HOMOMORPHISMS BETWEEN REPRESENTATIONS OF CLANS

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ABSTRACT. We give a description of the homomorphisms between representations of a clan [CB2], more or less as it was done in [Kr] for string-algebras. As a consequence of our findings we can establish the irreducibility of all maps which are expected to be so. This allows us, after defining a natural exact structure on the category of representations, to calculate in many situations Auslander-Reiten sequences like in the string-algebra case.

INTRODUCTION

In [CB2] Crawley-Boevey introduced clans, a class of problems which includes several well-known problems from representation theory, like the Gelfand-problem [Gf], self-reproducing systems [NRo] or semi-dihedral algebras [CB3]. In [CB2] Crawley-Boevey gives also a complete classification of the indecomposable representations of such a clan, similar to the well known classification of indecomposable modules for string-algebras (see for example [BR]), basically by certain words. The original description of the indecomposable representations can be improved by the work of Bondarenko [Bd] and Deng [De].

For string algebras there is moreover a well established description of the homomorphisms between representations, see [CB4] and [Kr], as well as a handy description of Auslander-Reiten sequences. In the present paper we supply analogous results for representations of clans and clannish algebras.

In section 1 we recall the definition of a clan $\mathcal{C}$ and of the category of its representations $\mathsf{R}(\mathcal{C})$, and introduce by the way some conventions about (bi-) quivers, words and letters.

Section 2 is dedicated to the description of the indecomposable representations of the clan $\mathcal{C}$. We follow essentially [CB2], though the proof that this particularly nice description works is due to [Bd] or [De]. We define a set $\mathfrak{X}$ of words, and functors $S_{\mathfrak{w}} : A_{\mathfrak{w}}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$ which parameterize the indecomposable representations. An important feature of this description is, that the functors $S_{\mathfrak{w}}$ are symbolized by certain almost linear quivers $H^{\mathfrak{w}}$ for $\mathfrak{w} \in \mathfrak{X}$ plus some extra data.

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In section 3 we present the system of linear equations to solve for $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$, codify this data in a quiver $H^{\mathfrak{w},\mathfrak{v}}$ (see 3.3) and state several auxiliary combinatorial facts. In section 4 we present and prove our findings, in fact it turns out, that the morphisms in $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ can be read off conveniently from the quiver $H^{\mathfrak{w},\mathfrak{v}}$ (see 3.3) plus some extra data. Section 5 is dedicated to Auslander-Reiten theory: Despite the fact, that in case of asymmetric bands we can not write down an explicit base, we have enough information to prove that that all "candidates" for irreducible morphisms are in fact irreducible. After introducing a natural exact structure on $\mathsf{R}(\mathcal{C})$ we present canonical exact sequences involving the irreducible maps, and obtain thus the existence of Auslander-Reiten sequences under reasonable hypothesis on $\mathcal{C}$.

The remaining two sections contain the proofs of the more technical results from 3, linear algebra in 6 and combinatorics in 7.

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1. Basic constructions

1.1. **Quivers and letters.** A *biquiver* Q is a tuple (Q_0, Q_1, Q_2, s, t) with Q_0 the vertices, Q_1 the arrows, Q_2 the special arrows, and $s, t\colon (Q_1 \cup Q_2) \to Q_0$ defining the start- and terminal point of the arrows, thus $s(a) \xrightarrow{a} t(a)$ for each arrow $a \in Q_1$ and $s(b) \overset{b}{\dashrightarrow} t(b)$ for each $b \in Q_2$.

For technical reasons we need frequently a *polarization* of Q, i.e. a pair of functions $\sigma, \tau\colon (Q_1 \cup Q_2) \to \{-1, 1\} \subset \mathbb{Z}$.

The set of *letters* Q_* of the quiver Q consists of the following three types of symbols l

type of l	symbol l		$s(l)$	$t(l)$	$\sigma(l)$	$\tau(l)$
direct	$\underline{\alpha}$	$\alpha \in Q_1$	$s(\alpha)$	$t(\alpha)$	$\sigma(\alpha)$	$\tau(\alpha)$
inverse	$\underline{\beta}^{-1}$	$\beta \in Q_1$	$t(\beta)$	$s(\beta)$	$\tau(\beta)$	$\sigma(\beta)$
special	$\underline{\varepsilon}^{*}$	$\varepsilon \in Q_2$	$s(\varepsilon)$	$t(\varepsilon)$	$\sigma(\varepsilon)$	$\tau(\varepsilon)$

We have also indicated the definition of the functions s, t, σ and τ for the letters.

1.2. **Definition.** A *clan* $\mathcal{C}$ is a tuple $(\mathrm{k}, Q, (\sigma, \tau) \geq)$ where k is a field, Q a biquiver with polarization (σ, τ), such that $s(\varepsilon) = t(\varepsilon)$ and $\sigma(\varepsilon) = \tau(\varepsilon)$ for all $\varepsilon \in Q_2$, and $\geq$ is a linear order on each set of letters

$$Q_*(p, \rho) := \{l \in Q_* \mid t(l) = p, \tau(l) = \rho\}.$$

A *representation* M of $\mathcal{C}$ consists of the following data:

(R1) a k-vector-space M_p for each $p \in Q_0$,

(R2) a filtration of vector-spaces

$$M_p = l_1^+(M) \geq l_1^-(M) = l_2^+(M) \geq \cdots \\ \cdots \geq l_{n-1}^-(M) = l_n^+(M) \geq l_n^-(M) = 0$$

for each of the linearly ordered sets $Q_*(p,\rho) = \{l_1 > l_2 > \cdots > l_n\}$,

(R3) an isomorphism

$$M_\alpha \colon (\underline{\alpha}^{-1})^+(M)/(\underline{\alpha}^{-1})^-(M) \xrightarrow{\sim} (\underline{\alpha})^+(M)/(\underline{\alpha})^-(M)$$

for every arrow $\alpha \in Q_1$

(R4) An *idempotent* endomorphism M_ε of $(\underline{\varepsilon}^*)^+(M)/(\underline{\varepsilon}^*)^-(M)$ for each special loop $\varepsilon \in Q_2$.

A *homomorphism* $\mathbf{f}$ between representations M and N of $\mathcal{C}$ consists of a collection of linear maps $\mathbf{f}_p \colon M_p \to N_p$ for each vertex $p \in Q_0$ such that

(H1) $\mathbf{f}_{t(l)}(l^+(M)) \leq l^+(N)$ for each letter $l \in Q_*$, (i.e. $\mathbf{f}$ respects the filtration (R2)), and thus inducing linear maps

$$\mathbf{f}_l \colon l^+(M)/l^-(M) \to l^+(N)/l^-(N),$$

(H2) $N_\alpha \mathbf{f}_{\underline{\alpha}^{-1}} = \mathbf{f}_{\underline{\alpha}} M_\alpha$ for each arrow $\alpha \in Q_1$,

(H3) $N_\varepsilon \mathbf{f}_{\underline{\varepsilon}^*} = \mathbf{f}_{\underline{\varepsilon}^*} M_\varepsilon$ for each special loop $\varepsilon \in Q_2$.

We denote by $\mathsf{R}(\mathcal{C})$ the category of representations of $\mathcal{C}$ with the above described homomorphisms. This is a Krull-Schmidt k-category, [CB2, 1.4].

Remark. We usually describe a clan, by displaying the corresponding (bi-) quiver Q and the linearly ordered sets of letters $Q_*(q,\rho)$ which contain more than one element.

1.3. **Words.** A *word* $\mathfrak{w}$ in Q is a formal sequence $w_1 w_2 \cdots w_n$ of letters in Q_* with $s(w_i) = t(w_{i+1})$ and $\sigma(w_i) = -\tau(w_{i+1})$ for $i = 1, 2, \ldots, n-1$. We define $s(\mathfrak{w}) := s(w_n), t(\mathfrak{w}) := t(w_1), \sigma(\mathfrak{w}) := \sigma(w_1), \tau(\mathfrak{w}) := \tau(w_1)$. We write $l(\mathfrak{w}) := n$. We also allow *two* trivial words $1_{(q,\rho)}$ with $\rho \in \{-1, 1\}$ for each vertex $q \in Q_0$ so that $l(1_{(q,\rho)}) = 0$, and $s(1_{(q,\rho)}) = q = t(1_{(q,\rho)})$, $\sigma(1_{(q,\rho)}) = \rho = -\tau(1_{(q,\rho)})$.

We define the *product* $\mathfrak{w}\mathfrak{v}$ of two words $\mathfrak{w} = w_1 \cdots w_n$ and $\mathfrak{v} = v_1 \cdots v_m$ by $\mathfrak{w}\mathfrak{v} := w_1 \cdots w_n v_1 \cdots v_m$ if this is again a word i.e. if $s(\mathfrak{w}) = t(\mathfrak{v})$ and $\sigma(\mathfrak{w}) = -\tau(\mathfrak{v})$. If $\mathfrak{w}\mathfrak{w}$ exists, one can define arbitrary powers $\mathfrak{w}^n$ of $\mathfrak{w}$.

We say that a word $\mathfrak{w}$ is *left* (resp. *right*) *inextensible* if there is no letter $x \in Q_*$ with $\mathfrak{w}$ (resp. $\mathfrak{w}x$) a word.

For $0 < i \leq n$ we write $\mathfrak{w}_{[i]} := w_1 \cdots w_i$; for $i > n$ we set $\mathfrak{w}_{[i]} := \mathfrak{w}$, while for $i \leq 0$, we set $\mathfrak{w}_{[i]} := 1_{(t(\mathfrak{w}), -\tau(\mathfrak{w}))}$. We define $\mathfrak{w}^{[i]}$ to be the unique word with $\mathfrak{w} = \mathfrak{w}_{[i]}\mathfrak{w}^{[i]}$, and for $j \geq i$ we define $\mathfrak{w}^{[i]}_{[j]}$ to be the unique word with $\mathfrak{w}_{[i]}\mathfrak{w}^{[i]}_{[j]}\mathfrak{w}^{[j]} = \mathfrak{w}$. By a *rotation* of $\mathfrak{w}$ we mean a word of the form $\mathfrak{w}^{[i]}\mathfrak{w}_{[i]}$

Finally, if all arrows in $\varepsilon \in Q_2$ are loops (i.e. $s(\varepsilon) = t(\varepsilon)$) and fulfill $\sigma(\varepsilon) = \tau(\varepsilon)$, we define a formal *inverse* l^{-1} for each letter l as follows:

$$l^{-1} := \begin{cases} \underline{\alpha}^{-1} & \text{if } l = \underline{\alpha} \text{ is a direct letter} \\ \underline{\alpha} & \text{if } l = \underline{\alpha}^{-1} \text{ is an inverse letter} \\ \underline{\varepsilon}^* & \text{if } l = \underline{\varepsilon}^* \text{ is a special letter} \end{cases}$$

In this situation we define the *inverse word* $\mathfrak{w}^{-1} := w_1^{-1} \cdots w_n^{-1}$ and $1_{p,\rho}^{-1} := 1_{p,-\rho}$.

If there is a (linear) order $>$ on each of the sets $Q_*(p,\rho) = \{w \in Q_* \mid t(w) = p, \tau(w) = \rho\}$ there is a partial ordering $\geq$ on the set of words defined by $\mathfrak{w} > \mathfrak{w}'$ if and only if $\mathfrak{w} = \mathfrak{x}y\mathfrak{u}$ and $\mathfrak{w}' = \mathfrak{x}z\mathfrak{v}$ where $\mathfrak{x}, \mathfrak{u}$ and $\mathfrak{v}$ are words and y and z are letters with $y > z$; in this situation we define

$$\Delta(\mathfrak{w}, \mathfrak{w}') := l(\mathfrak{x})$$

Observe, that the sets

$$\mathfrak{W}(p,\rho) := \{\mathfrak{w} \text{ word in } Q_* \mid t(\mathfrak{w}) = p,\ \tau(\mathfrak{w}) = \rho,\ \mathfrak{w} \text{ right inextensible}\}$$

are linearly ordered for all vertices $p \in Q_0$ and $\rho \in \{-1, 1\}$.

2. Description of the indecomposable representations

2.1. **Strings and bands.** Let $\mathfrak{S}$ be a set of representatives of words $\mathfrak{w}$ which are both right and left inextensible, under the equivalence relation which identifies $\mathfrak{w}$ with $\mathfrak{w}^{-1}$. We call the elements of $\mathfrak{S}$ *strings* and say that a string $\mathfrak{w}$ is *symmetric* if $\mathfrak{w} = \mathfrak{w}^{-1}$, and otherwise *asymmetric*.

Let $\mathfrak{B}$ be a set of representatives of nontrivial words $\mathfrak{w}$ such that $\mathfrak{w}\mathfrak{w}$ is defined and such that $\mathfrak{w}$ is not a power under the equivalence relation which identifies all rotations of $\mathfrak{w}$ and $\mathfrak{w}^{-1}$. We call the elements of $\mathfrak{B}$ *bands* and say that a band $\mathfrak{w}$ is *symmetric* if $\mathfrak{w}$ is equal to some rotation of $\mathfrak{w}^{-1}$, and otherwise *asymmetric*.

We define $\mathfrak{X} := \mathfrak{S} \cup \mathfrak{B}$. For each $\mathfrak{w} \in \mathfrak{X}$, we shall define an algebra $A_{\mathfrak{w}}$ and a functor $S_{\mathfrak{w}} \colon A_{\mathfrak{w}}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$.

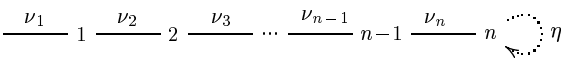
2.2. **From words to quivers.** We assign to each $\mathfrak{w} \in \mathfrak{X}$ a quiver $H^{\mathfrak{w}}$, a quiver morphism $G_{\mathfrak{w}} \colon H^{\mathfrak{w}} \to Q$, a k-algebra $A_{\mathfrak{w}}$ and a map $U_{\mathfrak{w}} \colon H_1^{\mathfrak{w}} \cup H_2^{\mathfrak{w}} \to A_{\mathfrak{w}}$. These data symbolize a functor $S_{\mathfrak{w}} \colon A_{\mathfrak{w}}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$, as we will explain in 2.4.

2.2.1. If $\mathfrak{w} = w_1 \cdots w_n$ is an *asymmetric string*, let $H^{\mathfrak{w}}$ be the quiver

$$0 \xrightarrow{\nu_1} 1 \xrightarrow{\nu_2} 2 \xrightarrow{\nu_3} \cdots \xrightarrow{\nu_{n-1}} n-1 \xrightarrow{\nu_n} n$$

in which, if w_i is a direct letter $\underline{\alpha}$, then ν_i points to the left, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; if w_i is an inverse letter $\underline{\alpha}^{-1}$, then ν_i points to the right, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; while if w_i is a special letter $\underline{\varepsilon}^*$ we set $G_{\mathfrak{w}}(\nu_i) = \varepsilon \in Q_2$ and ν_i points to the left if and only if $\mathfrak{w}_{[i-1]}^{-1} > \mathfrak{w}^{[i]}$. We define $A_{\mathfrak{w}} := \mathrm{k}$ and $U_{\mathfrak{w}}(\nu_i) = 1$.

