbf{Q}[v, v^{-1}]$-subalgebra of U generated by all divided powers $E_i^{(a)},\ F_i^{(a)}$ and $K_i,\ K_i^{-1}$. We refer to [L4] for the definitions, noting that for defining the divided powers we need to choose integers $d_i \in \{1, 2, 3\}$ such that $(d_i a_{ij})$ is symmetric.

Let $\xi \in \mathbf{C}^*$. Regard $\mathbf{Q}(\xi)$ as an $\mathcal{A}$-algebra by specializing v to ξ. Consider the quantized hyperalgebra $U_\xi = U_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbf{Q}(\xi)$. For simplicity, the images in U_ξ of $E_i^{(a)},\ F_i^{(a)},\ K_i$, K_i^{-1}, $\begin{bmatrix} K_i, c \\ a \end{bmatrix}$ (see [L4] for the definition), etc. will be denoted by the same notations respectively. The algebra U_ξ inherits a Hopf algebra structure from that of $U_{\mathcal{A}}$. The tensor product of two U_ξ-modules then has a natural U_ξ-module structure by means of the coproduct. We denote by $U_\xi^+,\ U_\xi^0,\ U_\xi^-$ the positive part, zero part, negative part, respectively. The algebra U_ξ is noetherian. One may see this by a standard argument (similar to that for the universal enveloping algebra of a semisimple Lie algebra).

When $\xi^2 = 1$, U_ξ is generated by all E_i, F_i, K_i. In this case, U_ξ is not far from the universal enveloping algebra $\mathfrak{U}$ (over $\mathbf{Q}$) corresponding to the Cartan matrix. More precisely, U_ξ contains a big subalgebra isomorphic to $\mathfrak{U}$, see [X3, ch. 11]

1.2. Let $R \subset \mathbf{Z}^n$ be the root system with simple roots $\alpha_i = (a_{1i}, a_{2i}, ..., a_{ni})$. For $\mu = (\mu_1, \mu_2, ..., \mu_n) \in \mathbf{Z}^n$, we also write $\langle \mu, \alpha_i^\vee \rangle$ for μ_i. Define $s_i : \mathbf{Z}^n \to \mathbf{Z}^n$ by $s_i\mu = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i$. The reflections $s_1, s_2, ..., s_n$ generate the Weyl group W of the root system R.

For $\alpha \in R^+$ (the set of positive roots in R), let $E_\alpha \in U_\xi^+$ and $F_\alpha \in U_\xi^-$ be the corresponding root vectors defined by a fixed reduced expression of the longest element of W and the automorphisms $T'_{i,-1}$ of U_ξ in [L5, 37.1.3]. We define the divided powers $E_\alpha^{(a)},\ F_\alpha^{(a)}$ as usual.

Let l_i be the order of ξ^{2d_i} and let $l_\alpha = l_i$ if $\alpha = w(\alpha_i)$ for some $w \in W$. Let $\mathbf{u}$ be the subalgebra of U_ξ generated by $E_\alpha, F_\alpha, K_i, K_i^{-1}$ for all α in $R_l^+ = \{\alpha \in R^+ \mid l_\alpha \geq 2\}$ and $i = 1, 2, ..., n$. We call $\mathbf{u}$ the Frobenius kernel of U_ξ. The Frobenius kernel $\mathbf{u}$ is a Hopf subalgebra of U_ξ. The tensor product of two $\mathbf{u}$-modules then has a natural $\mathbf{u}$-module structure and for a $\mathbf{u}$-module M we may define its dual modules M^* and $M^\star$ by means of the antipode and its inverse respectively.

1.3. Let $\gamma \in \mathbf{Z}R$. An element x in U_ξ is said to have degree γ if $K_i x K_i^{-1} = \xi^{\langle \gamma, \alpha_i^\vee \rangle d_i} x$ and $\begin{bmatrix} K_i, c \\ a \end{bmatrix} x = x \begin{bmatrix} K_i, c + \langle \gamma, \alpha_i^\vee \rangle \\ a \end{bmatrix}$ for $i = 1, 2, ..., n$, $c \in \mathbf{Z}$, $a \in \mathbf{N}$. We also call x a homogeneous element (of degree γ) and write $\deg(x) = \gamma$.

Let U'_ξ be a subalgebra of U_ξ containing U_ξ^0 and let M be a U'_ξ-module. For $\lambda \in \mathbf{Z}^n$ we define the λ-weight space M_λ of M as usual (see for examples, [APW, L]). An element $m \in M_\lambda$ is said to have weight λ. We call λ a weight of M if M_λ is not zero. If an element x in U'_ξ has degree γ, then obviously $xM_\lambda \subseteq M_{\lambda+\gamma}$. A nonzero element m in M is called primitive if $m \in M_\lambda$ for some $\lambda \in \mathbf{Z}^n$ and $xm = 0$ for all $x \in U'_\xi$ with positive degree.

Denote by $M(\lambda)$ be the Verma module of U_ξ with highest weight $\lambda \in \mathbf{Z}^n$. Let 1_λ be a non-zero element in $M(\lambda)_\lambda$. Denote by $L(\lambda)$ the unique irreducible quotient module. Write $\bar{v}_\lambda$ for the image in $L(\lambda)$ of 1_λ.

We say that an element $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{Z}_+^n$ is $\mathbf{l}$-restricted if $\lambda_1 \leq l_1 - 1, ..., \lambda_n \leq l_n - 1$. We shall denote by $\mathbf{Z}_{+,\mathbf{l}}^n$ the set of all $\mathbf{l}$-restricted elements in $\mathbf{Z}_+^n$. For each $\mu = (\mu_1, \mu_2, ..., \mu_n) \in \mathbf{Z}^n$ we set $\mathbf{l}\mu := (l_1\mu_1, l_2\mu_2, ..., l_n\mu_n)$. If λ is $\mathbf{l}$-restricted, then the restriction to $\mathbf{u}$ of $L(\lambda)$ is an irreducible $\mathbf{u}$-module [L3, Prop. 5.11].

1.4. Now we introduce certain monomials, which play a central role in the paper. Let $s_{i_1} s_{i_2} \cdots s_{i_k}$ be a reduced expression of an element $w \in W$. For $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{Z}_+^n$ and $j \in [1, k]$, let $a_j = \langle s_{i_{j-1}} \cdots s_{i_1}, \alpha_{i_j}^\vee \rangle$, then define

$$x_{\lambda,w} = F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)}, \qquad x'_{\lambda,w^{-1}} = F_{i_k}^{(a_k)} F_{i_{k-1}}^{(a_{k-1})} \cdots F_{i_1}^{(a_1)} \in U_\xi^-$$

and $\dot{x}_{\lambda,w} = E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} \cdots E_{i_k}^{(a_k)} \in U_\xi^+$. When w is the longest element w_0, we simply write x_λ, x'_λ, $\dot{x}_\lambda$ for $x_{\lambda,w}$, $x'_{\lambda,w}$, $\dot{x}_{\,w}$ respectively. These monomials are well defined [X2, Lemma 1.4].

Set $\kappa = (l_1 - 1, l_2 - 1, ..., l_n - 1)$, and write E_κ, F_κ for $E_{\beta_1}^{(l_{\beta_1}-1)} E_{\beta_2}^{(l_{\beta_2}-1)} \cdots E_{\beta_r}^{(l_{\beta_r}-1)} = \dot{x}_\kappa$, $F_{\beta_r}^{(l_{\beta_r}-1)} F_{\beta_{r-1}}^{(l_{\beta_{r-1}}-1)} \cdots F_{\beta_1}^{(l_{\beta_1}-1)} = x_\kappa$, respectively, where β_i are the positive roots defined through the reduced expression used to define the root vectors.

2. A Unified Proof of the Main Results in [X1, §5]

In this section we will give a unified proof for the main results in [X1, §5]. We first introduce the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand for U_ξ and discuss some simple properties. Then we show that $M(-\rho)$ is injective and projective in the category. From this we deduce that if l is $\mathbf{l}$-restricted then x_λ is contained in the negative part $\mathbf{u}^-$ of $\mathbf{u}$ (see 2.14). This is a key to the main results in loc.cit.

2.1.The Category $\mathcal{O}$ It is routine to define the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand. The objects of the category $\mathcal{O}$ are (left) U_ξ-modules M having the following properties:

1) M is a finitely generated U_ξ-module;

2) M is the direct sum of its weight spaces, i.e. $M = \oplus_{\lambda\in\mathbf{Z}^n} M_\lambda$;

3) M is U_ξ^+-finite, that is, $\dim U_\xi^+ f < \infty$ for any $f \in M$.

The morphisms of $\mathcal{O}$ are arbitrary U_ξ module morphisms (see [BGG]).

Obviously the Verma modules, Weyl modules and the irreducible modules $L(\lambda)$ are objects of $\mathcal{O}$. Let us list the basic properties of $\mathcal{O}$ and of Verma modules.

(i) $\mathcal{O}$ is an abelian category with finite direct sums. The space $\mathrm{Hom}(M, M')$ is finite-dimensional for any $M, M' \in \mathcal{O}$. (Note that U_ξ is noetherian, so any submodule of M is an object of $\mathcal{O}$ if M is an object of $\mathcal{O}$.)

(ii) Let $M \in \mathcal{O}$, $\lambda \in \mathbf{Z}^n$. Then $\dim M_\lambda < \infty$. Let $P(M) = \{\lambda \in \mathbf{Z}^n \mid M_\lambda \neq 0\}$. Then there exists a finite number of weights $\lambda_1, \lambda_2, ..., \lambda_k$ such that $P(M) \subset \cup(\lambda_i + \mathbf{N}R^-)$, for example, $P(M(\lambda)) = \lambda + \mathbf{N}R^-$. In particular, if $M \neq 0$, then $P(M)$ contains at least one maximal weight λ, i.e., such that $P(M) \cap \{\lambda + \mathbf{N}R^+\} = \lambda$.

(iii) The modules $L(\lambda)$ ($\lambda \in \mathbf{Z}^n$) are pairwise nonisomorphic and are exhausted by all the simple objects in $\mathcal{O}$. $L(\lambda)$ is finite-dimensional if and only if $\lambda \in \mathbf{Z}^n_+$.

(iv) Let M, N be objects of $\mathcal{O}$. If M is finite dimensional, then both $M \otimes N$ and $N \otimes M$ are objects of $\mathcal{O}$.

