Primitive and Quasi Primitive Elements in Weyl Modules for Type A_2

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The submodule structure of Weyl modules for SL_3 has been determined in [DS, K, I] and by Cline (unpublished). The main technique used in [DS] is an analysis of the interwining homomorphisms between certain Weyl modules. The approach in [I, K] is essentially a direct calculation. Also Jantzen's translation principal plays an important role in these works. For a few special cases, Irving gives some (quasi) primitive elements in Verma modules of the Frobenius kernel and Weyl modules (see [I]). In this paper we will find out all (quasi) primitive elements in Verma modules of the quantized Frobenius kernel and Weyl modules for type A_2, see §§2-3. Thus we understand the structure of the Verma modules and the Weyl modules more explicitly. Our method is based on the main results in [X1] and is rather simple. In this paper we only work with quantized enveloping algebras at roots of 1 (Lusztig version), for hyperalgebras the approach is completely similar. The method works for type B_2 (see [X2]). The author also did some computations for type A_3 by using the method, although the amount of the computation is large, it seems still not difficult to work out the structure of Weyl modules for type A_3.

1. Primitive and Quasi Primitive Elements

In this section we fix notation and give the definition for (quasi) primitive elements. We refer to [L1-4, X1] for additional information.

1.1. Let U_ξ be a quantized enveloping algebra (over $\mathbf{Q}(\xi)$) at a root ξ of 1. We assume that the order of $\xi \geq 3$. As usual, the generators of U_ξ are denoted by $E_i^{(a)}, F_i^{(a)}, K_i, K_i^{-1}$, etc. Let $\mathbf{u}$ be the Frobenius kernel and $\tilde{\mathbf{u}}$ the subalgebra of U_ξ generated by all elements in $\mathbf{u}$ and in the zero part of U_ξ. For $\lambda \in \mathbf{Z}^n$ and a U_ξ-module or a $\tilde{\mathbf{u}}$-module M we denote by M_λ the λ-weight space of M. An element in M_λ will be called an element of weight λ or a weight vector. We call m primitive if m is a nonzero weight vector and $E_i^{(a)}m = 0$

for all i and $a \geq 1$. Let $m \in M$ be a weight vector. If there exist two submodules M_1, M_2 of M such that the image in M_1/M_2 of m is primitive, then we call m a quasi primitive element. Obviously, primitive elements are quasi primitive. We have:

(a) Let $m \in M$ be a weight vector and let M_1 be the submodule of M generated by m and let N be a maximal submodule of M. Then m is quasi primitive if and only if the image in M/N of m is primitive.

(b) If m is a quasi primitive element of weight λ, then $L(\lambda)$ (or $\tilde{L}(\lambda)$) is a composition factor of M (depending on M is a U_ξ-module or a $\tilde{\mathbf{u}}$-module). Here and later we write $L(\lambda)$ (resp. $\tilde{L}(\lambda)$) for an irreducible U_ξ-module (resp. $\tilde{\mathbf{u}}$-module) of highest weight λ.

(c) Let M and N be modules and $\phi : M \to N$ a homomorphism. Let m be a weight vector in M. If $\phi(m)$ is a quasi primitive element of N, then m is a quasi primitive element of M.

(d) Let M, N, ϕ, m be as in (iii). If m is a (quasi) primitive element of M and $\phi(m) \neq 0$, then $\phi(m)$ is a (quasi) primitive element of N.

We shall denote by $\tilde{Z}(\lambda)$ the Verma module of $\tilde{\mathbf{u}}$ with highest weight λ and denote by $\tilde{1}_\lambda$ a nonzero element in $\tilde{Z}(\lambda)_\lambda$. Recall that to define U_ξ we need to choose $a_i \in \{1, 2, 3\}$ such that $(a_i a_{ij})$ is symmetric, where (a_{ij}) is the concerned Cartan matrix. Assume the rank of the Cartan matrix is n. Let l_i be the order of ξ^{2d_i}. For $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbf{Z}^n$ we set $\mathbf{l}\lambda = (l_1\lambda_1, ..., l_n\lambda_n)$. We call λ is $\mathbf{l}$-restricted if $0 \leq \lambda_i \leq l_i - 1$ for all i. Denote by $\mathbf{Z}^n_{+,\mathbf{l}}$ the set of all $\mathbf{l}$-restricted elements in $\mathbf{Z}^n$.

(e) Let $\lambda' \in \mathbf{Z}^n_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^n$. Set $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^n$. Then $F_i^{(\lambda'_i+1)}\tilde{1}_\lambda$ is primitive if $\lambda'_i \neq l_i - 1$.

2. Structure of $\tilde{Z}(\lambda)$ for Type A_2

2.1. From now on we assume that U_ξ is of type A_2. In this section we determine the (quasi) primitve elements in $\tilde{Z}(\lambda)$ (or equivalently in any highest weight module of $\tilde{\mathbf{u}}$) and the submodule structure of $\tilde{Z}(\lambda)$, see 2.2-7. We choose $d_i = 1$ and let $l' = l_1 = l_2$. The following result is a special case of [X1, 4.2 (ii)].

(a) Assume $0 \le a, b \le l' - 1$, $c, d \in \mathbf{Z}$, and let $\mu = (l'c - 1 + a, l'd - 1 + b)$. Then the element $F_2^{(a)} F_1^{(a+b)} F_2^{(b)} \tilde{1}_\mu = F_1^{(b)} F_2^{(a+b)} F_1^{(a)} \tilde{1}_\mu$ is primitive in $\tilde{Z}(\mu)$ and generates the unique irreducible submodule of $\tilde{Z}(\mu)$. The irreducible submodule is isomorphic to $\tilde{L}(l'c - 1 - b, l'd - 1 + b)$.

Let $\alpha_1 = (2, -1)$ and $\alpha_2 = (-1, 2)$. The set of positive roots is $R^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. Let W be the Weyl group generated by the simple reflections s_i corresponding to α_i. Let $\lambda = (a, b) \in \mathbf{Z}^2_{+,1}$ such that $\langle \lambda + \rho, \beta^\vee \rangle \ne l'$ for all $\beta \in R^+$. We consider the W-orbit of λ (dot action), which consists of the following 6 elements,

λ, $s_1.\lambda = \lambda - (a+1)\alpha_1$, $s_2.\lambda = \lambda - (b+1)\alpha_2$, $s_2 s_1.\lambda = \lambda - (a+1)\alpha_1 - (a+b+2)\alpha_2$, $s_1 s_2.\lambda = \lambda - (a+b+2)\alpha_1 - (b+1)\alpha_2$, $s_1 s_2 s_1.\lambda = \lambda - (a+b+2)\alpha_1 - (a+b+2)\alpha_2$.

