On the Power Method in Max Algebra

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Abstract

The eigenvalue problem for an irreducible nonnegative matrix $A = [a_{ij}]$ in the max algebra system is $A \otimes x = \lambda x$, where $(A \otimes x)_i = \max\limits_j (a_{ij} x_j)$ and λ turns out to be the maximum circuit geometric mean, $\mu(A)$. A power method algorithm is given to compute $\mu(A)$ and eigenvector x. This method generalizes and simplifies an algorithm due to Braker and Olsder. The algorithm is developed by using results on the convergence of max powers of A. These results are proved using nonnegative matrix theory. Some asymptotic formulas relating $\mu(A)$, the spectral radius and norms are also given.

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1 Introduction

For an $n \times n$ matrix A, the conventional eigenequation for eigenvalue λ and corresponding eigenvector x is $Ax = \lambda x$. In the max algebra system, for $A = [a_{ij}]$ nonnegative, the eigenequation is $\max_j(a_{ij}x_j) = \lambda x_i$, for $i = 1, 2, \ldots, n$. Our aim is to give a power method algorithm to compute the eigenvalue and corresponding eigenvectors in the max algebra.

To motivate this aim and to place it in context, we need some definitions and notation. As in [2], [14, IV3] the *max algebra* system consists of the set of nonnegative numbers with sum $a \oplus b = \max\{a, b\}$ and the standard product ab for $a, b \geq 0$. Thus, for compatible nonnegative matrices A, B, their product is denoted by $A \otimes B$ where $(A \otimes B)_{ik} = \max_j(a_{ij}b_{jk})$. In particular, for a vector $x \geq 0$, the product $(A \otimes x)_i = \max_j(a_{ij}x_j)$. The usual associative and distributive laws hold in this algebra.

The *max-plus algebra* system consists of the set $\mathbf{R} \cup \{-\infty\}$ with operations of maximization and addition; see, e.g. [1, 3, 6, 9], and, in particular, [4] for a power algorithm. The max-plus algebra is isomorphic by the exponential map to the max algebra. As illustrated in [2], the max algebra enables us to use powerful tools of nonnegative matrices, monotonicity and analysis. The isomorphism can then be used to give results in the max-plus algebra where the eigenequation arises in the study of discrete event dynamical systems [1].

Throughout Sections 1-3 we assume that $A = [a_{ij}]$ is an $n \times n$ nonnegative matrix. The weighted directed graph $\mathcal{G}(A)$ associated with A has vertex set $\{1, 2, \ldots, n\}$ and an edge (i, j) from i to j with weight a_{ij} if and only if $a_{ij} > 0$. A *circuit* of length k is a sequence of k edges $(i_1, i_2), (i_2, i_3), \ldots, (i_{k-1}, i_k), (i_k, i_1)$ where $i_1, i_2, \ldots, i_k$ are distinct. This has *circuit product* $a_{i_1,i_2}a_{i_2,i_3} \cdots a_{i_k,i_1}$, with the positive k-th root as the *circuit geometric mean.* The maximum circuit geometric mean in $\mathcal{G}(A)$ is denoted by $\mu(A)$ and plays an important role. A circuit with circuit product equal to $\mu(A)$ is called a *critical circuit*, and vertices on critical circuits are called *critical vertices.* The *critical matrix* of A, denoted by $A^{\mathcal{C}} = [a^{\mathcal{C}}_{ij}]$, is formed from the principal submatrix of A on the rows and columns corresponding to critical vertices by setting $a^{\mathcal{C}}_{ij} = a_{ij}$ if (i, j) is in a critical circuit, and $a^{\mathcal{C}}_{ij} = 0$ otherwise. Thus the critical graph $\mathcal{G}(A^{\mathcal{C}})$ has vertex set $V^{\mathcal{C}} = \{\text{critical vertices}\}$.

The following basic spectral results are well known in the max-plus algebra (see, e.g. [1, 3, 6, 9, 14]), and are proved in [2] for the max algebra.