2.2. It is convenient to introduce the formal characters of U_ξ-modules for studying the the category $\mathcal{O}$.

Let $\mathfrak{X}$ be the set of maps $f : \mathbf{Z}^n \to \mathbf{Z}$ such that the support $\{\lambda \in \mathbf{Z}^n \mid f(\lambda) \neq 0\}$ of f is contained in a finite union of the sets $\{\mu - \mathbf{N}R^+\}$ $(\mu \in \mathbf{Z}^n\}$. We define $(f+g)(\lambda) = f(\lambda) + g(\lambda)$ and $(fg)(\lambda) = \sum_{\mu+\nu=\lambda} f(\mu)g(\nu)$. Then $\mathfrak{X}$ is a commutative ring. Denote by e_λ the characteristic function $e_\lambda(\lambda) = 1,\ e_\lambda(\mu) = 0$ if $\mu \neq \lambda$. Then $e_\lambda e_\mu = e_{\lambda+\mu}$. We often write the elements in $\mathfrak{X}$ as $\mathbf{Z}$-linear combinations of e_λ $(\lambda \in \mathbf{Z}^n)$. We also write 1 for e_0.

For a U_ξ-module M, we define $\mathrm{ch}M = \sum_{\lambda\in\mathbf{Z}^n} \dim_{\mathbf{Q}(\xi)}(M_\lambda)e_\lambda$. When M is an object of $\mathcal{O}$, $\mathrm{ch}M$ is contained in $\mathfrak{X}$. We set $\mathrm{ch}U_\xi^- = \sum_{\mu\in\mathbf{N}R^+} e_{-\mu}$, $\mathrm{ch}U_{\xi,l}^- = \sum_{\mu\in\mathbf{N}R^+} e_{-\mathbf{l}\mu}$ and $\mathrm{ch}\mathbf{u}^- = \prod_{\alpha\in R^-}(1 + e_\alpha + \cdots + e_{(l_\alpha-1)\alpha})$. We have $\mathrm{ch}M(\lambda) = \sum_{\mu\in\mathbf{N}R^+} e_{\lambda-\mu} = e_\lambda \sum_{\mu\in\mathbf{N}R^+} e_{-\mu} = e_\lambda \mathrm{ch}U_\xi^-$. The following properties are trivial.

(i) $\mathrm{ch}U_\xi^-$=$\mathrm{ch}U_{\xi,l}^-\mathrm{ch}\mathbf{u}^-$, see [L4, L5].

(ii) $\mathrm{ch}L(\mathbf{l}\mu) = e_{\mathbf{l}\mu}\mathrm{ch}U_{\xi,l}^-$ for all $\mu \in -\mathbf{Z}_+^n - \rho$, see [L4, L5]. As usual, $\rho = (1,1,...,1) \in \mathbf{Z}_+^n$.

(iii) $\mathrm{ch}V(\kappa) = \mathrm{ch}L(\kappa) = e_\kappa \mathrm{ch}\mathbf{u}^-$, where $\kappa = (l_1-1, l_2-1, ..., l_n-1)$, see [X2, Prop. 2.8].

(iv) Let M be an object in $\mathcal{O}$. If $\mathrm{ch}M$=$\mathrm{ch}L$ for some simple object in $\mathcal{O}$, then M is isomorphic to L, in particular, M is simple.

Some Verma modules are simple, more precisely we have

Lemma 2.3. Let $\mu \in \mathbf{Z}_+^n$. Then the Verma module $M(-\mathbf{l}\mu - \rho)$ is simple, so is equal to $L(-\mathbf{l}\mu - \rho)$.

Proof. Using 2.2 (i)-(iii) and tensor product theorem [L2] we get $\mathrm{ch}M(-\mathbf{l}\mu-\rho)$= $e_{-\mathbf{l}\mu-\rho}\mathrm{ch}U_\xi^-$=$(e_{-\mathbf{l}(\mu+\rho)}\mathrm{ch}U_{\xi,l}^-)(e_\kappa\mathrm{ch}\mathbf{u}^-)$ =$\mathrm{ch}L(-\mathbf{l}(\mu+\rho))\mathrm{ch}L(\kappa)$=$\mathrm{ch}L(-\mathbf{l}(\mu+\rho)+\kappa)$ = $\mathrm{ch}L(-\mathbf{l}\mu-\rho)$. By 2.2 (iv) we see the lemma is true.

We now show that each object in $\mathcal{O}$ has a finite Jordan-Hölder compositiom series.

Lemma 2.4. Let M be a U_ξ-module such that M is equal to the sum of its weight spaces. Assume that $\mathrm{ch}M = \sum_{\lambda\in\mathbf{Z}^n} a_\lambda \mathrm{ch}L(\lambda)$ $(a_\lambda \in \mathbf{N})$, where only finitely many a_λ's are nonzero. Then M has finite length (i.e. has a finite Jordan-Hölder compositiom series), and the composition factors of M are just those $L(\lambda)$ counted with multiplicity a_λ. As a consequence, M is an object of $\mathcal{O}$.

Proof. We use induction on $a_M = \sum_{\lambda \in \mathbf{Z}^n} a_\lambda$. When a_M is 0,1, the assertion is obvious. Now assume that $a_M \geq 2$. In $\{\lambda \in \mathbf{Z}^n \mid a_\lambda \neq 0\}$ we choose a maximal element μ. Let $m \in M_\mu$ be a nonzero element. Then m is primitive and $N = U_\xi m$ has a unique maximal submodule N_1, the quotient N/N_1 is isomorphic to $L(\mu)$. Let $P = M/N \oplus N_1$. Then $P = \sum_{\lambda \in \mathbf{Z}^n} P_\lambda$ and $\mathrm{ch}P = (a_\mu - 1)\mathrm{ch}L(\mu) + \sum_{\lambda \in \mathbf{Z}^n - \{\mu\}} a_\lambda \mathrm{ch}L(\lambda)$. Note that $a_P = a_M - 1$. By induction hypothesis, P has finite length, so both M/N and N_1 have finite length. Therefore M has finite length.

2.5. Assume that the Cartan matrix (a_{ij}) is indecompable. For $1 \leq i \leq n$, let $\alpha_i^* = \alpha_i$ if $l_1 = l_2 = \cdots = l_n$, and let $\alpha_i^* = (a_{i1}, a_{i2}, ..., a_{in}) \in \mathbf{Z}^n$ if some l_k, l_m are not equal. Define $s_i^* : \mathbf{Z}^n \to \mathbf{Z}^n$ by $s_i^* \mu = \mu - \langle \mu, \alpha_i^\vee \rangle \alpha_i^*$. The reflections $s_1^*, ..., s_n^*$ generate a finite group W^*, which is W when all l_i are equal, and is the Weyl group corresponding to the transpose of (a_{ij}) when some l_k, l_m are not equal.

Let $\lambda \in \mathbf{Z}^n$. Define $W_\lambda^* = \{w \in W^* \mid w(\lambda + \rho) = \lambda + \rho\}$. Then W_λ^* is a Coxeter subgroup of W^*, denote its longest element by w_λ^*. For $y \in W^*$ we define $\lambda_y = y(\lambda + \rho) - \rho$. We denote $l(w)$ the length of an element w of W^* and use $\leq$ for the Bruhat order of W^*.

Lemma 2.6. Assume that (a_{ij}) is indecompable. Let $\lambda \in \mathbf{Z}_+^n - \rho$ and let $y \in W^*$ be such that $l(yw_\lambda^*) = l(y) + l(w_\lambda^*)$. Then

$$e_{\mathbf{l}\lambda_y} \mathrm{ch}U_{\xi,l}^- = \sum_{\substack{w \in W^* \\ l(ww_\lambda^*) = l(w) + l(w_\lambda^*) \\ yw_\lambda^* \leq ww_\lambda^*}} P_{yw_\lambda^*, ww_\lambda^*}(1) \mathrm{ch}L(\mathbf{l}\lambda_w),$$

where $P_{yw_\lambda^*, ww_\lambda^*}$ is the Kazhdan-Lusztig polynomial.

Proof. Let $\mathfrak{g}^*$ be the simple Lie algebra (over $\mathbf{Q}$) corresponding to the root system $R^* = W^*\{\alpha_1^*, ..., \alpha_n^*\}$ and let $\mathfrak{U}(\mathfrak{g}^*)^-$ be the negative part of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g}^*)$ of $\mathfrak{g}^*$. Let R_+^* be the set of positive roots of R^*. Define $\mathrm{ch}\mathfrak{U}(\mathfrak{g}^*)^- = \sum_{\mu \in \mathbf{N}R_+^*} e_{-\mu} \in \mathfrak{X}$. For any $\sum a_\mu e_\mu \in \mathfrak{X}$ we define $\mathbf{l} * \sum a_\mu e_\mu = \sum a_\mu e_{\mathbf{l}\mu}$. Then we have (see [L4, L5]

$$e_{\mathbf{l}\lambda_y} \mathrm{ch}U_{\xi,l}^- = \mathbf{l} * (e_{\lambda_y} \mathrm{ch}\mathfrak{U}(\mathfrak{g}^*)^-).$$

Thus we only need to prove the following identity

$$e_{\lambda_y}\mathrm{ch}\mathfrak{U}(\mathfrak{g}^*)^- = \sum_{\substack{w\in W^* \\ l(ww^*_\lambda)=l(w)+l(w^*_\lambda) \\ yw^*_\lambda \le ww^*_\lambda}} P_{yw^*_\lambda, ww^*_\lambda}(1)\mathrm{ch}L'(\lambda_w),$$

where $L'(\lambda_w)$ is the irreducible $\mathfrak{U}(\mathfrak{g}^*)$ module with highest weight λ_w and the character $\mathrm{ch}L'(\lambda_w)$ is defined in an obvious way, see 1.5 (iii) and 1.3.

According to [J, 2.11] we may assume that λ is regular, i.e. $W^*_\lambda = \{1\}$. By [BGG] we may assume that $\lambda = 0$. Now the above identity, which was conjectured by Kazhdan and Lusztig [KL1], follows from [BK] and [BB]. The lemma is proved.

Proposition 2.7. Each Verma module has finite length. Therefore each object of $\mathcal{O}$ has finite length.

