Analysis and geometry on configuration spaces: The Gibbsian case

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Abstract

Using a natural "Riemannian–geometry–like" structure on the configuration space Γ over $\mathbb{R}^d$, we prove that for a large class of potentials ϕ the corresponding canonical Gibbs measures on Γ can be completely characterized by an integration by parts formula. That is, if ∇^Γ is the gradient of the Riemannian structure on Γ one can define a corresponding divergence div_ϕ^Γ such that the canonical Gibbs measures are exactly those measures μ for which ∇^Γ and div_ϕ^Γ are dual operators on $L^2(\Gamma, \mu)$.

One consequence is that for such μ the corresponding Dirichlet forms $\mathcal{E}_\mu^\Gamma$ are defined. In addition, each of them is shown to be associated with a conservative diffusion process on Γ with invariant measure μ. The corresponding generators are extensions of the operator $\Delta_\phi^\Gamma := \mathrm{div}_\phi^\Gamma \nabla^\Gamma$. The diffusions can be characterized in terms of a martingale problem and they can be considered as a Brownian motion on Γ perturbed by a singular drift. Another main result of this paper is the following: if μ is a canonical Gibbs measure, then it is extreme (or a "pure phase") if and only if the corresponding weak Sobolev space $W^{1,2}(\Gamma, \mu)$ on Γ is irreducible. As a consequence we prove that for extreme canonical Gibbs measures the above mentioned diffusions are time–ergodic. In particular, this holds for tempered grand canonical Gibbs measures("Ruelle measures") provided the activity constant is small enough. We also include a complete discussion of the free case (i.e., $\phi \equiv 0$) where the underlying space $\mathbb{R}^d$ is even replaced by a Riemannian manifold X.

Contents

1 Introduction

In [AKR 97b] we have introduced a natural (infinite dimensional) differential geometry on the configuration space $\Gamma := \Gamma_X$ over a Riemanian manifold X, i.e.,

$$\Gamma_X := \{\gamma \subset X \mid \gamma \cap X \text{ is a finite set for every compact } K \subset X\}.$$

This geometry was constructed by a "lifting procedure" and is completely determined by the Riemannian structure on X. Let us now briefly describe the latter (for details we refer to [AKR 97b, in particular, Subsection 3.4]): The "tangent bundle" $T(\Gamma)$ of Γ is given as follows:

$$T_\gamma(\Gamma) := L^2(X \to T(X), \gamma), \ \gamma \in \Gamma,$$

i.e., the space of all γ–square integrable sections in the tangent bundle $T(X)$ of X. As usual here γ is identified with the positive Radon measure $\sum_{x\in\gamma} \varepsilon_x$ on X (where $\varepsilon_x :=$ Dirac measure in x). Since each $T_\gamma(\Gamma)$ is thus a Hilbert space (endowed with the corresponding L^2–inner product $\langle\ ,\ \rangle_{T_\gamma(\Gamma)}$ coming from the measure γ), Γ obtains a Riemannian–type structure which is non–trivial (i.e., varies with γ) even when $X = I\!R^d$. The gradient ∇^X and divergence div^X on X lift in a natural way (cf. [AKR 97b, Subsection 3.4]) to a gradient ∇^Γ resp. divergence div^Γ on Γ and are defined on the smooth finitely based cylinder functions resp. vector fields, denoted by $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, $\mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ respectively. "In coordinates" ∇^Γ and div^Γ have the following form: let $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, $V \in \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, i.e., by definition F resp. V have the (non–unique !) representations

$$\Gamma \ni \gamma \mapsto F(\gamma) = g_F(\langle\varphi_1,\gamma\rangle,\ldots,\langle\varphi_N,\gamma\rangle),$$

$N \in I\!N$, $\varphi_1,\ldots,\varphi_N \in \mathcal{D} := C_0^\infty(X) =$, $g_F \in C_b^\infty(I\!R^N)$, and

$$\Gamma \ni \gamma \mapsto V(\gamma) = \sum_{j=1}^N F_j(\gamma)\ v_j\ ,$$

$N \in I\!N$, $F_1,\ldots,F_N \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, $v_1,\ldots,v_N \in V_0(X)$. Here $C_0^\infty(X)$, $V_0(X)$ denote the set of all smooth functions resp. vector fields on X with compact support, $C_b^\infty(I\!R^N)$ the set of all functions on $I\!R^N$ with all derivatives of any order bounded, and for $\varphi \in C_0^\infty(X)$

$$\langle \varphi, \gamma \rangle := \int \varphi \; d\gamma = \sum_{x \in \gamma} \varphi(x).$$

Then for all $\gamma \in \Gamma$

$$\nabla^\Gamma F(\gamma) = \sum_{j=1}^N \partial_j g_F(\langle \varphi_1, \gamma \rangle, \ldots, \langle \varphi_N, \gamma \rangle) \; \nabla^X \varphi_j \;\; (\in T_\gamma(\Gamma)) \tag{1.1}$$

and

$$\operatorname{div}^\Gamma V(\gamma) = \sum_{j=1}^N \left(\langle \nabla^\Gamma F_j(\gamma), v_j \rangle_{T_\gamma(\Gamma)} + F_j(\gamma) \; \langle \operatorname{div}^X v_j, \gamma \rangle \right). \tag{1.2}$$

We recall that among all positive Radon measures the volume element m on X is the (up to constant multiples) unique measure ν on X such that the gradient ∇^X and divergence div^X on X become dual operators on $L^2(X, \nu)$ (w.r.t. $\langle \; , \; \rangle_{T(X)}$). One of the main results in [AKR 97b] is the identification of all "volume elements" for ∇^Γ, $\operatorname{div}^\Gamma$ on Γ, namely

(1) Among all probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ with mean measure being Radon, the mixed Poisson measures with mean proportional to m (cf. Subsection 2.4 below) are exactly those measures μ for which ∇^Γ (with domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$) and $\operatorname{div}^\Gamma$ (with domain $\mathcal{VF}C_b^\infty(\mathcal{D}, \Gamma)$) are dual operators on $L^2(\Gamma, \mu)$ (w.r.t. $\langle \; , \; \rangle_{T(\gamma)}$).

As shown in [AKR 97b] this result, shortly called "integration by parts characterization", has a number of important consequences. One is:

(2) For any "volume element" μ specified in (1) the corresponding pre–Dirichlet form is closable on $L^2(\Gamma, \mu)$. In particular, by [AKMR 97] and [MR 92] there is an associated conservative diffusion process on Γ with invariant measure μ. Its generator is an extension of the *Laplacian* Δ^Γ on Γ defined by $\Delta^\Gamma := \operatorname{div}^\Gamma \nabla^\Gamma$ with (initial) domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$, i.e., this diffusion is a *Brownian motion* on Γ.

Since the said Dirichlet forms will also play a major role in the present paper, let us recall their precise definition: let μ be a mixed Poisson measure with mean proportional to m. Define $(\mathcal{E}^{\Gamma}_{\mu}, D(\mathcal{E}^{\Gamma}_{\mu}))$ as the closure on $L^2(\Gamma,\mu)$ of the pre–Dirichlet form

$$\mathcal{E}^{\Gamma}_{\mu}(F,G) := \int \langle \nabla^{\Gamma} F(\gamma), \nabla^{\Gamma} G(\gamma)\rangle_{T_{\gamma}(\Gamma)}\, \mu(d\gamma), \tag{1.3}$$

$F, G \in \mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)$. In analogy to the finite dimensional case (i.e., X replaces Γ) we define

$$H_0^{1,2}(\Gamma,\mu) := D(\mathcal{E}^{\Gamma}_{\mu})$$

to be the $(1,2)$–*Sobolev space* on Γ w.r.t. μ. Since $\operatorname{div}^{\Gamma}$ is dual to ∇^{Γ} on $L^2(\Gamma,\mu)$ (w.r.t. $\langle\ ,\ \rangle_{T(\Gamma)}$) we have for all $F, G \in \mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma)$

$$\mathcal{E}^{\Gamma}_{\mu}(F,G) = -\int \Delta^{\Gamma} F\ G\, d\mu,$$

so $(\mathcal{E}^{\Gamma}_{\mu}, \mathcal{F}C_b^{\infty}(\mathcal{D},\Gamma))$ is, indeed, closable on $L^2(\Gamma,\mu)$. Furthermore, the closure is a Dirichlet form (cf. e.g. [MR 92]).

Another important consequence of the integration by parts characterization was then the following (cf. [AKR 97b, Subsections 4.3 and 6.3]):

(3) For a mixed Poisson measure μ as above we have that μ is a (pure) Poisson measure with intensity m (cf. Subsection 2.1 below) if and only if $(\mathcal{E}^{\Gamma}_{\mu}, H_0^{1,2}(\Gamma,\mu))$ is *irreducible* if and only if the corresponding diffusion, when started with μ is *(time) ergodic* (cf. Subsection 6.1 below for the two notions in italics).

In [AKR 97b] it was announced that the above results can be extended to the case where μ is replaced by a *Gibbs measure of Ruelle type* and $\operatorname{div}^{\Gamma}$ above is modified correspondingly. To carry out this extension in detail is the purpose of this paper. We mainly consider interactions given by a pair potential determined by a function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ (so in this case $X = \mathbb{R}^d$), but also multi–body potentials can be handled by our methods in a totally analogous way. (Details on the latter will, however, be contained in a forthcoming paper.) Let us now describe our results precisely.

Let $X = \mathbb{R}^d$ and assume that $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ satisfies the familiar conditions (SS), (LR), (I), and, in addition, (D) specified in Subsection 4.2. In the main body of this paper a quite substantial part of the effort is spent on including also the case where the pair potential has infinite range. Here, however, in order to make the presentation as easy as possible, we confine ourselves to the case

$$\operatorname{supp}\phi \text{ is compact,} \tag{1.4}$$

where our results can be presented more concisely. We have to distinguish between corresponding *canonical* and *grand canonical tempered Gibbs measures* whose precise definition is given in Subsections 2.2 and 2.3 (see also Definition 4.1). Here the first we denote by $\mathcal{G}_c^t$ and the latter by $\mathcal{G}_{gc}^t(z)$. We note that the set of canonical Gibbs measures, unlike the grand canonical one, (essentially) does not depend on the *intensity parameter* z (cf. Remark 4.4 (i)). The elements in $\mathcal{G}_{gc}^t(z)$, $z>0$, are also called *Ruelle measures* (see [Ru 70]). Now we modify $\operatorname{div}^\Gamma$ and define for $V=\sum_{j=1}^N F_j v_j \in \mathcal{VFC}_b^\infty(\mathcal{D},\Gamma)$

$$\Gamma \ni \gamma \mapsto \operatorname{div}_\phi^\Gamma V(\gamma) := \sum_{j=1}^N \Big[\langle \nabla^\Gamma F_j(\gamma), v_j\rangle_{T_\gamma(\Gamma)} + F_j(\gamma)(\langle \operatorname{div} v_j,\gamma\rangle + L_{v_j}^\phi(\gamma))\Big] \tag{1.5}$$

where for $v\in V_0(I\!R^d)$, $\gamma\in\Gamma$,

$$L_v^\phi(\gamma) := \sum_{\{x,y\}\subset\gamma} \langle \nabla\phi(x-y), v(x)-v(y)\rangle_{I\!R^d}.$$

For $q\in[1,\infty)$ let $\mathcal{P}_q$ denote the set of all tempered probability measures μ on $(\Gamma,\mathcal{B}(\Gamma))$ with finite q–th moment such that $L_v^\phi\in L^q(\Gamma,\mu)$ for all $v\in V_0(I\!R^d)$. Set

$$\mathcal{G}_{c,q} := \mathcal{G}_c^t \cap \mathcal{P}_q\ .$$

Then $\mathcal{G}_{gc}^t(z)\subset\mathcal{G}_{c,2}$ for all $z>0$ (cf. Lemmas 4.1, 4.2 below). One of our main results is then the following integration by parts characterization for $\mathcal{G}_{c,2}$ (cf. Theorem 4.3 below), which is the analogue of (1) in the interacting case:

(1)' Among all probability measures in $\mathcal{P}_1$, $\mathcal{G}_{c,1}$ are exactly those measures μ for which ∇^Γ (with domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$) and $\operatorname{div}_\phi^\Gamma$ (with domain $\mathcal{VFC}_b^\infty(\mathcal{D},\Gamma)$) become dual operators on $L^2(\Gamma,\mu)$ (w.r.t. $\langle\ ,\ \rangle_{T(\Gamma)}$).

Defining $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ as in (1.3) for $\mu\in\mathcal{G}_{c,1}$, as above we obtain as a consequence from (1)' (cf. Subsection 5.2):

(2)' If $\mu \in \mathcal{G}_{c,2}$, then $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$ is closable on $L^2(\Gamma,\mu)$ and its closure $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ is a Dirichlet form. In particular, by [AKMR 97] and [MR 92] there exists an associated conservative diffusion process on Γ with invariant measure μ. Its generator is an extension of the ϕ–*Laplacian* $\Delta_\phi^\Gamma := \mathrm{div}_\phi^\Gamma \nabla^\Gamma$ with (initial) domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$, i.e., this diffusion is the Brownian motion on Γ mentioned under (2) with a drift given by L_v^ϕ, $v \in V_0(\mathbb{R}^d)$.

In fact, the diffusion in (2)' can be characterized as the solution of a corresponding martingale problem (cf. Theorem 5.3 below). The study of diffusions as those under (2)' has been initiated by R. Lang [L 77] who considered the case $\phi \in C_0^3(\mathbb{R}^d)$ using approximation and stochastic differential equations. This approach had been further developed in the work by J.Fritz [F 87], still in the case of smooth compact support potentials and under the additional restriction $d \le 4$. (See also the recent paper by H.Tanemura [T 96] for the hard balls case). More singular ϕ as in the present paper which are of particular interest in physics (as the Lenard–Jones potential, see Example 4.1 below) have been treated by H. Osada and also M. Yoshida (see [O 96], [Y 96]). The latter two authors were the first who used Dirichlet forms to construct such processes. But the view–point of both these papers is, in contrast to ours, non–geometric, in particular, the explicit formula for L_v^ϕ was not given there and only $\mu \in \mathcal{G}_{gc}^t(z)$, $z > 0$, could be handled.

The main consequence of (1)' is, however, that for the first time the difficult long–standing open question whether these diffusions are (time) ergodic can be settled. But in order to have an analogue of (3) above, we have to replace $H_0^{1,2}(\Gamma,\mu)$ by the larger *weak* (1,2)–*Sobolev space* $W^{1,2}(\Gamma,\mu)$ on Γ and consider a corresponding extension $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$ of $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$. The definition is due to A. Eberle (cf. [Eb 95]) and given in Subsection 6.1 below. As a consequence of (1)' we then obtain (cf. Subsections 6.2, 6.3):

(3)' For $\mu \in \mathcal{G}_{c,2}$ we have that $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$ is irreducible if and only if μ is an extreme point of the convex set $\mathcal{G}_{c,2}$. In this case also $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ is irreducible which is equivalent to the corresponding diffusion (from (2)'), when started with μ, is (time) ergodic. In particular, for small enough $z > 0$ (the unique) $\mu \in \mathcal{G}_{gc}^t(z)$ is an extreme point of $\mathcal{G}_{c,2}$, so the latter applies.

Unfortunately, it is not clear in general whether $H_0^{1,2}(\Gamma,\mu) = W^{1,2}(\Gamma,\mu)$ (unless $\phi \equiv 0$, cf. [AKR 97b, Theorem 4.2] and Proposition 6.2, Remark 6.2

below). Therefore, we do not have the precise analogue of (3) in (3)'. We expect, however, that for a large class of pair potentials ϕ the two Sobolev spaces, in fact, do coincide.

The above results (2)' and (3)' are of a quite universal nature and hold in much more general cases. For example, X need not be equal to $\mathbb{R}^d$ and the type of interaction only enters via (1)'. We, therefore, deduce both (2)' and (3)' from results in Section 5 and, in particular, Subsection 6.2, which are proved in more general situations in order to illustrate and isolate the underlying concepts and ideas.

We also include a complete proof of (1) (i.e., the "free case"), however, on a Riemannian manifold X and even with the role of the volume element m replaced by a (non–finite) positive Radon measure $\sigma = \rho \cdot m$ with much weaker assumptions on ρ than those made in [AKR 97b]. Also the proof of one half of the equivalence is completely different form the proof of "(i) $\Rightarrow$ (ii)" of the corresponding Theorem 3.2 in [AKR 97b]. The method here is more probabilistic (cf. Subsection 4.1). Another reason why the free case is included throughout the paper is of pedagogical nature. Studying the canonical Gibbs measures $\mathcal{G}_c^t$ above is technically very complicated, so the much simpler free case serves as a guideline.

So far, we have described the essential contents of Sections 2, 4, 5, 6. It remains to add that in Section 3 we recall the basics about the Riemannian structure on Γ from [AKR 97b] which are used subsequently.

All the results of this paper have been announced in [AKR 97b]. They have also been presented in talks at the Universities of Bielefeld, Bonn, Erlangen, Göttingen and Moscow, and at international conferences in Barcelona in July 1997, Crete in June 1997, Kiev in May 1997 and Madeira in August 1997.

2 Specifications for canonical and grand canonical Gibbs measures

In this section we precisely describe our framework and recall all necessary definitions for the reader's convenience. The presentation is very much oriented along the beautiful work by C. Preston (see [P 79], but also [P 76] and [Ge 79]), however, specialized to the models we analyze in this paper. Therefore, we adopt the notation in [AKR 97b].

2.1 Configuration space and Poisson measure

Let X be a connected, oriented C^∞ (non-compact) Riemannian manifold. For each point $x \in X$, the tangent space to X at x will be denoted by $T_x(X)$, and the tangent bundle endowed with its natural differentiable structure will be denoted $T(X) = \cup_{x\in X} T_x(X)$. The Riemannian metric on X associates to each $x \in X$ an inner product on $T_x(X)$, which we denote by $< \cdot, \cdot >_{T_x}$. The associated norm will be denoted by $|\cdot|_{T_x}$. Let m denote the volume element.

$\mathcal{O}(X)$ is defined as the family of all non-empty open subsets of X and $\mathcal{B}(X)$ denotes the corresponding Borel σ - algebra. $\mathcal{O}_c(X)$ and $\mathcal{B}_c(X)$ denote the systems of all elements in $\mathcal{O}(X), \mathcal{B}(X)$ respectively, which have compact closures.

The configuration space $\Gamma := \Gamma_X$ over the manifold X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma \ := \ \{\gamma \subset X \mid |\gamma \cap K| < \infty \ \text{ for any compact } \ K \subset X\}. \tag{2.1}$$

Here (and below) $|A|$ denotes the cardinality of a set A.