Theorem 2.2. *Let $\lambda' = (a, b) \in \mathbf{Z}^2_{+,1}$, $\lambda'' \in \mathbf{Z}^2$. Set $\lambda = \lambda' + 1\lambda'' \in \mathbf{Z}^2$. Assume that $a + b + 2 < l'$. Then*

(i) *The following 6 elements are primitive in $\tilde{Z}(\lambda)$:*

$$\tilde{1}_\lambda, \qquad F_1^{(a+1)} \tilde{1}_\lambda, \qquad F_2^{(b+1)} \tilde{1}_\lambda, \qquad F_2^{(a+b+2)} F_1^{(a+1)} \tilde{1}_\lambda,$$

$$F_1^{(a+b+2)} F_2^{(b+1)} \tilde{1}_\lambda, \qquad F_2^{(a+1)} F_1^{(a+b+2)} F_2^{(b+1)} \tilde{1}_\lambda.$$

(ii) *The following 3 elements are quasi primitive elements in $\tilde{Z}(\lambda)$ but not primitive:*

$$(F_1^{(a+b+2)} F_2^{(l')} - \xi^{l'(a+b+2)} F_2^{(l')} F_1^{(a+b+2)}) F_2^{(b+1)} \tilde{1}_\lambda,$$

$$(F_2^{(a+b+2)} F_1^{(l')} - \xi^{l'(a+b+2)} F_1^{(l')} F_2^{(a+b+2)}) F_1^{(a+1)} \tilde{1}_\lambda,$$

$$F_1^{(l'-a-1)} F_2^{(l')} F_1^{(a+1)} \tilde{1}_\lambda$$

(We also can choose the third one to be $F_2^{(l'-b-1)} F_1^{(l')} F_2^{(b+1)} \tilde{1}_\lambda$.) Moreover there are no primitive elements in $\tilde{Z}(\lambda)$ which have the same weight with any of above three elements.

(iii) *The (quasi) primitive elements in (i-ii) provide nine composition factors of $\tilde{Z}(\lambda)$, which are*

$$\tilde{L}(\lambda), \qquad \tilde{L}(\lambda - (a+1)\alpha_1), \qquad \tilde{L}(\lambda - (b+1)\alpha_2),$$

$$\tilde{L}(\lambda - (a+1)\alpha_1 - (a+b+2)\alpha_2), \qquad \tilde{L}(\lambda - (a+b+2)\alpha_1 - (b+1)\alpha_2)$$

$$\tilde{L}(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2), \qquad \tilde{L}(\lambda-(l'+a+1)\alpha_1-(a+b+2)\alpha_2),$$

$$\tilde{L}(\lambda-(a+b+2)\alpha_1-(l'+b+1)\alpha_2), \qquad \tilde{L}(\lambda-l'\alpha_1-l'\alpha_2).$$

Moreover, $\tilde{Z}(\lambda)$ has only the nine composition factors.

(iv) *The submodule lattice of $\tilde{Z}(\lambda)$ is as follows (see [DS, K, I]):*

$$\tilde{L}(\lambda)$$

$$\tilde{L}(\lambda-(a+1)\alpha_1) \qquad \tilde{L}(\lambda-(b+1)\alpha_2)$$

$$\tilde{L}(\lambda-\gamma_1) \qquad \tilde{L}(\lambda-\gamma_2) \qquad \tilde{L}(\lambda-\gamma_3) \qquad \tilde{L}(\lambda-\gamma_4) \qquad \tilde{L}(\lambda-\gamma_5)$$

$$\tilde{L}(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2)$$

Where $\gamma_1=(l'+a+1)\alpha_1+(a+b+2)\alpha_2$, $\gamma_2=(a+1)\alpha_1+(a+b+2)\alpha_2$, $\gamma_3=l'\alpha_1+l'\alpha_2$, $\gamma_4=(a+b+2)\alpha_1+(b+1)\alpha_2$, $\gamma_5=(a+b+2)\alpha_1+(l'+b+1)\alpha_2$.

Proof: According to 1(d-e), we see (i) is true.

Using 1 (d-e) and 2.1 (a) we get

(1) Let $\mu=\lambda+(l'-a-1)\alpha_1$. The following elements are primitive in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \qquad F_1^{(l'-a-1)}\tilde{1}_\mu, \qquad F_2^{(a+b+2)}\tilde{1}_\mu, \qquad F_2^{b+1)}F_1^{(l'-a-1)}\tilde{1}_\mu,$$

$$F_1^{(b+1)}F_2^{(a+b+2)}\tilde{1}_\mu, \qquad F_2^{(l'-a-1)}F_1^{(l'+b+1)}F_2^{(a+b+2)}\tilde{1}_\mu.$$

(2) Let $\mu=\lambda+(l'-b-1)\alpha_2$. The following elements are primitive in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \qquad F_1^{(a+b+2)}\tilde{1}_\mu, \qquad F_2^{(l'-b-1)}\tilde{1}_\mu, \qquad F_2^{a+1)}F_1^{(a+b+2)}\tilde{1}_\mu,$$

$$F_1^{(b+1)}F_2^{(l'-b-1)}\tilde{1}_\mu, \qquad F_1^{(l'-b-1)}F_2^{(l'+b+1)}F_1^{(a+b+2)}\tilde{1}_\mu.$$

(3) Let $\mu=\lambda+(l'-a-1)\alpha_1+(l'-a-b-2)\alpha_2$. The following elements are primitive in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \qquad F_1^{(b+1)}\tilde{1}_\mu, \qquad F_2^{(l'-a-b-2)}\tilde{1}_\mu, \qquad F_2^{l'-a-1)}F_1^{(b+1)}\tilde{1}_\mu,$$
$$F_1^{(l'-a-1)}F_2^{(l'-a-b-2)}\tilde{1}_\mu, \qquad F_2^{(b+1)}F_1^{(l'-a-1)}F_2^{(l'-a-b-2)}\tilde{1}_\mu.$$

(4) Let $\mu=\lambda+(l'-a-b-2)\alpha_1+(l'-b-1)\alpha_2$. The following elements are primitive in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \qquad F_1^{(l'-a-b-2)}\tilde{1}_\mu, \qquad F_2^{(a+1)}\tilde{1}_\mu, \qquad F_2^{l'-b-1)}F_1^{(l'-b-2)}\tilde{1}_\mu,$$
$$F_1^{(l'-b-1)}F_2^{(a+1)}\tilde{1}_\mu, \qquad F_1^{(a+1)}F_2^{(l'-b-1)}F_1^{(l'-a-b-2)}\tilde{1}_\mu.$$

(5) Let $\mu=\lambda+(l'-a-b-2)\alpha_1+(l'-a-b-2)\alpha_2$. The following elements are primitive in $\tilde{Z}(\mu)$:

$$\tilde{1}_\mu, \qquad F_1^{(l'-b-1)}\tilde{1}_\mu, \qquad F_2^{(l'-a-1)}\tilde{1}_\mu, \qquad F_2^{l'-a-b-2)}F_1^{(l'-b-1)}\tilde{1}_\mu,$$
$$F_1^{(l'-a-b-2)}F_2^{(l'-a-1)}\tilde{1}_\mu, \qquad F_1^{(l'-a-1)}F_2^{(2l'-a-b-2)}F_1^{(l'-b-1)}\tilde{1}_\mu.$$

Now we consider the homomorphism:

$$\tilde{Z}(\lambda)\to\tilde{Z}(\lambda+(l'-a-1)\alpha_1)), \quad \tilde{1}_\lambda\to s=F_1^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-1)\alpha_1};$$

Let

$$x=(F_1^{(a+b+2)}F_2^{(l')}-\xi^{l'(a+b+2)}F_2^{(l')}F_1^{(a+b+2)})F_2^{(b+1)}\in\mathbf{u}^-.$$

Note that

$$F_1^{(a+b+2)}F_2^{(l'+b+1)}s=F_1^{(a+b+2)}F_2^{(l'+b+1)}F_1^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-1)\alpha_1}=\xi^{l'(b+1)}xs.$$

Using 1 (c) we see that $x\tilde{1}_\lambda$ is a quasi primitive element of weight $\lambda-(a+b+2)\alpha_1-(l'+b+1)\alpha_2$. It is easy to see that $x\tilde{1}_\lambda$ is not a primitive element, namely $E_2x\tilde{1}_\lambda\neq 0$.