Theorem 1.1 [2] *Let A be an $n \times n$ nonnegative irreducible matrix. Then there exists a unique positive eigenvalue $\mu(A)$, and a positive eigenvector. This eigenvector is unique (up to scalar multiples) if and only if $\mathcal{G}(A^{\mathcal{C}})$ is strongly connected.*

The notation $A^2_\otimes$ means $A \otimes A$, $A^k_\otimes$ means the $\otimes$ k-th power of A, and $\sum_\oplus$ is used for $\oplus$ summation. We denote the identity matrix by I, and the matrix with every entry 1 by J, the order is usually clear from the context. Vector e has every entry 1, thus $J = ee^T$. Inequalities are in the component-wise sense, for example, $A \leq J$ means that $a_{ij} \leq 1$ for all $1 \leq i, j \leq n$. For x, y nonnegative vectors, $x \otimes y^T = xy^T$, and $x \leq y$ implies that $A \otimes x \leq A \otimes y$. The following properties are useful in the sequel. For any permutation matrix P, $P \otimes A \otimes P^T = PAP^T$, and for any positive diagonal matrix D, $D \otimes A \otimes D^{-1} = DAD^{-1}$.

Section 2 is devoted to results on the convergence of the max powers, $A^k_\otimes$. We find conditions under which $A^{t+d}_\otimes = A^t_\otimes$ for sufficiently large t and fixed d, and study in detail the case $d = 1$, i.e., when the max powers of A converge in finitely many steps. Here d is the cyclicity of A as defined later in (2.14). These results are needed for our power method algorithm. We note that equivalent results in the max-plus algebra are known; nevertheless we choose to give proofs, as our approach using basic tools of nonnegative matrices is quite different. Our results are also more general as we do not need to assume that $A^d_\otimes$ is irreducible (cf. [1, Th. 3.112]). Our power method, given in Section 3 simplifies and generalizes the algorithm given in [4, Alg.3.1]. Our assumptions on A are more general, and an eigenvector is found directly. Finally in Section 4, we give some asymptotic formulas that stress analogies between the standard product, the Hadamard product and the max product, and also between the spectral radius, $\mu(A)$ and norms.

2 Results on $A^k_\otimes$

The following result is needed for the proofs of the main theorems of this section.

Lemma 2.1 *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

be a nonnegative $n \times n$ matrix where $A \leq J$, A_{11} is $p \times p$ with $p \geq 1$ and each diagonal entry equal to 1, and $A_{22} < J$. Then $lim_{k\to\infty} A_\otimes^k$, denoted by A_∞, exists. If, in addition, either A is irreducible or $p = n$, then there exists k_0 such that $A_\otimes^k = A_\infty$ for all $k \geq k_0$.

Proof. Let

$$A_\otimes^k = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{22}^{(k)} \end{bmatrix}.$$

The assumption that $A_{11} \geq I$, and the identities $A_\otimes^{k+1} = A \otimes A_\otimes^k = A_\otimes^k \otimes A$, give the following inequalities:

$$\begin{aligned} A_{11}^{(k+1)} &= A_{11} \otimes A_{11}^{(k)} \oplus A_{12} \otimes A_{21}^{(k)} \geq A_{11}^{(k)}, & (2.1) \\ A_{12}^{(k+1)} &= A_{11} \otimes A_{12}^{(k)} \oplus A_{12} \otimes A_{22}^{(k)} \geq A_{12}^{(k)}, & (2.2) \\ A_{21}^{(k+1)} &= A_{21}^{(k)} \otimes A_{11} \oplus A_{22}^{(k)} \otimes A_{21} \geq A_{21}^{(k)}. & (2.3) \end{aligned}$$

Hence the sequences $\{A_{11}^{(k)}\}$, $\{A_{12}^{(k)}\}$, and $\{A_{21}^{(k)}\}$ are each monotonic increasing and bounded above by J. Thus they converge to matrices $\tilde{A}_{11}$, $\tilde{A}_{12}$, and $\tilde{A}_{21}$, respectively.

Define

$$\tilde{A}_\otimes^k = \begin{bmatrix} A_{11}^{(k)} & A_{12}^{(k)} \\ A_{21}^{(k)} & A_{21}^{(k)} \otimes A_{12}^{(k)} \end{bmatrix} \text{ and } A_\infty = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{21} \otimes \tilde{A}_{12} \end{bmatrix}.$$

We show that there is an $\epsilon > 0$ such that

$$\tilde{A}_\otimes^k \neq A_\infty \Rightarrow ||\tilde{A}_\otimes^k - A_\infty||_F \geq \epsilon. \tag{2.4}$$