Proof. Consider the Verma module $M(\mathbf{l}\mu+\lambda)$, $\mu \in \mathbf{Z}^n,\ \lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$. We have

$$\mathrm{ch}M(\mathbf{l}\mu+\lambda) = (e_{\mathbf{l}\mu}\mathrm{ch}U^-_{\xi,l})(e_\lambda \mathrm{ch}\mathbf{u}^-).$$

According to [X2, 2.2 (vi) and 2.4],

$$e_\lambda\mathrm{ch}\mathbf{u}^- = \sum_{\substack{\tau\in\mathbf{Z}^n \\ \delta\in\mathbf{Z}^n_{+,\mathbf{l}}}} a_{\lambda,\mathbf{l}\tau+\delta}e_{\mathbf{l}\tau}\mathrm{ch}L(\delta), \qquad a_{\lambda,\mathbf{l}\tau+\delta}\in\mathbf{N},$$

where only finitely many $a_{\lambda,\mathbf{l}\tau+\delta}$'s are nonzero. Thus

$$\mathrm{ch}M(\mathbf{l}\mu+\lambda) = \sum_{\substack{\tau\in\mathbf{Z}^n \\ \delta\in\mathbf{Z}^n_{+,\mathbf{l}}}} a_{\lambda,\mathbf{l}\tau+\delta}e_{\mathbf{l}\mu+\mathbf{l}\tau}\mathrm{ch}U^-_{\xi,l}\mathrm{ch}L(\delta).$$

Note that $\mathrm{ch}L(\mathbf{l}\nu+\delta) = \mathrm{ch}L(\mathbf{l}\nu)\mathrm{ch}L(\delta)$ for all $\nu\in\mathbf{Z}^n,\ \delta\in\mathbf{Z}^n_{+,\mathbf{l}}$. Using Lemma 2.6 we see

$$\mathrm{ch}M(\mathbf{l}\mu+\lambda) = \sum_{\nu\in\mathbf{Z}^n} a_\nu\mathrm{ch}L(\nu), \qquad a_\nu\in\mathbf{N},$$

where only finitely many a_ν's are not zero. By Lemma 2.4, $M(\mathbf{l}\mu+\lambda)$ has finite length.

2.8. We are going to show that $M(-\rho)$ is projective and injective as well in the category $\mathcal{O}$, then show that $M(\lambda-\rho)$ has a unique irreducible submodule for any $\lambda\in\mathbf{Z}^n_+$. Taking $\lambda\in\mathbf{Z}^n_{+,\mathbf{l}}$, we deduce the main result (Theorem 2.14) of this section.

Let M be an object in $\mathcal{O}$. Following [BGG] we define its dual $D(M) \in \mathcal{O}$ as follows. Let Ψ be the antiautomorphism of U_ξ defined by $\Psi(E_i^{(a)}) = F_i^{(a)}$, $\Psi(F_i^{(a)}) = E_i^{(a)}, \Psi(K_i^{\pm 1}) = K_i^{\pm 1}$. Let M^* be the space of linear functionals on M. We define the action of U_ξ on M^* by $(x\theta)(m) = \theta(\Psi(x)m)$, $\theta \in M^*$, $m \in M$, $x \in U_\xi$. As $D(M)$ we take submodule of M^* generated by the vectors charateristic with respect to U_ξ^0. It can be proved that $D(M) \in \mathcal{O}$, and that $\mathrm{Hom}(M,N) = \mathrm{Hom}(D(N), D(M))$. In addition, $D(D(M))$ is naturally isomorphic to M. It is clear from the construction that $\dim D(M)_\lambda = \dim M_\lambda$ for all $\lambda \in \mathbf{Z}^n$. In particular, $D(L(\lambda)) = L(\lambda)$.

Given a finite dimensional U_ξ-module E in $\mathcal{O}$, we define the dual modules E^*, $E^\star$ as in [APW, 1.18] by means of the antipode of U_ξ and its inverse respectively. Then (loc.cit)

(i) We have $(E^*)^\star \simeq E \simeq (E^\star)^*$.

(ii) For any U_ξ-modules M, N, one has

$$\mathrm{Hom}_{U_\xi}(M, N \otimes E) \simeq \mathrm{Hom}_{U_\xi}(M \otimes E^\star, N),$$

$$\mathrm{Hom}_{U_\xi}(E^* \otimes M, N) \simeq \mathrm{Hom}_{U_\xi}(M, E \otimes N).$$

Lemma 2.9. Let M be an object of $\mathcal{O}$.

(i) Assume that $M(-\rho)$ is a submodule of M. Then $M(-\rho)$ is a direct summand of M, i.e. there exists a submodule M' of M such that M is isomorphic to $M(-\rho) \oplus M'$.

(ii) Assume that $M(-\rho)$ is a quotient module of M. Then $M(-\rho)$ is a direct summand of M.

(iii) Assume that $M(-\rho)$ is a composition factor of M. Then $M(-\rho)$ is a direct summand of M.

Proof. Since $\mathrm{Hom}(M,N) = \mathrm{Hom}(D(N), D(M))$ for any N in $\mathcal{O}$ and $D(D(M))$ is isomorphic to M, (i) and (ii) are equivalent. We give a proof of (i). By induction on the number of composition factors (counted with multiplicities) of M, we may assume that $M/M(-\rho)$ is irreducible. Recall that $P(M) = \{\lambda \in \mathbf{Z}^n \mid M_\lambda \neq 0\}$. One of the following three cases must happen.

(a) There is a maximal weight λ in $P(M)$ such that $\lambda \neq -\rho$.

(b) $-\rho$ is the unique maximal weight in $P(M)$ and $\dim_{\mathbf{Q}(\xi)} M_{-\rho} = 1$.

(c) $-\rho$ is the unique maximal weight in $P(M)$ and $\dim_{\mathbf{Q}(\xi)} M_{-\rho} = 2$.

Case (a) Choose a nonzero element m in M_λ, then m is a primitive element. Let M' be the submodule of M generated by m. By 2.2 (iii), Lemma 2.6 and the proof of Prop. 2.7 we see that $M(-\rho)$ is not a composition factor of M'. Thus $M' \cap M(-\rho) = 0$. So M' is irreducible and M is isomorphic to $M' \oplus M(-\rho)$.

Case (b) Let L be the quotient module $M/M(-\rho)$. Then L is irreducible. So $D(L) = L$ is an irreducible submodule of $D(M)$. Note that $D(M)_{-\rho} \cap L = 0$ and $-\rho$ is the unique maximal element in $P(D(M)) = P(M)$. Choose a nonzero element m in $M_{-\rho}$. Then m is primitive and generates an irreducible submodule M' of $D(M)$ with highest weight $-\rho$. Thus $D(M)$ is isomorphic to $L \oplus M(-\rho)$. So M is isomorphic to $L \oplus M(-\rho)$.

Case (c) In this case we have $M \simeq M(-\rho) \oplus M(-\rho)$.

Part (i) is proved. Part (iii) is a consequence of (i) and (ii).

Corollary 2.10. (i) $M(-\rho)$ is projective and injective as well in $\mathcal{O}$.

(ii) For each $\lambda \in \mathbf{Z}^n_+$, the U_ξ-module $M(-\rho) \otimes V(\lambda)$ is projective and injective as well in $\mathcal{O}$. (We use $V(\lambda)$ for the Weyl module of U_ξ wiht highest weight λ.)

Proof. (i) is a consequence of Lemma 2.9. (ii) follows from (i) and 2.8 (ii).

Corollary 2.11. Assume that $\lambda \in \mathbf{Z}^n_+$. Then $M(\lambda-\rho)$ has a unique irreducible submodule.

Proof. Let $M = M(-\rho) \otimes V(\lambda)$, then M is injective in $\mathcal{O}$. Obviously, the submodule of M generated by $1_{-\rho} \otimes v_\lambda$ is isomorphic to $M(\lambda-\rho)$. Let I be the indecomposable direct summand of M containing $1_{-\rho} \otimes v_\lambda$. Then $M(\lambda-\rho)$ is isomorphic to a submodule of I. This forces that $M(\lambda-\rho)$ has a unique irreducible submodule since I is an indecomposable injective of $\mathcal{O}$.

2.12. Now we are going to prove the main result of this section. Let $\lambda \in \mathbf{Z}^n_{+,l}$. According to [X2, Th. 3.7], there exists an element y'_λ in $\mathbf{u}^-$ of degree $-\lambda + w_0\lambda$ such that $y'_\lambda v_{l\mu+\kappa+\lambda}$

is primitive in $V(\mathbf{l}\mu+\kappa+\lambda)$ and generates the unique irreducible submodule of $V(\mathbf{l}\mu+\kappa+\lambda)$ for any $\mu \in \mathbf{Z}_+^n$.

Lemma 2.13. Assume that $\lambda \in \mathbf{Z}_{+,\mathbf{l}}^n$. Then $y'_\lambda 1_{\lambda-\rho}$ is primitive in $M(\lambda-\rho)$ and generates an irreducible submodule of $M(\lambda-\rho)$ with highest weight $w_0\lambda-\rho$.

Proof. We first prove that $y'_\lambda 1_{\lambda-\rho}$ is primitive in $M(\lambda-\rho)$. Note that all $z_1 = E_i y'_\lambda - y'_\lambda E_i$ and $z_2 = E_i^{(l_i)} y'_\lambda - y'_\lambda E_i^{(l_i)}$ are contained in $\mathbf{u}$. Moreover $zv_{\mathbf{l}\mu+\kappa+\lambda} \neq 0$ for all $z \in \mathbf{u}^-$ and $\mu \in \mathbf{Z}_+^n$. Since $y'_\lambda v_{\mathbf{l}\mu+\kappa+\lambda}$ is primitive in $V(\mathbf{l}\mu+\kappa+\lambda)$ for all $\mu \in \mathbf{Z}_+^n$, taking $\mu \in \mathbf{Z}_+^n$ such that each $\langle \mu, \alpha_i^\vee \rangle$ is odd, using PBW theorem for $\mathbf{u}$ and U_ξ we see z_1 and z_2 are contained in $I_{\mathbf{l}\mu+\kappa+\lambda} \cap \mathbf{u} \subseteq I^+_{\lambda-\rho} \cap \mathbf{u}$. Therefore $y'_\lambda 1_{\lambda-\rho}$ is primitive in $M(\lambda-\rho)$. (Definition: We denote $\{x \in U_\xi \mid xM(\mu)_\mu = 0\}$ by I^+_μ for $\mu \in \mathbf{Z}^n$ and denote $\{x \in U_\xi \mid xV(\mu)_\mu = 0\}$ by I_μ for $\mu \in \mathbf{Z}_+^n$.)