Via the identification (which we shall use below at various occasions without further notice)

$$\gamma \ \equiv \ \sum_{x\in\gamma} \varepsilon_x, \tag{2.2}$$

where ε_x denotes Dirac measure in x, Γ becomes a subspace of the set of all positive Radon measures on X, hence is naturally topologized by the vague topology. Let $\mathcal{B}(\Gamma)$ denote the corresponding Borel σ - algebra, and $\mathcal{B}_b(\Gamma)$, the set of all bounded $\mathcal{B}(\Gamma)$ - measurable functions $G : \Gamma \to I\!\!R$. If for $B \in \mathcal{B}(X)$ we define, as usual, $N_B : \Gamma \to Z\!\!\!Z_+ \cup \{+\infty\}$ by

$$N_B(\gamma) \ := \ \gamma(B), \tag{2.3}$$

then

$$\mathcal{B}(\Gamma) \ = \ \sigma(\{N_\Lambda | \Lambda \in \mathcal{O}_c(X)\}). \tag{2.4}$$

In particular, for all $f \in C_0(X)$ (:= the set of all continuous functions on X with compact support)

$$\gamma \ \mapsto < f, \gamma > := \int f \ d\gamma \ = \ \sum_{x \in \gamma} f(x) \tag{2.5}$$

is $\mathcal{B}(\Gamma)$ - measurable.

For $A \in \mathcal{B}(X)$ we define

$$\mathcal{B}_A(\Gamma) \ := \ \sigma(\{N_B | B \in \mathcal{B}_c(X), B \subset A\}).$$

For later use we recall the "localized" description of Γ: for $\Lambda \in \mathcal{B}_c(X)$ define

$$\Gamma_\Lambda \ := \ \{\gamma \in \Gamma \mid \gamma(X \setminus \Lambda) = 0\}, \tag{2.6}$$

and for $n \in \mathbb{Z}_+$

$$\Gamma_\Lambda^{(n)} \ := \ \{\gamma \in \Gamma_\Lambda \mid \gamma(\Lambda) = n\}. \tag{2.7}$$

There is a bijection

$$\tilde{\Lambda}^n / S_n \ \to \Gamma_\Lambda^{(n)}, \tag{2.8}$$

where

$$\tilde{\Lambda}^n \ := \ \{(x_1, \ldots, x_n) \in \Lambda^n | x_k \neq x_j, \ j \neq k\}, \tag{2.9}$$

and S_n is the permutation group over $\{1, \ldots, n\}$. If $\Lambda \in \mathcal{O}_c(X)$, this bijection defines a locally compact metrizably Hausdorff topology on $\Gamma_\Lambda^{(n)}$, hence a corresponding (sum) topology on

$$\Gamma_\Lambda \ = \ \bigcup_{n=0}^{\infty} \Gamma_\Lambda^{(n)}. \tag{2.10}$$

If Γ_Λ is equipped with the associated Borel σ - algebra then $(\Gamma, \mathcal{B}(\Gamma))$ is the projective limit of the measurable spaces $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ as $\Lambda \nearrow X$.

Let $\rho \in L^1_{\mathrm{loc}}(X, m)$ be such that $\rho^{\frac{1}{2}} \in H^{1,2}_{loc}(X, m)$, i.e., $\rho^{\frac{1}{2}}$ has locally m-square integrable weak derivatives. Assume that ρ is not globally m-integrable. Define

$$\sigma \ := \ \rho \, m \tag{2.11}$$

and let π_σ denote *Poisson measure* on $(\Gamma, \mathcal{B}(\Gamma))$ with intensity measure σ, i.e.,

$$\int_\Gamma e^{<f,\gamma>} \pi_\sigma(d\gamma) \ = \ \exp(\int_X (e^f - 1)\, d\sigma) \text{ for all } f \in C_0(X).$$

The respective "local" description of π_σ is as follows: for $\Lambda \in \mathcal{O}_c(X)$ and $n \in I\!N$ the product measure $\sigma^{\otimes n}$ can be considered as a (finite) measure on $\tilde{\Lambda}_n$. Let

$$\sigma_{\Lambda,n} \ := \sigma^{\otimes n} \cdot (s^n_\Lambda)^{-1}$$

be the corresponding measure on $\Gamma^{(n)}_\Lambda$ where $s^n_\Lambda : \tilde{\Lambda}^n \to \Gamma^{(n)}_\Lambda$, $s^n_\Lambda((x_1, \ldots, x_n)) := \sum_{i=1}^n \varepsilon_{x_i}$. Define $p_\Lambda : \Gamma \to \Gamma_\Lambda$ by

$$p_\Lambda(\gamma) \ := \ \gamma_\Lambda \tag{2.12}$$

where

$$\gamma_\Lambda \ := \ \gamma \cap \Lambda \ \equiv \ 1_\Lambda \cdot \gamma. \tag{2.13}$$

Then

$$\pi_\sigma \circ p_\Lambda^{-1} \ = \ e^{-\sigma(\Lambda)} \sum_{n=0}^\infty \frac{1}{n!} \, \sigma_{\Lambda,n}, \tag{2.14}$$

where $\sigma_{\Lambda,0} := \varepsilon_\emptyset$ on $\Gamma^{(0)}_\Lambda = \{\emptyset\}$.

We refer to [AKR 97b] and, in particular, to the references therein for more details, proofs, and historical remarks.

2.2 Grand canonical Gibbs measures

Let Φ be a *potential*, i.e., a function $\Phi : \Gamma \to I\!R \cup \{+\infty\}$ such that

$$\begin{aligned} \Phi \ &= \ 1_{\{N_X < \infty\}} \Phi, \ \Phi(\emptyset) \ = \ 0, \ \text{ and} \\ \gamma \ &\mapsto \ \Phi(\gamma_\Lambda) \ \text{ is } \ \mathcal{B}_\Lambda(\Gamma) - \text{measurable} \\ &\text{for all } \ \Lambda \in \mathcal{O}_c(X). \end{aligned} \tag{2.15}$$

For $\Lambda \in \mathcal{O}_c(X)$ the *conditional energy* $E^\Phi_\Lambda : \Gamma \to I\!R \cup \{+\infty\}$ is defined by

$$E_\Lambda^\Phi(\gamma) := \begin{cases} \sum_{\gamma'\subset\gamma,\gamma'(\Lambda)>0} \Phi(\gamma') & \text{if } \sum_{\gamma'\subset\gamma,\gamma'(\Lambda)>0} |\Phi(\gamma')| < \infty, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.16}$$

(where the sum of the empty set is defined to be zero).

Now we can define grand canonical Gibbs measures:

Definition 2.1 *For $\Lambda \in \mathcal{O}_c(X)$ define for $\gamma \in \Gamma, \Delta \in \mathcal{B}(\Gamma)$*

$$\begin{aligned} \Pi_\Lambda^{\sigma,\Phi}(\gamma, \Delta) := \ & 1_{\{Z_\Lambda^{\sigma,\Phi}<\infty\}}(\gamma)[Z_\Lambda^{\sigma,\Phi}(\gamma)]^{-1} \int_\Gamma 1_\Delta(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda) \\ & \times \exp[-E_\Lambda^\Phi(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)] \ \pi_\sigma(d\gamma'), \end{aligned}$$

where

$$Z_\Lambda^{\sigma,\Phi}(\gamma) := \int_\Gamma \exp[-E_\Lambda^\Phi(\gamma_{X\setminus\Lambda} + \gamma'_\Lambda)] \ \pi_\sigma(d\gamma').$$

A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a grand canonical Gibbs measure with interaction potential Φ if

$$\mu \Pi_\Lambda^{\sigma,\Phi} = \mu \ \textit{for all} \ \Lambda \in \mathcal{O}_c(X). \tag{2.17}$$

Let $\mathcal{G}_{gc}(\sigma, \Phi)$ denote the set of all such probability measures μ.

Remark 2.1 *(i) It is well-known that $(\Pi_\Lambda^{\sigma,\Phi})_{\Lambda\in\mathcal{O}_c(X)}$ is a $(\mathcal{B}_{X\setminus\Lambda}(\Gamma))_{\Lambda\in\mathcal{O}_c(X)}$ - specification in the sense of [Fö 75], [P 76] (cf. [P 76, Section 6] or [P 79]), i.e., for all $\Lambda, \Lambda' \in \mathcal{O}_c(X)$*

(S 1) $\Pi_\Lambda^{\sigma,\Phi}(\gamma, \Gamma) \in \{0, 1\}$ *for all* $\gamma \in \Gamma$.

(S 2) $\Pi_\Lambda^{\sigma,\Phi}(\cdot, \Delta)$ *is* $\mathcal{B}_{X\setminus\Lambda}(\Gamma)$*-measurable for all* $\Delta \in \mathcal{B}(\Gamma)$.

(S 3) $\Pi_\Lambda^{\sigma,\Phi}(\cdot, \Delta' \cap \Delta) = 1_{\Delta'}\Pi_\Lambda(\cdot, \Delta)$ *for all* $\Delta \in \mathcal{B}(\Gamma), \Delta' \in \mathcal{B}_{X\setminus\Lambda}(\Gamma)$.

(S 4) $\Pi_{\Lambda'}^{\sigma,\Phi} = \Pi_{\Lambda'}^{\sigma,\Phi}\Pi_\Lambda^{\sigma,\Phi}$ *if* $\Lambda \subset \Lambda'$.

Here for $\gamma \in \Gamma, \Delta \in \mathcal{B}(\Gamma)$

$$(\Pi_{\Lambda'}^{\sigma,\Phi}\Pi_{\Lambda}^{\sigma,\Phi})(\gamma,\Delta) \;:=\; \int \Pi_{\Lambda}^{\sigma,\Phi}(\gamma',\Delta)\;\Pi_{\Lambda'}^{\sigma,\Phi}(\gamma,d\gamma')$$

(and $\mu\Pi_{\Lambda}^{\sigma,\Phi}$ in (2.17) is defined correspondingly). (2.17) above are called Dobrushin-Lanford-Ruelle equations. We also note that as a direct consequence of (S 3) we have for all $\Lambda \in \mathcal{O}_c(X)$ that

$$\mu(\{Z_{\Lambda}^{\sigma,\Phi} < \infty\}) \;=\; 1$$

for all $\mu \in \mathcal{G}_{gc}(\sigma,\Phi)$.

(ii) It is an easy exercise (see e.g. [P 80, Proposition 10.3]) to show that because of (2.14) for all $\Lambda \in \mathcal{O}_c(X)$, $\gamma \in \Gamma$, $\Delta \in \mathcal{B}(\Gamma)$

$$\Pi_{\Lambda}^{\sigma,\Phi}(\gamma,\Delta) = 1_{\{\tilde{Z}_{\Lambda}^{\sigma,\Phi}<\infty\}}(\gamma)[\tilde{Z}_{\Lambda}^{\sigma,\Phi}(\gamma)]^{-1}[1_{\Delta}(\gamma_{X\setminus\Lambda})+$$

$$\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda}\dots\int_{\Lambda} 1_{\Delta}(\gamma_{X\setminus\Lambda}+\sum_{k=1}^{n}\varepsilon_{x_k})\exp[-E_{\Lambda}^{\Phi}(\gamma_{X\setminus\Lambda}+\sum_{k=1}^{n}\varepsilon_{x_k})]\;\sigma(dx_1)\dots\sigma(dx_n)],$$

where

$$\tilde{Z}_{\Lambda}^{\sigma,\Phi}(\gamma) := 1+\sum_{n=1}^{\infty}\frac{1}{n!}\int_{\Lambda}\dots\int_{\Lambda}\exp[-E_{\Lambda}^{\Phi}(\gamma_{X\setminus\Lambda}+\sum_{k=1}^{n}\varepsilon_{x_k})]\;\sigma(dx_1)\dots\sigma(dx_n)$$

$$= \exp(\sigma(\Lambda))Z_{\Lambda}^{\sigma,\Phi}(\gamma).$$

(iii) Clearly, by properties (S 2),(S 3) a probability measure μ on $(\Gamma,\mathcal{B}(\Gamma))$ is a grand canonical Gibbs measure if and only if for all $\Lambda \in \mathcal{O}_c(X)$ and all $G \in \mathcal{B}_b(\Gamma)$

$$E_{\mu}[G \mid \mathcal{B}_{X\setminus\Lambda}(\Gamma)] \;=\; \Pi_{\Lambda}^{\sigma,\Phi}G \;\; \mu - a.e.,$$

where for a sub-σ-algebra $\Sigma \subset \mathcal{B}(\Gamma)$, $E_{\mu}[\cdot\,|\Sigma]$ denotes the conditional expectation w.r.t. μ given Σ and for $G \in \mathcal{B}_b(\Gamma)$ we set

$$\Pi_{\Lambda}^{\sigma,\Phi}(\gamma,G) \;:=\; (\Pi_{\Lambda}^{\sigma,\Phi}G)(\gamma) \;=\; \int G(\gamma')\;\Pi_{\Lambda}^{\sigma,\Phi}(\gamma,d\gamma'). \tag{2.18}$$

(iv) Above as well as in the next section we may always replace $\mathcal{O}_c(X)$ by $\mathcal{B}_c(X)$ without any changes (in particular, $\mathcal{G}_{gc}(\sigma,\Phi)$ remains unchanged). We chose $\Lambda \in \mathcal{O}_c(X)$ only for simplicity. We will use E_{Λ}^{Φ} for $\Lambda \in \mathcal{B}_c(X)$ in Section 4.2 below.

2.3 Canonical Gibbs measures

Consider the situation of Subsection 2.2.

Definition 2.2 *For $\Lambda \in \mathcal{O}_c(X)$ define for $\gamma \in \Gamma, \Delta \in \mathcal{B}(\Gamma)$*

$$\hat{\Pi}_\Lambda^{\sigma,\Phi}(\gamma,\Delta) \;:=\; \begin{cases} \frac{\Pi_\Lambda^{\sigma,\Phi}(\gamma,\Delta\cap\{N_\Lambda=\gamma(\Lambda)\})}{\Pi_\Lambda^{\sigma,\Phi}(\gamma,\{N_\Lambda=\gamma(\Lambda)\})} & \text{if } \Pi_\Lambda^{\sigma,\Phi}(\gamma,\{N_\Lambda=\gamma(\Lambda)\})>0 \\ 0 & \text{otherwise} \end{cases} \tag{2.19}$$

A probability measure μ on $(\Gamma,\mathcal{B}(\Gamma))$ is called a canonical Gibbs measure with interaction potential Φ if

$$\mu\hat{\Pi}_\Lambda^{\sigma,\Phi} \;=\; \mu \;\; \textit{for all} \;\; \Lambda \in \mathcal{O}_c(X). \tag{2.20}$$

Let $\mathcal{G}_c(\sigma,\Phi)$ denote the set of all such probability measures μ.

Remark 2.2 *For $\Lambda \in \mathcal{O}_c(X)$ let $\hat{\mathcal{B}}_{X\setminus\Lambda}(\Gamma)$ denote the σ - algebra generated by $\mathcal{B}_{X\setminus\Lambda}(\Gamma)$ and $\sigma(N_\Lambda)$. It has been shown in [P 79] that $(\hat{\Pi}_\Lambda^{\sigma,\Phi})_{\Lambda\in\mathcal{O}_c(X)}$ is indeed a $(\hat{\mathcal{B}}_{X\setminus\Lambda}(\Gamma))_{\Lambda\in\mathcal{O}_c(X)}$ - specification (i.e., satisfies (S 1) - (S 4) in Remark 2.1 (i) with $\hat{\Pi}_\Lambda^{\sigma,\Phi}$ replacing $\Pi_\Lambda^{\sigma,\Phi}$ and $\hat{\mathcal{B}}_{X\setminus\Lambda}(\Gamma)$ replacing $\mathcal{B}_{X\setminus\Lambda}(\Gamma)$ for $\Lambda \in \mathcal{O}_c(X)$). So, the above definition makes sense. In particular, for all $\Lambda \in \mathcal{O}_c(X)$ we have that*

$$\mu(\{\gamma \in \Gamma \mid \Pi_\Lambda^{\sigma,\Phi}(\gamma,\{N_\Lambda=\gamma(\Lambda)\})>0\}) \;=\; 1 \tag{2.21}$$

for all $\mu \in \mathcal{G}_c(\sigma,\Phi)$. Clearly, also the analogue of Remark 2.1 (iii) holds.

It can be checked that

$$\mathcal{G}_{gc}(\sigma,\Phi) \;\subset \mathcal{G}_c(\sigma,\Phi) \tag{2.22}$$

(cf. [P 79, Proposition 2.1]). The precise relation between the convex sets $\mathcal{G}_c(\sigma,\Phi)$ and $\mathcal{G}_{gc}(\sigma,\Phi)$ resp. between their extreme points $ex\mathcal{G}_c(\sigma,\Phi)$ and $ex\mathcal{G}_{gc}(\sigma,\Phi)$ is one of the major problems in Statistical Mechanics, known under the key word *equivalence of ensembles* (cf. [P 79], [Ge 79]).

Remark 2.1 *By a general result due to E.B. Dynkin and H. Föllmer (cf. [Dy 78], [Fö 75], and for a beautiful exposition [P76, Theorem 2.2]) we have* $\mathrm{ex}\mathcal{G}_c(\sigma,\Phi) \neq \emptyset$, $\mathrm{ex}\mathcal{G}_{gc}(\sigma,\Phi) \neq \emptyset$ *provided* $\mathcal{G}_c(\sigma,\Phi) \neq \emptyset$, $\mathcal{G}_{gc}(\sigma,\Phi) \neq \emptyset$ *respectively, and that any* μ *in* $\mathcal{G}_c(\sigma,\Phi)$ *resp. in* $\mathcal{G}_{gc}(\sigma,\Phi)$ *has an integral representation in terms of the respective extreme points (see Theorem 2.1 below for the special case* $\Phi \equiv 0$*).*

In the sequel we shall mainly analyze $\mathcal{G}_c(\sigma,\Phi)$, but using known results on the above problem we shall deduce new results also for $\mathcal{G}_{gc}(\sigma,\Phi)$ (cf. Subsection 6.3 (c) below).

2.4 Preliminaries on the free case

In this subsection we consider the same situation as before, but assume $\Phi \equiv 0$, i.e., we are in the *free case*. We set $\mathcal{G}_c(\sigma) := \mathcal{G}_c(\sigma,0)$, $\mathcal{G}_{gc}(\sigma) := \mathcal{G}_{gc}(\sigma,0)$. Here we have the following well known result (cf. [NZ 77, Sect. 2] or [Ge 79, Sect. 4.2], resp. [MKM 78]).