To show this, consider an entry of A_∞ that differs from the corresponding entry of $\tilde{A}_\otimes^k$. As $0 \leq \tilde{A}_\otimes^k \leq A_\infty$, this entry is positive. Since all entries of $\tilde{A}_\otimes^k$ are finite products of entries of A, they have no accumulation point except zero, so for each positive entry of A_∞ there is a punctured neighbourhood of it containing no other product of entries of A. Take ϵ to be the smallest radius of these neighbourhoods, and (2.4) follows. The contrapositive of (2.4) implies that, as $\tilde{A}_\otimes^k \to A_\infty$, there exists q such that

$$A_{11}^{(k)} = \tilde{A}_{11}, \ A_{12}^{(k)} = \tilde{A}_{12}, \ A_{21}^{(k)} = \tilde{A}_{21} \ \text{ for all } k \geq q. \tag{2.5}$$

In fact it can be proved that $q \leq n$, but this is not needed in the sequel. If $p = n$, then $A_\infty = \tilde{A}_{11}$, and the proof is complete with $q = k_0$. We now claim also in the case $p < n$ that A_∞ is the limit of the matrices $A_\otimes^k$. By the above, this is obviously true for the entries in the first p rows and columns, so it remains to consider the block in the last $n - p$ rows and columns. Considering the $(1, 2)$ block of $A_\otimes^q \otimes A = A_\otimes^{q+1}$, and using (2.5) gives $(A_\otimes^{q+1})_{12} = \tilde{A}_{12} = \tilde{A}_{11} \otimes A_{12} \oplus \tilde{A}_{12} \otimes A_{22}$. Hence

$$\tilde{A}_{12} \otimes A_{22} \leq \tilde{A}_{12}. \tag{2.6}$$

The first step with $k = 1$ of an induction proof of the inequality

$$A_{22}^{(k)} \leq \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_\otimes^k, \quad \text{for } k \geq 1, \tag{2.7}$$

is obvious. To see that (2.7) holds in general, assume that it is true for k, and consider the case for $k + 1$. Thus, by multiplication and using the induction hypothesis,

$$\begin{aligned} A_{22}^{(k+1)} &= A_{21}^{(k)} \otimes A_{12} \oplus A_{22}^{(k)} \otimes A_{22} \\ &\leq \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (\tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_\otimes^k) \otimes A_{22} \\ &= \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus \tilde{A}_{21} \otimes \tilde{A}_{12} \otimes A_{22} \oplus (A_{22})_\otimes^{k+1} \\ &\leq \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_\otimes^{k+1}, \text{ by (2.6)}, \\ &= \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_\otimes^{(k+1)}, \end{aligned}$$

as required to show (2.7) is true for all $k \geq 1$. On the other hand, for $k \geq 2q$, writing $k = t + s$ with $s, t \geq q$,

$$\begin{aligned} A_{22}^{(k)} &= A_{21}^{(s)} \otimes A_{12}^{(t)} \oplus A_{22}^{(s)} \otimes A_{22}^{(t)} \\ &\geq A_{21}^{(s)} \otimes A_{12}^{(t)} \\ &= \tilde{A}_{21} \otimes \tilde{A}_{12}, \text{ from (2.5)}. \end{aligned}$$

Thus for $k \geq 2q$, from the above and (2.7),

$$\tilde{A}_{21} \otimes \tilde{A}_{12} \leq A_{22}^{(k)} \leq \tilde{A}_{21} \otimes \tilde{A}_{12} \oplus (A_{22})_\otimes^k.$$

Since $A_{22} < J$, it follows that $\lim_{k\to\infty} (A_{22})_\otimes^k = 0$, and thus $\lim\limits_{k\to\infty} A_{22}^{(k)} = \tilde{A}_{21} \otimes \tilde{A}_{12}$; completing the claim that $\lim_{k\to\infty} A_\otimes^{(k)} = A_\infty$.

If A is irreducible, then due to the fact that the diagonal entries of A_{11} are nonzero, the matrix A is primitive, i.e., in the standard product $A^m > 0$ for some positive integer m. But, since for all $A \geq 0$,

$$\frac{A^m}{n^{m-1}} \leq A_\otimes^m \leq A^m,$$

it is also true that $A_{\otimes}^{m} > 0$. Thus $\tilde{A}_{21} \otimes \tilde{A}_{12} > 0$, and the same argument as for (2.4) shows that $A_{22}^{(k)} = \tilde{A}_{21} \otimes \tilde{A}_{12}$ for sufficiently large k, i.e. $k > k_0$. Thus the limit A_∞ is reached after finitely many steps. $\square$

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Note that the last statement of Lemma 2.1 is not in general true for reducible A if $p < n$. This can be seen from the reducible example $A = \mathrm{diag}(1, 2^{-1})$, which satifies all the assumptions of the first part of Lemma 2.1 with $p = 1$, has $A_\infty = \mathrm{diag}(1, 0)$, but $A_{22}^{(k)} = 2^{-k} \neq 0$ for finite k.