Denote by M_1 the submodule of $M(\lambda-\rho)$ generated by $y'_\lambda 1_{\lambda-\rho}$. According to [X2, Corollary 2.7] and 2.2 (i)-(ii), $\mathrm{ch}M_1 = \mathrm{ch}U^-_{\xi,l}(e_{-\mathbf{l}\rho}\mathrm{ch}L(\kappa+w_0\lambda))= \mathrm{ch}L(-\mathbf{l}\rho)\mathrm{ch}L(\kappa+w_0\lambda)$=$\mathrm{ch}L(w_0\lambda-\rho)$. So M_1 is irreducible with highest weight $w_0\lambda-\rho$.

Now we can state and prove the main result of this section.

Theorem 2.14. Assume that $\lambda \in \mathbf{Z}_{+,\mathbf{l}}^n$. Then x'_λ and x_λ are contained in $\mathbf{u}^-$.

Proof. Obviously, $x'_\lambda 1_{\lambda-\rho}$ is primitive in $M(\lambda-\rho)$. Note that x'_λ has degree $-\lambda+w_0\lambda$. By Corollary 2.11 and Lemma 2.13 we see there exists a nonzero number $a \in \mathbf{Q}(\xi)$ such that $x'_\lambda = ay'_\lambda$. Note that $x_\lambda = x'_{-w_0\lambda}$. The theorem is proved.

Remark: By the above theorem we see the results in [X2, §5] hold for any Cartan matrix. Namely we have the following results.

Theorem 2.15. Assume that $\lambda \in \mathbf{Z}_{+,\mathbf{l}}^n$. Denote by $\mathbf{u}(\lambda)$ the left ideal of $\mathbf{u}$ generated by the elements $E_\alpha, K_i - \xi^{\lambda_i d_i}$ ($\alpha \in R_l^+$, $i = 1, 2, ..., n$) and elements $F \in \mathbf{u}^-$ such that $Fx_{\kappa-\lambda} = 0$. Then

(i) As $\mathbf{u}^-$-modules, $L(\lambda)$ is isomorphic to $\mathbf{u}^- x_{\kappa-\lambda}$.

(ii) $\mathbf{u}/\mathbf{u}(\lambda)$ is an irreducible $\mathbf{u}$-module. Moreover, as $\mathbf{u}$-modules, $L(\lambda)$ is isomorphic to $\mathbf{u}/\mathbf{u}(\lambda)$.

(iii) For any $\gamma \in -\mathbf{N}R^+$, denote by $\mathbf{u}_\gamma^-$ the set of all elements in $\mathbf{u}^-$ of degree γ, and set $\mathbf{n}_\gamma := \{F \in \mathbf{u}_\gamma^- \mid Fx_{\kappa-\lambda} = 0\}$. Then

$$\dim L(\lambda) = \dim \mathbf{u}_\gamma^- - \dim \mathbf{n}_\gamma.$$

In particular, we have

$$\dim L(\lambda) = \prod_{\alpha \in R^+} l_\alpha - \dim\{F \in \mathbf{u}^- \mid Fx_{\kappa-\lambda} = 0\}.$$

Theorem 2.16. Let $\lambda, \mu \in \mathbf{Z}_+^n$ and assume that $\lambda \in \mathbf{Z}_{+,\mathbf{l}}^n$. Denote by $J_{\mathbf{l}\mu+\lambda}$ the left ideal of U_ξ generated by all elements in $I_{\mathbf{l}\mu+\lambda}$ (see the proof of 2.13 for the definition) and elements $F \in \mathbf{u}^-$ such that $Fx_{\kappa-\lambda} = 0$. Then $U_\xi/J_{\mathbf{l}\mu+\lambda} \simeq L_\xi(\mathbf{l}\mu + \lambda, \sigma)$.

In next section we will give some applications for the results above.

2.17. Let $\mathfrak{k}$ be an algebraically closed field of prime characteristic and let $\mathfrak{U}_\mathfrak{k}$ be the hyperalgebra over $\mathfrak{k}$ and associated to the Cartan matrix (a_{ij}). It is easy to define the category $\mathcal{O}_\mathfrak{k}$ of $\mathfrak{U}_\mathfrak{k}$ modules. The Verma module $M_\mathfrak{k}(-\rho)$ of $\mathfrak{U}_\mathfrak{k}$ with highest weight $-\rho$ remains irreducible [H]. But the category $\mathcal{O}_\mathfrak{k}$ is not abelian since the maximal submodule of a Verma module may be not finitely generated. As an example, one may consider type A_1 and the Verma module of highest weight 0 (see [J2]). However it is likely that $M_\mathfrak{k}(-\rho)$ is projective and injective as well in the category. Of course we may suitably enlarge the category $\mathcal{O}_\mathfrak{k}$ to get an abelian category $\mathcal{O}'_\mathfrak{k}$ such that the simple objects of $\mathcal{O}'_\mathfrak{k}$ are the same as those of $\mathcal{O}_\mathfrak{k}$. It remains likely that $M_\mathfrak{k}(-\rho)$ is projective and injective as well in the category $\mathcal{O}'_\mathfrak{k}$.

3. The Socle of u and a Realization of Two Irreducible u-Modules

Through left multiplication we regard U_ξ (resp. $\mathbf{u}$) as a $\mathbf{u}$-module. The $\mathbf{u}$-module will be denoted by U_ξ^L (resp. $\mathbf{u}^L$). By means of Theorem 2.15 we can describe the socles of U_ξ^L and $\mathbf{u}^L$ and realize the tensor product of two irreducible $\mathbf{u}$-modules as a submodule of $\mathbf{u}^L$. For more applications of Theorem 2.15 see next section and [X4-5]. In [K, 6.5], through certain elements $R(\bar{\lambda})$ in $\mathfrak{u}_1$ (the first classical Frobenius kernel) Koppinen realizes the tensor product of two irreducible $\mathfrak{u}_1$-modules as a submodule of the regular $\mathfrak{u}_1$-module. The purpose and arguments of [8, 6.5] are different from here and the elements $R(\bar{\lambda})$ are interesting but look not easy to compute.

3.1. Some u modules Given $\lambda \in \mathbf{Z}^n$, $\sigma \in \{\pm 1\}^n$, we denote by $Z(\lambda, \sigma)$ the corresponding Verma module of $\mathbf{u}$ (see [L3, X1]). Let $1_{\lambda,\sigma}$ be a canonical generator of $Z(\lambda, \sigma)$. The unique irreducible quotient module of $Z(\lambda, \sigma)$ will be denoted by $L_{\mathbf{u}}(\lambda, \sigma)$. The image of $1_{\lambda,\sigma}$ will be denoted by $v_{\lambda,\sigma}$. The map $(\lambda, \sigma) \to L_{\mathbf{u}}(\lambda, \sigma)$ defines a bijection between the set $\mathbf{Z}^n_{+,\mathbf{l}} \times \{\pm 1\}^n$ and the set of isomorphism classes of irreducible $\mathbf{u}$-modules.

Let M be a $\mathbf{u}$-module. A non zero element $m \in M$ is called primitive if $E_\alpha m = 0$ for all $\alpha \in R_l^+$ and we can find $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\sigma \in \{\pm 1\}^n$ such that $K_i m = \sigma_i \xi^{d_i \lambda_i} m$.

(a) Up to scalars, $v_{\lambda,\sigma}$ is the unique primitive elements in $L_{\mathbf{u}}(\lambda, \sigma)$.

(b) Each $Z(\lambda, \sigma)$ has a unique irreducible submodule, which is generated by $F_\kappa 1_{\lambda,\sigma}$. If $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, then $x_{\kappa-\lambda} 1_{2\kappa+w_0\lambda,\sigma}$ is primitive and generates the unique irreducible submodule of $Z(2\kappa + w_0\lambda, \sigma)$, which is isomorphic to $L_{\mathbf{u}}(\lambda, \sigma)$. We have $Z(\kappa, \sigma) \simeq L_{\mathbf{u}}(\kappa, \sigma)$. [X2].

For convention we shall denote by $\lambda_i, \mu_i, \ldots$ the ith components of any elements $\lambda, \mu, \ldots$ in $\mathbf{Z}^n$ respectively. Given $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\sigma \in \{\pm 1\}^n$, we denote by $\omega_{\lambda,\sigma} \in \mathbf{u}^0$ the corresponding idempotent element. We have, a) $1 = \sum_{\substack{\lambda \in \mathbf{Z}^n_{+,\mathbf{l}} \\ \sigma \in \{\pm 1\}^n}} \omega_{\lambda,\sigma}$, b) $K_i \omega_{\lambda,\sigma} = \sigma_i \xi^{d_i \lambda_i} \omega_{\lambda,\sigma}$, c) $\omega_{\lambda,\sigma} \omega_{\mu,\tau} = 0$ for different (λ, σ), $(\mu, \tau) \in \mathbf{Z}^n_{+,\mathbf{l}} \times \{\pm 1\}^n$. For any $\mu \in \mathbf{Z}^n$ and $\tau \in \{\pm 1\}^n$, there exist unique $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\sigma \in \{\pm 1\}^n$ such that $\tau_i \xi^{d_i \mu_i} = \sigma_i \xi^{d_i \lambda_i}$ for all i, then we define $\omega_{\mu,\tau}$ to be $\omega_{\lambda,\sigma}$.

(c) For $\lambda \in \mathbf{Z}^n$, $\sigma \in \{\pm 1\}^n$, and $\alpha \in R^+$, $k \in \mathbf{N}$, we have $E_\alpha^{(k)} \omega_{\lambda,\sigma} = \omega_{\lambda+k\alpha,\sigma} E_\alpha^{(k)}$ and $F_\alpha^{(k)} \omega_{\lambda,\sigma} = \omega_{\lambda-k\alpha,\sigma} F_\alpha^{(k)}$. In particular, $\dot{x}_\mu \omega_{\lambda,\sigma} = \omega_{\lambda+\mu-w_0\mu,\sigma} \dot{x}_\mu$ and $x_\mu \omega_{\lambda,\sigma} = \omega_{\lambda+w_0\mu-\mu,\sigma} x_\mu$ ∎

for any $\mu \in \mathbf{Z}^n_{+,\mathbf{l}}$.