Theorem 2.1 *(i) Let* μ *be a probability measure on* $(\Gamma,\mathcal{B}(\Gamma))$*. Then* $\mu \in \mathcal{G}_c(\sigma)$ *if and only if there exists a probability measure* λ *on* $(\mathbb{R}_+,\mathcal{B}(\mathbb{R}_+))$ *such that*

$$\mu \;=\; \int_{\mathbb{R}_+} \pi_{z\sigma}\ \lambda(dz) \tag{2.23}$$

(i.e., if and only if μ *is a mixed Poisson measure with mean proportional to* m*). Here* $\pi_{0\sigma}$ *denotes the Dirac measure on* Γ *with mass in* $\gamma = \emptyset$*.*

(ii) $\quad ex\mathcal{G}_c(\sigma) = \{\pi_{z\sigma} \mid z \in \mathbb{R}_+\}$.

(iii) $\quad ex\mathcal{G}_c(\sigma) = \cup_{z\in\mathbb{R}_+} ex\mathcal{G}_{gc}(z\sigma)$.

We note that since σ is diffuse, the set $\ddot{\mathfrak{M}}$ in [NZ 77, Sect. 2] (resp. M in [Ge 79, Sect. 4.2]) can really be replaced by Γ as done in Theorem 2.1 above.

3 Geometry of configuration spaces

In this section we recall results used below from [AKR 97b] (resp. [AKR 96a]) to which we refer for the corresponding proofs and more details.

3.1 Flows, directional derivatives and integration by parts

By $\mathrm{Diff}_0(X)$ let us denote the group of all diffeomorphisms $\psi : X \to X$ which are trivial outside a compact. Any such ψ defines a transformation of Γ by

$$\Gamma \ni \gamma \mapsto \psi(\gamma) = \{\psi(x) \mid x \in \gamma\} = \sum_{x \in \gamma} \varepsilon_x \in \Gamma.$$

This mapping is measurable and the image $\pi_\sigma \circ \psi^{-1}$ under ψ is well defined. It is not hard to see that

$$\pi_\sigma \circ \psi^{-1} = \pi_{\sigma \circ \psi^{-1}}.$$

As a result (cf. e.g. [AKR 97b, Proposition 2.2]), the Poisson measure is quasi-invariant with respect to the group $\mathrm{Diff}_0(X)$.

Let $V_0(X)$ denote the space of all smooth vector fields on X with compact support. Any vector field $v \in V_0(X)$ defines (via the exponential mapping) a one-parameter group $\psi_t^v \in \mathrm{Diff}_0(X)$, $t \in I\!R$.

Definition 3.1 *For $F : \Gamma \to I\!R$ we define the directional derivative along the vector field v as*

$$(\nabla_v^\Gamma F)(\gamma) = \frac{d}{dt} F(\psi_t^v(\gamma))|_{t=0},$$

(provided the right hand side exists).

This definition applies to F in the following class of so-called smooth cylinder functions. Let $\mathcal{D} := C_0^\infty(X)$ (the set of all smooth functions on X with compact support). We define $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ as the set of all functions on Γ of the form

$$\gamma \mapsto F(\gamma) = g_F(<\varphi_1, \gamma>, \ldots, <\varphi_N, \gamma>), \tag{3.1}$$

where $\varphi_1, \ldots, \varphi_N \in \mathcal{D}$ and $g_F \in C_b^\infty(I\!R^N)$. Clearly, $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ is dense in $L^2(\pi_\sigma) := L^2(\Gamma, \mathcal{B}(\Gamma), \pi_\sigma)$. For any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we have

$$(\nabla_v^\Gamma F)(\gamma) = \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(<\varphi_1, \gamma>, \ldots, <\varphi_N, \gamma>) \ <\nabla_v^X \varphi_j, \gamma>, \tag{3.2}$$

where $x \mapsto (\nabla_v^X \varphi)(x) =< \nabla^X \varphi(x), v(x) >_{T_x(X)}$ is the usual directional derivative on X along the vector field v and ∇^X denotes the gradient on X. We obtain a differential operator

$$\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma) \ni F \mapsto \nabla_v^\Gamma F \in L^2(\pi_\sigma)$$

on $L^2(\pi_\sigma)$.

The *logarithmic derivative* of the measure σ is given by the vector field $\beta^\sigma := \nabla^X \log \rho = \nabla^X \rho / \rho$ (where $\beta^\sigma := 0$ on $\{\rho = 0\}$). From now on we *fix* an m-version of $\nabla^X \rho$ and of β^σ. Then for $v \in V_0(X)$ the logarithmic derivative of σ along v is the function $X \ni x \mapsto \beta_v^\sigma(x) := < \beta^\sigma(x), v(x) >_{T_x(X)} + \mathrm{div}^X v(x)$, where div^X denotes divergence on X (w.r.t. the volume element m). We note that $< \beta^\sigma, v >_{T(X)}$ belongs to $L^2(\sigma) := L^2(X, \sigma)$ (hence to $L^1(\sigma)$ because it has compact support) and that therefore the same is true for β_v^σ. Because the dual $(\nabla_v^X)^*$ of ∇_v^X on $L^2(\sigma)$ is given by

$$(\nabla_v^X)^* \;=\; -\nabla_v^X \;+\; \beta_v^\sigma$$

on $\mathcal{D}$, we define analogously

$$\mathrm{div}_\sigma^X(v) \;:=\; \mathrm{div}^X(v) + \;< \beta^\sigma, v >_{T(X)} = \beta_v^\sigma.$$

Definition 3.2 *For any $v \in V_0(X)$ we define the logarithmic derivative of π_σ along v as the following function on Γ:*

$$\Gamma \ni \gamma \mapsto B_v^{\pi_\sigma}(\gamma) :=< \beta_v^\sigma, \gamma >= \tag{3.3}$$
$$\int_X [< \beta^\sigma(x), v(x) >_{T_x(X)} \;+\; div^X v(x)] \;\gamma(dx).$$

We note that if we consider another m-version of $\nabla^X \rho$ and hence β^σ, $B_v^{\pi_\sigma}$ will not change except on a π_σ-zero set. Hence the following result is independent of the m-version chosen for $\nabla^X \rho$ above.

Theorem 3.1 *For all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ and any $v \in V_0(X)$ the following integration by parts formula holds:*

$$\int_\Gamma \nabla_v^\Gamma F \; G \; d\pi_\sigma = -\int_\Gamma F \; \nabla_v^\Gamma G \; d\pi_\sigma \;-\; \int_\Gamma F \; G \; B_v^{\pi_\sigma} \; d\pi_\sigma, \tag{3.4}$$

or

$$(\nabla_v^\Gamma)^* \;=\; -\nabla_v^\Gamma \;-\; B_v^{\pi_\sigma},$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ in $L^2(\pi_\sigma)$.

3.2 Tangent bundle and gradient

Definition 3.3 *We introduce the tangent space $T_\gamma(\Gamma)$ to the configuration space Γ at the point $\gamma \in \Gamma$ as the Hilbert space of measurable γ - square-integrable sections (measurable vector fields) $V_\gamma : X \longrightarrow T(X)$ with the scalar product*

$$< V_\gamma^1, V_\gamma^2 >_{T_\gamma(\Gamma)} = \int < V_\gamma^1(x), V_\gamma^2(x) >_{T_x(X)} \gamma(dx) \tag{3.5}$$

$V_\gamma^1, V_\gamma^2 \in T_\gamma(\Gamma)$. *The corresponding tangent bundle is*

$$T(\Gamma) = \bigcup_{\gamma \in \Gamma} T_\gamma(\Gamma).$$

Let us stress that any $v \in V_0(X)$ can be considered as a "constant" vector field on Γ such that

$$\Gamma \ni \gamma \ \mapsto \ V_\gamma(\cdot) = v(\cdot) \in T_\gamma(\Gamma),$$
$$< v, v >_{T_\gamma(\Gamma)} = \int_X |v(x)|^2_{T_x(X)} \gamma(dx) \ < \ \infty.$$

As usual in Riemannian geometry, having directional derivatives and a Hilbert space as the tangent space we can introduce the gradient.

Definition 3.4 *We define the intrinsic gradient of a function $F : \Gamma \longrightarrow I\!R$ as the mapping*

$$\Gamma \ni \gamma \ \mapsto \ (\nabla^\Gamma F)(\gamma) \in T_\gamma(\Gamma)$$

such that for any $v \in V_0(X)$

$$(\nabla_v^\Gamma F)(\gamma) \ = < \nabla^\Gamma F, v >_{T_\gamma(\Gamma)} . \tag{3.6}$$

Note that (3.6), in particular, implies that $\nabla_v^\Gamma F$ is the directional derivative along the "constant" vector field v on Γ. Furthermore, by (3.2) for any $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ of the form (3.1) we have

$$(\nabla^\Gamma F)(\gamma; x) \ = \ \sum_{j=1}^N \frac{\partial g_F}{\partial s_j}(< \varphi_1, \gamma >, \ldots, < \varphi_N, \gamma >) \nabla^X \varphi_j(x), \tag{3.7}$$
$$\gamma \in \Gamma, x \in X.$$

3.3 Divergence

Let us now recall the definition of the corresponding divergence.

Definition 3.5 *For a vector field*

$$V : \Gamma \ni \gamma \ \mapsto \ V_\gamma \in T_\gamma(\Gamma)$$

the divergence $div^\Gamma_{\pi_\sigma} V$ *is defined via the duality relation*

$$\int < V_\gamma, \nabla^\Gamma F(\gamma) >_{T_\gamma(\Gamma)} \pi_\sigma(d\gamma) \ = \ -\int_\Gamma F(\gamma)(div^\Gamma_{\pi_\sigma} V)(\gamma)\pi_\sigma(d\gamma) \qquad (3.8)$$

for all $F \in \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)$, *provided it exists (i.e., provided*

$$F \mapsto \int < V_\gamma, \nabla^\Gamma F(\gamma) >_{T_\gamma(\Gamma)} \pi_\sigma(d\gamma)$$

is continuous on $L^2(\pi_\sigma)$*).*

As a consequence of Theorem 3.1 we immediately get

Corollary 3.1 *For any vector field*

$$V_\gamma(x) \ = \ \sum_{j=1}^N G_j(\gamma) v_j(x), \ \ \gamma \in \Gamma, x \in X, \qquad (3.9)$$

with $G_j \in \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma), v_j \in V_0(X), j = 1, \ldots, N$, *we have*

$$\begin{aligned} (div^\Gamma_{\pi_\sigma} V)(\gamma) \ &= \ \sum_{j=1}^N (\nabla^\Gamma_{v_j} G_j)(\gamma) + \sum_{j=1}^N B^{\pi_\sigma}_{v_j}(\gamma) G_j(\gamma) \\ = \ \sum_{j=1}^N &< \nabla^\Gamma G_j(\gamma), v_j >_{T_\gamma(\Gamma)} + \sum_{j=1}^N < \beta^\sigma_{v_j}, \gamma > G_j(\gamma). \qquad (3.10) \end{aligned}$$

Remark 3.1 *If* $\mathcal{V}\mathcal{F}C^\infty_b(\mathcal{D}, \Gamma)$ *denotes the set of all* V *of type (3.9), then Corollary 3.1 just states that* $(\nabla^\Gamma, \mathcal{F}C^\infty_b(\mathcal{D}, \Gamma))$ *and* $(div^\Gamma_{\pi_\sigma}, \mathcal{V}\mathcal{F}C^\infty_b(\mathcal{D}, \Gamma))$ *are dual operators on* $L^2(\mu)$ *if* $\mu = \pi_\sigma$*. In the next section we shall characterize all measures* μ *having this property. An analogous characterization for* $\mu \in \mathcal{G}_c(\sigma, \Phi)$ *for certain* $\Phi \neq 0$ *will be proved in Section 4 below.*

4 Integration by parts characterization of canonical Gibbs measures

4.1 Free case

Consider the free case of Subsection 2.3, i.e., $\Phi \equiv 0$. It is well–known (cf. e.g. [Ch 84]) that the volume element m of our Riemannian manifold X is up to a constant the unique positive *Radon* measure ν on X such that div^X is the dual operator on $L^2(\nu)$ of ∇^X (when considered on $V_0(X)$ resp. $C_0^\infty(X)$). We now want to investigate the analogous question for the Poisson measure π_m or more generally π_σ on Γ. Before we can formulate and prove the corresponding theorem we introduce the following notation for the mixed Poisson measures from Subsection 2.3: let λ be a probability measure on $(I\!R_+ \,, \mathcal{B}(I\!R_+))$. Then we set

$$\mu_{\lambda,\sigma} := \int_{I\!R_+} \pi_{z\sigma} \ \lambda(dz). \tag{4.1}$$

Let $\mathcal{M}_q(\Gamma)$, $q \in [1, \infty)$, denote the set of all probability measures μ on $(\Gamma, \mathcal{B}(\Gamma))$ such that

$$\int_\Gamma |\langle\varphi, \gamma\rangle|^q \mu(d\gamma) < \infty \text{ for all } \varphi \in C_0(X)\,,\ \varphi \geq 0\ . \tag{4.2}$$

Clearly, $\mu_{\lambda,\sigma} \in \mathcal{M}_q(\Gamma)$ if and only if

$$\int_{I\!R_+} z^q \ \lambda(dz) < \infty\ . \tag{4.3}$$

For abbreviation we define $(IbP)^\sigma$ to be the set of all $\mu \in \mathcal{M}_1(\Gamma)$ with the property that $\gamma \mapsto << \beta^\sigma, v >_{T(X)}, \gamma >$ is μ-integrable for all $v \in V_0(X)$ and which satisfy (3.4) with μ replacing π_σ for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma), v \in V_0(X)$. We note that (3.4) only makes sense for such measures (cf. (3.2)) and that $B_v^{\pi_\sigma}$ only depends on σ not on π_σ (cf. (3.3)). Obviously, since ∇_v^Γ obeys the product rule for all $v \in V_0(X)$ we may always take $G \equiv 1$. Furthermore, $(IbP)^\sigma$ is convex.

Theorem 4.1 *Suppose that σ satisfies the following assumption:*
(A) For every $\Lambda \in \mathcal{O}_c(X)$, the measures $z\sigma, z \geq 0$, are the only positive Radon measures ν on X such that div_σ^X is the dual operator on $L^2(\nu)$ of

∇^X when considered with domains $V_0(\Lambda)$ resp. $C_0^\infty(\Lambda)$ (i.e., the set of all $v \in V_0(X)$ resp. $\varphi \in \mathcal{D}$ with compact support in Λ).

Then the following assertions are equivalent:

(i) $\mu \in (IbP)^\sigma$.

(ii) $\mu = \mu_{\lambda,\sigma}$ for some probability measure λ on $(I\!R_+, \mathcal{B}(I\!R_+))$ satisfying (4.3) with $q = 1$.

Remark 4.1 *(i) We emphasize the fact that we restrict to $\mu \in \mathcal{M}_1(\Gamma)$ (i.e., μ has a Radon measure as mean measure). This is the exact analogue to confining condition (A) to Radon measures ν on X. We also stress the fact that condition (A) is not needed to prove Theorem 4.1, (ii) $\Rightarrow$ (i).*

(ii) By Corollary 3.1 the assertion in the above theorem can obviously be rephrased in terms of ∇^Γ, $div^\Gamma_{\pi_\sigma}$. Hence Theorem 4.1 just says that if σ on X satisfies (A), then this property (A) "lifts" in a natural way to Γ.

(iii) The assumption (A) on σ in Theorem 4.1 is fulfilled if $\beta_v^\sigma \in L^1_{loc}(m)$ for all $v \in V_0(X)$. This is an immediate consequence of the proof of Theorem 6.2 in [BR 95] which by [BR 95, Lemma 6.4] and [BKrR 96, Theorem 1 (i)] immediately generalizes to the situation considered here. Note that in contrast to (A) the condition that $\beta_v^\sigma \in L^1_{loc}(m)$ for all $v \in V_0(X)$ does not depend on the m-version chosen for β_v^σ above.

(iv) Theorem 4.1 is a strict extension of the corresponding result (i.e., Theorem 3.2) in [AKR 97b].

Proof of Theorem 4.1

(ii) $\Rightarrow$ (i): Note first that trivially $\pi_{0\sigma} \in (IbP)^\sigma$. Since $\beta^{z\sigma} = \beta^\sigma$, it follows from (3.3) that $B_\cdot^{\pi_\sigma} = B_\cdot^{\pi_{z\sigma}}$. Hence $\pi_{z\sigma} \in (IbP)^\sigma$ for all $z \in I\!R_+$ and, therefore, any $\mu_{\lambda,\sigma}$ of type (4.1) is in $(IbP)^\sigma$ by Fubini's theorem provided λ satisfies (4.3) with $q = 1$.

(i) $\Rightarrow$ (ii): By Theorem 2.1 (i) and Remark 2.1 (ii) (resp. Definition 2.2 and Remark 2.2) we have to show that for μ–a.e. $\gamma \in \Gamma$

$$
\begin{aligned}
&E_\mu[F \mid \widehat{\mathcal{B}}_{X\setminus\Lambda}(\Gamma)](\gamma) \\
=\ &\sigma(\Lambda)^{-\gamma(\Lambda)} \int_\Lambda \ldots \int_\Lambda F(\varepsilon_{x_1} + \ldots + \varepsilon_{x_{\gamma(\Lambda)}})\ \sigma(dx_1)\ldots\sigma(dx_{\gamma(\Lambda)}) \qquad (4.4)
\end{aligned}
$$

for all bounded $\mathcal{B}_\Lambda(\Gamma)$–measurable functions $F : \Gamma \to I\!R$ and all $\Lambda \in \mathcal{O}_c(X)$. Here the right hand side of (4.4) is understood to be equal to $F(\emptyset)$ if $\gamma(\Lambda) = 0$. So fix $\Lambda \in \mathcal{O}_c(X)$.

Claim. Let $v \in V_0(\Lambda)$ and let $K_1, \dots, K_M$ be compact subsets of $X \setminus \Lambda$, and $\varphi_1, \dots, \varphi_N \in C_0^\infty(\Lambda)$. Let $G := g_G(N_{K_1}, \dots, N_{K_M})$, $F := g_F(\langle \varphi_1, \cdot \rangle, \dots, \langle \varphi_N, \cdot \rangle)$ with $g_G \in C_b^\infty(I\!R^M)$, $g_F \in C_b^\infty(I\!R^N)$, and $n \in I\!N$. Then

$$\int G \; 1_{\{N_\Lambda = n\}} \nabla_v^\Gamma \; F \, d\mu = - \int G \; 1_{\{N_\Lambda = n\}} \; B_v^{\pi_\sigma} F \, d\mu \; .$$

To prove the claim for $l \in I\!N$, $1 \leq m \leq M$, we choose functions χ_l, $\chi_{ml} \in C_0^\infty(X)$ taking values in $[0,1]$ such that

a) $\chi_l = 1$ on a neighbourhood of supp v, $\chi_l \leq 1_\Lambda$, and $\chi_l \to 1_\Lambda$, $l \to \infty$.

b) $\chi_{ml} = 1$ on K_m, $\chi_{ml} = 0$ in a neighbourhood of supp v, and $\chi_{ml} \to 1_{K_m}, l \to \infty$.