As in the max plus algebra, we define

$$A^{+} = A \oplus A_{\otimes}^{2} \oplus \cdots A_{\otimes}^{n} \oplus A_{\otimes}^{n+1} \oplus \cdots \ ,$$

see, e.g., [1, p. 10]. If $\mu(A) \leq 1$, it can be shown that only max powers up to $A_{\otimes}^{n}$ contribute to the above sum. This also explains $q \leq n$, where q is given in (2.5). If $\mu(A) = 1$, then A^{+} is equal to A_∞ in the rows and columns of $V^{\mathcal{C}}$. The columns of A^{+} give the eigenvectors of A, see [2, Th. 5] for this result in the max algebra.

Theorem 2.2 *Assume that A is an $n \times n$ nonnegative matrix with $\mu(A) = 1$, and that its critical matrix $A^{\mathcal{C}}$ is the direct sum of primitive matrices. If either A is irreducible or $V^{\mathcal{C}} = \{1, 2, \ldots, n\}$, then $\lim_{k\to\infty} A_{\otimes}^{k}$, denoted by A_∞, exists and $A_{\otimes}^{k} = A_\infty$ for sufficiently large k.*

Proof. First note that this statement is invariant under diagonal similarities, and simultaneous permutations on rows and columns. Permute the critical vertices so that $A^{\mathcal{C}}$ is a direct sum, namely $A^{\mathcal{C}} = A_{11}^{\mathcal{C}} \dot{+} A_{22}^{\mathcal{C}} \dot{+} \cdots \dot{+} A_{rr}^{\mathcal{C}}$, defining a block structure on A. Now by a result of [11], see also [10], A can be scaled by a diagonal similarity so that $a_{ij} \leq 1$ for all $i, j = 1, 2, \ldots, n$, with $a_{ij} = 1$ if edge (i, j) is on a critical circuit. As all critical circuits are contained in blocks $A_{11}, A_{22}, \ldots, A_{rr}$, an additional diagonal scaling achieves $a_{ij} < 1$ outside these blocks. So A can be assumed to be in block form

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1r} & A_{1,r+1} \\ \vdots & \ddots & \vdots & \vdots \\ A_{r1} & \cdots & A_{rr} & A_{r,r+1} \\ A_{r+1,1} & \cdots & A_{r+1,r} & A_{r+1,r+1} \end{bmatrix} \tag{2.8}$$

where $A_{ij} < e_i e_j^T$ with the exception of $i = j = 1, 2, \ldots, r$, and $A_{ii}^{\mathcal{C}}$ primitive for $i = 1, 2, \ldots, r$. Here e_i is the vector with every entry 1 and dimension equal to the cardinality of the block A_{ii}. Thus there exists a positive integer p such that

$$A_\otimes^p = \begin{bmatrix} A_{11}^{(p)} & \cdots & A_{1r}^{(p)} & A_{1,r+1}^{(p)} \\ \vdots & \ddots & \vdots & \vdots \\ A_{r1}^{(p)} & \cdots & A_{rr}^{(p)} & A_{r,r+1}^{(p)} \\ A_{r+1,1}^{(p)} & \cdots & A_{r+1,r}^{(p)} & A_{r+1,r+1}^{(p)} \end{bmatrix}$$

has $A_{ii}^{(p)} = e_i e_i^T$ for $i = 1, 2, \ldots, r$, and $A_{ij}^{(p)} < e_i e_j^T$ otherwise. This equality follows from primitivity and monotonicity, namely $0 < (A_{ii}^{\mathcal{C}})_\otimes^p \leq (A_{ii})_\otimes^p \leq e_i e_i^T$, and the fact that $(A_{ii}^{\mathcal{C}})_\otimes^p$ is a (0,1) matrix. From Lemma 2.1, since A is irreducible or $A_{r+1,r+1}^{(p)} = \emptyset$, there exists k_0 such that $A_\otimes^{kp}$ is constant for all $k \geq k_0$. This constant matrix is denoted by A_∞, and can be written as

$$A_\infty = \begin{bmatrix} e_1 e_1^T & \tilde{A}_{12} & \cdots & \tilde{A}_{1r} & \tilde{A}_{1,r+1} \\ \tilde{A}_{21} & \ddots & & & \tilde{A}_{2,r+1} \\ \vdots & & \ddots & & \vdots \\ \tilde{A}_{r1} & & & e_r e_r^T & \tilde{A}_{r+1,r} \\ \tilde{A}_{r+1,1} & \tilde{A}_{r+1,2} & \cdots & \tilde{A}_{r+1,r} & \tilde{A}_{r+1,r+1} \end{bmatrix}.$$