Lemma 3.2. A non zero element x in U^L_ξ is primitive if and only if $x = \omega_{\lambda,\sigma}E_\kappa y$ for some non zero $y = \sum y'_h y''_h$, and $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\sigma \in \{\pm 1\}^n$, where $y'_h \in \mathbf{u}^-$ and y''_h are in the subalgebra $U_{\xi,l}$ of U_ξ generated by all $E_i^{(l_i)}$, $F_i^{(l_i)}$, $\left[{K_i,0 \atop l_i}\right]$.

In particular, $x \in \mathbf{u}^L$ is primitive if and only if $x = \omega_{\lambda,\sigma}E_\kappa F$ for some non zero $F \in \mathbf{u}^-$ and $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\sigma \in \{\pm 1\}^n$.

Proof. One direction is obvious. Now we consider another direction. According to [L4, L5], the multiplication in U_ξ defines a $\mathbf{Q}(\xi)$-linear isomorphism between $\mathbf{u} \otimes U_{\xi,l}$ and U_ξ. Let $\{E^A\}$ (resp. $\{F^{A'}\}$) be the PBW base of $\mathbf{u}^+$ (resp. $\mathbf{u}^-$) and $\{Y_h\}$ a base of $U_{\xi,l}$. Then $x = \sum g(\lambda,\sigma,A,A',h)\omega_{\lambda,\sigma}E^A F^{A'} Y_h$, $g(\lambda,\sigma,A,A',h) \in \mathbf{Q}(\xi)$, where only finitely many coefficients are non zero. Since each K_i acts on x by a scalar multiplication, there are unique $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\sigma \in \{\pm 1\}^n$ such that all $g(\mu,\tau,A,A',h) = 0$ if $(\mu,\tau) \in \mathbf{Z}^n_{+,\mathbf{l}} \times \{\pm 1\}^n$ is different from (λ,σ). Since $E_\alpha x = 0$ for all $\alpha \in R^+_l$, we see all $g(\lambda,\sigma,A,A',h) = 0$ if $E^A \neq E_\kappa$ (see the proof of Lemma 2.5 in [X2]). Therefore $x = \omega_{\lambda,\sigma}E_\kappa \sum g(\lambda,\sigma,A_0,A',h)F^{A'}Y_h$, where A_0 is defined by $E^{A_0} = E_\kappa$. The lemma is proved.

The following theorem describes the socles of $\mathbf{u}^L$ and U^L_ξ.

Theorem 3.3. In (i)-(iii), $\lambda \in \mathbf{Z}^n_{+,\mathbf{l}}$ and $\sigma \in \{\pm 1\}^n$.

(i) We have $\omega_{\lambda,\sigma}E_\kappa x_{\kappa-\lambda} = \omega_{\lambda,\sigma}x_{\kappa-\lambda}E_\kappa$.

(ii) Each nonzero element in $\omega_{\lambda,\sigma}E_\kappa x_{\kappa-\lambda}\mathbf{u}^- U_{\xi,l}$ generates an irreducible submodule of $\mathbf{u}^L$, which is isomorphic to $L_{\mathbf{u}}(\lambda,\sigma)$. (Convention: $\mathbf{u}^- U_{\xi,l}$ is the linear space spanned by all elements $y'y''$, $y' \in \mathbf{u}^-$, $y'' \in U_{\xi,l}$.)

(iii) Let L be an irreducible submodule of U^L_ξ. If L is isomorphic to $L_{\mathbf{u}}(\lambda,\sigma)$, then L is generated by a non zero element in $\omega_{\lambda,\sigma}E_\kappa x_{\kappa-\lambda}\mathbf{u}^- U_{\xi,l}$.

(iv) The socle of $\mathbf{u}^L$ (i.e. the maximal completely reducible submodule of $\mathbf{u}^L$) is

$$\sum_{\substack{\lambda \in \mathbf{Z}^n_{+,\mathbf{l}} \\ \sigma \in \{\pm 1\}^n}} \mathbf{u}^- \omega_{\lambda,\sigma}E_\kappa x_{\kappa-\lambda}\mathbf{u}^-.$$

Moreover, for different (λ,σ), $(\mu,\tau)\in \mathbf{Z}^n_{+,\mathrm{l}}\times\{\pm1\}^n$, we have

$$\mathbf{u}^-\omega_{\lambda,\sigma}E_\kappa x_{\kappa-\lambda}\mathbf{u}^-\cap\mathbf{u}^-\omega_{\mu,\tau}E_\kappa x_{\kappa-\mu}\mathbf{u}^-=\{0\}.$$

The socle of U_ξ^L is

$$\sum_{\substack{\lambda\in\mathbf{Z}^n_{+,\mathrm{l}}\\ \sigma\in\{\pm1\}^n}}\mathbf{u}^-\omega_{\lambda,\sigma}E_\kappa x_{\kappa-\lambda}\mathbf{u}^-U_{\xi,l}.$$

Proof. (i) Obviously, the submodule M of $\mathbf{u}^L$ generated by $\omega_{2\kappa+w_0\lambda,\sigma}E_\kappa$ is isomorphic to $Z(2\kappa+w_0\lambda,\sigma)$. Using 3.1 (b) we get $E_\alpha x_{\kappa-\lambda}\omega_{2\kappa+w_0\lambda,\sigma}E_\kappa=0$ for all α in R_l^+. By 3.1 (c) and Lemma 3.2 we see $x_{\kappa-\lambda}\omega_{2\kappa+w_0\lambda,\sigma}E_\kappa=\omega_{\lambda,\sigma}x_{\kappa-\lambda}E_\kappa=\omega_{\lambda,\sigma}E_\kappa x_{\kappa-\lambda}$.

(ii) Using (i), 3.1 (b) and 3.1 (a).

(iii) It is no harm to assume that $L\subset\mathbf{u}^L$ since U_ξ^L is canonically isomorphic to a direct sum of copies of $\mathbf{u}^L$. Since $\mathbf{u}$ is symmetric, L appears in the Jordan-Hölder series of the socle of $\mathbf{u}^L$ with multiplicities $\dim L$. By (ii), the submodule of $\mathbf{u}^L$ generated by $\omega_{\lambda,\sigma}E_\kappa x_{\kappa-\lambda}\mathbf{u}^-$ is completely reducible and is isomorphic to the direct sum of $\dim x_{\kappa-\lambda}\mathbf{u}^-$ copies of $L_{\mathbf{u}}(\lambda,\sigma)$. Thus we only need to prove that $\dim x_{\kappa-\lambda}\mathbf{u}^-=\dim L_{\mathbf{u}}(\lambda,\sigma)$. Let Ψ be the anti-automorphism of U_ξ defined by $E_i^{(a)}\to E_i^{(a)}$, $F_i^{(a)}\to F_i^{(a)}$, $K_i\to K_i^{-1}$. Then $\Psi(x_{\kappa-\lambda}\mathbf{u}^-)=\mathbf{u}^-x_{\kappa+w_0\lambda}$. By 3.1 (b), $\dim\mathbf{u}^-x_{\kappa+w_0\lambda}=\dim L_{\mathbf{u}}(-w_0\lambda,\sigma)=\dim L_{\mathbf{u}}(\lambda,\sigma)^*=\dim L_{\mathbf{u}}(\lambda,\sigma)$, where $L_{\mathbf{u}}(\lambda,\sigma)^*(\simeq L_{\mathbf{u}}(-w_0\lambda,\sigma))$ is the dual module of $L_{\mathbf{u}}(\lambda,\sigma)$ defined through the antipode of $\mathbf{u}$. (iii) is proved.

Part (iv) follows from (ii), (iii) and its proof.

Proposition 3.4. Let $\lambda\in\mathbf{Z}^n_{+,\mathrm{l}}$, $\sigma\in\{\pm1\}^n$ and $y\in\mathbf{u}^-U_{\xi,l}-\{0\}$. The submodule M of U_ξ^L generated by $\omega_{2\kappa+w_0\lambda,\sigma}E_\kappa y$ is isomorphic to $Z(2\kappa+w_0\lambda,\sigma)$ if and only if $x_{\kappa-\lambda}y\neq0$.

Proof. There is a unique surjective $\mathbf{u}$-homomorphism from $Z(2\kappa+w_0\lambda,\sigma)$ to M such that $1_{2\kappa+w_0\lambda,\sigma}$ maps to $\omega_{2\kappa+w_0\lambda,\sigma}E_\kappa y$. According to 3.1 (b), $x_{\kappa-\lambda}1_{2\kappa+w_0\lambda,\sigma}$ generates the unique irreducible submodule of $Z(2\kappa+w_0\lambda,\sigma)$. Therefore M is isomorphic to $Z(2\kappa+w_0\lambda,\sigma)$ if and only if $x_{\kappa-\lambda}\omega_{2\kappa+w_0\lambda,\sigma}E_\kappa y\neq0$. But $x_{\kappa-\lambda}\omega_{2\kappa+w_0\lambda,\sigma}=\omega_{\lambda,\sigma}x_{\kappa-\lambda}$, by using Theorem 3.3 (i) and the $\mathbf{Q}(\xi)$-linear isomorphism $\mathbf{u}\otimes U_{\xi,l}\simeq U_\xi$ $(x\otimes y\to xy)$ we see the proposition is true.

The following theorem realizes the tensor product of two irreducible $\mathbf{u}$-modules as a submodule of $\mathbf{u}^L$.

Theorem 3.5. Let (λ, σ) and (μ, τ) be elements in $\mathbf{Z}^n_{+,\mathbf{l}} \times \{\pm 1\}^n$.

(i) The submodule of $\mathbf{u}^L$ generated by $\omega_{w_0\lambda+\mu,\sigma\tau}\dot{x}_{\kappa+w_0\lambda}x_{\kappa-\mu}$ is isomorphic to $L_{\mathbf{u}}(\lambda,\sigma) \otimes L_{\mathbf{u}}(\mu,\tau)$, where $\sigma\tau = (\sigma_1\tau_1, ..., \sigma_n\tau_n)$ if $\sigma = (\sigma_1, ..., \sigma_n)$ and $\tau = (\tau_1, ..., \tau_n)$.

(ii) The submodule M of $\mathbf{u}^L$ generated by $\omega_{\lambda+w_0\mu,\sigma\tau}x_{\kappa-\lambda}\dot{x}_{\kappa+w_0\mu}$ is isomorphic to $L_{\mathbf{u}}(\lambda,\sigma) \otimes L_{\mathbf{u}}(\mu,\tau)$.