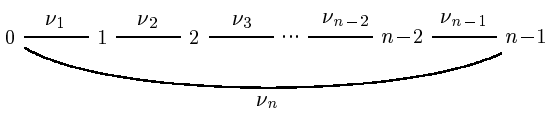
2.2.2. If $\mathfrak{w}$ is a *symmetric string*, then $\mathfrak{w} = \mathfrak{z}\,\underline{\delta}^*\,\mathfrak{z}$ for a unique word $\mathfrak{z} = z_1 \cdots z_n$. Let $H^{\mathfrak{w}}$ be the quiver



in which, if z_i is a direct letter $\underline{\alpha}$, then ν_i points to the left, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; if z_i is an inverse letter $\underline{\alpha}^{-1}$, then ν_i points to the right, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; while if z_i is a special letter $\underline{\varepsilon}^*$ we set $G_{\mathfrak{w}}\nu_i = \varepsilon \in Q_2$ and ν_i points to the left if and only if $\mathfrak{z}_{[i-1]}^{-1} > \mathfrak{z}^{[i]}\underline{f}^*\mathfrak{z}$. Finally, $G_{\mathfrak{w}}(\eta) := \delta$

We define $A_{\mathfrak{w}} := \mathrm{k}[z]/(z^2 - z)$ and $U_{\mathfrak{w}}(\nu_i) = 1$ for $i = 1, \ldots n$, and $U_{\mathfrak{w}}(\eta) = z$.

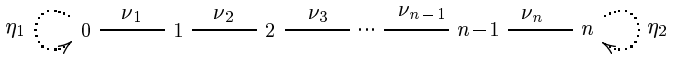
2.2.3. If $\mathfrak{w} = w_1 \cdots w_n$ is an *asymmetric band*, let $H^{\mathfrak{w}}$ be the quiver



in which, if w_i is a direct letter $\underline{\alpha}$, then ν_i points anti-clockwise, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; if w_i is an inverse letter $\underline{\alpha}^{-1}$, then ν_i points clockwise, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$. If w_i is a special letter $\underline{\varepsilon}^*$ then $G_{\mathfrak{w}}(\nu_i) = \varepsilon \in Q_2$ and ν_i points anti-clockwise if and only if $\mathfrak{w}_{[i-1]}^{-1}(\mathfrak{w}^{[i]})^{-1} > \mathfrak{w}^{[i]}\mathfrak{w}_{[i-1]}$.

We define $A_{\mathfrak{w}} := \mathrm{k}[x, x^{-1}]$ and $U_{\mathfrak{w}}(\nu_i) = 1$ for $i \neq 1$ and $U_{\mathfrak{w}}(\nu_1) = x$ if ν_1 is anti-clockwise, $U_{\mathfrak{w}}(\nu_1) = x^{-1}$ otherwise.

2.2.4. If $\mathfrak{w}$ is a *symmetric band*, it can be written uniquely in the form $\mathfrak{t}\underline{\delta}_2^*\,\mathfrak{t}^{-1}\mathfrak{r}^{-1}\,\underline{\delta}_1^*\,\mathfrak{r}$ where δ_1, δ_2 are special loops, and $\mathfrak{r}, \mathfrak{t}$ are words. Set $\mathfrak{z} := \mathfrak{r}\mathfrak{t} = z_1 \cdots z_n$ and let $H^{\mathfrak{w}}$ be the quiver



in which, if z_i is a direct letter $\underline{\alpha}$, then ν_i points to the left, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; if z_i is an inverse letter $\underline{\alpha}^{-1}$, then ν_i points to the right, and $G_{\mathfrak{w}}(\nu_i) = \alpha \in Q_1$; while if z_i is a special letter $\underline{\varepsilon}^*$, we set $G_{\mathfrak{w}}(\nu_i) = \beta \in Q_2$ and ν_i points to the left if and only if

$$\mathfrak{z}_{[i-1]}^{-1}\,\underline{\delta}_1^*\,\mathfrak{z}\,\underline{\delta}_2^*\,(\mathfrak{z}^{[i]})^{-1} > \mathfrak{z}_{[i]}\underline{\delta}_2^*\,\mathfrak{z}^{-1}\underline{\delta}_1^*\,\mathfrak{z}i{-}1.$$

Finally, $G_{\mathfrak{w}}(\eta_1) = \delta_1,\ G_{\mathfrak{w}}(\eta_2) = \delta_2$

We define $A_{\mathfrak{w}} := \mathrm{k}\langle x, y\rangle/(x^2 - x, y^2 - y)$ and $U_{\mathfrak{w}}(\nu_i) = 1$ for $i = 1, \ldots n$, and $U_{\mathfrak{w}}(\eta_1) = y,\ U_{\mathfrak{w}}(\eta_2) = x$.

2.3. **Note.** For $\mathfrak{w} \in \mathfrak{X}$ we consider the quiver $H^{\mathfrak{w}}$ as a biquiver, with $H_1^{\mathfrak{w}}$ consisting of the solid arrows and $H_2^{\mathfrak{w}}$ consisting of the dotted loops. Moreover $H^{\mathfrak{w}}$ has the following induced polarization:

$$\sigma(\nu) := \sigma(G_{\mathfrak{w}}(\nu)) \text{ and } \tau(\nu) := \tau(G_{\mathfrak{w}}(\nu))$$

for all arrows in $\nu \in H_1^{\mathfrak{w}} \cup H_2^{\mathfrak{w}}$. Thus, our conventions about letters and words apply for $H^{\mathfrak{w}}$. In particular, the quiver homomorphism $G_{\mathfrak{w}}: H^{\mathfrak{w}} \to Q$ induces a map between the corresponding sets of letters: $G_{\mathfrak{w}}^*: H_*^{\mathfrak{w}} \to Q_*$

$$\begin{aligned}
\underline{\nu} &\mapsto \begin{cases} \underline{G_{\mathfrak{w}}(\nu)} & \text{if } G_{\mathfrak{w}}(\nu) \in Q_1 \\ \underline{G_{\mathfrak{w}}(\nu)}^* & \text{if } G_{\mathfrak{w}}(\nu) \in Q_2 \end{cases} \\
\underline{\nu}^{-1} &\mapsto \begin{cases} \underline{G_{\mathfrak{w}}(\nu)}^{-1} & \text{if } G_{\mathfrak{w}}(\nu) \in Q_1 \\ \underline{G_{\mathfrak{w}}(\nu)}^* & \text{if } G_{\mathfrak{w}}(\nu) \in Q_2 \end{cases} \\
\underline{\eta}^* &\mapsto \underline{G_{\mathfrak{w}}(\eta)}^*
\end{aligned}$$

$G_{\mathfrak{w}}^*$ extends naturally to a map between the corresponding words.

Observe, that for any $\mathfrak{w} \in \mathfrak{X}$ and $\mathfrak{h}$ a word in $H^{\mathfrak{w}}$, there is at most one letter $h \in H_*^{\mathfrak{w}}$ such that $\mathfrak{h}\, h$ is again a word. Thus, given $(j, \rho) \in H_0^{\mathfrak{w}} \times \{-1, 1\}$ there is a unique, right inextensible word $\mathfrak{h}^{\mathfrak{w}}(j, \rho)$ with $t(\mathfrak{h}^{\mathfrak{w}}(j, \rho)) = j$ and $\tau(\mathfrak{h}^{\mathfrak{w}}(j, \rho)) = \rho$. Note, in case $\mathfrak{w}$ is a band, $\mathfrak{h}^{\mathfrak{w}}(j, \rho)$ is right infinite.

Remark. For $s \xrightarrow{\nu} t$ an arrow in $H_1^{\mathfrak{w}}$ with $G_{\mathfrak{w}}(\nu) \in Q_2$ let $\rho := \sigma(\nu)$, then trivially also $\rho = \tau(\nu)$. By the construction of $H^{\mathfrak{w}}$ we have

$$G_{\mathfrak{w}}^*(\mathfrak{h}^{\mathfrak{w}}(t, -\rho)) > G_{\mathfrak{w}}^*(\mathfrak{h}^{\mathfrak{w}}(s, -\rho)).$$

See 3.2 and 7.2 for related constructions.

2.4. **How to interpret the quivers.** For each letter $l \in Q_*$ we define

$$\begin{aligned}
\mathbf{t}(\mathfrak{w}, l) &:= \{t(x) \in H_0^{\mathfrak{w}} \mid x \in H_*^{\mathfrak{w}},\ G_{\mathfrak{w}}^*(x) = l\} \\
\mathbf{s}(\mathfrak{w}, l) &:= \{s(x) \in H_0^{\mathfrak{w}} \mid x \in H_*^{\mathfrak{w}},\ G_{\mathfrak{w}}^*(x) = l\},
\end{aligned}$$

thus $\mathbf{t}(\mathfrak{w}, l) = \mathbf{s}(\mathfrak{w}, l^{-1})$.

If $W \in A_w$-mod, we define the representation $S_{\mathfrak{w}}(W)$ by the following data:
For every $p \in Q_0$ let

$$S_{\mathfrak{w}}(W)_p := \bigoplus_{j \in G_{\mathfrak{w}}^{-1}(p)} W_j, \text{ where each } W_j = W.$$

For each letter $l \in Q_*$

$$\begin{aligned}
l^+(S_{\mathfrak{w}}(W)) &:= \bigoplus_{l' \leq l} \bigoplus_{j \in \mathbf{t}(\mathfrak{w}, l')} W_j \\
l^-(S_{\mathfrak{w}}(W)) &:= \bigoplus_{l' < l} \bigoplus_{j \in \mathbf{t}(\mathfrak{w}, l')} W_j
\end{aligned}$$

and identify

$$l^+(S_{\mathfrak{w}}(W))/l^-(S_{\mathfrak{w}}(W)) \text{ with } \bigoplus_{j \in \mathbf{t}(\mathfrak{w}, l)} W_j = \bigoplus_{j' \in \mathbf{s}(\mathfrak{w}, l^{-1})} W_j$$

For each (ordinary) arrow $\alpha \in Q_1$ we define

$$S_{\mathfrak{w}}(W)_\alpha \colon \bigoplus_{j \in \mathbf{s}(\mathfrak{w},\underline{a})} W_j \to \bigoplus_{j' \in \mathbf{t}(\mathfrak{w},\underline{a})} W_{j'}$$

by

$$W_{s(\nu)} \ni w \mapsto U_{\mathfrak{w}}(\nu)w \in W_{t(\nu)} \text{ for all } \nu \in G_{\mathfrak{w}}^{-1}(\alpha)$$

For each special loop $\varepsilon \in Q_2$ we define

$$S_{\mathfrak{w}}(W)_\varepsilon \colon \bigoplus_{j \in \mathbf{s}(\mathfrak{w},\underline{\varepsilon}^*)} W_j \to \bigoplus_{j' \in \mathbf{t}(\mathfrak{w},\underline{\varepsilon}^*)} W_{j'}$$

by

$$W_{s(\nu)} \ni w \mapsto U_{\mathfrak{w}}(\nu)w \in W_{t(h)} \text{ for all } h \in G_{\mathfrak{w}}^{-1}(\beta)$$

and

$$W_{t(\nu)} \ni w \mapsto w \in W_{t(\nu)} \text{ for all } \nu \in G_{\mathfrak{w}}^{-1}(\beta) \text{ with } s(\nu) \neq t(\nu)$$

2.5. By the work of Bondarenko [Bd] and Deng [De] we have the following result:

Theorem 1. *If $\mathcal{C} = (\mathrm{k}, Q, (\sigma, \tau), >)$ is a clan, the representations $S_{\mathfrak{w}}(W)$ with $\mathfrak{w}$ running through $\mathfrak{X}$, and W running through a complete set of non-isomorphic indecomposable $A_{\mathfrak{w}}$-modules, provide a complete set of non-isomorphic indecomposable representations of $\mathcal{C}$.*

Our exposition is essentially that of Crawley-Boevey [CB2], rather than that of Deng, the transfer is straight forward, though somewhat cumbersome.

3. Reduction to Combinatorics

3.1. **The system of equations.** Our aim is essentially to find a basis of the linear space $\mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ for $v, w \in \mathfrak{X}$ and V resp. W in $A_{\mathfrak{v}}$-mod resp $A_{\mathfrak{w}}$-mod. To this end we write down explicitly the corresponding system of linear equations to solve.

Proposition 1. *Consider for $\mathfrak{v}, \mathfrak{w} \in \mathfrak{X}$ the representations $S_{\mathfrak{v}}(V)$ and $S_{\mathfrak{w}}(W)$ of $\mathcal{C}$ constructed in 2.2 and 2.4. Let $\mathbf{f} = (\mathbf{f}_{(j,i)})_{(j,i) \in H_0^{\mathfrak{w},\mathfrak{v}}}$, with $\mathbf{f}_{(j,i)} \in \mathsf{Hom}_{\mathrm{k}}(V_i, W_j)$ and*

$$H_0^{\mathfrak{w},\mathfrak{v}} := \{(j,i) \in H_0^{\mathfrak{w}} \times H_0^{\mathfrak{v}} \mid G_{\mathfrak{w}}(j) = G_{\mathfrak{v}}(i)\}.$$

We have: $\mathbf{f} \in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ if and only if $\mathbf{f}$ fulfills the following set of equations:

$$\mathbf{f}_{(t(\nu),t(\mu))} = 0 \tag{1}$$

for all $(\nu,\mu)\in H_*^{\mathfrak{w}}\times H_*^{\mathfrak{v}}$ *with* $G_{\mathfrak{w}}^*(\nu)>G_{\mathfrak{v}}^*(\mu)$.

$$\mathbf{f}_{(t(\nu),t(\mu))}U_{\mathfrak{v}}(\mu)-U_{\mathfrak{w}}(\nu)\mathbf{f}_{(s(\nu),s(\mu))}=0 \tag{2}$$

for all $(\nu,\mu)\in H_1^{\mathfrak{w}}\times H_1^{\mathfrak{v}}$ *with* $G_{\mathfrak{w}}(\nu)=G_{\mathfrak{v}}(\mu)\in Q_1$.

$$\mathbf{f}_{(t(\nu),t(\mu))}U_{\mathfrak{v}}(\mu)-U_{\mathfrak{w}}(\nu)\mathbf{f}_{(s(\nu),s(\mu))}=\mathbf{f}_{(t(\nu),s(\mu))}\ \textit{and} \tag{3a}$$

$$\mathbf{f}_{(s(\nu),t(\mu))}=0 \tag{3a'}$$

for all $(\nu,\mu)\in H_1^{\mathfrak{w}}\times H_1^{\mathfrak{v}}$ *with* $G_{\mathfrak{w}}(\nu)=G_{\mathfrak{v}}(\mu)\in Q_2$.

$$\mathbf{f}_{(j,t(\mu))}U_{\mathfrak{v}}(\mu)-U_{\mathfrak{w}}(\eta)\mathbf{f}_{(j,s(\mu))}=0 \tag{3b}$$

for all $(\eta,\mu)\in H_2^{\mathfrak{w}}\times H_1^{\mathfrak{v}}$ *with* $G_{\mathfrak{w}}(\eta)=G_{\mathfrak{v}}(\mu)\in Q_2$ $(j:=s(\eta)=t(\eta))$.

$$\mathbf{f}_{(t(\nu),i)}(U_{\mathfrak{v}}(\zeta)-1)-U_{\mathfrak{w}}(\nu)\mathbf{f}_{(s(\nu),i)}=0 \tag{3c}$$

for all $(\nu,\zeta)\in H_1^{\mathfrak{w}}\times H_2^{\mathfrak{v}}$ *with* $G_{\mathfrak{w}}(\nu)=G_{\mathfrak{v}}(\zeta)\in Q_2$ $i:=s(\zeta)=t(\zeta)$.

$$\mathbf{f}_{(j,i)}U_{\mathfrak{v}}(\zeta)-U_{\mathfrak{w}}(\eta)\mathbf{f}_{(j,i)}=0 \tag{3d}$$

for all $(\eta,\zeta)\in H_2^{\mathfrak{w}}\times H_2^{\mathfrak{v}}$ *with* $G_{\mathfrak{w}}(\nu)=G_{\mathfrak{v}}(\mu)\in Q_2$, $(j:=s(\nu),i:=s(g))$.