Similarly we consider the homomorphism:

$$\tilde{Z}(\lambda)\to\tilde{Z}(\lambda+(l'-b-1)\alpha_2)), \quad \tilde{1}_\lambda\to t=F_2^{(l'-b-1)}\tilde{1}_{\lambda+(l'-b-1)\alpha_2};$$

Let

$$y=(F_2^{(a+b+2)}F_1^{(l')}-\xi^{l'(a+b+2)}F_1^{(l')}F_2^{(a+b+2)})F_1^{(a+1)}\in\mathbf{u}^-.$$

Note that

$$F_2^{(a+b+2)}F_1^{(l'+a+1)}t=F_2^{(a+b+2)}F_1^{(l'+a+1)}F_2^{l'-b-1)}\tilde{1}_{\lambda+(l'-b-1)\alpha_2}=\xi^{l'(a+1)}yt.$$

Using 1 (c) we see that $y\tilde{1}_\lambda$ is a quasi primitive element of weight $\lambda-(l'+a+1)\alpha_1-(a+b+2)\alpha_2$. It is easy to see that $y\tilde{1}_\lambda$ is not a primitive element.

Since $F_2^{(l'-a-b-2)}F_1^{(l'-b-1)}\in F_1^{(a+1)}\mathbf{u}^-$, we have a surjective homomorphism

$$\mathbf{u}F_1^{(a+1)}\tilde{1}_\lambda\to\mathbf{u}F_2^{(l'-a-b-2)}F_1^{(l'-b-1)}\tilde{1}_{\lambda+(l'-a-b-2)\alpha_1+(l'-a-b-2)\alpha_2}.$$

Let $z'=(F_1^{(l'-a-1)}F_2^{(l')}-\xi^{l'(l'-a-1)}F_2^{(l')}F_1^{(l'-a-1)})\in\mathbf{u}^-$. Then

$$F_1^{(l'-a-1)}F_2^{(2l'-a-b-2)}F_1^{(l'-b-1)}=\xi^{l'(l'-a-b-2)}zF_2^{l'-a-b-2)}F_1^{(l'-b-1)}.$$

Let $z=z'F_1^{(a+1)}$. According 1 (c) we see $z\tilde{1}_\lambda$ is a quasi primitive element of weight $\lambda-l'\alpha_1-l'\alpha_2$. We also can choose z to be $F_2^{(l'-b-1)}F_1^{(l')}F_2^{(b+1)}$ by considering the following homomorphism

$$\mathbf{u}F_2^{(b+1)}\tilde{1}_\lambda\to\mathbf{u}F_1^{(l'-a-b-2)}F_2^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-b-2)\alpha_1+(l'-a-b-2)\alpha_2}.$$

Since all the weights of $x\tilde{1}_\lambda$, $y\tilde{1}_\lambda$, $z\tilde{1}_\lambda$ are smaller than $\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2$ and $F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}\tilde{1}_\lambda$ generates the unique irreducible submodule of $\tilde{Z}(\lambda)$, we see there are no primitive elements in $\tilde{Z}(\lambda)$ which have the same weight with $x\tilde{1}_\lambda$ or $y\tilde{1}_\lambda$ or $x\tilde{1}_\lambda$.

We have proved (ii).

According to 1 (b) we see $\tilde{Z}(\lambda)$ has the nine composition factors. Note that the dimensions of irreducible $\tilde{\mathbf{u}}$-modules and U_ξ-modules are known. Comparing the dimensions we know $\tilde{Z}(\lambda)$ has only the above composition factors. Thus (iii) is proved.

Part (iv) is due to [DS, K, I], which also can be check directly by using (i-ii).

The theorem is proved.

Theorem 2.3. *Let* $\lambda' = (a,b) \in \mathbf{Z}^2_{+,\mathrm{l}}$, $\lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathrm{l}\lambda'' \in \mathbf{Z}^2$. *Assume that* $0 < a+1, b+1 < l'$ *and* $a+b+2 > l'$. *Then*

(i) *The following 7 elements are primitive in* $\tilde{Z}(\lambda)$:

$$\tilde{1}_\lambda, \qquad F_1^{(a+1)}\tilde{1}_\lambda, \qquad F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_2^{(a+b+2-l')}F_1^{(a+1)}\tilde{1}_\lambda,$$

$$F_1^{(a+b+2-l')}F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}\tilde{1}_\lambda,$$

$$\sum_{0\le r\le a+b+2-l'} \begin{bmatrix} b+1-r \\ l'-a-1 \end{bmatrix}_\xi^{-1} \xi^{(a+b+2-l')(b+1-r)} F_2^{(a+b+2-l'-r)} {F'_{12}}^{(r)} F_1^{(a+b+2-l'-r)}\tilde{1}_\lambda.$$

(ii) *The following 2 elements are quasi primitive elements in* $\tilde{Z}(\lambda)$ *but not primitive:*

$$(F_1^{(a+1)}F_2^{(l')} - \xi^{l'(a+1)}F_2^{(l')}F_1^{(a+1)})\tilde{1}_\lambda,$$

$$(F_2^{(b+1)}F_1^{(l')} - \xi^{l'(b+1)}F_1^{(l')}F_2^{(b+1)})\tilde{1}_\lambda.$$

Moreover there are no primitive elements in $\tilde{Z}(\lambda)$ *which have the same weight with any of above two elements.*

(iii) *The (quasi) primitive elements in (i-ii) provide nine composition factors of* $\tilde{Z}(\lambda)$, *which are*

$$\tilde{L}(\lambda), \qquad \tilde{L}(\lambda-(a+1)\alpha_1), \qquad \tilde{L}(\lambda-(b+1)\alpha_2),$$

$$\tilde{L}(\lambda-(a+1)\alpha_1-(a+b+2-l')\alpha_2), \qquad \tilde{L}(\lambda-(a+b+2-l')\alpha_1-(b+1)\alpha_2)$$

$$\tilde{L}(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2), \qquad \tilde{L}(\lambda-(a+b+2-l')\alpha_1-(a+b+2-l')\alpha_2),$$

$$\tilde{L}(\lambda-(a+1)\alpha_1-l'\alpha_2), \qquad \tilde{L}(\lambda-l'\alpha_1-(b+1)\alpha_2).$$