Furthermore, let $g \in C_0^\infty(I\!R)$ such that $1_{\{n\}} \leq g \leq 1_{]n-\frac{1}{2}, n+\frac{1}{2}[}$. Then for every $\gamma \in \Gamma$

$$G_l(\gamma) := g_G(\langle \chi_{1l}, \gamma \rangle, \dots, \langle \chi_{Ml}, \gamma \rangle = G(\gamma)$$

and

$$g_l(\gamma) := g(\langle \chi_l, \gamma \rangle) = 1_{\{N_\Lambda = n\}}(\gamma)$$

for all $l > l(\gamma)$. Furthermore, for all $l \in I\!N$, $1 \leq m \leq M$, we have that $\nabla_v^X \chi_l \; \equiv 0 \equiv \nabla_v^X \; \chi_{ml}$, hence by (3.7)

$$\nabla_v^\Gamma \, g_l = 0 = \nabla_v^\Gamma \, G_l \; .$$

Consequently,

$$\int G \; 1_{\{N_\Lambda = n\}} \; \nabla_v^\Gamma \; F \, d\mu = \lim_{l \to \infty} \int G_l \, g_l \, \nabla_v^\Gamma \; F \, d\mu$$

$$= \; - \lim_{l \to \infty} \left[\int \left(g_l \, \nabla_v^\Gamma \, G_l + G_l \, \nabla_v^\Gamma \, g_l \right) F \, d\mu - \int G_l \, g_l \, F \, B_v^{\pi_\sigma} \, d\mu \right]$$

$$= \; - \int G \, 1_{\{N_\Lambda = n\}} \, F \, B_v^{\pi_\sigma} d\mu \; ,$$

(where we used that $\mu \in (IbP)^\sigma$ in the second step) and the claim is proven.

The claim immediately implies that for F, v as in the claim

$$E_\mu \left[\nabla_v^\Gamma \, F \mid \widehat{\mathcal{B}}_{X \setminus \Lambda}(\Gamma) \right] = - E_\mu \left[B_v^{\pi_\sigma} \, F \mid \widehat{\mathcal{B}}_{X \setminus \Lambda}(\Gamma) \right] \; \mu\text{–a.e. .} \qquad (4.5)$$

Now set $A_1 := \Lambda$ and pick $A_i \in \mathcal{B}_c(X \setminus \Lambda)$, $i \in I\!N$, $i \geq 2$, closed under finite intersections such that

$$T := (N_{A_i})_{i \in I\!N} : \Gamma \longrightarrow Z\!\!Z_+^\infty$$

generates $\widehat{\mathcal{B}}_{X\setminus\Lambda}(\Gamma)$. Disintegrating μ w.r.t. T there exists a probability kernel $\tilde{\mu} : Z\!\!Z_+^\infty \times \mathcal{B}(\Gamma) \to [0,1]$ such that

$$\mu = \tilde{\mu}(\underline{n}, d\gamma)(\mu \circ T^{-1})(d\underline{n}) \ , \tag{4.6}$$

i.e., $\tilde{\mu}(\underline{n}, d\gamma)$ is a *regular* conditional probability corresponding to $E_\mu[\,\cdot \mid T = \underline{n}]$, $\underline{n} \in Z\!\!Z_+^\infty$. From now on all statements depending on $\underline{n}$ below are meant as statements which are true $(\mu \circ T^{-1})$–a.e. It follows by (4.5) and (3.3) that

$$\int \nabla_v^\Gamma F(\gamma)\, \tilde{\mu}(\underline{n}, d\gamma) = -\int \langle \beta_v^\sigma, \gamma \rangle\, F(\gamma)\, \tilde{\mu}(\underline{n}, d\gamma) \tag{4.7}$$

for all $v \in V_0(\Lambda)$ and all F as in the claim. Furthermore, since clearly $\tilde{\mu}(\underline{n}, d\gamma)$ is supported by $\{T = \underline{n}\}$ and since for $n \in I\!N$ $\{\gamma \subset \Lambda \mid \gamma(\Lambda) = n\}$ is isomorphic (as a measurable space) to $\tilde{\Lambda}^n / S_n$, there exists a probability measure $\mu_{\underline{n}}$ on $\tilde{\Lambda}^n$ invariant under S_n such that for all positive $\mathcal{B}(\Gamma)$–measurable functions G on Γ

$$\int G(\gamma)\ \tilde{\mu}(\underline{n}, d\gamma) = \int_{\Lambda^n} G(\sum_{k=1}^n \varepsilon_{x_k} + \tilde{\gamma}_{X\setminus\Lambda})\ \mu_{\underline{n}}(dx_1 \dots dx_n), \tag{4.8}$$

where $\underline{n} = (n, n_2, n_3, \dots)$ and $\tilde{\gamma}_{X\setminus\Lambda}$ is the unique element in $\Gamma_{X\setminus\Lambda}$ such that $\tilde{\gamma}_{X\setminus\Lambda}(A_i) = n_i$ for all $i \geq 2$. By the invariance under S_n it suffices to determine $\mu_{\underline{n}}$ on just one of the $n!$ connected components of $\tilde{\Lambda}^n$. Let $\tilde{\Lambda}^n_\circ$ denote this component. Hence it suffices to determine $\mu_{\underline{n}}$ on $M := \Lambda_1 \times \dots \times \Lambda_n \subset \tilde{\Lambda}^n_\circ$, $\Lambda_i, 1 \leq i \leq n$, pairwise disjoint open subsets of Λ. Therefore, by assumption (A) on σ and a simple disintegration argument, it is enough to show that for all $\varphi_i \in C_0^\infty(\Lambda_i)$ and all $v_i \in V_0(\Lambda_i)$, $1 \leq i \leq n$,

$$\int \varphi_1(x_1) \dots \langle \nabla^X \varphi_i, v_i \rangle_{T(X)}(x_i) \dots \varphi_n(x_n)\ \mu_{\underline{n}}(dx_1 \dots dx_n) \tag{4.9}$$

$$= -\int \beta_{v_i}^\sigma(x_i)\, \varphi_1(x_1) \dots \varphi_n(x_n)\ \mu_{\underline{n}}(dx_1 \dots dx_n)\ .$$

Because then by assumption (A) we see that $\mu_{\underline{n}}$ is up to a constant equal to σ^n on $\Lambda_1 \times \ldots \times \Lambda_n$, hence since $\mu_{\underline{n}}$ is a probability measure on $\tilde{\Lambda}^n$, it follows that

$$\mu_{\underline{n}} = \sigma(\Lambda)^{-n}\,\sigma^n\ ,$$

and (4.4) follows from (4.6) since the case $n = 0$ is trivial. To show (4.9) we fix $i \in \{1, \ldots, n\}$. Choosing F in (4.7) so that $F := \prod_{j=1}^{n} \langle \varphi_j, \cdot\ \rangle$ on $\{\gamma \subset \Lambda | \gamma(\Lambda) = n\}$ we conclude that

$$\int \langle \varphi_1, \gamma\rangle \ldots \langle \nabla^X_{v_i}\varphi_i, \gamma\rangle \ldots \langle \varphi_n, \gamma\rangle\ \tilde{\mu}(\underline{n}, d\gamma)$$

$$= -\int \langle \beta^\sigma_{v_i}, \gamma\rangle \langle \varphi_1, \gamma\rangle \ldots \langle \varphi_n, \gamma\rangle\ \tilde{\mu}(\underline{n}, d\gamma)\ .$$

Hence by (4.8)

$$\int_{\Lambda^n} \prod_{j=1, j\neq i}^{n} (\varphi_j(x_1) + \ldots + \varphi_j(x_n)) \sum_{k=1}^{n} \nabla^X_{v_i}\varphi_i(x_k)\ \mu_{\underline{n}}(dx_1 \ldots dx_n) \tag{4.10}$$

$$= -\int_{\Lambda^n} \sum_{k=1}^{n} \beta^\sigma_{v_i}(x_k) \prod_{j=1}^{n} (\varphi_j(x_1) + \ldots + \varphi_j(x_n))\ \mu_{\underline{n}}(dx_1 \ldots dx_n).$$

Since both integrands are invariant under S_n (4.10) also holds if we only take the integrals over $\tilde{\Lambda}^n_\circ$. But then (4.10) directly turns into (4.9), since φ_i, v_i have support in Λ_i, $1 \leq i \leq n$, which in turn are pairwise disjoint. □

4.2 Ruelle measures

In this subsection we will describe (following essentially [Ru 70]) a class of grand canonical Gibbs measures which appears in Classical Statistical Mechanics of continuous systems, see also [Do 70], [KY 93], [Ru 69]. To this end we consider the case $X = \mathbb{R}^d$ with a measure $\sigma = zm$, where m denotes Lebesgue measure on $\mathbb{R}^d$ and $z > 0$ is the activity parameter. We simplify notations by setting $\pi^z := \pi_{zm}$.

A pair potential is a Lebesgue measurable function $\phi : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ such that $\phi(-x) = \phi(x)$. Any pair potential ϕ defines a potential $\Phi = \Phi_\phi$ as follows: we set $\Phi(\gamma) := 0$, $|\gamma| \neq 2$ and $\Phi(\gamma) := \phi(x - y)$ for $\gamma = \{x, y\} \subset \mathbb{R}^d$.

It is useful to rewrite the conditional energy E_Λ^Φ (see (2.16)) in the following form for $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$:

$$E_\Lambda^\Phi(\gamma) = E_\Lambda^\Phi(\gamma_\Lambda) \; + \; W(\gamma_\Lambda|\gamma_{\Lambda^c}), \tag{4.11}$$

where the term

$$W(\gamma_\Lambda|\gamma_{\Lambda^c}) \; := \sum_{x\in\gamma_\Lambda, y\in\gamma_{\Lambda^c}} \phi(x-y)$$

describes the *interaction energy* between γ_Λ and γ_{Λ^c}. (Here as usual $\Lambda^c := \mathbb{R}^d \setminus \Lambda$). Analogously, we define $W(\gamma'|\gamma'')$ when γ', γ'' are located in disjoint regions.

For every $r = (r^1, \dots, r^d) \in \mathbb{Z}^d$ we define a cube

$$Q_r \; := \; \{x \in \mathbb{R}^d \mid r^i - \frac{1}{2} \leq x^i < r^i + \frac{1}{2}\}.$$

These cubes form a partition of $\mathbb{R}^d$. For any $\gamma \in \Gamma$ we set $\gamma_r := \gamma_{Q_r} = \gamma \cap Q_r, \; r \in \mathbb{Z}^d$. For $N \in \mathbb{N}$ let Λ_N be the cube with side length $2N-1$ centered at the origin in $\mathbb{R}^d$; Λ_N is then a union of $(2N-1)^d$ unit cubes of the form Q_r.

Now we are able to formulate conditions on the interaction.

(S) (*Stability*) There exists $B \geq 0$ such that for any $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and for all $\gamma \in \Gamma_\Lambda$

$$E_\Lambda^\Phi(\gamma) \; \geq \; - B|\gamma|. \tag{4.12}$$

A condition stronger than stability is the following.

(SS) (*Superstability*) There exist $A > 0, \; B \geq 0$ such that if $\gamma \in \Gamma_{\Lambda_N}$ for some N, then

$$E_{\Lambda_N}^\Phi(\gamma) \; \geq \; \sum_{r\in\mathbb{Z}^d} [A|\gamma_r|^2 \; - \; B|\gamma_r|]. \tag{4.13}$$

(LR) (*Lower regularity*) There exists a decreasing positive function $a : \mathbb{N} \to \mathbb{R}_+$ such that

$$\sum_{r\in\mathbb{Z}^d} a(\|r\|) \; < \; \infty \tag{4.14}$$

and for any Λ', Λ'' which are each finite unions of unit cubes of the form Q_r and disjoint, with $\gamma' \in \Gamma_{\Lambda'}$, $\gamma'' \in \Gamma_{\Lambda''}$,

$$W(\gamma'|\gamma'') \geq - \sum_{r', r'' \in \mathbb{Z}^d} a(\|r' - r''\|) \, |\gamma'_{r'}| \cdot |\gamma''_{r''}|. \tag{4.15}$$

Here $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^d$

We say that the pair potential ϕ is *stable* (resp. *superstable*, *lower regular*) if (S) (resp (SS), (LR)) is satisfied. In [Ru 70] there are given many criteria for stability, superstability and lower regularity of a pair potential. For example, if $\phi_1 \geq 0$ is continuous and $\phi_1(0) > 0$, then ϕ_1 is superstable. If ϕ_2 is stable than $\phi = \phi_1 + \phi_2$ is superstable etc. Let us mention also the following Dobrushin-Fisher-Ruelle criterium.

Proposition 4.1 *Let $0 < d_1 < d_2 < +\infty$ and let*

$$s_1 : [0, d_1] \to \mathbb{R} \cup \{+\infty\}, \quad s_2 : [d_2, +\infty) \to \mathbb{R}$$

be positive, decreasing and such that

$$\int_0^{d_1} t^{d-1} s_1(t) dt = +\infty, \; \int_{d_2}^{\infty} t^{d-1} s_2(t) dt < +\infty. \tag{4.16}$$

If the pair potential ϕ is bounded below and satisfies

$$\phi(x) \;\geq\; s_1(\|x\|) \;\; for \;\; \|x\| \leq d_1,$$

$$|\phi(x)| \;\leq\; s_2(\|x\|) \;\; for \;\; \|x\| \geq d_2$$

then ϕ is superstable and lower regular.

Definition 4.1 *A probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ is called tempered if μ is supported by*

$$S_\infty \;=\; \cup_{m=1}^{\infty} S_m,$$

where

$$S_m = \{\gamma \in \Gamma \mid \forall N \in \mathbb{N} \sum_{r \in \Lambda_N \cap \mathbb{Z}^d} |\gamma_r|^2 \leq m^2 |\Lambda_N \cap \mathbb{Z}^d| \, \}. \tag{4.17}$$

By $\mathcal{G}_{gc}^t(z,\phi) \subset \mathcal{G}_{gc}(z,\phi) := \mathcal{G}_{gc}(zm, \Phi_\phi)$ (cf. Subsection 2.3) we denote the set of all tempered grand canonical Gibbs measures (*Ruelle measures* for short). Due to [Ru 70, Sect. 5], $\mathcal{G}_{gc}^t(z,\phi) \subset \cap_{q\in\mathbb{N}} \mathcal{M}_q(\Gamma)$ and it is non-empty for all $z > 0$ and any super-stable, lower regular potential ϕ which satisfies the *integrability condition* (I):

$$\int_{\mathbb{R}^d} |1 - e^{-\phi(x)}| m(dx) \ < \ +\infty. \tag{4.18}$$

Let us note that if ϕ is semi-bounded from below then condition (I) is equivalent to the integrability of ϕ on the set $\mathbb{R}^d \setminus \{\phi \geq 1\}$ if $\{\phi \geq 1\}$ has finite Lebesgue measure. Of course, the stability condition (S) implies the semi-boundedness of ϕ from below.

For a given probability measure μ on $(\Gamma, \mathcal{B}(\Gamma))$ and any $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ we can introduce a probability measure μ_Λ on $(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda))$ as $\mu_\Lambda = \mu \circ p_\Lambda^{-1}$, cf. (2.14). Since there is a decomposition of Γ_Λ analogous to (2.10) for all $\Lambda \in \mathcal{B}_c(\mathbb{R}^d)$ (and not only for $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$), we have an induced decomposition of μ_Λ: $\mu_\Lambda \ = \ \sum_{n=0}^\infty \mu_\Lambda^{(n)}$. In the case of a Ruelle measure μ which corresponds to a potential ϕ with properties (SS), (LR) and (I) we have the representations

$$\mu_\Lambda^{(n)}(d\gamma_n) = \frac{1}{n!} \tau_{\Lambda,\mu}^n(\gamma_n) m_{\Lambda,n}(d\gamma_n),\ \gamma_n \in \Gamma_\Lambda^{(n)},\ n \geq 0,$$

where $\tau_{\Lambda,\mu}^n(\cdot)$ are positive measurable functions on $\Gamma_\Lambda^{(n)}$. They can be considered as positive Lebesgue integrable functions $\tau_{\Lambda,\mu}^n(x_1,\ldots,x_n)$ on Λ^n invariant w.r.t. all permutations from the group S_n. These functions are called the *system of density distributions.* We refer to [Ru 70] for details.

The following theorem gives necessary properties of the so-called correlation functions associated to Ruelle measures, for the proof see [Ru 70].

Theorem 4.2 *Let ϕ be a pair potential satisfying conditions (SS), (LR) and (I). For a given $\mu \in \mathcal{G}_{gc}^t(\phi,z)$ let $\{\tau_{\Lambda,\mu}^n | n \geq 0\}$ be the associated system of density distributions. We define correlation functions by*

$$\varrho_\mu(x_1,\ldots,x_n) = \sum_{k=0}^\infty \frac{1}{k!} \int_{\Lambda^n} \tau_{\Lambda,\mu}^{n+k}(x_1,\ldots,x_{n+k}) m(dx_{n+1}) \ldots m(dx_{n+k}) \tag{4.19}$$

for $x_1, \ldots, x_n \in \Lambda$*. The correlation functions satisfy the Mayer equations*

$$\varrho_\mu(x_1, \ldots, x_n) = \exp[-\sum_{i<j} \phi(x_j - x_i)] \times \tag{4.20}$$

$$\sum_{p=0}^{\infty} \frac{1}{p!} K((x_1, \ldots, x_n), (y_1, \ldots, y_p)) \varrho_\mu(y_1, \ldots, y_p) m(dy_1) \ldots m(dy_p),$$

where

$$K((x_1, \ldots, x_n), (y_1, \ldots, y_p)) = \prod_{j=1}^{p} K((x_1, \ldots, x_n), y_j), \tag{4.21}$$

$$K((x_1, \ldots, x_n), y) = \exp[-\sum_{i=1}^{n} \phi(y - x_j)] \; - \; 1. \tag{4.22}$$

Moreover, there exists a constant $\xi > 0$ *such that for all* n *and* $x_1, \ldots, x_n \in \mathbb{R}^d$ *we have*

$$\varrho_\mu(x_1, \ldots, x_n) \; \leq \; \xi^n. \tag{4.23}$$

4.3 Integration by parts characterization in the interacting case

In this section we shall prove an integration by parts formula for a certain convex subset of canonical Gibbs measures, to be introduced below, which contains the Ruelle measures. In fact, we shall characterize this subset by integration by parts in a way analogous to the free case in Theorem 4.1.