Remarks. (1) We understand here $U_{\mathfrak{w}}(\nu)$ as an k-endomorphism of W and $U_{\mathfrak{v}}(\mu)$ as a k-endomorphism of V.

(2) As a general rule, $U_{\mathfrak{w}}(\nu)$ (resp. $U_{\mathfrak{v}}(\mu)$) is *invertible* while ν (resp. μ) is *not* a loop; $U_{\mathfrak{w}}(\eta)$ (resp. $U_{\mathfrak{v}}(\zeta)$) is idempotent, while η (resp. ζ) is a loop.

The proof of this proposition will be given in section 6.

3.2. **A device.** For technical reasons we need the following: Let $\mathcal{C}=(\mathrm{k},Q,(\sigma,\tau),>)$ be a clan. In this situation we define the quiver $\widehat{Q}=(\widehat{Q}_0,\widehat{Q}_1,\widehat{Q}_2,\hat{s},\hat{t})$ by $\widehat{Q}_0:=Q_0,\widehat{Q}_1:=Q_1\cup Q_2,\widehat{Q}_2:=\emptyset,\hat{s}=s,\hat{t}=t$, i.e. we just convert all special loops into ordinary loops. Moreover $\hat{Q}$ obtains "the same" polarization as Q.

There is an obvious quiver homomorphism $\gamma:\widehat{Q}\to Q$, which induces a map $\gamma_*:\widehat{Q}_*\to Q_*$ between the corresponding sets of letters. Observe, $\gamma_*^{-1}(\underline{\varepsilon}^*)=\{\underline{\varepsilon},\underline{\varepsilon}^{-1}\}$ for all special letters $\underline{\varepsilon}^*\in Q_*$. Moreover we define a linear order $\succ$ on the sets of letters $\widehat{Q}_*(p,\rho)$ by

$$l\succ l'\Longleftrightarrow\begin{cases}\gamma_*(l) & >\ \gamma_*(l') \quad \text{or}\\ l=\underline{\varepsilon}^{-1} & \text{and}\ \ l'=\underline{\varepsilon},\ \varepsilon\in Q_2\end{cases}$$

We write also $\succ$ for the induced order on the words in $\widehat{Q}_*$, see 1.3. Compare 2.3 and see also 7.2.

If $\mathfrak{w} \in \mathfrak{X}$, the quiver homomorphism $G_{\mathfrak{w}}$ induces two maps $\widehat{G}^{\rho}_{\mathfrak{w}} \colon H^{\mathfrak{w}}_{*} \to \widehat{Q}_{*}$, $\rho \in \{-,+\}$ defined by

$$\begin{aligned} \underline{\nu} &\mapsto \underline{G_{\mathfrak{w}}(\nu)} \\ \underline{\nu}^{-1} &\mapsto \underline{G_{\mathfrak{w}}(\nu)}^{-1} \\ \underline{\eta}^{*} &\mapsto \underline{G_{\mathfrak{w}}(\eta)}^{\rho\,1} \end{aligned}$$

Here, by abuse of notation we understand $G_{\mathfrak{w}} \colon H^{\mathfrak{w}} \to \widehat{Q}$.

Remark. Trivially, $\widehat{G}^{+}_{\mathfrak{w}} = \widehat{G}^{-}_{\mathfrak{w}}$ if and only if $\mathfrak{w}$ is asymmetric. Also these maps extend to maps between the corresponding words, thus $\widehat{G}^{-}_{\mathfrak{w}}(\mathfrak{h}) \succeq \widehat{G}^{+}_{\mathfrak{w}}(\mathfrak{h})$ always.

3.3. **The quiver $H^{\mathfrak{w},\mathfrak{v}}$.** We visualize the equations of Proposition 1 in a (bi)-quiver $H^{\mathfrak{w},\mathfrak{v}}$ and a function $U_{\mathfrak{w},\mathfrak{v}} \to A_{\mathfrak{w}} \times A_{\mathfrak{v}}$:

$$\begin{aligned}
H^{\mathfrak{w},\mathfrak{v}}_{0} :=& \quad \{(j,i) \in H^{\mathfrak{w}}_{0} \times H^{\mathfrak{v}}_{0} \quad | \; G_{\mathfrak{w}}(j) = G_{\mathfrak{v}}(i)\} \\
H^{\mathfrak{w},\mathfrak{v}}_{1} :=& \{ \; (s(\nu), s(\mu)) \xrightarrow{(\nu,\mu)} (t(\nu), t(\mu)) \; | \; (\nu,\mu) \in H^{\mathfrak{w}}_{1} \times H^{\mathfrak{v}}_{1}, \; G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\mu)\} \\
& \cup \{ \; (j_0, i_0) \xrightarrow{(\eta,\zeta)_o} (j_0, i_0) \quad | \; (\eta,\zeta) \in H^{\mathfrak{w}}_{2} \times H^{\mathfrak{v}}_{2}, \; G_{\mathfrak{w}}(\eta) = G_{\mathfrak{v}}(\zeta), \\
& \qquad\qquad\qquad\qquad\qquad\qquad j_0 := s(\nu), i_0 := s(g)\} \\
H^{\mathfrak{w},\mathfrak{v}}_{2,-} :=& \{ \; (t(\nu), s(\mu)) \overset{(\nu,\mu)_+}{\dashrightarrow} (t(\nu), t(\mu)) \; | \; (\nu,\mu) \in H^{\mathfrak{w}}_{1} \times H^{\mathfrak{v}}_{1}, \; G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\mu)\} \\
& \cup \{ \; (t(\nu), s(\mu)) \overset{(\nu,\mu)_-}{\dashrightarrow} (s(\nu), s(\mu)) \; | \; (\nu,\mu) \in H^{\mathfrak{w}}_{1} \times H^{\mathfrak{v}}_{1}, \; G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\mu)\} \\
H^{\mathfrak{w},\mathfrak{v}}_{2,o} :=& \quad \{ \; (j_0, s(\mu)) \overset{(\eta,\mu)_\oplus}{\dashrightarrow} (i_0, s(\mu)) \quad | \; (\eta,\mu) \in H^{\mathfrak{w}}_{2} \times H^{\mathfrak{v}}_{1}, \; G_{\mathfrak{w}}(\eta) = G_{\mathfrak{v}}(\mu), \\
& \qquad\qquad\qquad\qquad\qquad\qquad j_0 := s(\eta) = t(\eta)\} \\
& \cup \{ \; (t(\nu), i_0) \overset{(\nu,\zeta)_\ominus}{\dashrightarrow} (s(\nu), i_0) \quad | \; (\nu,\zeta) \in H^{\mathfrak{w}}_{1} \times H^{\mathfrak{v}}_{2}, \; G_{\mathfrak{w}}(\nu) = G_{\mathfrak{v}}(\zeta), \\
& \qquad\qquad\qquad\qquad\qquad\qquad i_0 := s(\zeta) = t(\zeta)\} \\
H^{\mathfrak{w},\mathfrak{v}}_{2} :=& \quad H^{\mathfrak{w},\mathfrak{v}}_{2,-} \cup H^{\mathfrak{w},\mathfrak{v}}_{2,o}
\end{aligned}$$

The vertices

$$\{(t(y), t(x)) \in H^{\mathfrak{w},\mathfrak{v}}_{0} \mid (y,x) \in H^{\mathfrak{w}}_{*} \times H^{\mathfrak{v}}_{*}, \widehat{G}^{+}_{\mathfrak{w}}(y) \succ \widehat{G}^{-}_{\mathfrak{v}}(x)\}$$

are called *0-points*. These vertices are labeled by a "0" in $H^{\mathfrak{w},\mathfrak{v}}$. This "codifies" the equations (1) and (3a'). (See 3.2 for the definition of $\widehat{G}^{\rho}_{\mathfrak{w}}$.)

The function $U_{\mathfrak{w},\mathfrak{v}} = (U^{(1)}_{\mathfrak{w},\mathfrak{v}}, U^{(2)}_{\mathfrak{w},\mathfrak{v}})$ is defined as follows:

$$\begin{array}{lll}
(\nu,\mu) & \mapsto (U_{\mathfrak{w}}(\nu), U_{\mathfrak{v}}(\mu)) & \text{(compare (1))} \\
(\eta,\zeta)_o & \mapsto (U_{\mathfrak{w}}(\eta), U_{\mathfrak{v}}(\zeta)) & \text{(compare (3d))} \\
(\nu,\mu)_+ & \mapsto (1, U_{\mathfrak{v}}(\mu)) & \text{(compare (3a))} \\
(\nu,\mu)_- & \mapsto (U_{\mathfrak{w}}(\nu), -1) & \text{(compare (3a))} \\
(\eta,\mu)_\oplus & \mapsto (U_{\mathfrak{w}}(\eta), U_{\mathfrak{v}}(\mu)) & \text{(compare (3b))} \\
(\nu,\zeta)_\ominus & \mapsto (U_{\mathfrak{w}}(\nu), U_{\mathfrak{v}}(\zeta) - 1) & \text{(compare (3c))}
\end{array}$$

Finally we put on $H^{\mathfrak{w},\mathfrak{v}}$ the polarization induced by $H^{\mathfrak{w}}$ and $H^{\mathfrak{v}}$.

Definition. (1) The connected components of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}})$ are called *lines*. We denote the set of lines by $\mathcal{L}(\mathfrak{w},\mathfrak{v})$,
(2) A line L is a *h-line*, if for some vertex $(j,i) \in L_0$ we have:

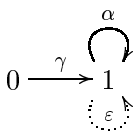
$$\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,\rho)) \preceq \widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho)) \text{ for } \rho \in \{-1,1\}$$

(see 3.2 and 2.3 for details).
(3) A line Z is a *z-line*, if for some vertex $(j,i) \in Z_0$ we have:

$$\begin{array}{l}
\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j,\rho)) \succ \widehat{G}^+_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho)) \text{ or} \\
\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(i,\rho)) \succ \widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i,\rho)) \text{ for some } \rho \in \{-1,1\}
\end{array}$$

(4) A h-line L is a *real h-line* if it is *not* a z-line, otherwise it is a *virtual h-line*. We denote the set of real h-lines by $\mathcal{H}(\mathfrak{w},\mathfrak{v})$. Similarly, a z-line is a *real z-line* if it is *not* a h-line. We denote the set of real z-lines by $\mathcal{Z}(\mathfrak{w},\mathfrak{v})$.

3.4. **Example.** Let $\mathcal{C}$ be the clan given by

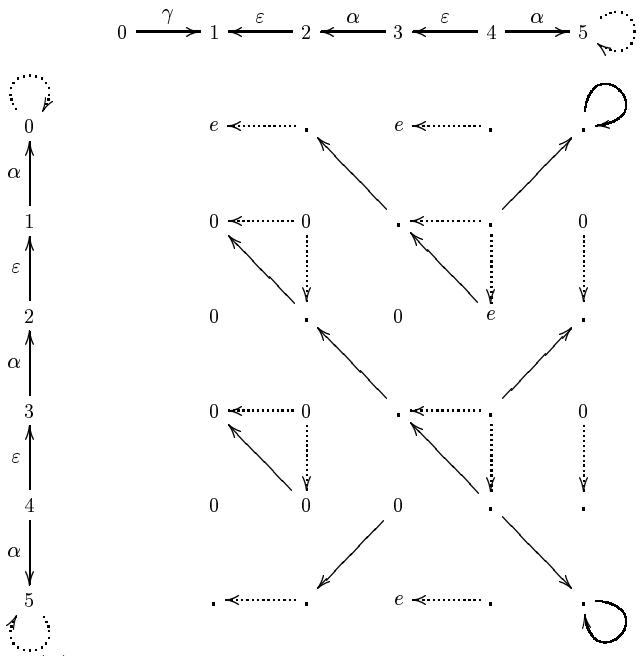


and $Q_*(1,-1) := \{\underline{\alpha}^{-1} > \underline{\gamma} > \underline{\alpha}\}$.

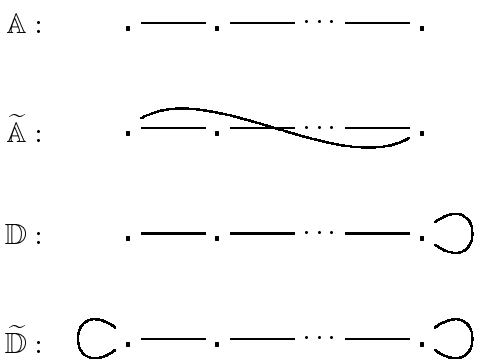
This clan represents the "problem of a an idempotent and a nilpotent matrix" [CB1]. Look at the following two words:

$$\mathfrak{v} := \underline{\gamma}^{-1}\underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}^{-1}\underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}^{-1}\underline{\varepsilon}^*\underline{\gamma} \text{ and } \mathfrak{w} := \underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}^{-1}\underline{\varepsilon}^*\underline{\alpha}\ \underline{\varepsilon}^*\underline{\alpha}^{-1}\underline{\varepsilon}^*\underline{\alpha}^{-1}$$

The first one is a symmetric string, the second a symmetric band. We write down the quivers $H^{\mathfrak{w}}, H^{\mathfrak{v}}$ and $H^{\mathfrak{w},\mathfrak{v}}$.