Moreover, $\tilde{Z}(\lambda)$ *has only the nine composition factors.*

(iv) *The submodule lattice of $\tilde{Z}(\lambda)$ is as follows (see [DS, K, I]):*

$$\tilde{L}(\lambda)$$

$$\tilde{L}(\lambda-\gamma_1) \qquad \tilde{L}(\lambda-\gamma_2) \qquad \tilde{L}(\lambda-\gamma_3) \qquad \tilde{L}(\lambda-\gamma_4) \qquad \tilde{L}(\lambda-\gamma_5)$$

$$\tilde{L}(\lambda-(a+1)\alpha_1-(a+b+2-l')\alpha_2) \qquad \tilde{L}(\lambda-(a+b+2-l')\alpha_1-(b+1)\alpha_2)$$

$$\tilde{L}(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2)$$

Where $\gamma_1=(a+1)\alpha_1$, $\gamma_2=(a+1)\alpha_1+l'\alpha_2$, $\gamma_3=(a+b+2-l')\alpha_1+(a+b+2-l')\alpha_2$, $\gamma_4=l'\alpha_1+(b+1)\alpha_2$, $\gamma_5=(b+1)\alpha_2$.

Proof: According to 1(d-e) and 2.1 (a) we see the first 6 elemets in (i) are primitive.

Consider the homomorphism:

$$\tilde{Z}(\lambda)\to\tilde{Z}(\lambda+(l'-a-1)\alpha_1)), \quad \tilde{1}_\lambda\to s=F_1^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-1)\alpha_1}.$$

Let

$$x=\sum_{0\le r\le a+b+2-l'}\begin{bmatrix} b+1-r \\ l'-a-1\end{bmatrix}_\xi^{-1}\xi^{(a+b+2-l')(b+1-r)}F_2^{(a+b+2-l'-r)}F_{12}'^{(r)}F_1^{(a+b+2-l'-r)}\in\mathbf{u}^-.$$

Note that

$$F_1^{(b+1)}F_2^{(a+b+2-l')}\tilde{1}_{\lambda+(l'-a-1)\alpha_1}=xF_1^{(l'-a-1)}\tilde{1}_{\lambda+(l'-a-1)\alpha_1}.$$

Using 1 (c) we see that $x\tilde{1}_\lambda$ is a quasi primitive element of weight $\lambda-(a+b+2-l')\alpha_1-(a+b+2-l')\alpha_2$. We will prove that $x\tilde{1}_\lambda$ is primitive after establishing (iv).

Now we consider the homomorphism:

$$\tilde{Z}(\lambda)\to\tilde{Z}(\lambda+(2l'-a-b-2)\alpha_1+(l'-b-1)\alpha_2),$$

$$\tilde{1}_\lambda \to t = F_2^{(l'-b-1)} F_1^{(2l'-a-b-2)} \tilde{1}_{\lambda+(2l'-a-b-2)\alpha_1+(l'-b-1)\alpha_2}.$$

Let

$$y = (F_1^{(a+1)} F_2^{(l')} - \xi^{l'(a+1)} F_2^{(l')} F_1^{(a+1)}) \in \mathbf{u}^-.$$

Note that

$$yt = \xi^{l'} F_1^{(a+1)} F_2^{(2l'-b-1)} F_1^{(2l'-a-b-2)} \tilde{1}_{\lambda+(2l'-a-b-2)\alpha_1+(l'-b-1)\alpha_2}$$

is a primitive element of the image of $\tilde{Z}(\lambda)$, using 1 (c) we see that $y\tilde{1}_\lambda$ is a quasi primitive element of weight $\lambda - (a+1)\alpha_1 - l'\alpha_2$.

Similarly by considering the homomorphism:

$$\tilde{Z}(\lambda) \to \tilde{Z}(\lambda + (l' - a - 1)\alpha_1 + (2l' - a - b - 2)\alpha_2)),$$

$$\tilde{1}_\lambda \to t = F_1^{(l'-a-1)} F_2^{(2l'-a-b-2)} \tilde{1}_{\lambda+(l'-a-1)\alpha_1+(2l'-a-b-2)\alpha_2},$$

we see that $z\tilde{1}_\lambda = (F_2^{(b+1)} F_1^{(l')} - \xi^{l'(b+1)} F_1^{(l')} F_2^{(b+1)})\tilde{1}_\lambda$ is a quasi primitive element. We will argue in the end of the proof for the second assertion of (ii).

The nine (quasi) primitive elements provide nine composition factors listed in (iii). Comparing the dimensions we see $\tilde{Z}(\lambda)$ has only the nine composition factors.

By a direct computation we get (iv).

Now we can see easily that the 7th element in (i) is primitive. A simple computation shows that $E_i y\tilde{1}_\lambda$ and $E_i z\tilde{1}_\lambda$ $(i = 1, 2)$ are contained in the submodule of $\tilde{Z}(\lambda)$ generated by $F_2^{(a+b+2-l')} F_1^{(a+1)} \tilde{1}_\lambda$ and $F_1^{(a+b+2-l')} F_2^{(b+1)} \tilde{1}_\lambda$. Thus $\lambda-(a+b+2-l')\alpha_1-(a+b+2-l')\alpha_2$ is a maximal weight of the maximal submodule of $\tilde{Z}(\lambda)$, so the 7th element in (i) is primitive.

Now we prove the second assertion of (ii). Assume m is a primitive element in $\tilde{Z}(\lambda)$ of weight $\lambda-(a+1)\alpha_1-l'\alpha_2$. Then $\mathbf{u}m$ contains $y\tilde{1}_\lambda = (F_1^{(a+1)} F_2^{(l')} - \xi^{l'(a+1)} F_2^{(l')} F_1^{(a+1)})\tilde{1}_\lambda$ since $\tilde{Z}(\lambda)$ has only one composition factor of highest weight $\lambda - (a+1)\alpha_1 - l'\alpha_2$. It is easy to see that $\mathbf{u}y\tilde{1}_\lambda$ contains the primitive element $m' = F_2^{(a+b+2-l')} F_1^{(a+1)} \tilde{1}_\lambda$. But this impossible since the weight of m' is bigger than the weight of $y\tilde{1}_\lambda$. Similarly we see there are no primitive elements in $Z(\lambda)$ of weight $\lambda - l'\alpha_1 - (b+1)\alpha_2$.

The theorem is proved.

Completely as the arguments for Theorems 2.2-3 we get the following results.