Let us introduce the following condition on the potential ϕ:

(D) (*Differentiability*) $e^{-\phi}$ is weakly differentiable on $\mathbb{R}^d$, ϕ is weakly differentiable on $\mathbb{R}^d \setminus \{0\}$ and the weak gradient $\nabla\phi$ (which is a locally m-integrable function on $\mathbb{R}^d \setminus \{0\}$) considered as an m-a.e. defined function on $\mathbb{R}^d$ satisfies

$$\nabla\phi \in L^1(\mathbb{R}^d, e^{-\phi} m) \cap L^2(\mathbb{R}^d, e^{-\phi} m). \tag{4.24}$$

(We refer to [Y 96] for a similar condition). Note that for many typical potentials in Statistical Physics we have $\phi \in C^\infty(I\!R^d \setminus \{0\})$. For such "regular outside the origin" potentials condition (D) nevertheless does not exclude a singularity at the point $0 \in I\!R^d$.

Lemma 4.1 *Let ϕ be a pair potential satisfying conditions (SS), (LR), (I) and (D). For $n \in I\!N$ let Λ_n denote as before the cube with side $2n-1$ centered at the origin in $I\!R^d$. For any vector field $v \in V_0(I\!R^d)$ we consider the function*

$$\Gamma \ni \gamma \mapsto L^\phi_{v,n}(\gamma) \ := \ - \sum_{\{x,y\}\subset \gamma_{\Lambda_n}} < \nabla\phi(x-y), v(x) - v(y) >_{I\!R^d} . \quad (4.25)$$

Then for any Ruelle measure μ and all $v \in V_0(I\!R^d)$ we have that

$$L^{\phi,\mu}_v \ := \ \lim_{n\to\infty} L^\phi_{v,n} \quad (4.26)$$

exists in $L^2(\mu)$.

Proof. Let $\varphi^{(2)}(x,y) = \varphi^{(2)}(y,x), x,y \in I\!R^d$, be a smooth compactly supported symmetric kernel (e.g. from $\mathcal{D}(I\!R^{2d})$). To this kernel there corresponds the following function on Γ:

$$\Gamma \ni \gamma \mapsto F_{\varphi^{(2)}}(\gamma) := \sum_{\{x,y\}\subset\gamma} \varphi^{(2)}(x,y) \in I\!R.$$

Using a representation for correlation functions via Wick products on configuration spaces and Wick's Theorem (see e.g. [GR 96], [LRS 96], [Re 93]) we have

$$\int_\Gamma F^2_{\varphi^{(2)}}(\gamma)\ \mu(d\gamma) = I_1 + I_2 + I_3\,, \quad (4.27)$$

where

$$\begin{aligned}
I_1 \ &:= \ C_1 \int_{I\!R^{4d}} \varphi^{(2)}(x_1,x_2)\varphi^{(2)}(x_3,x_4)\varrho_\mu(x_1,\ldots,x_4)\ m(dx_1)\ldots m(dx_4) \\
I_2 \ &:= \ C_2 \int_{I\!R^{3d}} \varphi^{(2)}(x_1,x_2)\varphi^{(2)}(x_2,x_3)\varrho_\mu(x_1,x_2,x_3)\ m(dx_1)m(dx_2)m(dx_3) \\
I_3 \ &:= \ C_3 \int_{I\!R^{2d}} |\varphi^{(2)}(x_1,x_2)|^2\varrho_\mu(x_1,x_2)\ m(dx_1)m(dx_2),
\end{aligned}$$

where C_j are constants independent on $\varphi^{(2)}$. Via approximation arguments this formula can be extended to more general kernels $\varphi^{(2)}$ which satisfy existence conditions for integrals in (4.27). In the case considered first of all for all $v \in V_0(I\!R^d)$ we need to show the existence of these integrals for the kernels

$$\varphi_n^{(2)}(x,y) := | < 1_{\Lambda_n}(x) 1_{\Lambda_n}(y) \nabla\phi(x-y), v(x) - v(y) >_{I\!R^d} | \,, \tag{4.28}$$

because $F_{\varphi_n^{(2)}} = L_{v,n}^{\phi}$.

Let us remark that (4.20) (together with the integrability condition (4.18) and bound (4.23)) implies

$$|\varrho_\mu(x_1,\ldots,x_m)| \ \leq \ R_m \exp[-\sum_{i<j} \phi(x_j - x_i)] \tag{4.29}$$

for all $m \in I\!N, \ x_1,\ldots,x_m \in I\!R^d$.

In the case of I_1 we have for fixed $n \in I\!N$ (using the latter estimate and the semiboundness of ϕ)

$$|I_1| \ \leq \ C_1'[\int_{\Lambda_n^2} \|\nabla\phi(x_1 - x_2)\| \, \|v(x_1) - v(x_2)\| e^{-\phi(x_1-x_2)} m(dx_1) m(dx_2)]^2$$

$$\leq C_1(v) \ \ \|\nabla\phi\|^2_{L^1(\exp(-\phi)dm)}.$$

Analogously (using Young's inequality) we obtain

$$|I_2| \ \leq \ C_2(v) \ \ \|\nabla\phi\|^2_{L^1(\exp(-\phi)dm)}.$$

I_3 can be estimated as follows:

$$|I_3| \ \leq \ C_3' \int_{\Lambda_n^2} \|\nabla\phi(x_1 - x_2)\|^2 \|v(x_1) - v(x_2)\|^2 e^{-\phi(x_1-x_2)} \ m(dx_1) m(dx_2)$$

$$\leq C_3(v) \ \ \|\nabla\phi\|^2_{L^2(\exp(-\phi)dm)}.$$

(The right hand side is finite due to the assumptions of the lemma.) Since $\Lambda_n \nearrow I\!R^d$ as $n \to \infty$, it easily follows that $(L_{v,n}^{\phi})_{n \in I\!N}$ is a Cauchy sequence in $L^2(\mu)$. $\square$

Remark 4.2 *(i) An analysis of the proof shows that condition (D) can be weakened by replacing $L^1(e^{-\phi}m)$, $L^2(e^{-\phi}m)$ by $L^1_w(e^{-\phi}m)$, $L^2_w(e^{-\phi}m)$ respectively.*

(ii) It is an easy exercise to check that (D) implies that

$$\nabla e^{-\phi} \;=\; -\nabla\phi\; e^{-\phi} \;\; m - a.e. \text{ on } I\!\!R^d.$$

We shall also use the following condition:

(LS) (*Local summability*) For all $\Lambda \in \mathcal{O}_c(I\!\!R^d)$ and all $\gamma \in S_\infty$
$\lim_{n\to\infty} \sum_{y\in\gamma_{\Lambda_n\setminus\Lambda}} \nabla\phi(\cdot - y)$ exists in $L^1_{loc}(\Lambda, m)$.

Remark 4.3 *In view of the definition of S_∞, (LS) obviously is a condition on the behaviour of $\nabla\phi$ at infinity, which is trivially fulfilled if ϕ has compact support.*

In order to prove the main result of this section (i.e., Theorem 4.3 below) we need to introduce the convex set $\mathcal{P}_q$, $q \in [1,\infty)$, of all probability measures μ on $(\Gamma, \mathcal{B}(\Gamma))$ satisfying the following properties:

($\mathcal{P}$ 1) $\mu(S_\infty) = 1$, i.e., μ is tempered.

($\mathcal{P}$ 2) $\mu \in \mathcal{M}_q(\Gamma)$ (cf. (4.2)).

($\mathcal{P}$ 3) $\gamma \mapsto R(\gamma) := \exp[\sum_{r\in Z\!\!\!Z^d} |\gamma_r| a(\|r\|)] \in \cap_{p>1} L^p(\mu)$ (cf. Condition (LR) for the definition of $a(\cdot)$).

($\mathcal{P}$ 4) For all $v \in V_0(I\!\!R^d)$ there exists

$$L_v^{\phi,\mu} := \; L^1(\mu) - \lim_{n\to\infty} L^\phi_{v,n} \;\text{ and }\; L_v^{\phi,\mu} \in L^q(\mu),$$

where $L^\phi_{v,n}$ is as defined in (4.25).

For $\mu \in \mathcal{P}_q$ and $v \in V_0(I\!\!R^d)$ we define

$$B_v^{\phi,\mu} \;:=\; L_v^{\phi,\mu} + \;<\operatorname{div} v, \cdot>\;\; (\in L^1(\mu)). \tag{4.30}$$

Remark 4.4 *(i) If $\gamma \in S_\infty \cap \{R < \infty\}$, where R is as in ($\mathcal{P}$ 3), by [Ru70, Lemma 5.1 and the proof of Proposition 5.2(a)] we have that in (2.19) for $z > 0$*

$$\Pi_\Lambda^{zm,\Phi}(\gamma,\; \{N_\Lambda = \gamma(\Lambda)\}) \;>\; 0$$

for all $\Lambda \in \mathcal{O}_c(I\!\!R^d)$. Hence all $\hat\Pi_\Lambda^{zm,\Phi}(\gamma,\cdot)$ are measures independent of z. Hence $\mathcal{G}_c(zm,\Phi) \cap \mathcal{P}_1$ is independent of $z > 0$.

(ii) We also note that if supp$[|\phi|m]$ is compact, i.e., the interaction has finite range, then the sum in (4.25) has only finitely many summands also if γ_{Λ_n} is replaced by γ. Hence all $L_v^{\phi,\mu}$ and hence all $B_v^{\phi,\mu}$ (introduced in (4.30)) for all $\mu \in \mathcal{P}_1$, $v \in V_0(I\!R^d)$ have this finite sum as a common version. Furthermore, as is easily checked, we can drop ($\mathcal{P}$ 3) in the definition of $\mathcal{P}_q$ in this case and all results below remain true.

Lemma 4.2 *$\mathcal{G}_{gc}^t(z,\phi) \subset \mathcal{G}_c(m,\Phi) \cap \mathcal{P}_2$ for all $z > 0$.*

Proof. Let $z > 0$, $\mu \in \mathcal{G}_{gc}(z,\phi)$. We already know that $\mu \in \mathcal{G}_c(m,\Phi)$ by (2.21) and that μ satisfies ($\mathcal{P}$ 2). By Lemma 4.1 also ($\mathcal{P}$ 4) with $q = 2$ holds and ($\mathcal{P}$ 3) follows from [Ru 70, Proposition 5.2 (a) and its proof]. $\square$.

Now we are prepared to prove the main result of this section.

Theorem 4.3 *Let ϕ be a pair potential satisfying properties (SS), (LR), (I), (D) and (LS). Then for $\mu \in \mathcal{P}_1$ the following assertions are equivalent:*

(i) $\mu \in \mathcal{G}_c(m,\Phi)$

(ii) For all $v \in V_0(I\!R^d)$ and all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ the following integration by parts formula holds:

$$\int_\Gamma \nabla_v^\Gamma F\ G\ d\mu \ = \ -\int_\Gamma F\ \nabla_v^\Gamma G\ d\mu \ - \ \int_\Gamma F\ G\ B_v^{\phi,\mu}\ d\mu. \qquad (4.31)$$

In particular, any Ruelle measure (with interaction given by ϕ) satisfies (ii).

Remark 4.5 *(i) We note that if even $\mu \in \mathcal{P}_2$, then (ii) in Theorem 4.3 is equivalent to*

$$(\nabla_v^\Gamma)^* \ = \ - \ \nabla_v^\Gamma \ - \ B_v^{\phi,\mu} \quad \text{for all } v \in V_0(I\!R^d) \qquad (4.32)$$

as an operator equality on the domain $\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ in $L^2(\mu)$, where $(\nabla_v^\Gamma)^*$ denotes the adjoint of ∇_v^Γ on $L^2(\mu)$.

(ii) In analogy with Definition 3.2 we call $B_v^{\phi,\mu}$ the logarithmic derivative *of $\mu \in \mathcal{G}_c(m,\Phi)$ along v.*

Proof of Theorem 4.3. (i) $\rightarrow$ (ii). Suppose (i) holds and let $v \in V_0(I\!R^d)$. Since ∇_v^Γ satisfies the product rule we may assume that $G \equiv 1$. Let

$F = g_F(< \varphi_1, \cdot >, \dots, < \varphi_N, \cdot >) \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$. Choose $\Lambda \in \mathcal{O}_c(I\!R^d)$ such that

$$\cup_{i=1}^N \text{supp } \varphi_i \cup \text{supp } v \ \subset \ \Lambda.$$

Then by (2.20) and Remarks 2.1 (i), (ii), 2.2

$$\int \nabla_v^\Gamma F \, d\mu \ = \ \int \hat{\Pi}_\Lambda^{zm,\Phi}(\nabla_v^\Gamma F) \, d\mu = \tag{4.33}$$

$$\int \frac{\int_{\Lambda^{\gamma(\Lambda)}} \nabla_v^\Gamma F(\gamma_{\Lambda^c} + \sum_{k=1}^{\gamma(\Lambda)} \varepsilon_{x_k}) \exp[-E_\Lambda^\Phi(\gamma_{\Lambda^c} + \sum_{k=1}^{\gamma(\Lambda)} \varepsilon_{x_k})] \ m(dx_1)\dots m(dx_{\gamma(\Lambda)})}{\int_{\Lambda^{\gamma(\Lambda)}} \exp[-E_\Lambda^\Phi(\gamma_{\Lambda^c} + \sum_{k=1}^{\gamma(\Lambda)} \varepsilon_{x_k})] \ m(dx_1)\dots m(dx_{\gamma(\Lambda)})} \mu(d\gamma).$$

Fix $n \in I\!N$ and $\gamma \in \{\gamma' \in \Gamma \mid \gamma'(\Lambda) = n\} \cap \{R < \infty\} \cap S_\infty$ (where R is as in condition ($\mathcal{P}$ 3)). By [Ru 70, Lemma 5.1 and the proof of Proposition 5.2 (a)] we then know that (for the cubes Λ_N as before)

$$\exp[-W(\sum_{k=1}^{\gamma(\Lambda)} \varepsilon_{x_k} \mid \gamma_{\Lambda_N \setminus \Lambda})] \ \le \ (R(\gamma))^{\gamma(\Lambda)} \tag{4.34}$$

for all $N \in I\!N$ and all $x_1, \dots, x_{\gamma(\Lambda)} \in \Lambda$ and

$$\lim_{N\to\infty} W(\sum_{k=1}^{\gamma(\Lambda)} \varepsilon_{x_k} \mid \gamma_{\Lambda_N \setminus \Lambda}) \ = \ W(\sum_{k=1}^{\gamma(\Lambda)} \varepsilon_{x_k} \mid \gamma_{\Lambda^c}) \tag{4.35}$$

for all $x_1, \dots, x_{\gamma(\Lambda)} \in \Lambda$. Hence by Lebesgue's dominated convergence theorem the numerator of the integrand in (4.33) for such γ is equal to

$$\lim_{N\to\infty} \int_{\Lambda^n} \nabla_v^\Gamma F(\gamma_{\Lambda^c} + \sum_{k=1}^n \varepsilon_{x_k}) \exp[-E_\Lambda^\Phi(\gamma_{\Lambda_N \setminus \Lambda} + \sum_{k=1}^n \varepsilon_{x_k})] \ m(dx_1)\dots m(dx_n).$$

Since ϕ is lower bounded, by (4.11) and Remark 4.2 this is equal to

$$- \lim_{N\to\infty} \sum_{l=1}^n \int_{\Lambda^n} F(\sum_{k=1}^n \varepsilon_{x_k})(\text{div } v(x_l) +$$

$$< \nabla_{x_l}[\sum_{1 \le i < j}^n \phi(x_i - x_j) + \sum_{i=1}^n \sum_{y \in \gamma_{\Lambda_N \setminus \Lambda}} \phi(x_i - y)], v(x_l) >_{I\!R^d})$$

$$\times \exp[-E^{\Phi}_{\Lambda}(\gamma_{\Lambda_N\setminus\Lambda} + \sum_{k=1}^{n} \varepsilon_{x_k})] \; m(dx_1)\ldots m(dx_n) =$$

$$-\lim_{N\to\infty}\int_{\Lambda^n} F(\sum_{k=1}^{n}\varepsilon_{x_k})(\sum_{i=1}^{n}\mathrm{div}v(x_i) + \sum_{1\le i<j}^{n} < \nabla\phi(x_i - x_j), v(x_i) - v(x_j) >_{\mathbb{R}^d} +$$

$$\sum_{i=1}^{n}\sum_{y\in\gamma_{\Lambda_N\setminus\Lambda}} < \nabla\phi(x_i - y), v(x_i) >_{\mathbb{R}^d}) \exp[-E^{\Phi}_{\Lambda}(\gamma_{\Lambda_N\setminus\Lambda} + \sum_{k=1}^{n}\varepsilon_{x_k})] \; m(dx_1)\ldots m(dx_n) =$$

$$-\int_{\Lambda^n} F(\sum_{k=1}^{n}\varepsilon_{x_k})(\sum_{i=1}^{n}\mathrm{div}v(x_i) + \sum_{1\le i<j}^{n} < \nabla\phi(x_i - x_j), v(x_i) - v(x_j) >_{\mathbb{R}^d} +$$

$$\sum_{i=1}^{n}\sum_{y\in\gamma_{\Lambda^c}} < \nabla\phi(x_i - y), v(x_i) >_{\mathbb{R}^d}) \exp[-E^{\Phi}_{\Lambda}(\gamma_{\Lambda^c} + \sum_{k=1}^{n}\varepsilon_{x_k})] \; m(dx_1)...m(dx_n),$$

where we used (LS) in the last step and ∇_{x_l} denotes the gradient w.r.t. the variable $x_l \in \Lambda$. Thus by (4.33), (4.30) and ($\mathcal{P}$ 1), ($\mathcal{P}$ 2), ($\mathcal{P}$ 4)

$$\int \nabla^{\Gamma}_{v} F \, d\mu \; = \; -\int \hat{\Pi}^{zm,\Phi}_{\Lambda}(F \; B^{\phi,\mu}_{v}) \; d\mu \; = \; -\int F \; B^{\phi,\mu}_{v} \; d\mu,$$

and (ii) holds.