3.5. **Remarks.** (1) The underlying graph of a line L is exactly one of the following types:



Moreover, a line L is of type $\widetilde{\mathbb{A}}$ (resp. $\widetilde{\mathbb{D}}$) if and only if $\mathfrak{w} = \mathfrak{v}$ is an asymmetric (resp. symmetric) band and L is the diagonal.

(2) $\widehat{G}_{\mathfrak{v}}^{-/+}(\mathfrak{h}^{\mathfrak{v}}(p,\rho))$ and $\widehat{G}_{\mathfrak{w}}^{+/-}(\mathfrak{h}^{\mathfrak{w}}(q,\rho))$ are comparable for $\rho \in \{-1,+1\}$, thus every line is a h-line or a z-line. Moreover it is easy to see, that the definition is independent of the choice of the point on a line. Obviously, virtual h-lines appear only if $\mathfrak{v}$ or $\mathfrak{w}$ is symmetric.

(3) A line Z is a z-line if and only if Z contains a 0-point or if it contains the endpoint of an arrow from $H^{\mathfrak{w},\mathfrak{v}}_{2,o}$. Thus, also real h-lines are also easy to recognize.

A line L is a h-line if and only if the connected component of L in $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}} \cup H_{2,o}^{\mathfrak{w},\mathfrak{v}})$ contains no 0-point, otherwise it is a real z-line. This is a direct consequence of Corollary 1 and Proposition 3 (a) below.

3.6. **The combinatorial results.** The quiver $H^{\mathfrak{w},\mathfrak{v}}$ as defined above has some interesting properties: Consider on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$ the following relation:

$$L' >' L \Longleftrightarrow \text{there is an arrow } \; s \dashrightarrow t \; \text{ in } H_2^{\mathfrak{w},\mathfrak{v}} \text{ with } s \in L_0', t \in L_0$$

Write $>$ for the transitive relation generated by $>'$.

Proposition 2. *Suppose, that $\mathfrak{v}$ and $\mathfrak{w}$ are both not asymmetric bands, then $>$ is a partial order on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$.*

The proof is quite easy, see 7.1, but it does not work in general, if asymmetric bands are involved.

We conjecture, that the proposition is true in general. A partial result in this direction is Proposition 3 (c) below.

Corollary 1. *Consider on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$ the following weaker relation:*

$$L' >' L \Longleftrightarrow \textit{there is an arrow } \; s \dashrightarrow t \; \textit{ in } H_{2,o}^{\mathfrak{w},\mathfrak{v}} \textit{ with } s \in L_0', t \in L_0$$

The transitive relation generated by $>''$ is always a partial order on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$.

This is a trivial consequence of Proposition 2 if no asymmetric bands are involved. The remaining cases are easy to control (for example, if $\mathfrak{v}$ and $\mathfrak{w}$ are both asymmetric, then $>''$ is empty).

Proposition 3. *We have the following:*

(a) *Let $L' >' L \in \mathcal{L}(\mathfrak{w},\mathfrak{v})$. If L' is a h-line, then L is also a h-line of type $\mathbb{A}$ or $\mathbb{D}$.*

(b) *If L is a h-line, then there is at most one arrow from $s \overset{(h,g)_\rho}{\dashrightarrow} t$ in $H_{2,o}^{\mathfrak{w},\mathfrak{v}}$ with $t \in L_0$; if L is additionally of type $\mathbb{D}$, there is no such arrow.*

(c) *The set of real z-lines $(\mathcal{Z}(\mathfrak{w},\mathfrak{v}), >)$ is a poset.*

(d) *If $(j,i) \in H_0^{\mathfrak{w},\mathfrak{v}}$ lies on a h-line, holds*

$$G_{\mathfrak{w}}^*(\mathfrak{h}^{\mathfrak{w}}(j,\rho)) \leq G_{\mathfrak{w}}^*(\mathfrak{h}^{\mathfrak{v}}(i,\rho))$$

for $\rho = -1, 1$.

The proof of this result will be given in section 7.

Corollary 2. *Let $\mathfrak{v},\mathfrak{w} \in \mathfrak{X}$ and suppose, that $(j,i) \in H_0^{\mathfrak{w},\mathfrak{v}}$ lies on a h-line. If $\mathfrak{v} \neq \mathfrak{w}$ or $(i \neq j)$ then $(i,j) \in H_0^{\mathfrak{v},\mathfrak{w}}$ lies on a real z-line.*

This follows at once from Proposition 3 (d) above. This looks quite innocent, but together with part (a) of our Theorem 2 this is fundamental for the description of irreducible homomorphisms in 5.1. We call a vertex $q \in H^{\mathfrak{w},\mathfrak{v}}$ an *e-point* if

- p is *not* a zero-point,
- p is not start-point of an arrow from $H_2^{\mathfrak{w},\mathfrak{v}}$.

- p is endpoint of a connected component of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}}, H_{2,o}^{\mathfrak{w},\mathfrak{v}})$.

Corollary 3. *Suppose, that $(\mathcal{L}(\mathfrak{w},\mathfrak{v}),>)$ is a poset. Let $p \in H_0^{\mathfrak{w},\mathfrak{v}}$ be a vertex lying on a h-line not of type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$ and $\rho \in \{-1,1\}$. Then there is a word $\mathfrak{h}$ in $H^{\mathfrak{w},\mathfrak{v}}$ with $s(\mathfrak{h}) = p$, $\sigma(\mathfrak{h}) = \rho$ and $t(\mathfrak{h})$ an e-point.*

This is a direct consequence of the above proposition. Usually, there are many choices for these words.

4. Description of the Homomorphisms

Theorem 2. *Let $\mathcal{C}$ be a clan, and $\mathfrak{v},\mathfrak{w} \in \mathfrak{X}$, $V \in A_{\mathfrak{v}}$-mod and $W \in A_{\mathfrak{w}}$-mod.*

(a) *For $\mathbf{f} = (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}} \in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ we have $\mathbf{f}_{(j,i)} = 0$ if (j,i) lies on a real z-line in $H^{\mathfrak{w},\mathfrak{v}}$.*

(b) *If $(\mathcal{L}(\mathfrak{w},\mathfrak{v}),>)$ is a poset (see Proposition 2), there is a* k*-linear isomorphism*

$$\mathbf{\Phi}\colon \oplus_{L\in\mathcal{H}(\mathfrak{w},\mathfrak{v})} M_L \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$

with

$$M_L := \begin{cases} \mathsf{Hom}_{\mathrm{k}}(V,W) & \text{if } L \text{ is of type } \mathbb{A} \\ \mathsf{Hom}_{\mathrm{k}[z]/(z^2-z)}(V,W) & \text{if } L \text{ is of type } \mathbb{D} \\ \mathsf{Hom}_{A_{\mathfrak{v}}}(V,W) & \text{if } L \text{ is of type } \widetilde{\mathbb{A}} \text{ or } \widetilde{\mathbb{D}} \end{cases}$$

Remark. If L is of type $\mathbb{D}$, let $(\eta,\zeta)_o$ be the unique loop of L, then the algebra homomorphisms $\mathrm{k}[z]/(z^2-z) \to A_{\mathfrak{v}}$ and $\mathrm{k}[z]/(z^2-z) \to A_{\mathfrak{w}}$ defined by $z \mapsto U_{\mathfrak{v}}(\zeta)$ and $z \mapsto U_{\mathfrak{w}}(\eta)$ induce a natural $\mathrm{k}[z]/(z^2-z)$-module structure on V and W, thus M_L is well defined in this case.

If L is of type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$, we have (see 3.5 (1)) $\mathfrak{w} = \mathfrak{v}$ a band, thus also in this situation M_L is well-defined.

Of course, in order to prove (b), we have to define $\mathbf{\Phi}$ explicitly; this will be done essentially in terms of the quiver $H^{\mathfrak{w},\mathfrak{v}}$.

4.1. **Construction of the basis.** For every $L \in \mathcal{L}(\mathfrak{w},\mathfrak{v})$ we construct a k-linear map

$$\boldsymbol{\varphi}_L\colon M_L \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$

with $(\boldsymbol{\varphi}_L(f))_{(j,i)_L} = f$ for a previously chosen vertex $(j,i)_L$ of L.

4.1.1. If L is a (real) h-line of type $\widetilde{\mathbb{D}}$ or $\widetilde{\mathbb{A}}$, recall that $\mathfrak{w} = \mathfrak{v}$ is a band, and L the diagonal. Define for $f \in \mathsf{Hom}_{\mathrm{k}}(V,W)$ and $q \in H_0^{\mathfrak{w},\mathfrak{v}}$

$$\mathbf{f}_{(j,i)} := \begin{cases} f & \text{if } (j,i) \in L_0 \\ 0 & \text{else} \end{cases}$$

$$\boldsymbol{\varphi}_L(f) := (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}}$$

It is an easy exercise to check, that this actually defines an injective k-linear map

$$\boldsymbol{\varphi}_L\colon \mathsf{Hom}_{A_\mathfrak{v}}(V,W) \to \mathsf{Hom}_{\mathcal{C}}(S_\mathfrak{v}(V), S_\mathfrak{w}(W))$$

In fact, $\boldsymbol{\varphi}_L(f) = S_\mathfrak{w}(f)$ and trivially $(\boldsymbol{\varphi}_L(f))_{(j,i)_L} = f$ for all vertices $(j,i)_L$ of L.

4.1.2. If L is a real h-line of type $\mathbb{D}$, we take $(i,j)_L =: s((\eta,\zeta)_o)$, where $(\eta,\zeta)_o$ is the unique loop in L. By Corollary 3 we find a word $\mathfrak{x} = x_m \cdots x_1$ in $H^{\mathfrak{w},\mathfrak{v}}$ with $x_1 \neq (\eta,\zeta)_o$ and $t(\mathfrak{x})$ an e-point. Set $H(\mathfrak{x}) := \{(j_n,i_n),\cdots,(j_1,i_1),(j_0,i_0)\}$ with $(j_k,i_k) := t(x_k)$ and $(j_0,i_0) = (i,j)_L$. For $f \in \mathsf{Hom}_{\mathrm{k}[z]/(z^2-z)}(V,W)$ (see Theorem 2) we define inductively

$$\mathbf{f}_{(j_0,i_0)} := f$$

$$\mathbf{f}_{(j_k,i_k)} := \begin{cases} U^{(1)}_{\mathfrak{w},\mathfrak{v}}(x_k)\,\mathbf{f}_{(j_{k-1},i_{k-1})}\,(U^{(2)}_{\mathfrak{w},\mathfrak{v}}(x_k))^{-1} & \text{if } x_k \text{ is a direct letter or} \\ & \text{if } x_k = \underline{(g,h)}^*_\rho \text{ is special} \\ & \text{with } \rho \in \{+,\oplus\} \\ (U^{(1)}_{\mathfrak{w},\mathfrak{v}}(x_k))^{-1}\,\mathbf{f}_{(i_{k-1},j_{k-1})}\,U^{(2)}_{\mathfrak{w},\mathfrak{v}}(x_k) & \text{if } x_k \text{ is a inverse letter or} \\ & \text{if } x_i = \underline{(\nu,\mu)}^*_\rho \text{ is special} \\ & \text{with } \rho \in \{-,\ominus\} \end{cases}$$

$$\mathbf{f}_{(j,i)} := 0 \text{ for } (j,i) \in H_0^{\mathfrak{w},\mathfrak{v}} \setminus H(\mathfrak{x})$$

and finally

$$\boldsymbol{\varphi}_L(f) := (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}}$$

In order to see, that actually $\boldsymbol{\varphi}_L(f) \in \mathsf{Hom}_{\mathcal{C}}(S_\mathfrak{v}(V), S_\mathfrak{w}(W))$, we observe first $L_0 \subset H(\mathfrak{x})$ and $(i_k,j_k) = (i_l,j_l)$ if and only if $k = l$. Moreover, the elements of $\mathbb{M}(\mathfrak{x})$ lie all on h-lines (Proposition 3 (b)), thus they are all no 0-points. Finally note, that if there is an arrow $(i_k,j_k) \overset{x}{\dashrightarrow} (i_l,j_l)$ in $H_2^{\mathfrak{w},\mathfrak{v}}$ then $x = h_l$ and $k = j-1$; this makes it very easy to verify the equations (1)–(3d). Trivially, $\boldsymbol{\varphi}_L$ is injective.

4.1.3. If L is a real h-line of type $\mathbb{A}$, we choose an arbitrary vertex $(j_0,i_0) = (j,i)_L$ in L and two words $\mathfrak{y} = y_n \cdots y_1$, $\mathfrak{x} = x_m \cdots x_1$ in $H^{\mathfrak{w},\mathfrak{v}}$ with $s(\mathfrak{y}) = (j_0,i_0) = s(\mathfrak{x})$, $\sigma(\mathfrak{y}) = 1 = -\sigma(\mathfrak{x})$, both ending in an e-point as in Corollary 3. Set $H(\mathfrak{x}) := \{(j_m,i_m)\cdots,(j_0,i_0)\}$ with $(j_k,i_k) := t(x_k)$ as before.

For $f \in \mathsf{Hom}_{\mathrm{k}}(V,W)$ (see Theorem 2) we define (inductively)

$$\mathbf{f}^-_{(j_0,i_0)} := f$$

$$\mathbf{f}^-_{(j_k,i_k)} := \begin{cases} U^{(1)}_{\mathfrak{w},\mathfrak{v}}(x_k)\,\mathbf{f}^-_{(j_{k-1},i_{k-1})}\,(U^{(2)}_{\mathfrak{w},\mathfrak{v}}(x_k))^{-1} & \text{if } x_k \text{ is a direct letter or} \\ & \text{if } x_k = \underline{(\nu,\mu)}^*_\rho \text{ is special} \\ & \text{with } \rho \in \{+,\oplus\} \\ (U^{(1)}_{\mathfrak{w},\mathfrak{v}}(x_i))^{-1}\,\mathbf{f}^-_{(j_{k-1},i_{k-1})}\,U^{(2)}_{\mathfrak{w},\mathfrak{v}}(x_k) & \text{if } x_k \text{ is a inverse letter or} \\ & \text{if } x_i = \underline{(g,h)}^*_\rho \text{ is special} \\ & \text{with } \rho \in \{-,\ominus\} \end{cases}$$

$$f^-_{(j,i)} := 0 \text{ for } (j,i) \in H_0^{\mathfrak{w},\mathfrak{v}} \setminus H(\mathfrak{x})$$

and similarly, starting with $\mathfrak{y}$ we define $\mathbf{f}^+_{(j,i)}$ for $(j,i) \in H_0^{\mathfrak{w},\mathfrak{v}}$,then

$$\mathbf{f}_{(j,i)} := \begin{cases} f & \text{if } q = p \\ f^+_{(j,i)} + f^-_{(j,i)} & \text{else} \end{cases}$$

$$\phi_L(f) := (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}}$$

It is essentially the same argument as in the case $\mathbb{D}$ to see that this defines a injective k-linear map

$$\boldsymbol{\varphi}_L \colon \mathsf{Hom}_{\mathrm{k}}(V,W) \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$

From Proposition 3 (a) and from the construction of the $\boldsymbol{\varphi}_L$ is is clear, that

$$\boldsymbol{\Phi} := \oplus_{L\in\mathcal{H}}\ \boldsymbol{\varphi}_L \colon \oplus_{L\in\mathcal{H}} M_L \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$

is injective (the point is, that words starting in $(j,i)_L$ can not end in a line L' with $L' > L$).