Theorem 2.4. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathrm{l}}$, $\lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathrm{l}\lambda'' \in \mathbf{Z}^2$. *Assume that* $0 < a+1, b+1 < l'$ *and* $a+b+2 = l'$. *Then*

(i) *The following 4 elements are primitive in* $\tilde{Z}(\lambda)$:

$$\tilde{1}_\lambda, \qquad F_1^{(a+1)}\tilde{1}_\lambda, \qquad F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_2^{(a+1)}F_1^{(l')}F_2^{(b+1)}\tilde{1}_\lambda.$$

(ii) *The primitive elements in (i) provide four composition factors of* $\tilde{Z}(\lambda)$, *which are* $\tilde{L}(\lambda)$, $\tilde{L}(\lambda-(a+1)\alpha_1)$, $\tilde{L}(\lambda-(b+1)\alpha_2)$, $\tilde{L}(\lambda-l'\alpha_1-l'\alpha_2)$. *Moreover,* $\tilde{Z}(\lambda)$ *has only the four composition factors.*

(iii) *The submodule lattice of* $\tilde{Z}(\lambda)$ *is as follows (see [DS, K, I]):*

$$\tilde{L}(\lambda)$$

$$\tilde{L}(\lambda-(a+1)\alpha_1) \qquad\qquad \tilde{L}(\lambda-(b+1)\alpha_2)$$

$$\tilde{L}(\lambda-l'\alpha_1-l'\alpha_2)$$

Theorem 2.5. *Let* $\lambda' = (l'-1, b) \in \mathbf{Z}^2_{+,\mathrm{l}}$, $\lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathrm{l}\lambda'' \in \mathbf{Z}^2$. *Assume that* $0 < b+1 < l'$. *Then*

(i) *The following 3 elements are primitive in* $\tilde{Z}(\lambda)$:

$$\tilde{1}_\lambda, \qquad F_2^{(b+1)}\tilde{1}_\lambda, \qquad F_1^{(b+1)}F_2^{(b+1)}\tilde{1}_\lambda.$$

(ii) *The element* $x = (F_2^{(b+1)}F_1^{(l')} - \xi^{l'(b+1)}F_1^{(l')}F_2^{(b+1)})\tilde{1}_\lambda$ *is quasi primitve. There are no primitive elements in* $\tilde{Z}(\lambda)$ *of weight* $\lambda - l'\alpha_1 - (b+1)\alpha_2$.

(iii) *The (quasi) primitive elements in (i-ii) provide four composition factors of* $\tilde{Z}(\lambda)$, *which are* $\tilde{L}(\lambda)$, $\tilde{L}(\lambda-(b+1)\alpha_2)$, $\tilde{L}(\lambda-(b+1)\alpha_1-(b+1)\alpha_2)$, $\tilde{L}(\lambda-l'\alpha_1-(b+1)\alpha_2)$. *Moreover,* $\tilde{Z}(\lambda)$ *has only the four composition factors.*

(iv) *The submodule lattice of* $\tilde{Z}(\lambda)$ *is as follows (see [DS, K, I]):*

$$\tilde{L}(\lambda)$$

$$\tilde{L}(\lambda-l'\alpha_1-(b+1)\alpha_2) \qquad\qquad \tilde{L}(\lambda-(b+1)\alpha_2)$$

$$\tilde{L}(\lambda-(b+1)\alpha_1-(b+1)\alpha_2)$$

Theorem 2.6. *Let* $\lambda'=(a,l'-1)\in\mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda''\in\mathbf{Z}^2$. *Set* $\lambda=\lambda'+\mathbf{l}\lambda''\in\mathbf{Z}^2$. *Assume that* $0<a+1<l'$. *Then*

(i) *The following 3 elements are primitive in* $\tilde{Z}(\lambda)$:

$$\tilde{1}_\lambda, \qquad F_1^{(a+1)}\tilde{1}_\lambda, \qquad F_2^{(a+1)}F_1^{(a+1)}\tilde{1}_\lambda.$$

(ii) *The element* $x=(F_1^{(a+1)}F_2^{(l')}-\xi^{l'(a+1)}F_2^{(l')}F_1^{(a+1)})\tilde{1}_\lambda$ *is quasi primitve. There are no primitive elements in* $\tilde{Z}(\lambda)$ *of weight* $\lambda-(a+1)\alpha_1-l'\alpha_2$.

(iii) *The (quasi) primitive elements in (i-ii) provide four composition factors of* $\tilde{Z}(\lambda)$, *which are* $\tilde{L}(\lambda)$, $\tilde{L}(\lambda-(a+1)\alpha_1)$, $\tilde{L}(\lambda-(a+1)\alpha_1-(a+1)\alpha_2)$, $\tilde{L}(\lambda-(a+1)\alpha_1-l'\alpha_2)$. *Moreover,* $\tilde{Z}(\lambda)$ *has only the four composition factors.*

(iv) *The submodule lattice of* $\tilde{Z}(\lambda)$ *is as follows (see [DS, K, I]):*

$$\tilde{L}(\lambda)$$

$$\tilde{L}(\lambda-(a+1)\alpha_1) \qquad\qquad \tilde{L}(\lambda-(a+1)\alpha_1-l'\alpha_2)$$

$$\tilde{L}(\lambda-(a+1)\alpha_1-(a+1)\alpha_2)$$

Theorem 2.7. *Let* $\lambda' = (l'-1, l'-1) \in \mathbf{Z}^2_{+,\mathbf{l}},\ \lambda'' \in \mathbf{Z}^2$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2$. *Then* $\tilde{Z}(\lambda)$ *is irreducible. (See [X1])*

3. Structure of Weyl Modules for Type A_2

For $\lambda \in \mathbf{Z}^2_+$ we denote by $V(\lambda)$ the Weyl module of highest weight λ and let v_λ be a nonzero element in $V(\lambda)_\lambda$. A similar discussion as in §2 leads to the following results.

Theorem 3.1. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}},\ \lambda'' = (c, d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $a+b+2 < l'$ *and* $1 \le c, d$. *Then*

(i) *The following 6 elements are primitive in* $V(\lambda)$*:*

$$v_\lambda, \qquad F_1^{(a+1)}v_\lambda, \qquad F_2^{(b+1)}v_\lambda, \qquad F_2^{(a+b+2)}F_1^{(a+1)}v_\lambda,$$

$$F_1^{(a+b+2)}F_2^{(b+1)}v_\lambda, \qquad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}v_\lambda.$$

(ii) *The following 3 elements are quasi primitive elements in* $V(\lambda)$ *but not primitive:*

$$(dF_1^{(a+b+2)}F_2^{(l')} - (d-1)\xi^{l'(a+b+2)}F_2^{(l')}F_1^{(a+b+2)})F_2^{(b+1)}v_\lambda,$$

$$(cF_2^{(a+b+2)}F_1^{(l')} - (c-1)\xi^{l'(a+b+2)}F_1^{(l')}F_2^{(a+b+2)})F_1^{(a+1)}v_\lambda,$$

$$F_1^{(l'-a-1)}F_2^{(l')}F_1^{(a+1)}v_\lambda$$

(We also can choose the third one to be $F_2^{(l'-b-1)}F_1^{(l')}F_2^{(b+1)}v_\lambda$*.) Moreover there are no primitive elements in* $V(\lambda)$ *which have the same weight with any of above three elements.*

(iii) *The (quasi) primitive elements in (i-ii) provide nine composition factors of* $V(\lambda)$*, which are*

$$L(\lambda), \qquad L(\lambda - (a+1)\alpha_1), \qquad L(\lambda - (b+1)\alpha_2),$$

$$L(\lambda - (a+1)\alpha_1 - (a+b+2)\alpha_2), \qquad L(\lambda - (a+b+2)\alpha_1 - (b+1)\alpha_2)$$

$$L(\lambda - (a+b+2)\alpha_1 - (a+b+2)\alpha_2), \qquad L(\lambda - (l'+a+1)\alpha_1 - (a+b+2)\alpha_2),$$

$$L(\lambda - (a+b+2)\alpha_1 - (l'+b+1)\alpha_2), \qquad L(\lambda - l'\alpha_1 - l'\alpha_2).$$

Moreover, $V(\lambda)$ has only the nine composition factors.