(iii) $\omega_{w_0\lambda+\mu,\sigma\tau}\dot{x}_{\kappa+w_0\lambda}x_{\kappa-\mu} = \omega_{w_0\lambda+\mu,\sigma\tau}x_{\kappa-\mu}\dot{x}_{\kappa+w_0\lambda}$.

We need some preparations to prove the theorem.

3.6. For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbf{Z}^n$, $\sigma = (\sigma_1, \sigma_2, ..., \sigma_n) \in \{\pm 1\}^n$, let $\mathbf{I}^-_{\lambda,\sigma}$ be the left ideal of $\mathbf{u}$ generated by the elements F_α, $K_i - \sigma_i\xi^{\lambda_i d_i}$ for $\alpha \in R^+_l$, $i = 1, 2, ..., n$. Define $Z(\lambda,\sigma)' = \mathbf{u}/\mathbf{I}^-_{\lambda,\sigma}$. The $\mathbf{u}$-module $Z(\lambda,\sigma)'$ has a unique irreducible quotient module, denoted by $L_{\mathbf{u}}(\lambda,\sigma)'$. The image in $Z(\lambda,\sigma)'$ of $1 \in \mathbf{u}$ will be denoted by $1'_{\lambda,\sigma}$ and the image in $L_{\mathbf{u}}(\lambda,\sigma)'$ of $1'_{\lambda,\sigma}$ will be denoted by $v'_{\lambda,\sigma}$. We have

(a) If $\lambda \in \mathbf{Z}^n_{-,\mathbf{l}} = \{(a_1, ..., a_n) \mid -l_i + 1 \le a_i \le 0 \text{ for all } \alpha \in R^+_l\}$, then $L_{\mathbf{u}}(\lambda,\sigma)' \simeq L_{\mathbf{u}}(w_0\lambda,\sigma)$. By abusing notation, we denote the image in $L_{\mathbf{u}}(\lambda,\sigma)'$ (resp. $L_{\mathbf{u}}(w_0\lambda,\sigma)$) of $v_{w_0\lambda,\sigma}$ (resp. $v'_{\lambda,\sigma}$) by the same notation.

(b) Each $Z(\lambda,\sigma)'$ has a unique irreducible submodule, which is generated by $E_\kappa 1'_{\lambda,\sigma}$. If $\lambda \in \mathbf{Z}^n_{-,\mathbf{l}}$, then $F_\alpha\dot{x}_{\kappa+\lambda}1'_{-2\kappa+w_0\lambda,\sigma} = 0$ for all $\alpha \in R^+_l$ and $x_{\kappa+\lambda}1'_{-2\kappa+w_0\lambda,\sigma}$ generate the unique irreducible submodule of $Z(-2\kappa + w_0\lambda,\sigma)'$, which is isomorphic to $L_{\mathbf{u}}(\lambda,\sigma)'(\simeq L_{\mathbf{u}}(w_0\lambda,\sigma))$. We have $Z(-\kappa,\sigma)' \simeq L_{\mathbf{u}}(-\kappa,\sigma)' \simeq Z(\kappa,\sigma)$.

(c) For $\lambda \in \mathbf{Z}^n$, $\sigma \in \{\pm 1\}^n$, we have $Z(\lambda,\sigma)' \simeq Z(-\lambda,\sigma)^*$.

The assertion (a) is obvious. The proof of (b) is similar to that of 2.2 (a). The assertion (c) follows from the definitions of $Z(\lambda,\sigma)'$ and of Verma module and the definition of dual module.

Lemma 3.7. Let $(\lambda,\sigma) \in \mathbf{Z}^n_{+,\mathbf{l}} \times \{\pm 1\}^n$ and $(\mu,\tau) \in \mathbf{Z}^n_{-,\mathbf{l}} \times \{\pm 1\}^n$.

(i) The element $1_{\lambda,\sigma}\otimes 1'_{\mu,\tau}$ generates $Z(\lambda,\sigma)\otimes Z(\mu,\tau)'$ as a $\mathbf{u}$-module.

(ii) Let M be the submodule of $\mathbf{u}^L$ generated by $\omega_{\lambda+\mu,\sigma\tau}$. Then the map $u\omega_{\lambda+\mu,\sigma\tau}\to u(1_{\lambda,\sigma}\otimes 1'_{\mu,\tau})$ defines a $\mathbf{u}$-isomorphism between M and $Z(\lambda,\sigma)\otimes Z(\mu,\tau)'$, where $u\in\mathbf{u}$.

Proof. (i) is obvious.

(ii) Let $\mathbf{k}_{\lambda+\mu,\sigma\tau}$ be the left ideal of $\mathbf{u}$ generated by all $K_i-\sigma_i\tau_i\xi^{d_i(\lambda_i+\mu_i)}$ and let $\bar{1}$ be the image in $M'=\mathbf{u}/\mathbf{k}_{\lambda+\mu,\sigma\tau}$ of $1\in\mathbf{u}$. There is a unique $\mathbf{u}$-isomorphism from M to M' such that $\omega_{\lambda+\mu,\sigma\tau}$ maps to $\bar{1}$. By (i) then we get a surjective $\mathbf{u}$-homomorphism from M to $Z(\lambda,\sigma)\otimes Z(\mu,\tau)'$ which maps $\omega_{\lambda+\mu,\sigma\tau}$ to $1_{\lambda,\sigma}\otimes 1'_{\mu,\tau}$. Since both M and $Z(\lambda,\sigma)\otimes Z(\mu,\tau)'$ have dimension $\prod_{\alpha\in R^+} l_\alpha^2$, the $\mathbf{u}$-homomorphism is actually an isomorphism. The lemma is proved.

3.8. *Proof of Theorem 3.1.* (i) By 3.6 (a) and 3.1 (b), $L_{\mathbf{u}}(\lambda,\sigma)(\simeq L_{\mathbf{u}}(w_0\lambda,\sigma)')$ is isomorphic to the submodule of $Z(-2\kappa+\lambda,\sigma)'$ generated by $\dot{x}_{\kappa+w_0\lambda}1'_{-2\kappa+\lambda,\sigma}$ and $L_{\mathbf{u}}(\mu,\tau)$ is isomorphic to the submodule of $Z(2\kappa+w_0\mu,\tau)$ generated by $x_{\kappa-\mu}1_{2\kappa+w_0\lambda,\tau}$. Thus $L_{\mathbf{u}}(\lambda,\sigma)\otimes L_{\mathbf{u}}(\mu,\tau)$ is isomorphic to the submodule of $Z(-2\kappa+\lambda,\sigma)'\otimes Z(2\kappa+w_0\mu,\tau)$ generated by $\dot{x}_{\kappa+w_0\lambda}1'_{-2\kappa+\lambda,\sigma}\otimes x_{\kappa-\mu}1_{2\kappa+w_0\mu,\tau}$. By 3.1 (c) we have

$(*)$ $\omega_{w_0\lambda+\mu,\sigma\tau}\dot{x}_{\kappa+w_0\lambda}x_{\kappa-\mu}=\dot{x}_{\kappa+w_0\lambda}x_{\kappa-\mu}\omega_{\lambda+w_0\mu,\sigma\tau}$.

Since $F_\alpha 1'_{-2\kappa+\lambda,\sigma}=0$ for all $\alpha\in R_l^+$, we get $x_{\kappa-\mu}(1'_{-2\kappa+\lambda,\sigma}\otimes 1_{2\kappa+w_0\mu,\tau})=1'_{-2\kappa+\lambda,\sigma}\otimes x_{\kappa-\mu}1_{2\kappa+w_0\mu,\tau}$. By 3.1 (b), $E_\alpha x_{\kappa-\mu}1_{2\kappa+w_0\mu,\tau}=0$ for all $\alpha\in R_l^+$. Therefore we have

$(\star)$ $\dot{x}_{\kappa+w_0\lambda}x_{\kappa-\mu}(1'_{-2\kappa+\lambda,\sigma}\otimes 1_{2\kappa+w_0\mu,\tau})=\dot{x}_{\kappa+w_0\lambda}1'_{-2\kappa+\lambda,\sigma}\otimes x_{\kappa-\mu}1_{2\kappa+w_0\mu,\tau}$.

Using Lemma 3.7, $(*)$ and $(\star)$, we see (i) is true.

(ii) By (i), M is isomorphic to $L_{\mathbf{u}}(\mu,\tau)\otimes L_{\mathbf{u}}(\lambda,\sigma)$. But $L_{\mathbf{u}}(\mu,\tau)\otimes L_{\mathbf{u}}(\lambda,\sigma)$ is isomorphic to $L_{\mathbf{u}}(\lambda,\sigma)\otimes L_{\mathbf{u}}(\mu,\tau)$ [L5, Coro. 32.1.6]. Hence (ii) is true. We also can prove (ii) directly by considering the embedding $L_{\mathbf{u}}(\lambda,\sigma)\otimes L_{\mathbf{u}}(\mu,\tau)\hookrightarrow Z(2\kappa+w_0\lambda,\sigma)\otimes Z(-2\kappa+\mu,\tau)'$.

(iii) It follows from Lemma 3.7(ii), $(\star)$ in the proof of (i) and the following identity $x_{\kappa-\mu}\dot{x}_{\kappa+w_0\lambda}(1'_{-2\kappa+\lambda,\sigma}\otimes 1_{2\kappa+w_0\mu,\tau})=\dot{x}_{\kappa+w_0\lambda}1'_{-2\kappa+\lambda,\sigma}\otimes x_{\kappa-\mu}1_{2\kappa+w_0\mu,\tau}$.

3.9. A possible application of Theorem 3.5 is based on the following consideration. The $\mathbf{u}$-module $Z(\kappa,\sigma)$ is actually a projective and injective $\mathbf{u}$-module. So for any $(\lambda,\sigma)\in$

$\mathbf{Z}^n_{+,\mathbf{l}} \times \{\pm 1\}^n$, the tensor product $Z(\kappa, \mathbf{1}) \otimes L_{\mathbf{u}}(\lambda, \sigma)$ is a projective $\mathbf{u}$-module and an injective $\mathbf{u}$-module as well, here and later we use $\mathbf{1}$ for $(1, ..., 1) \in \{\pm 1\}^n$. We have seen that the tensor product is isomorphic to the submodule of $\mathbf{u}^L$ generated by $\omega_{\kappa+w_0\lambda,\sigma}\dot{x}_{\kappa+w_0\lambda}$. Once we know the socle of $\mathbf{u}\omega_{\kappa+w_0\lambda,\sigma}\dot{x}_{\kappa+w_0\lambda}$, we know how to decompose it into the direct sum of its indecomposable components. By Theorem 3.3, this is equivalent to compute the dimension of the space

$$\mathbf{u}\omega_{\kappa+w_0\lambda,\sigma}\dot{x}_{\kappa+w_0\lambda} \cap \omega_{\mu,\tau}E_\kappa x_{\kappa-\mu}\mathbf{u}^-$$

for all $(\mu, \tau) \in \mathbf{Z}^n_{+,\mathbf{l}} \times \{\pm 1\}^n$.