4.2. **Proof of the theorem.** Both statements of the theorem follow now from the following lemma by induction on the *posets* $(\mathcal{Z}(\mathfrak{w},\mathfrak{v}),>)$ resp. $(\mathcal{L}(\mathfrak{w},\mathfrak{v}),>)$.

Lemma. *Let* $f = (\mathbf{f}_{(j,i)})_{(j,i)\in H_0^{\mathfrak{w},\mathfrak{v}}} \in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$ *and* $L \in \mathcal{L}(\mathfrak{w},\mathfrak{v})$. *Suppose, that* $L' >' L$ *implies* $\mathbf{f}_{(j',i')} = 0$ *for* $(j',i') \in L'_0$. *Then we have:*

(a) *If* L *is a z-line, then we have already* $\mathbf{f}_{(j,i)} = 0$ *for all* $(j,i) \in L_0$.
(b) *Otherwise,* L *is a real h-line (by definition),* $\mathbf{f}_{(j,i)_L} \in M_L$, *and* $\big(\mathbf{f} - \boldsymbol{\varphi}_L(\mathbf{f}_{(j,i)_L})\big)_{(j,i)} = 0$ *for all* $(j,i) \in L'_0$ *with* $L' \geq L$.

Proof. (1) Under the hypothesis of the lemma we have: $\mathbf{f}_{(j,i)} = 0$ for some $(j,i) \in L_0$ implies $\mathbf{f}_{q'} = 0$ for all $(j',i') \in L_0$. In fact, we may suppose, that there is an arrow $(j,i) \xrightarrow{h} (j',i')$ in $H_1^{\mathfrak{w},\mathfrak{v}}$, then we have $\mathbf{f}_{(j',i')} = U^{(1)}_{\mathfrak{w},\mathfrak{v}}(h)\mathbf{f}_{(j,i)}(U^{(2)}_{\mathfrak{w},\mathfrak{v}}(h))^{-1}$ either directly by equation (2), or by equation (3a), using the hypothesis .

(2) If L is a z-line, there is a 0-point $(j,i) \in L_0$, or there is an arrow $(j',i') \xrightarrow{h'} (j,i)$ with $h' \in H^{\mathfrak{w},\mathfrak{v}}_{2,o}$ and $(j,i) \in L_0$ (remark (3) in 3.5). In the first case we are immediately done by (1); in the second case (j',i') belongs to a line $L' > L$, and since $\mathbf{f}_{(j,i)} = U^{(1)}_{\mathfrak{w},\mathfrak{v}}(h')\mathbf{f}_{(j',i')}(U^{(2)}_{\mathfrak{w},\mathfrak{v}}(h'))^{-1} = 0$ by hypothesis, we are also done.

(3) If L is a real h-line, we have only to show that $\mathbf{f}_{(j,i)_L} \in M_L$, then we can apply (1) to $(\mathbf{f} - \boldsymbol{\varphi}_L(\mathbf{f}_{(j,i)_L}))$. This is clear if L is of type $\mathbb{A}$ or $\mathbb{D}$ (!). In the remaining two cases the hypothesis imply $\mathbf{f}_{(j,i)} = \mathbf{f}_{(j,i)_L}$ for all $(j,i) \in L_0$, the remaining equalities force $\mathbf{f}_{(j,i)_L} \in M_L$ □

5. Auslander-Reiten theory

Let $\mathcal{C} = (\mathrm{k}, Q, (\sigma,\tau), >)$ be a clan.

5.1. **Irreducible maps.** We start with two observations.

(1) Let $\mathfrak{w} \in \mathfrak{X}$ be a band, and $\mathfrak{v} \in \mathfrak{X}$ arbitrary. Consider the following composition of morphisms in $\mathsf{R}(\mathcal{C})$

$$S_{\mathfrak{w}}(W) \xleftarrow{\mathbf{f}} S_{\mathfrak{v}}(V) \xleftarrow{\mathbf{g}} S_{\mathfrak{w}}(W')$$

with $W, W' \in A_{\mathfrak{w}}$-mod and $V \in A_{\mathfrak{v}}$-mod. If $\mathfrak{v} \neq \mathfrak{w}$ or $\mathbf{f}_{(i,i)} = 0$, we have $(\mathbf{fg})_{(i,i)} = 0$ for all $i \in H^{\mathfrak{w}}_0$. In fact,

$$(\mathbf{fg})_{(i,i)} = \sum_{j \in H^{\mathfrak{v}}_0, G_{\mathfrak{v}}(j) = G_{\mathfrak{w}}(i)} \mathbf{f}_{(i,j)}\mathbf{g}_{(j,i)},$$

and $\mathbf{g}_{(j,i)} \neq 0$ implies $\mathbf{f}_{(i,j)} = 0$ by Corollary 2 and Theorem 2 (a).

(2) Let $\mathfrak{w} < \mathfrak{v}$ be inextensible, $\mathfrak{x} \in \mathfrak{X} \setminus \{\mathfrak{v}, \mathfrak{w}\}$ (we may suppose, $\mathfrak{v}, \mathfrak{w} \in \mathfrak{X}$) and consider the following composition of morphisms in $\mathsf{R}(\mathcal{C})$

$$S_{\mathfrak{w}}(W) \xleftarrow{\mathbf{f}} S_{\mathfrak{x}}(X) \xleftarrow{\mathbf{g}} S_{\mathfrak{v}}(V)$$

with $W \in A_{\mathfrak{w}}$-mod, $V \in A_{\mathfrak{v}}$-mod, $X \in A_{\mathfrak{x}}$-mod. If $(\mathbf{fg})_{(0,0)} \neq 0$, then $\mathfrak{x}$ is a string, and we may suppose $\mathfrak{v} > \mathfrak{x} > \mathfrak{w}$. In fact,

$$(\mathbf{fg})_{(0,0)} = \sum_{j \in H^{\mathfrak{x}}_0, G_{\mathfrak{x}}(j) = G_{\mathfrak{v}}(0)} \mathbf{f}_{(0,j)}\mathbf{g}_{(j,0)},$$

and, since $\mathfrak{v}$ is inextensible, $G_{\mathfrak{x}}(j) = G_{\mathfrak{v}}(0)$ for some $j \in H^{\mathfrak{x}}_0$ implies that $\mathfrak{x}$ is also inextensible, and we may suppose that it is comparable with $\mathfrak{v}$ and $\mathfrak{w}$. Finally, $\mathfrak{v} > \mathfrak{x}$ implies $\mathbf{g}_{(0,0)} = 0$ and $\mathfrak{x} > \mathfrak{w}$ implies $\mathbf{f}_{(0,0)}$ by a similar argument as in (1).

Thus, we have shown:

Proposition 4. *(1) For $\mathfrak{w}$ a band, the functor*

$$S_{\mathfrak{w}} : A_{\mathfrak{w}}\text{-mod} \to \mathsf{R}(\mathcal{C})$$

preserves irreducible maps.

(2) If $\mathfrak{w} < \mathfrak{v}$ are inextensible words, there is a real h-line L of type $\mathbb{A}$ that contains $(0,0) \in H_0^{\mathfrak{w},\mathfrak{v}}$. (Use the Lemma from 7.2!) Construct

$$\boldsymbol{\varphi}_L \colon \mathsf{Hom}_{\mathrm{k}}(V,W) \to \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{v}}(V), S_{\mathfrak{w}}(W))$$

as in 4.1. In case there is no inextensible word $\mathfrak{x}$ with $\mathfrak{v} > \mathfrak{x} > \mathfrak{w}$, the morphisms $\boldsymbol{\varphi}_L(f) \neq 0$ are irreducible in $\mathsf{R}(\mathcal{C})$.

Remark. Suppose there is no *symmetric* string $\mathfrak{x}$ with $\mathfrak{w} < \mathfrak{x} < \mathfrak{v}$, denote by L the (real) h-line containing $(0,0)$ and let $\tilde{L}$ be the connected component of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}} \cup H_{2,o}^{\mathfrak{w},\mathfrak{v}})$ that that contains L, (compare remark 3.5 (3) and Corollary 1). Than it is easy to see, that $\tilde{L}$ contains *two* e-points (one of them is just $(0,0)$). Thus, the construction of $\boldsymbol{\varphi}_L$ involves no choices in that case.

Notation. In case $\mathfrak{w} < \mathfrak{v}$ we denote by $L(\mathfrak{w} <_r \mathfrak{v})$ the real h-line in $H^{\mathfrak{w},\mathfrak{v}}$ that contains the vertex $(0,0)$. Similarly, in case $\mathfrak{w}^{-1} < \mathfrak{v}^{-1}$ there is a real h-line in $H^{\mathfrak{w},\mathfrak{v}}$ naturally associated with that relation; we denote this line by $L(\mathfrak{w} <_l \mathfrak{v})$.

5.2. **Exact structures.** We shall define on $\mathsf{R}(\mathcal{C})$ an exact structure ([GRo, 9.1]). Consider the class $\mathcal{E}$ of pairs $(\mathbf{i},\mathbf{d})$ of composable morphisms

$$M \xrightarrow{\mathbf{i}} E \xrightarrow{\mathbf{d}} N$$

in $\mathsf{R}(\mathcal{C})$ such, that the induced k-linear maps

$$(*) \qquad 0 \to l^+(M)/l^-(M) \xrightarrow{\mathbf{i}_l} l^+(E)/l^-(E) \xrightarrow{\mathbf{d}_l} l^+(N)/l^-(N) \to 0$$

(see 1.2) form short exact sequences for all letters $l \in Q_*$.

Remark. A morphism $\mathbf{d} : E \to N$ is a deflation with respect to $\mathcal{E}$ if (and only if) the induced k-linear maps $\mathbf{d}_l : l^+(E)/l^-(E) \to l^+(N)/l^-(N)$ are surjective for each letter $l \in Q_*$.

In fact, define $M_p := \ker \mathbf{d}_p \subset E_p$ for all $p \in Q_0$, and $\mathbf{i}_p$ the inclusion.

$$l^{+/-}(M) := l^{+/-}(E) \cap \ker \mathbf{d}_{t(l)}$$

for all $l \in Q_*$. Now, it is easy to construct the induced structure maps for M to make it a representation of $\mathcal{C}$, and our hypothesis on $\mathbf{d}$ implies $(*)$ for all letters $l \in Q_*$.

Dually, a morphism $\mathbf{i} \colon M \to E$ is an inflation if (and only if) the induced maps $\mathbf{i}_l$ are injective for each letter $l \in Q_*$.

With this observation it is straight forward to verify for $\mathcal{E}$ the axioms of an exact structure [GRo, 9.1] directly.

The following lemma seems to be well-known, but we could not find an explicit reference.

Lemma. *Let $(\mathcal{A},\mathcal{E})$ be an exact category, $(\mathbf{i},\mathbf{d}) \in \mathcal{E}$ with indecomposable end terms and $\mathbf{i},\mathbf{d}$ irreducible, then $(\mathbf{i},\mathbf{d})$ is an Auslander-Reiten sequence.*

Proof. Let $\mathbf{f}\colon X \to M$ be not a retraction and consider the pull-back diagram

$$\begin{array}{ccccc} M & \xrightarrow{\mathbf{i}'} & E' & \xrightarrow{\mathbf{d}'} & X \\ \| & & \downarrow{\scriptstyle \mathbf{f}'} & & \downarrow{\scriptstyle \mathbf{f}} \\ M & \xrightarrow{\mathbf{i}} & E & \xrightarrow{\mathbf{d}} & N \end{array}$$

If the first row splits, $\mathbf{f}$ factors trivially over $\mathbf{d}$, otherwise, since $\mathbf{i}$ is irreducible, $\mathbf{f}'$ admits a section $\mathbf{g}'$, thus $\mathbf{d} = \mathbf{f}(\mathbf{d}'\mathbf{g}')$. Since $\mathbf{d}$ is irreducible and $\mathbf{f}$ admits no section, $\mathbf{d}'\mathbf{g}'$ admits a retraction, and $\mathbf{f}$ factors again over $\mathbf{d}$. □

5.3. **Finite-dimensional clans.** We start with some combinatorial considerations.

Definition. We say, the clan $\mathcal{C} = (\mathrm{k}, Q, (\sigma, \tau), >)$ is *finite dimensional*, if for each $(q, \rho) \in Q_0 \times \{-1, 1\}$ the linearly ordered set $\mathfrak{W}(q, \rho)$ (see 1.3) has a (unique) minimal element $\mathfrak{i}_{(q,\rho)}$ and a (maximal) element $\mathfrak{p}_{(q,\rho)}$.

Remarks. Let $\mathcal{C}$ as above be finite-dimensional.

(1) In this situation each $\mathfrak{w} \in \mathfrak{W}(q, \rho) \setminus \{\mathfrak{i}_{(q,\rho)}\}$ admits a unique successor $\mathfrak{w}[1]$, in fact, we have a unique composition

$$\mathfrak{w} = \mathfrak{w}' l \mathfrak{i}_{(s(l),\sigma(l))}$$

for some letter $l \in Q_*(s(l), -\sigma(l))$ which is not minimal in that set; with l' the successor of l in $Q_*(s(l), -\sigma(l))$ we have

$$\mathfrak{w}[1] = \mathfrak{w}' l' \mathfrak{p}_{(s(l'),-\sigma(l'))}$$

Dually, each $\mathfrak{w} \in \mathfrak{W}(q, \rho) \setminus \{\mathfrak{p}_{(q,\rho)}\}$ has a unique predecessor $\mathfrak{w}[-1]$.

In case $\mathfrak{w}$ is left inextensible we set $[i]\mathfrak{w} := (\mathfrak{w}^{-1}[i])^{-1}$ whenever this makes sense.