The equality $A_\infty A_\infty = A_\infty$ leads to

$$\begin{array}{l} \tilde{A}_{ij} = \alpha_{ij} e_i e_j^T \ \text{ for } i, j \leq r \text{ with } \alpha_{ii} = 1,\ \alpha_{ij} < 1 \text{ for } i \neq j, \\ \tilde{A}_{r+1,j} = \eta_j e_j^T,\ \tilde{A}_{j,r+1} = e_j \xi_j^T \text{ for } j \leq r, \\ \tilde{A}_{r+1,r+1} = \sum_{\substack{\oplus \\ j=1}}^{r} \eta_j \xi_j^T, \quad \text{where } \eta_j,\ \xi_j \text{ are nonnegative vectors.} \end{array} \tag{2.9}$$

For example, when $i, j \leq r,\ i \neq j$,

$$\begin{aligned} \tilde{A}_{ij} &= \sum_{\substack{\oplus \\ k=1}}^{r+1} \tilde{A}_{ik} \otimes \tilde{A}_{kj} \\ &\geq \tilde{A}_{ii} \otimes \tilde{A}_{ij} \\ &= e_i e_i^T \otimes \tilde{A}_{ij}. \end{aligned}$$

Similary $\tilde{A}_{ij} \geq \tilde{A}_{ij} \otimes \tilde{A}_{jj} = \tilde{A}_{ij} \otimes e_j e_j^T$. These two inequalities give part of the first equation of (2.9). Also,

$$\tilde{A}_{j,r+1} = \sum_{\substack{\oplus \\ k=1}}^{r+1} \tilde{A}_{jk} \otimes \tilde{A}_{k,r+1} \tag{2.10}$$

$$\begin{aligned} &\geq \tilde{A}_{jj} \otimes \tilde{A}_{j,r+1} \\ &= e_j e_j^T \otimes \tilde{A}_{j,r+1}. \end{aligned}$$

Thus, $\tilde{A}_{j,r+1} = e_j \xi_j^T$, where $\xi_j^T = e_j^T \otimes \tilde{A}_{j,r+1}$. Substituting in (2.10) gives

$$\begin{aligned} e_j \xi_j^T &= \sum_{\substack{\oplus \\ k=1}}^{r} \alpha_{jk} e_j e_k^T \otimes e_k \xi_k^T \oplus \tilde{A}_{j,r+1} \otimes \tilde{A}_{r+1,r+1} \\ &\geq \sum_{\substack{\oplus \\ k=1}}^{r} \alpha_{jk} e_j \xi_j^T \end{aligned}$$

with equality. So

$$e_j \xi_j^T = \sum_{\substack{\oplus \\ k=1}}^{r} \alpha_{jk} e_j \xi_k^T. \tag{2.11}$$

In a similar way, $\tilde{A}_{r+1,j} = \eta_j e_j^T$ for $j \leq r$, where $\eta_j = \tilde{A}_{r+1,j} e_j$. Now the last equation in (2.9) follows from Lemma 2.1, since $\tilde{A}_{r+1,r+1} = \sum_{\substack{\oplus \\ j=1}}^{r} \tilde{A}_{r+1,j} \otimes \tilde{A}_{j,r+1} = \sum_{\substack{\oplus \\ j=1}}^{r} \eta_j \xi_j^T$.