Since $\mathbf{u}$ is symmetric [X2], the injective hull of $L_{\mathbf{u}}(\lambda, \sigma)$ is also the projective cover $P(\lambda, \sigma)$ of $L_{\mathbf{u}}(\lambda, \sigma)$. Thus the dimension $d_{\lambda,\mu}$ of $\mathbf{u}\omega_{\kappa+w_0\lambda,\sigma}\dot{x}_{\kappa+w_0\lambda} \cap \omega_{\mu,\tau}E_\kappa x_{\kappa-\mu}\mathbf{u}^-$ is equal to the dimension of $\mathrm{Hom}_{\mathbf{u}}(Z(\kappa, \mathbf{1}) \otimes L_{\mathbf{u}}(\lambda, \sigma), L_{\mathbf{u}}(\mu, \tau))$=$\mathrm{Hom}_{\mathbf{u}}(Z(\kappa, \mathbf{1}), L_{\mathbf{u}}(\mu, \tau) \otimes L_{\mathbf{u}}(-w_0\lambda, \sigma))$. Using Theorem 3.5 and Theorem 3.3 we see $d_{\lambda,\mu}$ is equal to the dimension of the space

$$\omega_{\kappa,\mathbf{1}}E_\kappa\mathbf{u}^- \cap \mathbf{u}\omega_{\mu-\lambda,\sigma\tau}x_{\kappa-\mu}\dot{x}_{\kappa-\lambda}$$

(and equal to the dimension of the space

$$\omega_{\kappa,\mathbf{1}}E_\kappa\mathbf{u}^- \cap \mathbf{u}\omega_{\lambda-\mu,\sigma\tau}\dot{x}_{\kappa-\mu}x_{\kappa-\lambda}.)$$

The author has no idea how to computer the dimesion of the space.

4. Some Submodules and Quotient modules of the Verma Modules of u

The main results in §2 lead to the consideration of some submodules and quotients modules of the $\mathbf{u}$-module $Z(\lambda, \sigma)$, which seem interesting. Namely for each $w \in W$ we will construct a submodule and a quotient module for $Z(\lambda, \sigma)$. Based on the construction we compute certain irreducible characters for type A. The construction also can be used to find out some (quasi) primitive elements in $Z(\lambda, \sigma)$ (see [X4-5]). For simplicity, we shall assume that $\sigma = (1, 1, ..., 1)$ and will write $Z(\lambda)$, ω_λ, etc. for $Z(\lambda, \sigma)$, $\omega_{\lambda,\sigma}$, etc..

4.1. Quotient modules constructed from the elements $x'_{\lambda,w}$ Let $\lambda \in \mathbf{Z}^n_{+,1}$ and $w \in W$. In general, $x'_{\lambda,w}$ is not in $\mathbf{u}^-$, therefore $x'_{\lambda,w}1_{\kappa+\lambda}$ is not well defined in general. Note that the submodule of U_ξ^L generated by $\omega_{\kappa+\lambda}E_\kappa$ is isomorphic to $Z(\kappa+\lambda)$. Now the element $x_{\lambda,w}\omega_{\kappa+\lambda}E_\kappa \in U_\xi^L$ is well defined. Moreover, by [X2, 4.4 (iii)], $x_{\lambda,w}\omega_{\kappa+\lambda}E_\kappa \in U_\xi^L$ is primitive. Using 3..1 (c) and 3.2 we see $x_{\lambda,w}\omega_{\kappa+\lambda}E_\kappa = \omega_{\kappa+w\lambda}E_\kappa x_{\lambda,w} \in U_\xi^L$. Therefore the submodule M of U_ξ^L generated by $x_{\lambda,w}\omega_{\kappa+\lambda}E_\kappa = \omega_{\kappa+w\lambda}E_\kappa x_{\lambda,w}$ is a quotient module of $Z(\kappa+w\lambda)$. If $x_{\lambda,w} \in \mathbf{u}^-$, then M is isomorphic to a submodule of $Z(\kappa+\lambda)$, so its socle is isomorphic to $L(\kappa+w_0\lambda)$. If $x_{\lambda,w}$ is not contained in $\mathbf{u}^-$, then M may have a big socle. We give three examples.

For simplicity we assume that $l > 3$ is prime, then $l_i = l$ for all i.

Type A_2. Let $\lambda = (l-2, l-2)$ and $\mu = (1, l-2)$. Consider the submodule M of U_ξ^L generated by $m = F_2^{(2l-2)}F_1^{(l-1)}\omega_\lambda E_\kappa = \omega_\mu E_\kappa F_2^{(2l-2)}F_1^{(l-1)}$. Then M is a quotient of $Z(\mu)$. The socle of M is generated by $F_1^{(l-1)}m$ and $(F_1^{(l-1)}F_2^{(l)} - F_2^{(l)}F_1^{(l-1)})m$, so the socle is isomorphic to $L(0,0) \oplus L(0,0)$. Note that $Z(\mu)$ has only two composition factors isomorphic to $L(0,0)$.

Type B_2. Assume that $\alpha_1 = (2,-1)$ and $\alpha_2 = (-2,2)$. Let $\lambda = (l-2, l-2)$ and $\mu = (l-2, 1)$. Consider the submodule M of U_ξ^L generated by $m = F_2^{(2l-2)}F_1^{(3l-3)}F_2^{(l-1)}\omega_\lambda E_\kappa = \omega_\mu E_\kappa F_2^{(2l-2)}F_1^{(3l-3)}F_2^{(l-1)}$. Then M is a quotient of $Z(\mu)$. The submodule of M is generated by $F_1^{(l-1)}m$ and $(F_1^{(l-1)}F_2^{(l)} - F_2^{(l)}F_1^{(l-1)})m$ is isomorphic to $L(0,0) \oplus L(0,0)$. Let $x \in \mathbf{u}^-$ be such that $xF_2^{(l-2)}F_1^{(2l-3)}F_2^{(l-1)} = F_\kappa$, then xm generates an irreducible submodule of M isomorphic to $L(0,0)$ (for detail see [X5]). Therefore the socle of M has at least three composition factors isomorphic to $L(0,0)$. Note that $Z(\mu)$ has four composition factors isomorphic to $L(0,0)$.

Type A_3. Let $\lambda = (l-2, l-2, l-2)$ and $\mu = (l-2, 2, l-2)$. Consider the submodule M of U_ξ^L generated by

$$m = F_2^{(3l-3)}F_1^{(2l-2)}F_3^{(2l-2)}F_2^{(l-1)}\omega_\lambda E_\kappa = \omega_\mu E_\kappa F_2^{(3l-3)}F_1^{(2l-2)}F_3^{(2l-2)}F_2^{(l-1)}.$$

Then M is a quotient of $Z(\mu)$. The socle of M has at least nine composition factors isomorphic to $L(0,0,0)$. Note that $Z(\mu)$ has 11 composition factors isomorphic to $L(0,0,0)$. The detail will appear elsewhere.

4.2. Now we are going to construct some submodules of $Z(\lambda)$ by means of $x_{\lambda,w}$. First we recall the definition of $x_{\lambda,w}$. Let $s_{i_1}s_{i_2}\cdots s_{i_k}$ be a reduced expression of w. Define $a_j=\langle s_{i_{j-1}}s_{i_{j-2}}\cdots s_{i_1}\lambda,\alpha_{i_j}^\vee\rangle$ and then $x_{\lambda,w}=F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)}$.

Lemma 4.3. Keep the notation above and write r_j for l_{i_j}. Choose $\theta_j\in\mathbf{N}$ such that

$$y=F_{i_1}^{(a_1-\theta_1r_1)}F_{i_2}^{(a_2-\theta_2r_2)}\cdots F_{i_k}^{(a_k-\theta_kr_k)}\neq 0\quad\text{and}\quad\sum_{j=1}^k\theta_j\ \text{ is maximal.}$$

Then $y\in\mathbf{u}^-$. (Convention, $F_i^{(a)}=0$ if $a<0$.)

Proof. According to [L4, L5], $y=\sum z'_hz''_h,\ z'_h\in\mathbf{u}^-,\ z''_h\in U^-_{\xi,l}$, where $\{z''_h\}$ is a basis of $U^-_{\xi,l}$ consisting of homogeneous elements. If $z'_h\neq 0$ and $\deg(z''_h)\neq 0$ for some h, using commutation relations between root vectors we can find a suitable t such that

$$F_{i_1}^{(a_1-\theta_1r_1)}F_{i_2}^{(a_2-\theta_2r_2)}\cdots F_{i_{t-1}}^{(a_{t-1}-\theta_{t-1}r_{t-1})}F_{i_t}^{(a_t-\theta_tr_t-r_t)}F_{i_{t+1}}^{(a_{t+1}-\theta_{t+1}r_{t+1})}\cdots F_{i_k}^{(a_k-\theta_kr_k)}\neq 0.$$

This contradicts that the assumption $\sum_{j=1}^k\theta_j$ is maximal. The lemma is proved.

Example. Consider type A_3. Assume $s_1s_3=s_3s_1$. Let $l>3$ be prime and $\lambda=(2,l-1,l-2),\ w=s_2s_1s_3s_2$. Then $x_{\lambda,w}=F_2^{(l-1)}F_1^{(l+1)}F_3^{(2l-3)}F_2^{(2l-1)}$. We have

$$y_1=F_2^{(l-1)}F_1^{(l+1}F_3^{(l-3)}F_2^{(l-1)}\in\mathbf{u}^--\{0\},$$

$$y_2=F_2^{(l-1)}F_1F_3^{(2l-3)}F_2^{(l-1)}\in\mathbf{u}^--\{0\}.$$

This example shows that the y in the lemma may be not unique.