(2) Consider the following sets of inextensible words:

$$\mathfrak{I} := \{\mathfrak{i}_l := \mathfrak{i}^{-1}_{(t(l),-\tau(l))}\, l\, \mathfrak{i}_{(s(l),-\sigma(l)} \mid l \in Q_*\}$$
$$\mathfrak{P} := \{\mathfrak{p}_l := \mathfrak{p}^{-1}_{(t(l),-\tau(l))}\, l\, \mathfrak{p}_{(s(l),-\sigma(l)} \mid l \in Q_*\}$$

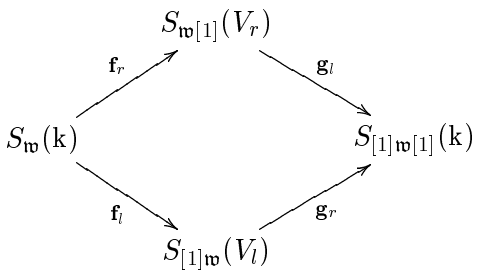
If $\mathfrak{w}$ is an inextensible word, *not* in $\mathfrak{I}$, then $[1](\mathfrak{w}[1])$ and $([1]\mathfrak{w})[1]$ are defined and equal, thus we write just $[1]\mathfrak{w}[1] \notin \mathfrak{P}$ in this situation. Dually, if $\mathfrak{w}$ is *not* in $\mathfrak{P}$, we can write just $[-1]\mathfrak{w}[-1] \notin \mathfrak{I}$.

Observe, $[-1]([1]\mathfrak{w}[1])[-1] = \mathfrak{w}$, thus $[1]\mathfrak{w}[1]$ is symmetric if and only if $\mathfrak{w}$ is symmetric.

(3) Let $\mathfrak{i}_l \in \mathfrak{I}$ and $V \in A_{\mathfrak{i}_l}$-mod, then $S_{\mathfrak{i}_l}(V)$ is injective in $(\mathsf{R}(\mathcal{C}), \mathcal{E})$. In fact, take $\mathfrak{w} \in \mathfrak{X} \setminus \{\mathfrak{i}_l\}$ (we may suppose $\mathfrak{i}_l \in \mathfrak{X}$) and consider $f \in \mathsf{Hom}_{\mathcal{C}}(S_{\mathfrak{i}_l}(V), S_{\mathfrak{w}}(W))$. Then follows easily from our previous results that $f_l(V_{t(x)}) = 0$ if x is the letter in $H^{\mathfrak{i}_l}_*$ corresponding to l. Thus every inflation starting in $S_{\mathfrak{i}_l}(V)$ splits. By a dual argument $S_{\mathfrak{p}}(W)$ is projective for each $\mathfrak{p} \in \mathfrak{P}$.

5.4. **Auslander-Reiten sequences.** Let $\mathcal{C}$ be a finite-dimensional clan.

5.4.1. If $\mathfrak{w}$ is an *asymmetric string*, not in $\mathfrak{I}$, then $[1]\mathfrak{w}[1]$ is also asymmetric, and we consider the following diagram:



In case $\mathfrak{w}[1]$ is asymmetric, we take $V_r := \mathrm{k}$, $\mathbf{f}_r := \boldsymbol{\varphi}_{L(\mathfrak{w}[1]<_r\mathfrak{w})}(1)$ and $\mathbf{g}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(-1)$.
Otherwise $\mathfrak{w}[1]$ is symmetric (thus $A_{\mathfrak{w}[1]} = \mathrm{k}[z]/(z^2 - z)$), we take $V_r = \mathrm{k}^2$ with $V_r(z) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $\mathbf{f}_r := \boldsymbol{\varphi}_{L(\mathfrak{w}[1]<_r\mathfrak{w})}\binom{1}{0}$ and $\mathbf{g}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(0,-1)$.

In case $[1]\mathfrak{w}$ is asymmetric, we take $V_l := \mathrm{k}$, $\mathbf{f}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}<_l\mathfrak{w})}(1)$ and $\mathbf{g}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_r[1]\mathfrak{w})}(1)$.
Otherwise $[1]\mathfrak{w}$ is symmetric (thus $A_{[1]\mathfrak{w}} = \mathrm{k}[z]/(z^2 - z)$), we take $V_l = \mathrm{k}^2$ with $V_r(z) = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\mathbf{f}_l := \boldsymbol{\varphi}_{L(\mathfrak{w}[1]<_r\mathfrak{w})}\binom{0}{1}$ and $\mathbf{g}_r := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(1,0)$.

By Proposition 4 the morphisms $\binom{\mathbf{f}_l}{\mathbf{f}_r}$ and $(\mathbf{g}_r, \mathbf{g}_l)$ are irreducible. It is a direct calculation to verify that $\left(\binom{\mathbf{f}_l}{\mathbf{f}_r}, (\mathbf{g}_r, \mathbf{g}_l)\right) \in \mathcal{E}$, the point is to use the remark in 5.1. Thus, by the lemma from 5.2 we have constructed an Auslander-Reiten sequence starting at $S_{\mathfrak{w}}(\mathrm{k})$ for every asymmetric string not in $\mathfrak{I}$. Similarly, we can construct an Auslander-Reiten sequence ending in $S_{\mathfrak{w}}(\mathrm{k})$ for every asymmetric string not in $\mathfrak{P}$.

5.4.2. If $\mathfrak{w}$ is a *symmetric string* not in $\mathfrak{I}$, $[1]\mathfrak{w}[1]$ is also symmetric and $A_{\mathfrak{w}} = A_{[1]\mathfrak{w}[1]} = \mathrm{k}[z]/(z^2 - z)$. Consider the following diagram:

$$S_{\mathfrak{w}}(\mathrm{k}_a) \xrightarrow{\mathbf{f}_r} S_{\mathfrak{w}[1]}(V_r) \xrightarrow{\mathbf{g}_l} S_{[1]\mathfrak{w}[1]}(\mathrm{k}_{(1-a)})$$

Here, we denote by k_a the one-dimensional $\mathrm{k}[z]/(z^2-z)$-module with $\mathrm{k}_a(z) = a$ for $a \in \{0, 1\}$.

In case $\mathfrak{w}[1]$ is asymmetric, we take $V_r := \mathrm{k}$, $\mathbf{f}_r := \boldsymbol{\varphi}_{L(\mathfrak{w}[1]<_r\mathfrak{w})}(1)$ and $\mathbf{g}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(1)$.
Otherwise $\mathfrak{w}[1]$ is symmetric (thus $A_{\mathfrak{w}[1]} = \mathrm{k}[z]/(z^2 - z)$), we take $V_r = \mathrm{k}^2$ with $V_r(z) = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, $\mathbf{f}_r := \boldsymbol{\varphi}_{L(\mathfrak{w}[1]<_r\mathfrak{w})}\binom{1}{0}$ and $\mathbf{g}_l := \boldsymbol{\varphi}_{L([1]\mathfrak{w}[1]<_l\mathfrak{w}[1])}(0,1)$.

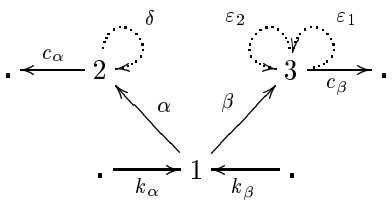
Now, we find similarly that $(\mathbf{f}_l, \mathbf{g}_r) \in \mathcal{E}$ with $\mathbf{f}_l, \mathbf{g}_r$ irreducible. Thus, we have constructed an Auslander-Reiten sequence starting at $S_{\mathfrak{w}}(\mathrm{k}_a)$ (with $a \in \{0, 1\}$) for every symmetric string $\mathfrak{w}$ not in $\mathfrak{I}$. Similarly, we can construct an Auslander-Reiten sequence ending in $S_{\mathfrak{w}}(\mathrm{k}_a)$ for every symmetric string $\mathfrak{w}$ not in $\mathfrak{P}$.

5.4.3. If $\mathfrak{w}$ is a *band*, the functor $S_{\mathfrak{w}} : A_{\mathfrak{w}}\text{-}\mathsf{mod} \to \mathsf{R}(\mathcal{C})$ is easily seen to be exact, and since it preserves irreducible maps (Proposition 4), it also preserves Auslander-Reiten sequences by the above lemma. Thus, we obtain in the asymmetric case a $\mathsf{Spec}(\mathrm{k}[T,T^{-1}])$-family of homogeneous tubes, and in the symmetric case a $\mathsf{Spec}(\mathrm{k}[T])$-family of tubular type $(2,2)$.

All together, we established for a finite-dimensional clan the global existence of Auslander-Reiten sequences. The sequences for bands exist for arbitrary clans.

We should point out that the exact structure $\mathcal{E}$ tends to be too restrictive if the clan $\mathcal{C}$ is obtained by "translating" for example a subspace-problem. Typically, one gets "too many" projectives and injectives and "looses" thus a finite number of Auslander-Reiten sequences from the original problem.

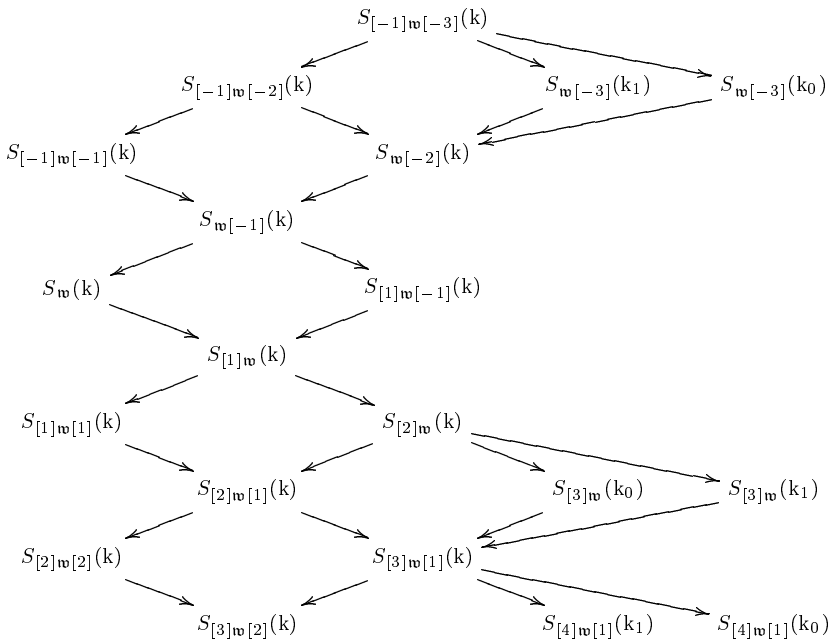
5.5. **Example.** Consider the following clan $\mathcal{C}$, given by



$$
\begin{aligned}
Q_*(2,-1) &= \{\underline{c}_\alpha^{-1} > \underline{\alpha}\} & \qquad Q_*(3,-1) &= \{\underline{c}_\beta^{-1} > \underline{\beta}\}\\
Q_*(2,\ \ 1) &= \{\underline{\delta}^*\} & \qquad Q_*(3,\ \ 1) &= \{\underline{\varepsilon}_2^* > \underline{\varepsilon}_1^*\}\\
Q_*(1,-1) &= \{\underline{\alpha}^{-1} > \underline{k}_\alpha\}\\
Q_*(1,\ \ 1) &= \{\underline{\beta}^{-1} > \underline{k}_\beta\}
\end{aligned}
$$

This is the "clannish version" of the Zavadskij-poset.

Compare [CB2, 2.8] and [ZN]. Obviously $\mathcal{C}$ is finite-dimensional. We exhibit a piece of the following particular component of the Auslander-Reiten quiver of $\mathsf{R}(\mathcal{C})$:



Where $\mathfrak{w} = \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_1^*\ \underline{\beta}\ \underline{\alpha}^{-1}\underline{\delta}^*\ \underline{\alpha}\ \underline{\beta}^{-1}\underline{\varepsilon}_2^*\ \underline{c}_\beta^{-1}$. For the convenience of the reader we give some more strings explicitly:

$$\mathfrak{w}[-3] = \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_2^*\ \underline{\beta}\ \underline{k}_\alpha \qquad \underline{c}_\beta\underline{\varepsilon}_1^*\ \underline{c}_\beta^{-1} = [3]\mathfrak{w}$$
$$\mathfrak{w}[-2] = \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_1^*\ \underline{c}_\beta^{-1} \qquad \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_2^*\ \underline{c}_\beta^{-1} = [2]\mathfrak{w}$$
$$\mathfrak{w}[-1] = \underline{k}_\alpha^{-1}\underline{\beta}^{-1}\underline{\varepsilon}_1^*\ \underline{\beta}\ \underline{\alpha}^{-1}\underline{\delta}^*\ \underline{c}_\alpha^{-1} \qquad \underline{k}_\beta^{-1}\underline{\alpha}^{-1}\underline{\delta}^*\ \underline{\alpha}\ \underline{\beta}^{-1}\underline{\varepsilon}_2^*\ \underline{c}_\beta^{-1} = [1]\mathfrak{w}$$

and

$$[1]\mathfrak{w}[-1] = \underline{k}_\beta^{-1}\underline{\alpha}^{-1}\underline{\delta}^*\ \underline{c}_\alpha^{-1}$$

Moreover we have:

$$\mathfrak{i}_{\underline{c}_\beta^{-1}} = \mathfrak{w}[-2], \quad \mathfrak{i}_{\underline{c}_\alpha^{-1}} = [1]\mathfrak{w}[-1] = \mathfrak{p}_{\underline{k}_\beta^{-1}}, \quad [2]\mathfrak{w} = \mathfrak{p}_{\underline{k}_\alpha^{-1}}$$

The postprojective and the preinjective component are well-known; the remaining components containing strings are of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$ or $\mathbb{Z}\mathbb{D}_\infty$, and they are parameterized by the asymmetric resp. symmetric strings of the form $\underline{c}_\beta\underline{\varepsilon}_1^*\ ?\ \underline{\varepsilon}_1^*\ \underline{c}_\beta^{-1}$ (use an argument like in [Ge]).

5.6. **Clannish algebras.** Let Q' be an ordinary (connected) quiver and Sp a subset of the loops of Q'. Call the elements of Sp "special loops" and the remaining arrows "ordinary arrows". Let $R = R^{\mathrm{Sp}} \cup Z$ be a set of relations for Q' consisting of $R^{\mathrm{Sp}} := \{\varepsilon^2 - \varepsilon \mid \varepsilon \in \mathrm{Sp}\}$ and Z a set of zero-relations.

Definition. [CB2] $\Lambda := \mathrm{k}Q'/(R)$ is a *clannish* algebra provided that the following holds:

(C0) None of the relations in Z do start or end with a special loop, or involve the square of a special loop.

(C1) At most two arrows terminate at any vertex, and at most two arrows start at any vertex.

(C2) For any *ordinary* arrow β in Q' there is at most one (*arbitrary*) arrow α with $\beta\alpha$ a path not in Z, and there is at most one (*arbitrary*) arrow γ in Q with $\gamma\beta$ a path not in Z.