(ii) $\to$ **(i)**: Assume (ii) holds. For the reader's convenience we shall keep the presentation of this part of the proof as close as possible to the proof of Theorem 4.1 (ii) $\to$ (i), also refered to below as "the free case". By Remark 2.1 (ii) (resp. Definition 2.2 and Remark 2.2) and Remark 4.1 (i) we have to show that for all $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$ and all $F = g_F(< \varphi_1, \cdot >, \ldots, < \varphi_N, \cdot >) \in \mathcal{F}C^{\infty}_{b}(\mathcal{D}, \Gamma)$ with $\varphi_1, \ldots, \varphi_N \in C^{\infty}_0(\Lambda)$ we have that for μ-a.e. $\gamma \in S_\infty$

$$E_\mu[F \mid \hat{\mathcal{B}}_{\Lambda^c}(\Gamma)](\gamma) = \tag{4.36}$$

$$\int_{\Lambda^{\gamma(\Lambda)}} F(\sum_{k=1}^{\gamma(\Lambda)}\varepsilon_{x_k}) \; c_\Lambda(\gamma_{\Lambda^c}) \; \exp[-E^{\Phi}_{\Lambda}(\gamma_{\Lambda^c} + \sum_{k=1}^{\gamma(\Lambda)}\varepsilon_{x_k})] \; m(dx_1)\ldots m(dx_{\gamma(\Lambda)}),$$

where

$$c_\Lambda(\gamma_{\Lambda^c})^{-1} := \int_{\Lambda^{\gamma(\Lambda)}} \exp[-E^{\Phi}_{\Lambda}(\gamma_{\Lambda^c} + \sum_{k=1}^{\gamma(\Lambda)}\varepsilon_{x_k})] \; m(dx_1)\ldots m(dx_{\gamma(\Lambda)})$$

and we have also used the fact that $\tilde{Z}_{\Lambda}^{zm,\Phi}(\gamma)$ in Remark 2.1 (ii) is finite for μ-a.e. $\gamma \in S_\infty$. The latter is true because (4.34) holds for μ-a.e. $\gamma \in S_\infty$ by ($\mathcal{P}$ 3) and ϕ is lower bounded.

Fix $\Lambda \in \mathcal{O}_c(\mathbb{R}^d)$, $v \in V_0(\Lambda)$ and F as in (4.36). By exactly the same arguments that proved (4.5) in the proof of Theorem 4.1, we obtain that

$$E_\mu[\nabla_v^\Gamma F \mid \hat{\mathcal{B}}_{\Lambda^c}(\Gamma)] \;=\; -E_\mu[B_v^{\phi,\mu}\; F \mid \hat{\mathcal{B}}_{\Lambda^c}(\Gamma)] \;\; \mu-\text{a.e.} \tag{4.37}$$

Now proceeding as in the free case, we introduce $T = (N_{A_i})_{i\in\mathbb{N}}$, $\tilde{\mu}(\underline{n}, d\gamma)$, $\underline{n} \in \mathbb{Z}_+^\infty$, such that

$$\mu \;=\; \tilde{\mu}(\underline{n}, d\gamma)(\mu \circ T^{-1})(d\underline{n}), \tag{4.38}$$

and $\mu_{\underline{n}}$ to obtain that

$$\int \nabla_v^\Gamma F(\gamma)\; \tilde{\mu}(\underline{n}, d\gamma) \;=\; -\int B_v^{\phi,\mu}(\gamma)\; F(\gamma)\; \tilde{\mu}(\underline{n}, d\gamma) \tag{4.39}$$

and

$$\int G(\gamma)\; \tilde{\mu}(\underline{n}, d\gamma) \;=\; \int_{\Lambda^n} G(\sum_{k=1}^n \varepsilon_{x_k} + \tilde{\gamma}_{\Lambda^c})\; \mu_{\underline{n}}(dx_1 \ldots dx_n), \tag{4.40}$$

where $\underline{n} = (n, n_2, n_3, \ldots)$ with n assumed to be non-zero and $\tilde{\gamma}_{\Lambda^c}$ is the unique element in Γ_{Λ^c} such that $\tilde{\gamma}_{\Lambda^c}(A_i) = n_i$ for all $i \geq 2$. G is any positive $\mathcal{B}(\Gamma)$-measurable function on Γ. (Again above all statements are meant to be true for $\mu \circ T^{-1}$-a.e. $\underline{n} \in \mathbb{Z}_+^\infty$). As in the free case it suffices to determine $\mu_{\underline{n}}$ on $M := \Lambda_1 \times \cdots \times \Lambda_n \subset \tilde{\Lambda}_0^n$, where $\Lambda_i, 1 \leq i \leq n$, are pairwise disjoint open subsets of Λ and $\tilde{\Lambda}_0^n$ is any fixed connected component of $\tilde{\Lambda}^n$.

Claim. For all $\varphi_i \in C_0^\infty(\Lambda_i)$ and all $v_i \in V_0(\Lambda_i)$, $1 \leq i \leq n$, we have

$$\int \varphi_1(x_1) \ldots < \nabla\varphi_i(x_i), v_i(x_i) >_{\mathbb{R}^d} \ldots \varphi_n(x_n)\; \mu_{\underline{n}}(dx_1 \ldots dx_n) = \tag{4.41}$$

$$-\int B_{v_i}^{\phi,\mu}(\tilde{\gamma}_{\Lambda^c} + \sum_{k=1}^n \varepsilon_{x_k})\; \varphi_1(x_1) \ldots \varphi_n(x_n)\; \mu_{\underline{n}}(dx_1 \ldots dx_n).$$

Suppose the claim is proven. Then since for μ-a.e. $\gamma \in S_\infty$ by ($\mathcal{P}$ 4), $(x_1, \ldots, x_n) \mapsto B_{v_i}^{\phi,\mu}(\gamma_{\Lambda^c} + \sum_{k=1}^n \varepsilon_{x_k})$ is in $L^1(\Lambda_1 \times \cdots \times \Lambda_n, \mu_{\underline{n}})$, it follows by standard regularity theory that $\mu_{\underline{n}}$ is absolutely continuous w.r.t. m^n on $\Lambda_1 \times \cdots \times \Lambda_n$ with weakly differentiable Radon-Nikodym derivative. By (D), (LS) for all $\gamma \in S_\infty$, $(x_1, \ldots, x_n) \mapsto B_{v_i}^{\phi,\mu}(\gamma_{\Lambda^c} + \sum_{k=1}^n \varepsilon_{x_k})$ is in $L^1_{loc}(\Lambda_1 \times \cdots \times \Lambda_n, m^n)$. Hence as an immediate consequence of the proof of [BR 95, Theorem 6.2 and Lemma 6.4]

$$\mu_{\underline{n}}(dx_1 \ldots dx_n) \;=\; c_\Lambda(\tilde{\gamma}_{\Lambda^c}) \exp[-E_\Lambda^\Phi(\tilde{\gamma}_{\Lambda^c} + \sum_{k=1}^n \varepsilon_{x_k})]\; m(dx_1) \ldots m(dx_n) \tag{4.42}$$

since by analogous considerations as in the proof of "(i) $\to$ (ii)", (4.41) also holds with $\mu_{\underline{n}}$ replaced by the probability measure on the right hand side of (4.42). But (4.42) immediately implies (4.36) by (4.38), since the case $n = 0$ is trivial.

To prove the claim we fix $i \in \{1, \ldots, n\}$. Choosing F in (4.39) so that $F = \prod_{j=1}^n < \varphi_j, \cdot >$ on $\{\gamma \subset \Lambda \mid \gamma(\Lambda) = n\}$ we conclude that

$$\int < \varphi_1, \gamma > \ldots < \nabla_{v_i}^\Gamma \varphi_i, \gamma > \ldots, < \varphi_n, \gamma > \tilde{\mu}(\underline{n}, d\gamma) =$$

$$- \int B_{v_i}^{\phi,\mu}(\gamma) \; < \varphi_1, \gamma > \ldots < \varphi_n, \gamma > \tilde{\mu}(\underline{n}, d\gamma).$$

Hence by (4.40)

$$\int_{\Lambda^n} \prod_{j=1, j\neq i}^n (\varphi_j(x_1) + \ldots + \varphi_j(x_n)) \sum_{k=1}^n \nabla_{v_i} \varphi_i(x_k)\; \mu_{\underline{n}}(dx_1 \ldots dx_n) = \tag{4.43}$$

$$- \int_{\Lambda^n} B_{v_i}^{\phi,\mu}(\tilde{\gamma}_{\Lambda^c} + \sum_{k=1}^n \varepsilon_{x_k}) \prod_{j=1}^n (\varphi_j(x_1) + \ldots + \varphi_j(x_n))\; \mu_{\underline{n}}(dx_1 \ldots dx_n)$$

and the claim follows (by exactly the same arguments as at the end of the proof of the free case). □.

Remark 4.6 *It is interesting and important to note that from the DLR-equation the quasi-invariance of Ruelle measures w.r.t. the group* $\mathit{Diff}_0(\mathbb{R}^d)$ *follows. More precisely, let* ϕ *satisfy the conditions of Theorem 4.3 and let* $\mu \in \mathcal{G}_{gc}^t(z, \phi)$. *Then by direct computations using the DLR-equation one can show that for any* $\psi \in \mathit{Diff}_0(\mathbb{R}^d)$ *the measures* $\mu \circ \psi$ *and* μ *are equivalent.*

Moreover, let $\Lambda \in \mathcal{O}_c(I\!R^d)$ be such that ψ is trivial outside Λ. Then

$$\frac{d(\mu \circ \psi)(\gamma)}{d\mu(\gamma)} = \exp[E_\Lambda^\Phi(\gamma) - E_\Lambda^\Phi(\psi(\gamma))] \frac{d(\pi^z \circ \psi)(\gamma)}{d\pi^z(\gamma)}, \tag{4.44}$$

where

$$\frac{d(\pi^z \circ \psi)(\gamma)}{d\pi^z(\gamma)} = \prod_{x \in \gamma} J_m^\psi(x) \exp[z \int (1 - J_m^\psi(x)) \, m(dx)], \tag{4.45}$$

and $J_m^\psi = det[\nabla \psi]$ is the Jacobian determinant of ψ (with respect to Lebesgue measure).

Let us note that the Radon-Nikodym derivative in Remark 4.6 depends explicitly on the activity parameter $z > 0$. It gives us the possibility to characterize *grand canonical* Gibbs measures via corresponding quasi-invariance properties (and Radon-Nikodym derivatives) under the action of the diffeomorphysm group. Note that in contrast to the previous, since the logarithmic derivatives $B_v^{\phi,\mu}$ along $v \in V_0(I\!R^d)$ in Theorem 4.3 do not depend on $z > 0$, an integration by parts characterization could only be proved for *canonical* Gibbs measures (see Remark 4.4 (i)).

We introduce the expression

$$E_{\text{rel}}^\Phi(\psi(\gamma)|\gamma) := E_\Lambda^\Phi(\psi(\gamma)) - E_\Lambda^\Phi(\gamma) = \sum_{\{x,y\} \subset \gamma} [\phi(\psi(x) - \psi(y)) - \phi(x - y)], \tag{4.46}$$

where we have used any $\Lambda \in \mathcal{O}_c(I\!R^d)$ such that ψ is trivial outside of Λ. In analogy with the lattice case (see e.g. [Gr 80]) it is natural to call E_{rel}^Φ the *relative energy* functional. From physical point of view this functional describes the variation of the (formal) potential energy functional $E^\Phi(\gamma) = \sum_{\{x,y\} \subset \gamma} \phi(x - y)$ when the configuration γ is locally deformed by the diffeomorpism ψ. As a result, formula (4.37) can be rewritten as

$$\frac{d(\mu \circ \psi)(\gamma)}{d\mu(\gamma)} = \exp[- E_{\text{rel}}^\Phi(\psi(\gamma)|\gamma)] \frac{d(\pi^z \circ \psi)(\gamma)}{d\pi^z(\gamma)}. \tag{4.47}$$

This new description of grand canonical Gibbs measures and its consequences will be the subject of detailed analysis in the forthcoming paper [AKR 97c].

Example 4.1 There are many (easily checkable) sufficient conditions on a pair potential ϕ which ensure properties (SS) and (LR) (see e.g. [Ru 69], [Ru 70]). Conditions (I) and (D) are formulated in terms of integrability properties of ϕ and $\nabla\phi$ respectively. But condition (LS) seems to be more complicated to check in applications. Let us show that the assumption

$$\|\nabla\phi(x)\| \;\leq\; \frac{C}{\|x\|^{\alpha}},\;\; \|x\| \geq R, \tag{4.48}$$

for some R, $C>0$, $\alpha > d+1$, together with (D) implies (LS).

To this end for a given $\Lambda \in \mathcal{O}_c(I\!\!R^d)$ we consider any compact $K \subset \Lambda$. We want to show that for any $\gamma \in S_\infty$ the series $\sum_{y\in\gamma_{\Lambda^c}} \nabla\phi(\cdot - y)$ converges absolutely in $L^1(K,m)$. Note that for any point $y \in \Lambda^c$ we have $\nabla\phi(\cdot - y) \in L^1(K,m)$ due to (D) (because $\mathrm{dist}(y,K) > 0$). Therefore, it is enough to consider the sum over $y \in \gamma_{\Lambda^c_{N_0}}$ for some $N_0 \in I\!\!N$ such that $N_0 - D_K > 0$ (where $D_K := \mathrm{diam} K$):

$$\sum_{n\geq N_0} \sum_{y\in\gamma_{\Lambda_{n+1}\setminus\Lambda_n}} \int_K \|\nabla\phi(x-y)\| m(dx) \leq$$

$$C_1 \sum_{n\geq N_0} (n-D_K)^{-\alpha} \left(\sum_{y\in\gamma_{\Lambda_{n+1}\setminus\Lambda_n}} 1 \right) \leq$$

$$C_2(\gamma,K) \sum_{n\geq N_0} \frac{n^d(\log(n+1))^{\frac{1}{2}}}{(n-D_K)^{\alpha}}$$

for suitable constants C_1, $C_2(\gamma,K)$, and the latter sum converges due to the assumption that $\alpha > d+1$. We have used here the following property of S_∞: for any $\gamma \in S_\infty$ there exists $M = M(\gamma) > 0$ such that $|\gamma_i| \leq M(\log_+ \|i\|)^{\frac{1}{2}}$, $i \in Z\!\!\!Z^d$, where $\log_+ r = \max\{1, \log r\}$ (see e.g. [KY 93]). This estimate gives then

$$|\gamma_{\Lambda_{n+1}\setminus\Lambda_n}| \;\leq\; |\gamma_{\Lambda_{n+1}}| \;\leq\; \sum_{i\in Z\!\!\!Z^d\cap\Lambda_{n+1}} |\gamma_i| \;\leq$$

$$M\; (2n+2)^d\; (\log(n+1))^{\frac{1}{2}}.$$

A concrete example of a pair potential which is especially important in atomic and molecular physics is given for $d = 3$ by the so-called Lennard-Jones potential

$$\phi_{a,b}(x) \;=\; \frac{a}{r^{12}(x)} \;-\; \frac{b}{r^{6}(x)},\;\; r(x) := (< x, x >_{\mathbb{R}^d})^{\frac{1}{2}}, x \in \mathbb{R}^d \setminus \{0\},$$

where $a, b > 0$. This potential obviously satisfies the conditions of Proposition 4.1 and is, therefore, super-stable and lower regular. Furthermore, we have

$$\nabla\phi_{a,b}(x) \;=\; \frac{6x}{r^{8}(x)} \;-\; \frac{12x}{r^{14}(x)}$$

and it is easy to see that (D) is also true. Also (4.48) holds with $\alpha := 7 > d+1 = 4$. So, the condition (LS) is fulfilled and we can apply Theorem 4.3 and all the consequences discussed in Section 6 below to $\phi = \phi_{a,b}$.

$\square$

5 Dirichlet forms and corresponding stochastic dynamics

In this section we again consider the general situation described in Subsection 2.1 (in particular, X is a C^∞–Riemannian manifold, so $\Gamma := \Gamma_X$), and we adopt the notation of Sections 2 and 3. Our aim now is to describe how the theory of Dirichlet forms can be applied to all cases discussed above.

For simplicity we shall formulate the results for the general class of probability measures in $\mathcal{M}_2(\Gamma)$ satisfying an integration by parts formula so that the measures in Section 4 are just special cases. So, we fix $\mu \in \mathcal{M}_2(\Gamma)$ (cf. (4.2)) such that for all $v \in V_0(X)$ there exists $B_v^\mu \in L^2(\mu)$ such that for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$

$$\int \nabla_v^\Gamma F \; G \; d\mu = -\int F \; \nabla_v^\Gamma G \; d\mu - \int F \; G \; B_v^\mu \; d\mu. \tag{5.1}$$

As before by the product rule for ∇_v^Γ we may take $G \equiv 1$ in (5.1).

5.1 Dirichlet forms on configuration spaces

Define for $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$

$$\mathcal{E}_\mu^\Gamma(F, G) := \int_\Gamma \langle \nabla^\Gamma F(\gamma), \nabla^\Gamma G(\gamma) \rangle_{T_\gamma(\Gamma)} \; \mu(d\gamma). \tag{5.2}$$

We emphasize that by (3.7), $\langle \nabla^\Gamma F, \nabla^\Gamma G \rangle_{T(\Gamma)} \in L^1(\mu)$ only because $\mu \in \mathcal{M}_2(\Gamma) \subset \mathcal{M}_1(\Gamma)$. By (5.1), ∇^Γ respects the μ–classes $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)^\mu$ determined by $\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ (i.e., $\nabla^\Gamma F = \nabla^\Gamma G$ μ–a.e. provided $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ satisfy $F = G$ μ–a.e.). Hence $(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)^\mu)$ is a densely defined positive definite symmetric bilinear form on $L^2(\mu)$.

In analogy with Corollary 3.1, and Remark 3.1 for $V = \sum_{j=1}^N G_j \cdot v_j \in \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we define

$$\mathrm{div}_\mu^\Gamma V := \sum_{j=1}^N \left(\nabla_{v_j}^\Gamma G_j + B_{v_j}^\mu \; G_j \right) \tag{5.3}$$

and for $F \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$

$$\Delta_\mu F := \mathrm{div}_\mu^\Gamma \nabla^\Gamma F. \tag{5.4}$$

(We note that $\nabla^\Gamma F \in \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ by (3.7)).

Proposition 5.1 *$(\mathcal{E}_\mu^\Gamma, \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)^\mu)$ is closable on $L^2(\mu)$ and its closure $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ is a symmetric Dirichlet form which is* conservative *(i.e., $1 \in D(\mathcal{E}_\mu^\Gamma)$ and $\mathcal{E}_\mu^\Gamma(1, 1) = 0$). Its generator, denoted by $-H_\mu^\Gamma$, is the Friedrichs' extension of Δ_μ.*

Proof. Because of (3.7) for all $F, G \in \mathcal{F}C_b^\infty(\mathcal{D}, \Gamma)$ we have by (5.1)–(5.3)

$$\int_\Gamma \langle \nabla^\Gamma F, \nabla^\Gamma G \rangle_{T(\Gamma)} \; d\mu = -\int_\Gamma \Delta_\mu F \; G \; d\mu$$

which implies closability and the last part of the assertion. The Dirichlet property immediately follows from the chain rule for ∇^Γ on $\mathcal{F}C_b^\infty$ and the conservativity is obvious. (We refer to [MR 92, Chap. I and Chap. II, Sections 2, 3] for the terminology and details.)

$\square$

Clearly, ∇^Γ extends to all of $D(\mathcal{E}_\mu^\Gamma)$. We shall denote this extension by ∇_μ^Γ.