(iv) *The submodule lattice of $V(\lambda)$ is as follows (see [DS, K, I]):*

$$L(\lambda)$$

$$L(\lambda-(a+1)\alpha_1) \qquad L(\lambda-(b+1)\alpha_2)$$

$$L(\lambda-\gamma_1) \qquad L(\lambda-\gamma_2) \qquad L(\lambda-\gamma_3) \qquad L(\lambda-\gamma_4) \qquad L(\lambda-\gamma_5)$$

$$L(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2)$$

Where $\gamma_1=(l'+a+1)\alpha_1+(a+b+2)\alpha_2$, $\gamma_2=(a+1)\alpha_1+(a+b+2)\alpha_2$, $\gamma_3=l'\alpha_1+l'\alpha_2$, $\gamma_4=(a+b+2)\alpha_1+(b+1)\alpha_2$, $\gamma_5=(a+b+2)\alpha_1+(l'+b+1)\alpha_2$.

Theorem 3.2. *Let $\lambda'=(a,b)\in\mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda''=(c,d)\in\mathbf{Z}^2_{+}$. Set $\lambda=\lambda'+\mathbf{l}\lambda''\in\mathbf{Z}^2_{+}$. Assume that $0<a+1,b+1<l'$, $a+b+2>l'$ and $1\le c,d$. Then*

(i) *The following 7 elements are primitive in $V(\lambda)$:*

$$v_\lambda, \qquad F_1^{(a+1)}v_\lambda, \qquad F_2^{(b+1)}v_\lambda, \qquad F_2^{(a+b+2-l')}F_1^{(a+1)}v_\lambda,$$

$$F_1^{(a+b+2-l')}F_2^{(b+1)}v_\lambda, \qquad F_2^{(a+1)}F_1^{(a+b+2)}F_2^{(b+1)}v_\lambda,$$

$$\sum_{0\le r\le a+b+2-l'}\begin{bmatrix}b+1-r\\ l'-a-1\end{bmatrix}_\xi^{-1}\xi^{(a+b+2-l')(b+1-r)}F_2^{(a+b+2-l'-r)}{F'_{12}}^{(r)}F_1^{(a+b+2-l'-r)}v_\lambda.$$

(ii) *The following 2 elements are quasi primitive elements in $V(\lambda)$ but not primitive:*

$$((d+1)F_1^{(a+1)}F_2^{(l')}-d\xi^{l'(a+1)}F_2^{(l')}F_1^{(a+1)})v_\lambda,$$

$$((c+1)F_2^{(b+1)}F_1^{(l')}-c\xi^{l'(b+1)}F_1^{(l')}F_2^{(b+1)})v_\lambda.$$

Moreover there are no primitive elements in $V(\lambda)$ which have the same weight with any of above two elements.

(iii) *The (quasi) primitive elements in (i-ii) provide nine composition factors of $V(\lambda)$, which are*

$$L(\lambda), \qquad L(\lambda-(a+1)\alpha_1), \qquad L(\lambda-(b+1)\alpha_2),$$

$$L(\lambda-(a+1)\alpha_1-(a+b+2-l')\alpha_2), \qquad L(\lambda-(a+b+2-l')\alpha_1-(b+1)\alpha_2)$$

$$L(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2), \qquad L(\lambda-(a+b+2-l')\alpha_1-(a+b+2-l')\alpha_2),$$

$$L(\lambda-(a+1)\alpha_1-l'\alpha_2), \qquad L(\lambda-l'\alpha_1-(b+1)\alpha_2).$$

Moreover, $V(\lambda)$ has only the nine composition factors.

(iv) *The submodule lattice of $V(\lambda)$ is as follows (see [DS, K, I]):*

$$L(\lambda)$$

$$L(\lambda-\gamma_1) \qquad L(\lambda-\gamma_2) \qquad L(\lambda-\gamma_3) \qquad L(\lambda-\gamma_4) \qquad L(\lambda-\gamma_5)$$

$$L(\lambda-(a+1)\alpha_1-(a+b+2-l')\alpha_2) \qquad L(\lambda-(a+b+2-l')\alpha_1-(b+1)\alpha_2)$$

$$L(\lambda-(a+b+2)\alpha_1-(a+b+2)\alpha_2)$$

Where $\gamma_1=(a+1)\alpha_1$, $\gamma_2=(a+1)\alpha_1+l'\alpha_2$, $\gamma_3=(a+b+2-l')\alpha_1+(a+b+2-l')\alpha_2$, $\gamma_4=l'\alpha_1+(b+1)\alpha_2$, $\gamma_5=(b+1)\alpha_2$.

Theorem 3.3. *Let $\lambda'=(a,b)\in\mathbf{Z}^2_{+,1}$, $\lambda''=(c,d)\in\mathbf{Z}^2_+$. Set $\lambda=\lambda'+\mathbf{l}\lambda''\in\mathbf{Z}^2_+$. Assume that $0<a+1,b+1<l'$, $a+b+2=l'$ and $1\le c,d$. Then*

(i) *The following 4 elements are primitive in $V(\lambda)$:*

$$v_\lambda, \qquad F_1^{(a+1)}v_\lambda, \qquad F_2^{(b+1)}v_\lambda, \qquad F_2^{(a+1)}F_1^{(l')}F_2^{(b+1)}v_\lambda.$$

(ii) *The primitive elements in (i) provide four composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda-(a+1)\alpha_1)$, $L(\lambda-(b+1)\alpha_2)$, $L(\lambda-l'\alpha_1-l'\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the four composition factors.*

(iii) *The submodule lattice of* $V(\lambda)$ *is as follows (see [DS, K, I]):*

$$L(\lambda)$$

$$L(\lambda-(a+1)\alpha_1) \qquad\qquad L(\lambda-(b+1)\alpha_2)$$

$$L(\lambda-l'\alpha_1-l'\alpha_2)$$

Theorem 3.4. *Let* $\lambda'=(l'-1,b)\in\mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda''=(c,d)\in\mathbf{Z}^2_{+}$. *Set* $\lambda=\lambda'+\mathbf{l}\lambda''\in\mathbf{Z}^2_{+}$. *Assume that* $0<b+1<l'$ *and* $1\le c,d$. *Then*

(i) *The following 3 elements are primitive in* $V(\lambda)$:

$$v_\lambda, \qquad F_2^{(b+1)}v_\lambda, \qquad F_1^{(b+1)}F_2^{(b+1)}v_\lambda.$$

(ii) *The element* $x=((c+1)F_2^{(b+1)}F_1^{(l')}-c\xi^{l'(b+1)}F_1^{(l')}F_2^{(b+1)})v_\lambda$ *is quasi primitve. There are no primitive elements in* $V(\lambda)$ *of weight* $\lambda-l'\alpha_1-(b+1)\alpha_2$.