According to [X2, 4.4(iv)] (or a slight generalization) we get

Proposition 4.4. Assume $\lambda\in\mathbf{Z}^n_{+,\mathbf{l}}$. Let w,y be as in Lemma 4.1.2. Then $y1_{\lambda-\rho}$ is a primitive element in $Z(\lambda-\rho)$.

Remark: If $l(w)=l(w_0)-1$ then $\mathbf{u}y1_{\lambda-\rho}$ should have two composition factors.

4.5. Let I be a subset of $\{1,2,...,n\}$ and let $\mathbf{u}_I$ be the subalgebra of $\mathbf{u}$ generated by all elements in $\mathbf{u}^+\cup\mathbf{u}^0$ and $F_i,\ i\in I$. For each $\lambda\in\mathbf{Z}^n_{+,\mathbf{l}}$ and $\sigma\in\{\pm 1\}^n$ we denote by

$L_{\mathbf{u},I}(\lambda,\sigma)$ the corresponding irreducible $\mathbf{u}_I$-module. Let w_I be the longest element of the parabolic subgroup $< s_i \mid i \in I >$ of W. Then $x_{\kappa-\lambda,w_I}$ and $x'_{\kappa-\lambda,w_I}$ are contained in $\mathbf{u}^-$. The $\mathbf{u}$-module $\mathbf{u}\omega_\lambda x_{\kappa-\lambda,w_I}E_\kappa$ is isomorphic to the induced module $\mathbf{u}\otimes_{\mathbf{u}_I} L_{\mathbf{u},I}(\lambda,\sigma)$.

Now we consider a special case for type A_n. We choose all d_i to be 1. Denote by l' the order of ξ^2. Let $\lambda = (a, l'-2, ..., l'-2) \in \mathbf{Z}^n_{+,\mathbf{l}}$. Denote by u the longest element of the parabolic subgroup $< s_2, s_3, ..., s_n >$ of W (the numbering is the standard one).

Let $a_i \in \mathbf{N}$ $(0 \le i \le n)$ be such that $0 \le a_i \le l'-1$ and $(n-i)(l'-1)+a+1-a_i \in l'\mathbf{N}$. Then either $a_i = a_{i-1}+1$ or $a_i = 0$. Set $m_0 = x_{\lambda+\rho,u}1_\lambda$. Then m_0 is a primitive element in $Z(\lambda)$. Let $M = M_0 = \mathbf{u}m_0$. We have $\dim M_0 = l'^n$.

Set $b_{ii} = 0$ for $1 \le i \le n$. For $1 \le j \le i-1$, set $b_{ij} = b_{i-1,j}$ if $a_i = 0$, $b_{ij} = b_{i-1,j}+1$ if $a_i = a_{i-1}+1$. By Prop. 4.4, the following elements are primitive in M, $m_i = F_i^{(a_i)}F_{i-1}^{(a_{i-1}+b_{i,i-1}l')}\cdots F_1^{(a_1+b_{i,1}l')}m_0$. Note that $m_i = m_{i-1}$ when $a_i = 0$. Let $M_i = \mathbf{u}m_i$.

Define $\lambda^i = (\lambda^i_1, ..., \lambda^i_n) \in \mathbf{Z}^n_{+,\mathbf{l}}$ $(0 \le i \le n)$ be defined by $\lambda^i_i = l'-a_i-1$, $\lambda^i_{i+1} = a_i$ and other components being 0. Then $K_j m_i = \xi^{d_j\lambda^i_j}m_i$. Therefore M_i is a quotinet module of $Z(\lambda^i)$.

Using induction on i we get easily the following assertion.

(a) For $0 \le j \le i-1$ we have $a_j + b_{ij}l' - a_{j+1} - b_{i,j+1}l' = l'-1$.

Denote by F_{1j} the root vector corresponding to $-\alpha_1-\alpha_2-\cdots-\alpha_j$. For $\mathbf{c} = (c_n, ..., c_2, c_1) \in \mathbf{Z}^n_{+,\mathbf{l}}$ we define $F^{\mathbf{c}} = F_{1n}^{(c_n)}\cdots F_{1,2}^{(c_2)}F_{1,1}^{(c_1)}$ and $c_{jn} = c_j + c_{j+1} + \cdots + c_n$. Noting that $F_i x_{\lambda+\rho,u} = 0$ if $i \ge 2$, we see

(b) $m_i = F_{1i}^{(a_i)}F_{1,i-1}^{(l'-1)}\cdots F_{1,1}^{(l'-1)}m_0$.

Using commutation relations between root vectors (see for example [X1]), we see the elements $F^{\mathbf{c}}m_0$ $(\mathbf{c} \in \mathbf{Z}^n_{+,\mathbf{l}},\ c_{jn} \ge a_j + b_{ij}l'$ for all $j \in [1,i])$ form a basis of $\mathbf{u}m_i$. By this and using (a) we get

(c) The elements $F^{\mathbf{c}}m_0$ $(\mathbf{c} \in \mathbf{Z}^n_{+,\mathbf{l}},\ c_{1n} \ge a_1 + b_{i1}l')$ form a basis of $\mathbf{u}m_i$.

(d) Therefore the image (in M/M_{i+1}) of $\{F^{\mathbf{c}}m_0 \mid \mathbf{c} \in \mathbf{Z}^n_{+,\mathbf{l}},\ c_{1n} \leq a_1 + b_{i+1,1}l' - 1\}$ is a basis of M/M_{i+1}.

Assume that $a_{i+1} > 0$. Then $a_i = a_{i+1} - 1$ and $m_{i+1} \neq m_i$, so $M_i/M_{i+1} \neq 0$. Let $\theta \in \mathbf{N}$ be such that $a_1 + i = \theta l' + a_i + 1$. Then we can easily see $b_{i+1,1} = i - \theta$. Therefore $a_1 + b_{i+1,1}l' - 1 = i(l' - 1) + a_i$. Hence the element $z_i == F^{(l'-1)}_{1n} \cdots F^{(l'-1)}_{1,n-i+1}F^{(a_i)}_{1,n-i}$ has the lowest degree among the elements $F^{\mathbf{c}}$ ($\mathbf{c} \in \mathbf{Z}^n_{+,\mathbf{l}},\ c_{1n} \leq a_1 + b_{i+1,1}l' - 1$). Let $\bar{m}'_i$ be the image in M/M_{i+1} of $z_i m_0$. Using (d) we can easily check that for any nonzero element $\bar{m} \in M/M_{i+1}$ there exists $x \in \mathbf{u}^-$ such that $x\bar{m} = \bar{m}'_i$. So M/M_{i+1} contains a unique irreducible submodule which is generated by $\bar{m}'_i$.

We have $x'_{\lambda^i} = F^{(a_i)}_{n-i}F^{(l'-1)}_{n-i+1} \cdots F^{(l'-1)}_{n} F^{(a_i)}_{n-i-1}F^{(l'-1)}_{n-i} \cdots F^{(l'-1)}_{n-1} \cdots F^{(a_i)}_{1}F^{(l'-1)}_{2} \cdots \times F^{(l'-1)}_{i+1}F^{(l'-a_i-1)}_{1}F^{(l'-a_i-1)}_{2} \cdots F^{(l'-a_i-1)}_{i}$. So $x'_{\lambda^i}m_i = \xi^e z_i m_0$ for some $e \in \mathbf{Z}$. Therefore M_i/M_{i+1} is the unique irreducible submodule of M/M_{i+1}. In summary (recalling that $L_{\mathbf{u}}(\lambda^i) \simeq L(\lambda^i)|_{\mathbf{u}}$), we have

Proposition 4.6. keep the notation in 4.5. Assume that $a_{i+1} \neq 0$ (this is equivalent to $\lambda^i \neq \lambda^{i+1}$). Then

(i) As $\mathbf{u}$-modules, M_i/M_{i+1} is isomorphic to $L(\lambda^i)$. In aprticular, we have

(ii) Each weight space of $L(\lambda^i)$ has dimension 1.

(iii) $\dim L(\lambda^i) = \#\{(c_n, ..., c_1) \in \mathbf{Z}^n_{+,\mathbf{l}} \mid (i-1)(l'-1) + a_i \leq c_n + \cdots + c_1 \leq i(l'-1) + a_i\}$.

Remark: Let $\mu \in \mathbf{Z}^n_{+,\mathbf{l}}$. Assume that μ and λ are in the same alcove. Using the translation principal we see the module sructure $\mathbf{u}x_{\mu+\rho,u}1_\mu$ is the same as that of M.

For GL_n, the above dimensions are computed in [BKS] and [MP].

4.7. Unfortunately the author has not found a simple formula for the dimension in 4.6. The following result maybe is useful for obtaining a formula for the dimension.

(a) The number of solutions of $c_n + \cdots + c_2 + c_1 = r$ in non-negative integers not exceeding a fixed positive integer k is $\sum_{i=0}^n (-1)^i C^i_n C^{r-(k+1)i}_{n+r-(k+1)i-1}$, where C^i_n are the binomial coefficients, see [GS, p. 1037].

When $l' > n$ and $a = l' - 2$, we have $a_i = l' + i - n - 1$ and

$$m_i = F_i^{(l'+i-n-1)} F_{i-1}^{(2l'+i-1-n-1)} \cdots F_1^{(il'-n)} x_{\lambda+\rho,u} 1_\lambda.$$

We have $\lambda^i_{i+1} = l' - (n-i) - 1$, $\lambda^i_i = n - i$ and other components of λ^i being zero. The following elements form a basis of $\mathbf{u}m_i$,

$$F_{\beta_n}^{(c_n)} \cdots F_{\beta_2}^{(c_2)} F_{\beta_1}^{(c_1)} x_{\lambda+\rho,u} 1_\lambda,$$

$0 \le c_i < l'$ and $c_n + c_{n-1} + \cdots + c_1 \ge il' - n$. Therefore we have

$$\mathrm{dim} L(\lambda^i) = \#\{(c_n, ..., c_2, c_1) \in \mathbf{Z}^n_{+,\mathbf{l}} \mid il' - n \le c_n + c_{n-1} + \cdots + c_1 \le (i+1)l' - n - 1\}.$$

Let $\mu \in \mathbf{Z}^n_{+,\mathbf{l}}$. Assume that μ and λ are in the same alcove. Using the translation principal we see the module sructure $\mathbf{u}x_{\mu+\rho,u}1_\mu$ is the same as that of M. In particular, $\mathbf{u}x_{\mu+\rho,u}1_\mu$ has $n+1$ composition factors.

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