In [CB2] is indicated, how Λ-mod is embedded in $\mathsf{R}(\mathcal{C})$ for an adequate clan $\mathcal{C}$. Since we want to calculate Auslander-Reiten sequences in Λ-mod for Λ a finite-dimensional clannish algebra, we have to make that explicitly:

Write $Z = Z' \cup Z''$, where Z'' is minimal with respect to condition (C2) above. Thus we can define two functions $\sigma, \tau \colon Q'_1 \to \{-1, 1\}$ subject to the following conditions: $-\ \sigma(\varepsilon) = \tau(\varepsilon)$ for $\varepsilon \in \mathrm{Sp}$
$-\ s(\alpha) = s(\beta)$ implies $\sigma(\beta) = -\sigma(\alpha)$
$-\ t(\alpha) = t(\beta)$ implies $\tau(\beta) = -\tau(\alpha)$
$-\ \beta\alpha$ a path not in Z'' implies $\sigma(\beta) = -\tau(\alpha)$
Now, we consider the biquiver $Q'' := (Q'_0, (Q'_1 \setminus \mathrm{Sp}), \mathrm{Sp})$ with the above polarization (σ, τ). Observe, that the sets $Q''_*(q, \rho)$ are of one of the following forms:

$$\emptyset,\ \{\underline{\alpha}\},\ \{\underline{\beta}^{-1}\},\ \{\underline{\beta}^{-1}, \underline{\alpha}\},\ \{\underline{\varepsilon}^*\}$$

for certain ordinary arrows α, β special loops ε. We shall extend Q'' to a biquiver Q by "adding kernels and co-kernels" to the ordinary arrows and determine by the way a linear order $>$ on the sets of $Q_*(q, \rho)$ with more than one element. This data defines our clan $\mathcal{C}$.

- In case $Q''(q, \rho) = \{\underline{\alpha}\}$ add an arrow $q \xrightarrow{c_\alpha} t(c_\alpha)$ with and set $Q_*(q, \rho) := \{\underline{c}_\alpha^{-1} > \underline{a}\}$.

- In case $Q''(q, \rho) = \{\underline{\beta}^{-1}\}$ add an arrow $s(k_\beta) \xrightarrow{k_\beta} q$ and set $Q_*(q, \rho) := \underline{\beta}^{-1} > \underline{k}_\beta$.

- In case $Q''(q, \rho) = \{\underline{b}^{-1}, \underline{a}\}$ add an arrow $s(\underline{z}_{\beta\alpha}) \xrightarrow{z_{\beta\alpha}} q$ and set $Q(q, \rho) := \underline{\beta}^{-1} > \underline{z}_{\beta\alpha} > \underline{\alpha}$. Moreover let in this situation formally $\underline{z}_{\beta\alpha} = \underline{c}_\beta^{-1} = \underline{k}_\alpha$.

We get an equivalence of categories $F \colon \mathsf{R}(\mathcal{C}) \xrightarrow{\sim} \mathsf{rep}_{\mathrm{k}}(Q', R^{\mathrm{Sp}} \cup Z'')$ defined by

$$(FM)_q := M_q \text{ for all } q \in Q'_0$$
$$(FM)(\varepsilon) := M_\varepsilon \text{ for all special loops } e$$
$$(FM)(\alpha) := \iota_\alpha M_\alpha \pi_\alpha \text{ for all ordinary arrows } \alpha$$

Note: For a special loop ε we have $(\underline{\varepsilon}^*)^+(M) = M_{s(\varepsilon)}$ and $(\underline{\varepsilon}^*)^- = 0$. While for an ordinary arrow α we have $(\underline{\alpha}^{-1})^+(M) = M_{s(\alpha)}$ and $(\underline{\alpha})^-(M) = 0$, thus $\pi_\alpha \colon M_{s(\alpha)} \to (\underline{\alpha}^{-1})^+(M)/(\underline{\alpha}^{-1})^-(M)$ and $\iota_\alpha \colon (\underline{\alpha})^+(M)/(\underline{\alpha})^-(M) \to M_{t(\alpha)}$ are well-defined.

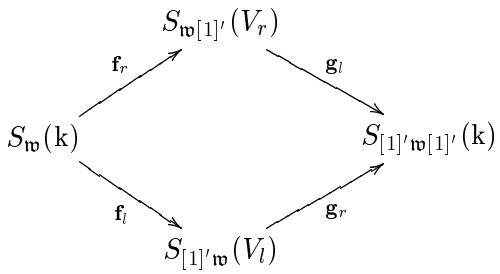
Let us point out, that F is exact in the sense, that it sends exact sequences from the standard exact structure $\mathcal{E}$ on $\mathsf{R}(\mathcal{C})$ to exact sequences in $\mathsf{rep}_{\mathrm{k}}(Q', R^{\mathrm{Sp}} \cup Z'')$, but usually in the latter category there are "more" exact sequences.

If $\mathfrak{w}$ is a string, and $0 \neq W \in A_{\mathfrak{w}}\text{-}\mathsf{mod}$, then $F(S_{\mathfrak{w}}(W))$ fulfills the relations from Z' if and only if $\mathfrak{w}$ is not of the form $\mathfrak{w}_l \mathfrak{z}^{\omega} \mathfrak{w}_r$ for some $\mathfrak{z} \in Z'$, some $\omega \in \{-1, 1\}$ and $\mathfrak{w}_l, \mathfrak{w}_r$ words.
Similarly, if $\mathfrak{w}$ is a band, the same is true if $\mathfrak{w}^n$ is not of the form $\mathfrak{w}_l \mathfrak{z}^{\omega} \mathfrak{w}_r$. Thus, we call a string (resp. band) $\mathfrak{w}$ *admissible* if $\mathfrak{w}$ (resp. $\mathfrak{w}^n$) is not of the above form.

In case the clannish algebra $\Lambda = \mathrm{k}Q/(R)$ is finite dimensional, the sets

$$\mathfrak{W}'(q, \rho) := \{\mathfrak{w} \in \mathfrak{W}(q, \rho) \text{ admissable } \}$$

contain maximal and minimal elements $\mathfrak{p}'_{(q,\rho)}$ (which contain no inverse letter from Q''_*) and $\mathfrak{i}'_{(q,\rho)}$ (which contain no direct letter from Q''_*). Consequently an element $\mathfrak{w} \in \mathfrak{W}'(q, \rho)$ has a predecessor $\mathfrak{w}[-1]'$ and a successor $\mathfrak{w}[1]'$ in $\mathfrak{W}'(q, \rho)$! After defining sets $\mathfrak{I}'$ (resp. $\mathfrak{P}'$ of injective (resp. projective) strings relative to Z' we can define exact sequences



for asymmetric admissible strings not in $\mathfrak{I}'$ and

$$S_{\mathfrak{w}}(\mathrm{k}_a) \xrightarrow{\mathbf{f}_r} S_{\mathfrak{w}[1]'}(V_r) \xrightarrow{\mathbf{g}_l} S_{[1]'\mathfrak{w}[1]'}(\mathrm{k}_{(1-a)})$$

for symmetric admissible strings not in $\mathfrak{I}'$, which become Auslander-Reiten sequences in $\mathsf{rep}_{\mathrm{k}}(Q', R)$. (The exactness follows basically as in 5.4, using the remark at the end of 5.1). Additionally we obtain for every *ordinary* arrow α in Q' an Auslander-Reiten sequence corresponding to the following sequence of asymmetric strings:

$$\mathfrak{i}'^{-1}_{(t(\alpha),-\tau(\alpha))}\, \underline{c}_{\alpha}^{-1} \rightarrow \mathfrak{i}'^{-1}_{(t(\alpha),-\tau(\alpha))}\, \underline{\alpha}\, \mathfrak{p}'_{(s(\alpha),-\sigma(\alpha))} \rightarrow \underline{k}_{\alpha}^{-1}\, \mathfrak{p}'_{(s(\alpha),-\sigma(\alpha))},$$

which come *not* from exact sequences in $\mathsf{R}(\mathcal{C})$, in fact, the end-terms of these sequences come precisely from the strings in $\mathfrak{I}'$ resp. $\mathfrak{P}'$ which give *not* rise to injective (resp. projective) modules. The admissible bands yield families of tubes as in the case of clans.

All together, we obtained just a description of the Auslander-Reiten sequences, as it could be expected (in case $\mathrm{char}\,\mathrm{k} \neq 2$) by transferring the

well-known results for string-algebras via [ReRd] to clannish algebras – despite the fact, that the description of the indecomposables obtained in that way is not directly compatible with our description.

6. The system of equations

Condition (H1) is for f trivially equivalent to equation (1).

Condition (H2) is equivalent to equation (2). Let us explain this: For each arrow $\alpha \in Q_1$ we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{j \in \mathbf{s}(\mathfrak{w},\underline{\alpha})} W_j & \xleftarrow{\mathbf{f}_{\underline{\alpha}^{-1}}} & \bigoplus_{i \in \mathbf{s}(\mathfrak{v},\underline{\alpha})} V_i \\ \downarrow{\scriptstyle S_{\mathfrak{w}}(W)_\alpha} & & \downarrow{\scriptstyle S_{\mathfrak{v}}(V)_\alpha} \\ \bigoplus_{j' \in \mathbf{t}(\mathfrak{w},\underline{\alpha})} W_{j'} & \xleftarrow[\mathbf{f}_{\underline{\alpha}}]{} & \bigoplus_{i' \in \mathbf{t}(\mathfrak{v},\underline{\alpha})} V_{i'} \end{array}$$

which decomposes into

$$\begin{array}{ccc} W_{s(\nu)} & \xleftarrow{\mathbf{f}_{(s(\nu),s(\mu))}} & V_{s(\nu)} \\ \downarrow{\scriptstyle U_{\mathfrak{w}}(\nu)} & & \downarrow{\scriptstyle U_{\mathfrak{v}}(\mu)} \\ W_{t(\nu)} & \xleftarrow[\mathbf{f}_{(t(\nu),t(\mu))}]{} & V_{t(\nu)} \end{array}$$

for all $(\nu,\mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\nu) = \alpha = G_{\mathfrak{v}}(\mu)$ Similarly, (H3) is equivalent to the equations (3a)–(3d): For each special loop $\varepsilon \in Q_2$ we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{j \in \mathbf{s}(\mathfrak{w},\underline{\varepsilon}^*)} W_j & \xleftarrow{\mathbf{f}_{\underline{\varepsilon}^*}} & \bigoplus_{i \in \mathbf{s}(\mathfrak{v},\underline{\varepsilon}^*)} V_i \\ \downarrow{\scriptstyle S_{\mathfrak{w}}(W)_\varepsilon} & & \downarrow{\scriptstyle S_{\mathfrak{v}}(V)_\varepsilon} \\ \bigoplus_{j' \in \mathbf{t}(\mathfrak{w},\underline{\varepsilon}^*)} W_{j'} & \xleftarrow[\mathbf{f}_{\underline{\varepsilon}^*}]{} & \bigoplus_{i' \in \mathbf{t}(\mathfrak{v},\underline{\varepsilon}^*)} V_{i'} \end{array}$$

which decomposes into the following four types of diagrams:

$$\begin{array}{ccc} W_{s(\nu)} \oplus W_{t(\nu)} & \xleftarrow{\begin{pmatrix} \mathbf{f}_{(s(\nu),s(\mu))} & \mathbf{f}_{(s(\nu),t(\mu))} \\ \mathbf{f}_{(t(\nu),s(\mu))} & \mathbf{f}_{(t(\nu),t(\mu))} \end{pmatrix}} & V_{s(\mu)} \oplus V_{t(\mu)} \\ \downarrow{\scriptstyle (U_{\mathfrak{w}}(\nu),\mathbf{1})} & & \downarrow{\scriptstyle (U_{\mathfrak{v}}(\mu),1)} \\ W_{t(\nu)} & \xleftarrow{\mathbf{f}_{(t(\nu),t(\mu))}} & V_{t(\mu)} \end{array}$$

for each pair of arrows $(\nu,\mu) \in H_1^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\nu) = \varepsilon = G_{\mathfrak{v}}(\mu)$. From this we can read off the equations (3a) and (3a') (use that $U_{\mathfrak{w}}(\nu)$ is invertible

in this situation).

$$\begin{array}{ccc} W_j & \xleftarrow{(\mathbf{f}_{(j,s(\mu))},\mathbf{f}_{(j,t(\mu))})} & V_{s(\mu)} \oplus V_{t(\mu)} \\ {\scriptstyle U_{\mathfrak{w}}(\eta)}\downarrow & & \downarrow{\scriptstyle (U_{\mathfrak{v}}(\mu),1)} \\ W_j & \xleftarrow{\mathbf{f}_{(j,t(\mu))}} & V_{t(\mu)} \end{array}$$

for each pair of arrows $(\eta,\mu) \in H_2^{\mathfrak{w}} \times H_1^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\eta) = \varepsilon = G_{\mathfrak{v}}(\mu)$ and $s(\eta) = j = t(\eta)$ From this we can read off the equations (3b) (use, that $U_{\mathfrak{w}}(\eta)$ is idempotent and $U_{\mathfrak{v}}\zeta$ is invertible in this situation).

$$\begin{array}{ccc} W_{s(\nu)} \oplus W_{t(\nu)} & \xleftarrow{\binom{\mathbf{f}_{(s(\nu),i)}}{\mathbf{f}_{(t(\nu),i)}}} & V_i \\ {\scriptstyle (U_{\mathfrak{w}}(\nu),1)}\downarrow & & \downarrow{\scriptstyle U_{\mathfrak{v}}(\zeta)} \\ W_{t(\nu)} & \xleftarrow{\mathbf{f}_{(t(\nu),i)}} & V_i \end{array}$$

for each pair of arrows $(\nu,\zeta) \in H_1^{\mathfrak{w}} \times H_2^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\nu) = \varepsilon = G_{\mathfrak{v}}(\zeta)$, $s(\zeta) = i = t(\zeta)$ is a loop. From this we can read off the equations (3c) (observe, that $1 - U_{\mathfrak{v}}(\zeta)$ is idempotent in this situation).

$$\begin{array}{ccc} W_j & \xleftarrow{\mathbf{f}_{(j,i)}} & V_i \\ {\scriptstyle U_{\mathfrak{w}}(\eta)}\downarrow & & \downarrow{\scriptstyle U_{\mathfrak{v}}(\zeta)} \\ W_j & \xleftarrow{\mathbf{f}_{(j,i)}} & V_i \end{array}$$

for each pair of arrows $(\eta,\zeta) \in H_2^{\mathfrak{w}} \times H_2^{\mathfrak{v}}$ with $G_{\mathfrak{w}}(\eta) = \varepsilon = G_{\mathfrak{v}}(\zeta)$ and $s(\eta) = j = t(\eta)$ and $s(\zeta) = i = t(\zeta)$. From this we can read off the equations (3d).