(iii) *The (quasi) primitive elements in (i-ii) provide four composition factors of* $V(\lambda)$, *which are* $L(\lambda)$, $L(\lambda-(b+1)\alpha_2)$, $L(\lambda-(b+1)\alpha_1-(b+1)\alpha_2)$, $L(\lambda-l'\alpha_1-(b+1)\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the four composition factors.*

(iv) *The submodule lattice of* $V(\lambda)$ *is as follows (see [DS, K, I]):*

$$L(\lambda)$$

$$L(\lambda-l'\alpha_1-(b+1)\alpha_2) \qquad\qquad L(\lambda-(b+1)\alpha_2)$$

$$L(\lambda-(b+1)\alpha_1-(b+1)\alpha_2)$$

Theorem 3.5. *Let* $\lambda' = (a, l'-1) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a+1 < l'$ *and* $1 \le c, d$. *Then*

(i) *The following 3 elements are primitive in* $V(\lambda)$*:*

$$v_\lambda, \qquad F_1^{(a+1)} v_\lambda, \qquad F_2^{(a+1)} F_1^{(a+1)} v_\lambda.$$

(ii) *The element* $x = ((d+1) F_1^{(a+1)} F_2^{(l')} - d\xi^{l'(a+1)} F_2^{(l')} F_1^{(a+1)} v_\lambda$ *is quasi primitve. There are no primitive elements in* $V(\lambda)$ *of weight* $\lambda - (a+1)\alpha_1 - l'\alpha_2$.

(iii) *The (quasi) primitive elements in (i-ii) provide four composition factors of* $V(\lambda)$*, which are* $L(\lambda)$, $L(\lambda - (a+1)\alpha_1)$, $L(\lambda - (a+1)\alpha_1 - (a+1)\alpha_2)$, $L(\lambda - (a+1)\alpha_1 - l'\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the four composition factors.*

(iv) *The submodule lattice of* $V(\lambda)$ *is as follows (see [DS, K, I]):*

$$L(\lambda)$$

$$L(\lambda - (a+1)\alpha_1) \qquad\qquad L(\lambda - (a+1)\alpha_1 - l'\alpha_2)$$

$$L(\lambda - (a+1)\alpha_1 - (a+1)\alpha_2)$$

Theorem 3.6. *Let* $\lambda' = (l'-1, l'-1) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Then* $V(\lambda)$ *is irreducible. (See [X1])*

Theorem 3.7. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, 0) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $a+b+2 < l'$ *and* $1 \le c$. *Then*

(i) *The following* v_λ, $F_1^{(a+1)} v_\lambda$, $F_2^{(a+b+2)} F_1^{(l'+a+1)} v_\lambda$ $(c \ge 2)$ *are primitive in* $V(\lambda)$.

(ii) *The primitive elements in (i) provide the folowing composition factors of* $V(\lambda)$, $L(\lambda)$, $L(\lambda - (a+1)\alpha_1)$, $L(\lambda - (a+b+2)\alpha_1 - (a+b+2)\alpha_2)$ $(c \ge 2)$. *Moreover,* $V(\lambda)$ *has only these composition factors. The submodule lattice of* $V(\lambda)$ *is trivial.*

Theorem 3.7'. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (0, d) \in \mathbf{Z}^2_{+}$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_{+}$. *Assume that* $a + b + 2 < l'$ *and* $1 \leq d$. *Then*

(i) *The following* v_λ, $F_2^{(b+1)} v_\lambda$, $F_1^{(a+b+2)} F_2^{(l'+b+1)} v_\lambda$ $(d \geq 2)$ *are primitive in* $V(\lambda)$.

(ii) *The primitive elements in (i) provide the folowing composition factors of* $V(\lambda)$, $L(\lambda)$, $L(\lambda - (b + 1)\alpha_2)$, $L(\lambda - (a + b + 2)\alpha_1 - (l' + b + 1)\alpha_2)$ $(d \geq 2)$. *Moreover,* $V(\lambda)$ *has only these composition factors. The submodule lattice of* $V(\lambda)$ *is trivial.*

Theorem 3.8. *Let* $\lambda' = (a, b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c, 0) \in \mathbf{Z}^2_{+}$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_{+}$. *Assume that* $0 < a + 1, b + 1 < l'$, $a + b + 2 > l'$ *and* $1 \leq c$. *Then*

(i) *The following 4 elements are primitive in* $V(\lambda)$:

$$v_\lambda, \qquad F_1^{(a+1)} v_\lambda, \qquad F_2^{(a+b+2-l')} F_1^{(a+1)} v_\lambda,$$

$$\sum_{0 \leq r \leq a+b+2-l'} \begin{bmatrix} b+1-r \\ l'-a-1 \end{bmatrix}_\xi^{-1} \xi^{(a+b+2-l')(b+1-r)} F_2^{(a+b+2-l'-r)} F'^{(r)}_{12} F_1^{(a+b+2-l'-r)} v_\lambda.$$

(ii) *The element* $F_2^{(b+1)} F_1^{(l')} v_\lambda$ *is quasi primitive elements in* $V(\lambda)$ *but not primitive. Moreover there are no primitive elements in* $V(\lambda)$ *of weight* $\lambda - l'\alpha_1 - (b + 1)\alpha_2$.

(iii) *The (quasi) primitive elements in (i-ii) provide five composition factors of* $V(\lambda)$, *which are*

$$L(\lambda), \qquad L(\lambda - (a + 1)\alpha_1), \qquad L(\lambda - (a + 1)\alpha_1 - (a + b + 2 - l')\alpha_2),$$

$$L(\lambda - (a + b + 2 - l')\alpha_1 - (a + b + 2 - l')\alpha_2), \qquad L(\lambda - l'\alpha_1 - (b + 1)\alpha_2).$$

Moreover, $V(\lambda)$ *has only the five composition factors.*

(iv) *The submodule lattice of* $V(\lambda)$ *is as follows (see [DS, K, I]):*

$$L(\lambda)$$

$$L(\lambda - \gamma_1) \qquad L(\lambda - \gamma_3) \qquad L(\lambda - \gamma_4)$$

$$L(\lambda - (a + 1)\alpha_1 - (a + b + 2 - l')\alpha_2)$$

Where $\gamma_1 = (a+1)\alpha_1, \quad \gamma_3 = (a+b+2-l')\alpha_1 + (a+b+2-l')\alpha_2, \quad \gamma_4 = l'\alpha_1 + (b+1)\alpha_2.$