5.2 Corresponding diffusions

The desired processes in general will live on the bigger state space $\ddot{\Gamma}_X$ consisting of all $\mathbb{Z}_+$-valued Radon measures on X (which is Polish, see e.g. [Ka 75]). To state the corresponding existence result precisely we need Definition 5.1 below. For abbreviation below we sometimes set $(\mathcal{E}, D(\mathcal{E})) := (\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$. Since $\Gamma \subset \ddot{\Gamma}_X$ and $\mathcal{B}(\ddot{\Gamma}_X) \cap \Gamma = \mathcal{B}(\Gamma)$, we can consider μ as a measure on $(\ddot{\Gamma}_X, \mathcal{B}(\ddot{\Gamma}_X))$ and correspondingly $(\mathcal{E}, D(\mathcal{E}))$ as a Dirichlet form on $L^2(\ddot{\Gamma}_X, \mu)$.

Definition 5.1 (*cf.* [MR 92, Ch.III, Definitions 2.1 and 3.2]).

(i) *A sequence $(F_n)_{n\in\mathbb{N}}$ of closed subsets of $\ddot{\Gamma}_X$ is called an $\mathcal{E}$-nest, if*

$$\bigcup_{n\in\mathbb{N}} \{F \in D(\mathcal{E}) | F = 0 \ \mu\text{-}a.e. \ on \ \ddot{\Gamma}_X \setminus F_n\}$$

is dense in $D(\mathcal{E})$ w.r.t. $\mathcal{E}_1^{1/2} (:= [\mathcal{E} + (\ , \ )_{L^2(\mu)}]^{1/2})$.

(ii) *A set $N \subset \ddot{\Gamma}_X$ is called $\mathcal{E}$-exceptional, if $N \subset \ddot{\Gamma}_X \setminus \bigcup_{n\in N} F_n$ for some $\mathcal{E}$-nest $(F_n)_{n\in\mathbb{N}}$. We say that a property of points in $\ddot{\Gamma}_X$ holds $\mathcal{E}$-quasi-everywhere (abbeviated $\mathcal{E}$-q.e.) if it holds outside some $\mathcal{E}$-exceptional set.*

(iii) *A function $f : \ddot{\Gamma}_X \to \mathbb{R}$ is called $\mathcal{E}$-quasi-continuous if there exists an $\mathcal{E}$-nest $(F_n)_{n\in\mathbb{N}}$ such that the restriction $f_{|F_n}$ of f to F_n is continuous for all $n \in \mathbb{N}$.*

We recall that the notion "$\mathcal{E}$-q.e." is much finer than "μ-a.e." (cf. [MR 92, Exercise 2.6 (ii)]).

The following is a special case of the main result in [AKMR 97].

Theorem 5.1 *$(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ is quasi–regular.*

As a consequence we obtain

Corollary 5.1 *$(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ is* local *(i.e., $\mathcal{E}_\mu^\Gamma(F, G) = 0$ provided $F, G \in D(\mathcal{E}_\mu^\Gamma)$ with $\mathrm{supp}(|F|\mu) \cap \mathrm{supp}(|G|\mu) = \emptyset$).*

Proof. Since ∇_μ^Γ satisfies the product rule on bounded functions in $D(\mathcal{E}_\mu^\Gamma)$, the proof for locality is exactly the same as that in [MR92, Examples 1.12 (ii)]. (We also refer to the beautiful work [Sch 95] for a complete discussion of locality of Dirichlet forms.)

$\square$

As a consequence of Theorem 5.1, Corollary 5.1 and [MR 92] we immediately obtain

Theorem 5.2 *There exists a conservative diffusion process*

$$\mathbf{M} = (\mathbf{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, (\mathbf{\Theta}_t)_{t\geq 0}, (\mathbf{X}_t)_{t\geq 0}, (\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}_X})$$

on $\ddot{\Gamma}_X$ (*cf.* [Dy 65]) *which is properly associated with* $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$, *i.e., for all (*$\mu$*-versions of)* $F \in L^2(\ddot{\Gamma}_X, \mu)$ *and all* $t > 0$ *the function*

$$\gamma \ \mapsto \ p_t F(\gamma) \ := \ \int_\Omega F(X_t)\ d\mathbf{P}_\gamma, \ \ \gamma \in \ddot{\Gamma}_X, \tag{5.5}$$

is an $\mathcal{E}_\mu^\Gamma$*-quasi-continuous version of* $\exp(-tH_\mu^\Gamma)F$. $\mathbf{M}$ *is up to* μ*-equivalence unique* (*cf.* [MR 92, Ch. IV, Sect. 6]). *In particular,* $\mathbf{M}$ *is* μ*–symmetric* (*i.e.,* $\int G\ p_t F d\mu = \int F\ p_t G\ d\mu$ *for all* $F, G : \ddot{\Gamma}_X \to I\!R_+, \mathcal{B}(\ddot{\Gamma}_X)$*-measurable*) *and has* μ *as an invariant measure.*

Proof. By Theorem 5.1 and Corollary 5.1 the proof follows directly from [MR 92, Ch.V, Theorem 1.11]. $\square$

In the above theorem $\mathbf{M}$ is canonical, i.e., $\mathbf{\Omega} = C([0,\infty) \to \ddot{\Gamma}_X)$, $\mathbf{X}_t(\omega) := \omega(t)$, $t \geq 0$, $\omega \in \mathbf{\Omega}, (\mathbf{F}_t)_{t\geq 0}$ together with $\mathbf{F}$ is the corresponding minimum completed admissible family (cf. [Fu 80, Sect.4.1]) and $\mathbf{\Theta}_t$, $t \geq 0$, are the corresponding natural time shifts.

Remark 5.1 (i) Theorems 5.1, 5.2 and Corollary 5.1 generalize corresponding results in [Y 96]

(ii) $\mathbf{M}$ is also called the corresponding *stochastic dynamics*.

Theorem 5.3 $\mathbf{M}$ from Theorem 5.1 is the (up to μ–equivalence, cf. [MR92, Definition 6.3]) unique diffusion process having μ as an invariant measure and solving the martingale problem for $(-H_\mu^\Gamma, D(H_\mu^\Gamma))$, i.e., for all $G \in D(H_\mu^\Gamma)$ $(\supset \mathcal{F}C_b^\infty(\mathcal{D},\Gamma))$

$$\tilde{G}(\mathbf{X}_t) - \tilde{G}(\mathbf{X}_0) + \int_0^t H_\mu^\Gamma G(\mathbf{X}_s)\, ds,\ t \geq 0,$$

is an $(\mathbf{F}_t)$–martingale under $\mathbf{P}_\gamma$ (hence starting at γ) for $\mathcal{E}_\mu^\Gamma$–q.e. $\gamma \in \ddot{\Gamma}_X$. (Here $\tilde{G}$ denotes a quasi–continuous version of G, cf. [MR92, Chap. IV, Proposition 3.3]).

Proof. This follows immediately from [AR 95, Theorem 3.5]. □

Remark 5.2 In fact, the uniqueness statement in Theorem 5.3 can be strengthened, as follows:
$\mathbf{M}$ is (up to μ–equivalence) unique among all right processes $\mathbf{M}' = (\mathbf{\Omega}', \mathbf{F}', (\mathbf{F}'_t)_{t\geq 0}, (\mathbf{\Theta}'_t)_{t\geq 0}, (\mathbf{X}')_{t\geq 0}, (\mathbf{P}'_\gamma)_{\gamma\in\ddot{\Gamma}_X})$ on $\ddot{\Gamma}_X$ having μ as a sub–invariant measure and being such that for all $G \in D(H_\mu^\Gamma)$

$$\tilde{G}(\mathbf{X}'_t) - \tilde{G}(\mathbf{X}'_0) + \int_0^t H_\mu^\Gamma G(\mathbf{X}'_s)\, ds,\ t \geq 0,$$

is and $(\mathbf{F}'_t)$–martingale under $\mathbf{P}'_\mu := \int \mathbf{P}'_\gamma\ \mu(d\gamma)$.

The proof is analogous to that of [AR 95, Theorem 3.5] except for its ending where one has to use the recent result of P. Fitzsimmons in [Fi 96] in order to be able to apply the arguments in [MR 92, Chap. IV, Section 6] to deduce the result.

By the results in Section 4 we obtain:

Corollary 5.2 *Theorems 5.1 – 5.3 and Corollary 5.1 hold if:*

(i) $\mu = \mu_{\lambda,\sigma}$ *where* σ, λ *are as in Section 4 such that* λ *satisfies (4.3) with* $l = 2$.

(ii) $X = \mathbb{R}^d$, $\Phi = \Phi_\phi$ *is as in Theorem 4.3 and* $\mu \in \mathcal{G}_c(m,\Phi) \cap \mathcal{P}_2$ $(\supset \mathcal{G}_{gc}^t(z,\phi) =$ *Ruelle measures for all* $z > 0)$.

Remark 5.3 *Corollary 5.2 (i) is essentially already contained in [AKR 97b, Section 6.1]*

6 Ergodicity

We are interested in conditions that imply the irreducibility of the Dirichlet form $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ or equivalently the time–ergodicity of its corresponding diffusion (when started with distribution μ) in the cases described in Corollary 5.2. We recall all necessary definitions in Subsection 6.1 and then prove a general result for Dirichlet forms on configuration spaces as in Subsection 5.1 characterizing irreducibility in terms of "extremality" (see below). In Subsection 6.3 we then apply this to the cases in Corollary 5.2 in particular, to Ruelle measures. In this chapter (except for Subsection 6.3 (b), (c)) we again consider the general situation with X a C^∞–Riemannian manifold as in Section 5.

6.1 Preliminaries: Irreducibility and weak Sobolev spaces

Fix μ as in Subsection 5.1 and let $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ be as defined there. Since $D(\mathcal{E}_\mu^\Gamma)$ is obtained as a completion w.r.t. the norm

$$D(\mathcal{E}_\mu^\Gamma) \ni F \mapsto \left(\mathcal{E}_\mu^\Gamma(F,F) + \int F^2 \, d\mu\right)^{1/2} =: \|F\|_{H_0^{1,2}(\Gamma,\mu)}, \tag{6.1}$$

we set

$$H_0^{1,2}(\Gamma,\mu) := D(\mathcal{E}_\mu^\Gamma), \tag{6.2}$$

i.e., the $(1,2)$–*Sobolev space on* Γ.

We recall that $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ is *irreducible* if for all $F \in H_0^{1,2}(\Gamma,\mu)$ with $\mathcal{E}_\mu^\Gamma(F,F) = 0$ (or equivalently $\nabla_\mu^\Gamma F = 0$), it follows that $F = \text{const}$.

Lemma 6.1 *$(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ is irreducible if and only if for all* bounded *$F \in H_0^{1,2}(\Gamma,\mu)$ with $\mathcal{E}_\mu^\Gamma(F,F) = 0$ it follows that $F = \text{const}$.*

Proof. Suppose $F \in H_0^{1,2}(\Gamma,\mu)$ with $\mathcal{E}_\mu^\Gamma(F,F) = 0$. Then (e.g. by [MR92, Chap. I, Proposition 4.17]) we have that for all $n \in I\!N$

$$F_n := (F \wedge n) \vee (-n) \in H_0^{1,2}(\Gamma,\mu)$$

and $F_n \underset{n\to\infty}{\longrightarrow} F$ w.r.t. $\| \ \|_{H_0^{1,2}(\Gamma,\mu)}$. Furthermore, $\mathcal{E}_\mu^\Gamma(F_n,F_n) \le \mathcal{E}_\mu^\Gamma(F,F)$ for all $n \in I\!N$ (cf. e.g. [MR 92, Chap. I, Theorem 4.12]). Now the assertion follows easily. $\square$

We recall the following known result characterizing the irreducibility of $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$. Let $\mathbf{M} = (\boldsymbol{\Omega}, \mathbf{F}, (\mathbf{F}_t)_{t\ge 0}, (\boldsymbol{\Theta}_t)_{t\ge 0}, (\mathbf{X}_t)_{t\ge 0}, (\mathbf{P}_\gamma)_{\gamma\in\ddot{\Gamma}_X})$ be the diffusion process associated with $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ (cf. Theorem 5.2). As usual we set

$$\mathbf{P}_\mu := \int \mathbf{P}_\gamma \ \mu(d\gamma) \ .$$

Theorem 6.1 *The following assertions are equivalent:*

(i) $(\mathcal{E}_\mu^\Gamma, D(\mathcal{E}_\mu^\Gamma))$ *is irreducible.*

(ii) $(e^{-H_\mu^\Gamma t})_{t>0}$ *is irreducible (i.e., if* $G \in L^2(\mu)$ *such that* $e^{-H_\mu^\Gamma t}(GF) = G\, e^{-H_\mu^\Gamma t} F$ *for all* $F \in L^\infty(\mu)$, $t > 0$, *then* $G = const.$*).*

(iii) If $F \in L^2(\mu)$ *such that* $e^{-H_\mu^\Gamma t}F = F$ *for all* $t > 0$, *then* $F = const.$*.*

(iv) $(e^{-H_\mu^\Gamma t})_{t>0}$ *is ergodic (i.e.,*

$$\int \left(e^{-H_\mu^\Gamma t}F - \int F \ d\mu \right)^2 d\mu \to 0 \ \textit{as } t \to \infty \textit{ for all } F \in L^2(\mu))$$

(v) If $F \in D(H_\mu^\Gamma)$ *with* $H_\mu^\Gamma F = 0$, *then* $F = const.$ *("uniqueness of ground state").*

(vi) $\mathbf{P}_\mu$ *is (time) ergodic (i.e., every bounded* $\mathbf{F}$*–measurable function* $G : \Omega \to I\!R$, *which is* $\boldsymbol{\Theta}_t$*–invariant for all* $t \ge 0$, *is constant* $\mathbf{P}_\mu$*–a.e.*

In this case, if $(p_t)_{t>0}$ is as defined in (5.5), then $\lim_{t\to\infty} p_t F = \int f \, d\mu$ $\mathcal{E}_\mu^\Gamma$–q.e. for all bounded $\mathcal{B}(\Gamma)$–measurable functions $F : \ddot{\Gamma} \to I\!R$.

Proof. The proof of the equivalence of (i)–(v) is exactly analogous to that of [AKR 97a, Proposition 2.3]. The equivalence with (vi) and the last assertion follow by the regularization method in [MR 92, Chap. VI, Sect. 1] directly from the main results in [Fu 82], [Fu 83]. $\square$

We shall now construct an extension of $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ which is the exact analogue on Γ of a weak Sobolev space on a finite dimensional manifold. This is essentially a special case of a construction by A. Eberle in [Eb 95] slightly modified.

We first note that (as is easy to check) $\mathcal{VFC}_b^\infty(\mathcal{D},\Gamma)$ is dense in $L^2(\Gamma \to T(\Gamma),\mu)$, i.e., the Hilbert space of μ–square integrable sections in $T(\Gamma)$. Let $((\mathrm{div}_\mu^\Gamma)^{*,\mu}, D((\mathrm{div}_\mu^\Gamma)^{*,\mu}))$ be the adjoint of $(\mathrm{div}_\mu^\Gamma, \mathcal{VFC}_b^\infty(\mathcal{D},\Gamma))$ as an operator from $L^2(\Gamma \to T(\Gamma),\mu)$ to $L^2(\mu)$.

Remark 6.1 As an adjoint the operator $((\mathrm{div}_\mu^\Gamma)^{*,\mu}, D(\mathrm{div}_\mu^\Gamma)^{*,\mu})$ is automatically closed, and by definition it is an operator from $L^2(\Gamma,\mu)$ to $L^2(\Gamma \to T(\Gamma),\mu)$. Furthermore, again by definition, $G \in L^2(\Gamma,\mu)$ belongs to $D((\mathrm{div}_\mu^\Gamma)^{*,\mu})$ if and only if there exists $V_G \in L^2(\Gamma \to T(\Gamma),\mu)$ such that

$$\int G \ \mathrm{div}_\mu^\Gamma \ V \ d\mu = -\int \langle V_G, V\rangle_{T(\Gamma)} \ d\mu \text{ for all } V \in \mathcal{VFC}_b^\infty(\mathcal{D},\Gamma). \tag{6.3}$$

In this case $(\mathrm{div}_\mu^\Gamma)^{*,\mu} G = V_G$.

Because of Remark 6.1 we set

$$W^{1,2}(\Gamma,\mu) := D((\mathrm{div}_\mu^\Gamma)^{*,\mu}), \quad d^\mu := (\mathrm{div}_\mu^\Gamma)^{*,\mu} \tag{6.4}$$

and think of $W^{1,2}(\Gamma,\mu)$ as a weak $(1,2)$–Sobolev space on Γ with norm

$$W^{1,2}(\Gamma,\mu) \ni G \mapsto \|G\|_{W^{1,2}(\Gamma,\mu)} := \left(\int \langle d^\mu G, d^\mu G\rangle_{T(\Gamma)} d\mu + \int G^2 \, d\mu\right)^{1/2}. \tag{6.5}$$

By (5.1) it follows from Remark 6.1 that

$$\mathcal{F}C_b^\infty(\mathcal{D},\Gamma)^\mu \subset W^{1,2}(\Gamma,\mu) \text{ and } d^\mu = \nabla^\Gamma \text{ on } \mathcal{F}C_b^\infty(\Gamma,\mu), \tag{6.6}$$

hence the densely defined positive definite symmetric bilinear form

$$(F,G) \mapsto \int \langle d^\mu F, d^\mu G\rangle_{T(\Gamma)} d\mu,$$

with domain $W^{1,2}(\Gamma,\mu)$ extends $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ and is, therefore, denoted by $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$.