7. Proof of the combinatorial results

7.1. **Proof of Proposition 2.** Let $\mathfrak{w} \in \mathfrak{X}$ not an asymmetric band, and $H_0^{\mathfrak{w}} = \{0, 1, \cdots, n\}$ as in 2.2. We define a function $\omega_{\mathfrak{w}} \colon H_0^{\mathfrak{w}} \to \mathbb{Z}$ uniquely by the following prescription:

$$\omega_{\mathfrak{w}}(0) = 0 \text{ and } \omega_{\mathfrak{w}}(t(\nu)) = \begin{cases} \omega_{\mathfrak{w}}(s(\nu)) & \text{if } G_{\mathfrak{w}}(\nu) \in Q_1 \\ \omega_{\mathfrak{w}}(s(\nu)) + 1 & \text{if } G_{\mathfrak{w}}(\nu) \in Q_2 \end{cases} \text{ for all } \nu \in H_1^{\mathfrak{w}}$$

Now, if $\mathfrak{w}, \mathfrak{v} \in \mathfrak{X}$ are not asymmetric bands, we set

$$\Omega_{\mathfrak{w},\mathfrak{v}} \colon H_0^{\mathfrak{w},\mathfrak{v}} \to \mathbb{Z},\ (j,i) \mapsto \omega_{\mathfrak{w}}(j) - \omega_{\mathfrak{v}}(i),$$

and observe, that $\Omega_{\mathfrak{w},\mathfrak{v}}$ is constant on (the vertices of) any line, while $L >' L'$ implies $\Omega_{\mathfrak{w},\mathfrak{v}}(L_0) = \Omega_{\mathfrak{w},\mathfrak{v}}(L_0') + 1$. Thus, in this situation the transitive relation $>$ on $\mathcal{L}(\mathfrak{w},\mathfrak{v})$ generated by $>'$ is an order relation.

Remark. If $\mathfrak{w}$ is an asymmetric band, we can define the function $\omega_{\mathfrak{w}}$ in this way only, if the number c of clockwise arrows $\nu \in H_1^{\mathfrak{w}}$ with $G_{\mathfrak{w}}(\nu) \in Q_2$ coincides with the number a of anti-clockwise arrows ν' with $G_{\mathfrak{w}}(\nu') \in Q_2$. In all examples of asymmetric bands $\mathfrak{w}$ I know, it is possible to define slight variations of this ω which work similarly if $\mathfrak{v}$ is not an asymmetric band.

7.2. For the proof of Proposition 3 we need the following technical result:

Lemma. *Let $\mathfrak{w} \in \mathfrak{X}$ and $H^{\mathfrak{w}}$ the corresponding quiver constructed in 2.2. If $s \xrightarrow{h} t$ is an arrow in $H_1^{\mathfrak{w}}$ with $G_{\mathfrak{w}}(\nu) \in Q_2$ and thus $\sigma(\nu) = \tau(\nu) =: -\rho$, we have*

$$(*) \qquad \begin{aligned} &G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)) > G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho)) \text{ and} \\ &\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)) \succ \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho)) \end{aligned}$$

Moreover, $\Delta\big(G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)), G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))\big) \geq \Delta\big(\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)), \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))\big)$.

Proof. The first inequality holds, as already observed, by the construction of $H^{\mathfrak{w}}$. Anyway, $\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho))$ and $\widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))$ are comparable, and by construction we have

$$d := \Delta(G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)), G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))) \geq \Delta(\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)), \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho))).$$

We will show $(*)$ only for the most complicated case of $\mathfrak{w}$ a symmetric band, the other cases are up to obvious changes the same argument. We may suppose moreover $\mathfrak{w} = \mathfrak{z}\,\underline{f}^*\mathfrak{z}^{-1}\underline{g}^*$ with $\mathfrak{w} = w_1 \cdots w_{2n+2}$ i.e. $\mathfrak{z} = w_1 \cdots w_n$. We define three (infinite) words, $\widetilde{\mathfrak{w}}$ in Q and $\hat{\mathfrak{w}}^\rho$, $\rho \in \{+,-\}$ in $\widehat{Q}$:

$$\widetilde{\mathfrak{w}} = (\tilde{w}_i)_{i\in\mathbb{Z}}, \quad \tilde{w}_{k(2n+2)+i} := w_i$$

$$\hat{\mathfrak{w}}^\rho = (\hat{w}^\rho_i)_{i\in\mathbb{Z}}, \quad \hat{w}^\rho_i := \begin{cases} \tilde{w}_i & \text{if } \tilde{w}_i \text{ is a direct or or inverse letter} \\ \underline{\varepsilon} & \text{if } \tilde{w}_i = \underline{\varepsilon}^* \text{ special, } \widetilde{\mathfrak{w}}_{[i-1]} > \widetilde{\mathfrak{w}}^{[i]} \\ & \text{or } \tilde{w}_i = \underline{\varepsilon}^* \text{ special, } \widetilde{\mathfrak{w}}_{[i-1]} = \widetilde{\mathfrak{w}}^{[i]} \text{ and } \rho\, i > 0. \\ \underline{\varepsilon}^{-1} & \text{if } \tilde{w}_i = \underline{\varepsilon}^* \text{ special, } \widetilde{\mathfrak{w}}_{[i-1]} < \widetilde{\mathfrak{w}}^{[i]} \\ & \text{or } \tilde{w}_i = \underline{\varepsilon}^* \text{ special, } \widetilde{\mathfrak{w}}_{[i-1]} = \widetilde{\mathfrak{w}}^{[i]} \text{ and } \rho\, i \leq 0. \end{cases}$$

If, say $t = i-1$ and $s = i$ in $H_0^{\mathfrak{w}}$, we have by construction of the objects

$$\begin{aligned} G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho)) &= \widetilde{\mathfrak{w}}^{[i]} & \qquad G^*_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)) &= \widetilde{\mathfrak{w}}^{-1}_{[i-1]} \\ \widehat{G}^\rho_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s,\rho)) &= (\hat{\mathfrak{w}}^\rho)^{[i]} & \qquad \widehat{G}^\rho_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t,\rho)) &= (\hat{\mathfrak{w}}^\rho)^{-1}_{[i-1]} \end{aligned}$$

Thus, by hypothesis we have

$$\text{(i)} \qquad \begin{aligned} &\widetilde{\mathfrak{w}}^{-1}_{[i-1]} > \widetilde{\mathfrak{w}}^{[i]} \text{ and} \\ &(\widetilde{\mathfrak{w}}^{[i-d-1]}_{[i-1]})^{-1} = \widetilde{\mathfrak{w}}^{[i]}_{[i+d]}, \text{ in particular} \end{aligned}$$

$$\text{(ii)} \qquad (\widetilde{\mathfrak{w}}^{[i-k-1]}_{[i-1]})^{-1} = \widetilde{\mathfrak{w}}^{[i]}_{[i+k]} \quad (0 < k \leq d)$$

and we have to show:

$$(\hat{\mathfrak{w}}^+_{[i-1]})^{-1} \succ (\hat{\mathfrak{w}}^-)^{[i]}$$

To this end it is enough, to see for $k \in \{1, \dots, d\}$, that if $\tilde{w}_{i+k} = \tilde{w}_{i-k}$ is special and $\hat{w}^-_{i+k}$ is inverse, then $\hat{w}^+_{i-k}$ is a direct letter. Now, recall that

$$\text{(iii)} \qquad \hat{w}^-_{i+k} \text{ inverse} \Longleftrightarrow \widetilde{\mathfrak{w}}^{-1}_{[i+k-1]} \leq \widetilde{\mathfrak{w}}^{[i+k]}$$

$$\text{(!)} \qquad \hat{w}^+_{i-k} \text{ direct} \Longleftrightarrow \widetilde{\mathfrak{w}}^{-1}_{[i-k-1]} > \widetilde{\mathfrak{w}}^{[i-k]}$$

From (i) and (ii) we get $\widetilde{\mathfrak{w}}^{-1}_{[i-k-1]} > \widetilde{\mathfrak{w}}^{[i+k]}$ and then, using this $\widetilde{\mathfrak{w}}^{-1}_{[i+k-1]} > \widetilde{\mathfrak{w}}^{[i-k]}$. These two inequalities, together with (iii) give (!). □

7.3. **Proof of Proposition 3 (a).** Let L' be a h-line in $H^{\mathfrak{w},\mathfrak{v}}$, and say $(t(\nu), s(\mu)) \xrightarrow{(\nu,g)_+} (t(\nu), t(\mu))$ an arrow from $H_2^{\mathfrak{w},\mathfrak{v}}$ starting in L'. Thus we have $\sigma(\nu) = \tau(\nu) = \sigma(\mu) = \tau(\mu) =: -\rho$, and

$$\begin{aligned}
&\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu), \rho)) \succeq \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu), \rho)) \ (L' \text{ is h-line})\\
&\widehat{G}^+_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu), \rho)) \succ \widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu), \rho)) \text{ and}\\
&\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu), \rho)) \succ \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu), \rho)) \text{ by the lemma from 7.2.}
\end{aligned}$$

Thus, in particular

$$\begin{aligned}
&\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu), \rho)) \succeq \widehat{G}^+_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu), \rho)) \succ \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu), \rho)) \text{ and}\\
&\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu), \rho)) \succ \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu), \rho)) \succeq \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu), \rho))
\end{aligned}$$

This means, that the line L, which contains $(s(\nu), s(\mu)) \xrightarrow{(\nu,\mu)} (t(\nu), t(\mu))$ is a h-line, but it can not be of type $\widetilde{\mathbb{A}}$ or $\widetilde{\mathbb{D}}$, because then we would have for example $\widehat{G}^+_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu), \rho)) = \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu), \rho))$. We leave it as an exercise, to show that L neither is of type $\mathbb{D}$.

7.4. **Proof of Proposition 3 (b).** Clearly, no line contains more then two endpoints of arrows from $H^{\mathfrak{w},\mathfrak{v}}_{2,o}$. Thus, suppose that L is a h-line, that contains the endpoints of the arrows from $H^{\mathfrak{w},\mathfrak{v}}_{2,o}$, say $(j, s(\mu)) \overset{(\eta,\mu)_\oplus}{\dashrightarrow} (j, t(\mu))$ and $(t(\nu), i) \overset{(\nu,\zeta)_\ominus}{\dashrightarrow} (s(\nu), i)$. Let $\rho := -\sigma(\mu)$ and $\rho' := -\sigma(\zeta)$. It is easy to see, that we have in this situation

$$\begin{aligned}
&\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu), \rho)) \succeq \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(j, \rho)), \qquad d_1 := \Delta(\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu), \rho)), \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(l, \rho)))\\
&\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i, \rho')) \succeq \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu), \rho)), \quad d_2 := \Delta(\widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(i, \rho')), \widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu), \rho'))).
\end{aligned}$$

Moreover, we have by the lemma from 7.2

$$\begin{aligned}
&\widehat{G}^+_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(t(\mu), \rho)) \succ \widehat{G}^-_{\mathfrak{v}}(\mathfrak{h}^{\mathfrak{v}}(s(\mu), \rho)) \text{and}\\
&\widehat{G}^+_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(t(\nu'), \rho)) \succ \widehat{G}^-_{\mathfrak{w}}(\mathfrak{h}^{\mathfrak{w}}(s(\nu'), \rho)),
\end{aligned}$$

thus in particular

$$d_1 \geq c_1 := \Delta(\widehat{G}_{\mathfrak{v}}^-(\mathfrak{h}^{\mathfrak{v}}(t(\mu),\rho)), \widehat{G}_{\mathfrak{w}}^+(\mathfrak{h}^{\mathfrak{w}}(j,\rho)))$$
$$d_2 \geq c_2 := \Delta(\widehat{G}_{\mathfrak{v}}^-(\mathfrak{h}^{\mathfrak{v}}(i,\rho')), \widehat{G}_{\mathfrak{w}}^-(\mathfrak{h}^{\mathfrak{w}}(s(\nu'),\rho))).$$

On the other hand we have by construction $c_1 > d_2$ and $c_2 > d_1$, a contradiction. Essentially the same argument shows, that for a h-line of type $\mathbb{D}$ there is no such arrow.

7.5. **Proof of Proposition 3 (c).** A real z-line Z contains at most one vertex $(j,i) \in H^{\mathfrak{w},\mathfrak{v}}$, which is starting point of an arrow from $H_2^{\mathfrak{w},\mathfrak{v}}$. In case there is no such vertex, trivially Z can not be contained in a cycle $Z_1 >' Z_2 >' \cdots >' Z_n >' Z_1$. Otherwise, there is a unique $\rho \in \{-1,1\}$ with $\widehat{G}_{\mathfrak{v}}^-(\mathfrak{h}^{\mathfrak{v}}(i,\rho)) \prec \widehat{G}_{\mathfrak{w}}^+(\mathfrak{h}^{\mathfrak{w}}(j,\rho))$, and

$$\delta(Z) := \Delta(\widehat{G}_{\mathfrak{v}}^-(\mathfrak{h}^{\mathfrak{v}}(i,\rho)), \widehat{G}_{\mathfrak{w}}^+(\mathfrak{h}^{\mathfrak{w}}(j,\rho))) \in \mathbb{N}_0$$

is well defined. Now, if Z' is an other line with $Z' >' Z$ (thus a real z-line), then $\delta(Z') < \delta(Z)$. In fact, this is trivial if the arrow from Z' to Z lies in $H_{2,o}^{\mathfrak{w},\mathfrak{v}}$, otherwise this follows easily from the lemma in 7.2.

7.6. **Proof of Proposition 3 (d).** Let (j,i) lie on the h-line L, and $\tilde{L}$ the connected component of $(H_0^{\mathfrak{w},\mathfrak{v}}, H_1^{\mathfrak{w},\mathfrak{v}} \cup H_{2,o}^{\mathfrak{w},\mathfrak{v}})$ that contains L, (compare remark 3.5 (3) and Corollary 1). If there is at most one line L' with $\tilde{L} >' L'$, our affirmation is directly to see. Thus suppose, there are two such lines, and $G_{\mathfrak{w}}^*(\mathfrak{h}^{\mathfrak{w}}(j,\rho)) > G_{\mathfrak{w}}^*(\mathfrak{h}^{\mathfrak{v}}(i,\rho))$ for say $\rho = 1$, this implies the same for $\rho = -1$. But now, we find a nontrivial word $\mathfrak{x}$ such that

$$\mathfrak{x}G_{\mathfrak{w}}^*(\mathfrak{h}^{\mathfrak{w}}(j,1)) > G_{\mathfrak{w}}^*(\mathfrak{h}^{\mathfrak{w}}(j,1)) > G_{\mathfrak{v}}^*(\mathfrak{h}^{\mathfrak{v}}(i,1)) > \mathfrak{x}G_{\mathfrak{v}}^*(\mathfrak{h}^{\mathfrak{v}}(i,1)),$$

which is absurd.

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