Theorem 3.8'. *Let* $\lambda' = (a,b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (0,d) \in \mathbf{Z}^2_{+}$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_{+}$. *Assume that* $0 < a+1, b+1 < l'$, $a+b+2 > l'$ *and* $1 \leq d$. *Then*

(i) *The following 4 elements are primitive in* $V(\lambda)$*:*

$$v_\lambda, \qquad F_2^{(b+1)} v_\lambda, \qquad F_1^{(a+b+2-l')} F_2^{(b+1)} v_\lambda,$$

$$\sum_{0 \leq r \leq a+b+2-l'} \begin{bmatrix} b+1-r \\ l'-a-1 \end{bmatrix}_\xi^{-1} \xi^{(a+b+2-l')(b+1-r)} F_2^{(a+b+2-l'-r)} {F'_{12}}^{(r)} F_1^{(a+b+2-l'-r)} v_\lambda.$$

(ii) *The element* $F_1^{(a+1)} F_2^{(l')} v_\lambda$ *is quasi primitive elements in* $V(\lambda)$ *but not primitive. Moreover there are no primitive elements in* $V(\lambda)$ *of weight* $\lambda - (a+1)\alpha_1 - l'\alpha_2$.

(iii) *The (quasi) primitive elements in (i-ii) provide five composition factors of* $V(\lambda)$*, which are*

$$L(\lambda), \qquad L(\lambda - (b+1)\alpha_2), \qquad L(\lambda - (a+b+2-l')\alpha_1 - (b+1)\alpha_2),$$

$$L(\lambda - (a+b+2-l')\alpha_1 - (a+b+2-l')\alpha_2), \qquad L(\lambda - (a+1)\alpha_1 - l'\alpha_2).$$

Moreover, $V(\lambda)$ *has only the five composition factors.*

(iv) *The submodule lattice of* $V(\lambda)$ *is as follows (see [DS, K, I]):*

$$L(\lambda)$$

$$L(\lambda - \gamma_2) \qquad\qquad L(\lambda - \gamma_3) \qquad\qquad L(\lambda - \gamma_4)$$

$$L(\lambda - (a+1)\alpha_1 - (a+b+2-l')\alpha_2)$$

Where $\gamma_2 = (a+1)\alpha_1 + l'\alpha_2, \quad \gamma_3 = (a+b+2-l')\alpha_1 + (a+b+2-l')\alpha_2, \quad \gamma_5 = (b+1)\alpha_2.$

Theorem 3.9. *Let* $\lambda' = (a,b) \in \mathbf{Z}^2_{+,\mathbf{l}}$, $\lambda'' = (c,0) \in \mathbf{Z}^2_{+}$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_{+}$. *Assume that* $0 < a+1, b+1 < l'$, $a+b+2 = l'$ *and* $1 \leq c$. *Then*

(i) *The two elements* $v_\lambda,\ F_1^{(a+1)}v_\lambda$ *are primitive in* $V(\lambda)$. *(ii)The primitive elements in (i) provide two composition factors of* $V(\lambda)$, *which are* $L(\lambda), \quad L(\lambda-(a+1)\alpha_1)$. *Moreover,* $V(\lambda)$ *has only the two composition factors and the submodule lattice of* $V(\lambda)$ *is trivial.*

Theorem 3.9'. *Let* $\lambda' = (a,b) \in \mathbf{Z}^2_{+,l},\ \lambda'' = (0,d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a+1, b+1 < l'$, $a+b+2 = l'$ *and* $1 \leq d$. *Then*

(i)The two elements $v_\lambda,\ F_2^{(b+1)}v_\lambda$ *are primitive in* $V(\lambda)$.

(ii)The primitive elements in (i) provide two composition factors of $V(\lambda)$, *which are* $L(\lambda), \quad L(\lambda-(b+1)\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the two composition factors and the submodule lattice of* $V(\lambda)$ *is trivial.*

Theorem 3.10. *Let* $\lambda' = (l'-1,b) \in \mathbf{Z}^2_{+,l},\ \lambda'' = (c,0) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < b+1 < l'$ *and* $1 \leq c$. *Then*

(i)The two elements $v_\lambda,\ F_2^{(b+1)}F_1^{(l')}v_\lambda$ *are primitive in* $V(\lambda)$.

(ii)The primitive elements in (i) provide two composition factors of $V(\lambda)$, *which are* $L(\lambda), \quad L(\lambda - l'\alpha_1 - (b+1)\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the two composition factor sand the submodule lattice of* $V(\lambda)$ *is trivial.*

Theorem 3.10'. *Let* $\lambda' = (l'-1,b) \in \mathbf{Z}^2_{+,l},\ \lambda'' = (0,d) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < b+1 < l'$ *and* $1 \leq d$. *Then*

(i)The three elements $v_\lambda,\ F_2^{(b+1)}v_\lambda,\ F_1^{b+1)}F_2^{(b+1)}v_\lambda$ *are primitive in* $V(\lambda)$.

(ii)The primitive elements in (i) provide three composition factors of $V(\lambda)$, *which are* $L(\lambda), \quad L(\lambda-(b+1)\alpha_2), \quad L(\lambda-(a+1)\alpha_1-(b+1)\alpha_2)$. *Moreover,* $V(\lambda)$ *has only the three composition factor sand the submodule lattice of* $V(\lambda)$ *is trivial.*

Theorem 3.12. *Let* $\lambda' = (a,l'-1) \in \mathbf{Z}^2_{+,l},\ \lambda'' = (c,0) \in \mathbf{Z}^2_+$. *Set* $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. *Assume that* $0 < a+1 < l'$ *and* $1 \leq c$. *Then*

(i)The following 3 elements are primitive in $V(\lambda)$:

$$v_\lambda, \qquad F_1^{(a+1)}v_\lambda, \qquad F_2^{(a+1)}F_1^{(a+1)}v_\lambda.$$

(ii)The primitive elements in (i) provide three composition factors of $V(\lambda)$, which are $L(\lambda)$, $L(\lambda-(a+1)\alpha_1)$, $L(\lambda-(a+1)\alpha_1-(a+1)\alpha_2)$. Moreover, $V(\lambda)$ has only the three composition factors and the module structure is trivial.

Theorem 3.12'. *Let $\lambda' = (a, l'-1) \in \mathbf{Z}^2_{+,l}$, $\lambda'' = (0, d) \in \mathbf{Z}^2_+$. Set $\lambda = \lambda' + \mathbf{l}\lambda'' \in \mathbf{Z}^2_+$. Assume that $0 < a+1 < l'$ and $1 \leq d$. Then*

(i)The two elements v_λ, $F_1^{(a+1)}F_2^{(l')}v_\lambda$ are primitive in $V(\lambda)$.

(ii)The primitive elements in (i) provide two composition factors of $V(\lambda)$, which are $L(\lambda)$, $L(\lambda-(a+1)\alpha_1-l'\alpha_2)$. Moreover, $V(\lambda)$ has only the two composition factors and the submodule lattice is trivial.

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