Proposition 6.1 *Let $F \in H_0^{1,2}(\Gamma,\mu) \cap L^\infty(\mu)$ and $G \in W^{1,2}(\Gamma,\mu) \cap L^\infty(\mu)$. Then $FG \in W^{1,2}(\Gamma,\mu)$ and*

$$d^\mu(FG) = F \ d^\mu G + G \ d^\mu F. \tag{6.7}$$

Proof. Let $V \in \mathcal{VFC}_b^\infty(\mathcal{D},\Gamma)$ and assume first that $F \in \mathcal{FC}_b^\infty(\mathcal{D},\Gamma)$. Then by the product rule for ∇^Γ and (5.3) for all $G \in W^{1,2}(\Gamma,\mu) \cap L^\infty(\mu)$

$$\begin{aligned}\int FG \operatorname{div}_\mu^\Gamma V \, d\mu &= \int G \operatorname{div}_\mu^\Gamma (FV)\, d\mu - \int G \langle \nabla^\Gamma F, V\rangle_{T(\Gamma)}\, d\mu \\ &= -\int \langle d^\mu G, FV\rangle_{T(\Gamma)}\, d\mu - \int \langle G d^\mu F, V\rangle_{T(\Gamma)}\, d\mu\end{aligned}$$

where we used (6.3), (6.4) and (6.6) in the last step. Since $\mathcal{VFC}_b^\infty(\mathcal{D},\Gamma)$ is dense in $L^2(\Gamma \to T(\Gamma),\mu)$, the assertion follows provided $F \in \mathcal{FC}_b^\infty(\mathcal{D},\Gamma)$. The case where $F \in H_0^{1,2}(\Gamma,\mu) \cap L^\infty(\mu)$ is then an immediate consequence, since $\mathcal{FC}_b^\infty(\mathcal{D},\Gamma)$ is dense in $H_0^{1,2}(\Gamma,\mu)$ w.r.t. $\| \ \|_{H_0^{1,2}(\Gamma,\mu)}$ by definition. □

Proposition 6.2 *Suppose that* $(\Delta_\mu, \mathcal{FC}_b^\infty(\mathcal{D},\Gamma))$ *(cf.* (5.4)*) is* essentially self–adjoint *on* $L^2(\Gamma,\mu)$ *(i.e., its closure is self–adjoint). Then*

$$H_0^{1,2}(\Gamma,\mu) = W^{1,2}(\Gamma,\mu).$$

Proof. For all $F \in \mathcal{FC}_b^\infty(\mathcal{D},\Gamma)$ we have that

$$\nabla^\Gamma F \subset \mathcal{VFC}_b^\infty(\mathcal{D},\Gamma).$$

Hence for all $G \in W^{1,2}(\Gamma,\mu)$ by (6.4) and (6.6)

$$\begin{aligned}\mathcal{E}_\mu^\Gamma(G,F) &= \int \langle d^\mu G, \nabla^\Gamma F\rangle_{T(\Gamma)} d\mu \\ &= -\int G \ \Delta_\mu F \ d\mu \ .\end{aligned}$$

This implies that the generator of $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$ on $L^2(\Gamma,\mu)$, which is self–adjoint, extends $(\Delta_\mu, \mathcal{FC}_b^\infty(\mathcal{D},\Gamma))$ (cf. e.g. [MR 92, Chap. I, Proposition 2.16]), hence it also extends its self–adjoint closure. Since self–adjoint operators cannot strictly extend each other, they must coincide. □

Remark 6.2 According to [AKR 97 b, Theorem 4.2 and Remark 4.4 (i)] $(\Delta_\mu, \mathcal{FC}_b^\infty(\mathcal{D},\Gamma))$ is essentially self–adjoint on $L^2(\Gamma,\mu)$, if $\mu = \mu_{\lambda,\sigma}$ as in Corollary 5.2 (i), X is complete and $|\beta^\sigma|_{T(X)} \in L^p_{loc}(m)$ for some $p > \dim(X)$. In case μ is a Ruelle measure we also expect the assumption in Proposition

6.2 to be satisfied for a large class of pair potentials ϕ. This will be the subject of study in a forthcoming paper.

In general it is not clear whether $H_0^{1,2}(\Gamma,\mu)$ coincides with $W^{1,2}(\Gamma,\mu)$. In view of Lemma 6.1 we define $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$ to be *irreducible* if for all bounded $F \in W^{1,2}(\Gamma,\mu)$ with $\mathcal{E}_\mu^\Gamma(F,F)=0$, it follows that $F = const$. However, unlike as for $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ we *cannot* drop the term "bounded" in this definition, since it is not clear whether the larger bilinear form $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$ is also a Dirichlet form. We emphasize here that the Dirichlet property was heavily used in the proof of Lemma 6.1. In particular, we do not know whether an analogue of Theorem 6.1 for $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$ holds

Remark 6.3 We obviously have that the irreducibility of $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$ implies that of $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$.

6.2 A general result on irreducibility

Suppose for every $v \in V_0(X)$ we are given a sequence $(B_{v,n})_{n\in\mathbb{N}}$ of $\mathcal{B}(\Gamma)$–measurable $\mathbb{R}$–valued functions on Γ. Let $(\mathrm{IbP})^B$ denote the set of all $\mu \in \mathcal{M}_2(\Gamma)$ such that

(IbP1) $B_v^\mu := L^1(\mu) - \lim\limits_{n\to\infty} B_{v,n}$ exists and $B_v^\mu \in L^2(\mu)$ for all $v \in V_0(X)$.

(IbP2) $\displaystyle\int \nabla_v^\Gamma F \, d\mu = -\int F \; B_v^\mu \, d\mu$ for all $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$ and all $v \in V_0(X)$.

Obviously, $(\mathrm{IbP})^B$ is a convex set. Let $\mathrm{ex}(\mathrm{IbP})^B$ denote the set of its extreme points. Clearly, if $\mu_1,\mu_2 \in (\mathrm{IbP})^B$ such that $\mu_1-\mu_2$ is a positive measure on $(\Gamma,\mathcal{B}(\Gamma))$, then $[(\mu_1-\mu_2)(\Gamma)]^{-1}(\mu_1-\mu_2) \in (\mathrm{IbP})^B$. For every $\mu \in (\mathrm{IbP})^B$ both $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ and $(\mathcal{E}, W^{1,2}(\Gamma,\mu))$ are defined according to Subsections 5.1 and 6.1.

Theorem 6.2 *Let* $\mu \in (\mathrm{IbP})^B$*. Then the following assertions are equivalent:*

(i) $\mu \in ex(\mathrm{IbP})^B$

(ii) $\{\nu \in (\mathrm{IbP})^B \mid \nu = \rho\cdot\mu$ *for some bounded,* $\mathcal{B}(\Gamma)$*–measurable function* $\rho : \Gamma \to \mathbb{R}_+\} = \{\mu\}$

(iii) $(\mathcal{E}_\mu^\Gamma, W^{1,2}(\Gamma,\mu))$ *is irreducible.*

Proof. (i) $\Rightarrow$ (ii): Assume (i) holds. Let $\rho : \Gamma \to I\!R_+$, bounded, $\mathcal{B}(\Gamma)$–measurable such that $\nu := \rho \cdot \mu \in (\mathrm{IbP})^B$, and let $M := \sup_{\gamma \in \Gamma} \rho(\gamma)$. Define

$$\mu_1 := \frac{M-\rho}{M-1} \cdot \mu \ .$$

Then $\mu_1 \in (\mathrm{IbP})^B$ and $\mu = \frac{M-1}{M}\mu_1 + \frac{1}{M}\nu$. By assumption (i) it follows that $\mu_1 = \nu$ which implies $\rho = 1$, and (ii) is proved.

(ii) $\Rightarrow$ (i): Assume (ii) holds. Let $\mu_1, \mu_2 \in (\mathrm{IbP})^B$ and $t \in (0,1)$ such that $\mu = t\mu_1 + (1-t)\mu_2$. Then $\mu_1, \mu_2 \in (\mathrm{IbP})^B$, and they are both absolutely continuous w.r.t. μ with bounded densities. By assumption (ii) it follows that $\mu_1 = \mu = \mu_2$. Consequently, $\mu \in \mathrm{ex}(\mathrm{IbP})^B$.

(ii) $\Rightarrow$ (iii): Assume (ii) holds. Let $G \in W^{1,2}(\Gamma,\mu) \cap L^\infty(\mu)$ such that $\mathcal{E}_\mu^\Gamma(G,G) = 0$. We have to show that $G = const$. Since $1 \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma) \subset W^{1,2}(\Gamma,\mu)$ and $d^\mu 1 = \nabla_\mu^\Gamma 1 = 0$, replacing G by $G - \mathrm{essinf}\, G$ we may assume that $G \geq 0$, and, in addition, that $\int G \, d\mu = 1$. Define $\nu := G \cdot \mu$. Then $\nu \in \mathcal{M}_2(\Gamma)$ and since $d^\mu G = 0$, Proposition 6.1 and (IbP2) imply that for all $v \in V_0(X)$ and all $F \in \mathcal{F}C_b^\infty(\mathcal{D},\Gamma)$

$$\begin{aligned} \int \nabla_v^\Gamma F \, d\nu &= \int \langle d^\mu(FG), v\rangle_{T(\Gamma)} \, d\mu \\ &= -\int F \, B_v^\mu \, d\nu \end{aligned} \tag{6.8}$$

where we used (6.3) in the last step. Since G is bounded, clearly $B_{v,n} \to B_v^\mu$ as $n \to \infty$ in $L^1(\nu)$ and $B_v^\mu \in L^2(\mu)$. So, $\nu \in (\mathrm{IbP})^B$ and assumption (ii) implies that $G = 1$, hence (iii) is proved.

(iii) $\Rightarrow$ (ii): Assume (iii) holds. Let $\rho : \Gamma \to I\!R_+$, bounded, $\mathcal{B}(\Gamma)$–measurable such that $\nu := \rho \cdot \mu \in (\mathrm{IbP})^B$. Then $B_{v,n} \to B_v^\mu$ as $n \to \infty$ in $L^1(\nu)$ and $B_v^\mu \in L^2(\mu)$, hence by (IbP2)

$$\int \mathrm{div}_\mu^\Gamma V \, \rho \, d\mu = 0 \quad \text{for all } V \in \mathcal{V}\mathcal{F}C_b^\infty(\mathcal{D},\Gamma).$$

Hence by Remark 6.1 and (6.4) it follows that

$$\rho \in W^{1,2}(\Gamma,\mu) \text{ and } d^\mu \rho = 0,$$

i.e., $\mathcal{E}_\mu^\Gamma(\rho,\rho)=0$. By assumption (iii) it follows that $\rho\equiv 1$ and (ii) is proved. $\square$

Corollary 6.1 *Let $\mu\in \mathrm{ex}(\mathrm{IbP})^B$. Then $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma,\mu))$ is irreducible and all equivalent assertions in Theorem 6.1 hold.*

Proof. Theorem 6.2 and Remark 6.3. $\square$

6.3 Applications to mixed Poisson measures, canonical Gibbs measures, and Ruelle measures

We start with the free case.

(a) Mixed Poisson measures
We consider the situation described in Subsection 4.1 (i.e., the general "manifold–case" with $\phi\equiv 0$). So, our canonical Gibbs measures are the mixed Poisson measures $\mu_{\lambda,\sigma}$ (see (4.1)).

We assume that σ satisfies condition (A) of Theorem 4.1, which is e.g. the case if $\beta_v^\sigma\in L^1_{loc}(m)$ (cf. Remark 4.1 (iii)).

Obviously, if $(\mathrm{IbP})^B$ is defined as in Subsection 6.2 with $B_{v,n}:=B_v^{\pi_\sigma}$, $n\in I\!N$, $v\in V_0(X)$, then

$$\begin{aligned}(\mathrm{IbP})^B=(\mathrm{IbP})_2^\sigma &= \{\mu_{\lambda,\sigma}\mid \lambda \text{ satisfies (4.3) with } l=2\}\\ &= \mathcal{G}_c(\sigma)\cap\mathcal{M}_2(\Gamma)\ , \end{aligned}\tag{6.9}$$

where $(\mathrm{IbP})_2^\sigma$ denotes the set of all $\mu\in(\mathrm{IbP})^\sigma\cap\mathcal{M}_2(\Gamma)$ such that $\gamma\mapsto\langle\langle\beta^\sigma,v\rangle_{T(X)},\gamma\rangle$ is μ–square integrable for all $v\in V_0(X)$ and where the second and third equalities immediately follow from Theorem 4.1 resp. Theorem 2.1.

As before, for a convex set $\mathcal{K}$ of measures on $(\Gamma,\mathcal{B}(\Gamma))$ we denote the set of its extreme points by $\mathrm{ex}\mathcal{K}$.

Lemma 6.2

$$\begin{aligned}\mathrm{ex}(\mathrm{IbP})^B &= \mathrm{ex}\{\mu_{\lambda,\sigma}\mid \lambda \textit{ satisfies (4.3) with } l=2\}\\ &= \{\pi_{z\sigma}\mid z\in I\!R_+\}\end{aligned}$$

Proof. The first equality follows from (6.9). To prove the second we note that since

$$\begin{aligned} & \{\mu_{\lambda,\sigma} \mid \lambda \text{ satisfies (4.3) with } l=2\} \\ = \ & \{\mu_{\lambda,\sigma} \mid \lambda \text{ any probability measure on } (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\} \cap \mathcal{M}_2(\Gamma), \end{aligned}$$

it is straightforward to check that

$$\begin{aligned} & \mathrm{ex}\{\mu_{\lambda,\sigma} \mid \lambda \text{ satisfies (4.3) with } l=2\} \\ \subset \ & \mathrm{ex}\{\mu_{\lambda,\sigma} \mid \lambda \text{ any probability measure on } (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))\}. \end{aligned}$$

Since $\pi_{z\sigma} \in \mathcal{M}_2(\Gamma)$ for all $z \in \mathbb{R}_+$, the second equality in the assertion follows by Theorem 2.1 (i) and (ii). □

Theorem 6.3 *Let* $\mu \in \mathcal{G}_c(\sigma) \cap \mathcal{M}_2(\Gamma)$ *(or by* (6.9) *equivalently* $\mu = \mu_{\lambda,\sigma}$ *as in (4.1) with* λ *satisfying (4.3) for* $l = 2$*). Then the following assertions are equivalent:*

(i) $\mu = \pi_{z\sigma}$ *for some* $z \in \mathbb{R}_+$.

(ii) $(\mathcal{E}^\Gamma, W^{1,2}(\Gamma, \mu))$ *is irreducible.*

Furthermore, the assertion in Corollary 6.1 hold for $\mu = \pi_{z\sigma}$ *and any* $z \in \mathbb{R}_+$.

Proof. (6.9), Lemma 6.2 and Theorem 6.2 □

Theorem 6.4 *Assume that the following condition holds:*

$$X \text{ is complete and for all } v \in V_0(X),\ \beta_v^\sigma \in L^p_{loc}(m) \text{ for some } p > \dim(X). \tag{6.10}$$

Let μ *be as in Theorem 6.3. Then* $H_0^{1,2}(\Gamma, \mu) = W^{1,2}(\Gamma, \mu)$. *In particular, equivalent are:*

(i) $\mu = \pi_{z\sigma}$ *for some* $z \in \mathbb{R}_+$.

(ii) $(\mathcal{E}_\mu^\Gamma, H_0^{1,2}(\Gamma, \mu))$ *is irreducible.*

Furthermore, all equivalent assertions in Theorem 6.1 hold for $\mu = \pi_{z\sigma}$, $z \in I\!R_+$.

Proof. [AKR 97b, Theorem 4.2 and Remark 4.4(i)] and Proposition 6.2 imply that $H_0^{1,2}(\Gamma,\mu) = W^{1,2}(\Gamma,\mu)$. The remaining assertions are then immediate by Theorem 6.3. $\square$

Remark 6.4 (6.10) can be relaxed by condition (C) in [AKR 97b, Proposition 4.1].

(b) Canonical Gibbs measures

We now consider the situation described in Subsections 4.2 and 4.3. In particular, $X = I\!R^d$. We assume that the pair potential ϕ satisfies (SS), (LR), (I), (D) and (LS).

Obviously, if $(\mathrm{IbP})^B$ is defined as in Subsection 6.2 with $B_{v,n} := L^{\phi}_{v,n} + \langle \operatorname{div} v, \cdot\rangle$, $n \in I\!N$, $v \in V_0(I\!R^d)$ (and $L^{\phi}_{v,n}$ as in (4.25)), then by Theorem 4.3

$$(\mathrm{IbP})_2^B = \mathcal{G}_c(m,\Phi)_2 \ , \tag{6.11}$$

where for a set $\mathcal{K}$ of probability measures on $(\Gamma, \mathcal{B}(\Gamma))$ we set

$$\mathcal{K}_2 := \mathcal{K} \cap \mathcal{P}_2$$

with $\mathcal{P}_2$ as in Subsection 4.3.

Lemma 6.3 *For any convex set* $\mathcal{K}$ *of probability measures on* $(\Gamma, \mathcal{B}(\Gamma))$

$$\mathrm{ex}(\mathcal{K} \cap \mathcal{P}_2) = \mathrm{ex}\mathcal{K} \cap \mathcal{P}_2 \ .$$

The proof of Lemma 6.3 is straightforward, hence omitted.

Theorem 6.5 *Let* $\mu \in \mathcal{G}_c(m,\Phi)_2$. *Then the following assertions are equivalent:*

(i) $\mu \in \mathrm{ex}\mathcal{G}_c(m,\Phi)$.

(ii) $(\mathcal{E}^{\Gamma}_{\mu}, W^{1,2}(\Gamma,\mu))$ *is irreducible.*

Furthermore, the assertion in Corollary 6.1 holds for $\mu \in \mathrm{ex}\mathcal{G}_c(m,\Phi)_2$.

Proof. (6.11), Lemma 6.3 and Theorem 6.2. $\square$.

Remark 6.5 Note that since $\emptyset \neq \mathcal{G}^t_{gc}(z,\phi) \subset \mathcal{G}_c(m,\Phi)_2$ for all $z > 0$, it immediately follows from Remark 2.3 that $\mathrm{ex}\mathcal{G}_c(m,\Phi)_2 \neq \emptyset$.

(c) Ruelle measures

We are still in the situation of Subsection 6.3 (b), but assume, in addition, the following:

$$\mathrm{ex}\mathcal{G}^t_{gc}(z,\phi) \subset \mathrm{ex}\mathcal{G}_c(m,\Phi) \text{ for all } z > 0. \tag{6.12}$$

Remark 6.6 By [Ge 79, Theorem 6.14] condition (6.12) is e.g. fulfilled if $\mathrm{supp}[|\phi| \cdot m]$ is compact.

Theorem 6.6 *Let $z > 0$ and $\mu \in \mathcal{G}^t_{gc}(z,\phi)$, i.e., μ is a Ruelle measure. Then the following assertions are equivalent:*

(i) $\mu \in \mathrm{ex}\mathcal{G}^t_{gc}(z,\phi)$.

(ii) $(\mathcal{E}^\Gamma_\mu, W^{1,2}(\Gamma,\mu))$ *is irreducible.*

Furthermore, the assertion in Corollary 6.1 holds for $\mu \in \mathrm{ex}\mathcal{G}^t_{gc}(z,\phi)$.

Proof. Since $\mathcal{G}^t_{gc}(z,\phi) \subset \mathcal{G}_c(m,\Phi)_2$ by Lemma 4.2, all assertions follow from Theorem 6.5 and condition (6.12). $\square$

By [Do 70] (see also [Ru 70]) there exists $z_0 > 0$ such that

$$\#\mathcal{G}^t_{gc}(z,\phi) = 1 \text{ for all } z < z_0.$$

As an immediate consequence of Theorem 6.6 we thus obtain:

Corollary 6.2 *Let $z < z_0$ and $\mu \in \mathcal{G}^t_{gc}(z,\phi)$. Then $(\mathcal{E}^\Gamma_\mu, W^{1,2}(\Gamma,\mu))$ is irreducible and the assertion in Corollary 6.1 holds for μ. In particular, the corresponding stochastic dynamics started with μ is (time) ergodic.*

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7 References

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