For $i = 1, 2, \ldots, r$, define $x_i = (\alpha_{1i} e_1^T, \alpha_{2i} e_2^T, \ldots, \alpha_{ri} e_r^T, \eta_i^T)^T$, and $y_i = (0, \ldots, 0, e_i^T, 0, \ldots, 0, \xi_i^T)^T$. Recall that $\alpha_{ii} = 1$. Consider the $(j, r+1)$ block of $\sum_{\substack{\oplus \\ k=1}}^{r} x_k \otimes y_k^T$. It is given by $\sum_{\substack{\oplus \\ k=1}}^{r} \alpha_{jk} e_j \xi_k^T = e_j \xi_j^T$ from (2.11). This is also equal to $\tilde{A}_{j,r+1}$, which is the $(j, r+1)$ block of A_∞. By also comparing blocks in columns $1, 2, \ldots, r$ and the $(r+1, r+1)$ block, it can be easily seen from (2.9) that

$$A_\infty = \sum_{\substack{\oplus \\ i=1}}^{r} x_i \otimes y_i^T. \tag{2.12}$$

Now we claim that $A \otimes A_\infty \geq A_\infty$. From $A \otimes A_\infty = A_\infty \otimes A$, it follows that

$$\sum_{\substack{\oplus \\ i=1}}^{r} (A \otimes x_i) \otimes y_i^T = \sum_{\substack{\oplus \\ i=1}}^{r} x_i \otimes (y_i^T \otimes A). \tag{2.13}$$

Define $z_j = (0, \ldots, 0, e_j^T, 0, \ldots, 0)^T$ for $j = 1, 2, \ldots, r$, then $y_i^T \otimes z_j = \delta_{ij}$ and $y_i^T \otimes A \otimes z_j \leq 1$ with equality when $i = j$. From (2.13)

$$\sum_{i=1}^{r} {}^{\oplus} (A \otimes x_i) \otimes y_i^T \otimes z_j = A \otimes x_j = \sum_{i=1}^{r} {}^{\oplus} x_i \otimes (y_i^T \otimes A \otimes z_j) \geq x_j,$$

and by (2.12),

$$A \otimes A_\infty = \sum_{i=1}^{r} {}^{\oplus} A \otimes x_i \otimes y_i^T \geq \sum_{i=1}^{r} {}^{\oplus} x_i \otimes y_i^T = A_\infty.$$

Thus our claim is proved, and can be used recursively to give

$$A_\infty \leq A \otimes A_\infty \leq A_\otimes^2 \otimes A_\infty \leq \cdots \leq A_\otimes^p \otimes A_\infty = A_\infty.$$

So $A \otimes A_\infty = A_\infty$ and $\lim_{k\to\infty} A_\otimes^k = A_\infty$, and in fact $A_\infty^k = A_\infty$ for $k \geq pk_0$. $\square$

Note that the assumption that $A_{ii}^{\mathcal{C}}$ are primitive cannot be dropped in general from Theorem 2.2. For example, consider the matrix

$$A = \begin{bmatrix} 2^{-2} & 1 & 2^{-1} \\ 1 & 2^{-2} & 2^{-1} \\ 2^{-1} & 2^{-1} & 2^{-2} \end{bmatrix}$$

where A_{11} is 2×2 irreducible, but $A_{11}^{\mathcal{C}}$ is not primitive. Then

$$A_\otimes^{2k} = \begin{bmatrix} 1 & 2^{-2} & 2^{-1} \\ 2^{-2} & 1 & 2^{-1} \\ 2^{-1} & 2^{-1} & 2^{-2} \end{bmatrix}$$

and $A_\oplus^{2k+1} = A$. Thus $\lim_{k\to\infty} A_\otimes^k$ does not exist.

In general $A^{\mathcal{C}}$ is the direct sum of irreducible matrices, namely $A^{\mathcal{C}} = A_{11}^{\mathcal{C}} \dot{+} A_{22}^{\mathcal{C}} \dot{+} \cdots \dot{+} A_{rr}^{\mathcal{C}}$ with indices of imprimitivity s_i for $i = 1, 2, \ldots, r$. Recall that the index of imprimitivity h of a nonnegative irreducible matrix B is the number of eigenvalues of modulus $\rho(B)$, where $\rho(B)$ is the spectral radius of B, and if $h = 1$ then B is primitive. The value h is also equal to the g.c.d. of all circuit lengths of the graph $\mathcal{G}(B)$ (see, e.g. [5]). In particular, it is known that B^h is the direct sum of h primitive matrices (e.g. [13, p.82, Cor.2],or [5, p.74, Thm.3.4.5]). So defining the *cyclicity* of A by

$$d = \text{l.c.m.}(s_1, s_2, \ldots, s_r), \tag{2.14}$$

then $(A^{\mathcal{C}})^d$ is the direct sum of $\sum_{i=1}^r s_i$ primitive matrices. We can now formulate a general result concerning the behaviour of the powers of A.

Theorem 2.3 *Assume that $A \geq 0$ is irreducible, $\mu(A) = 1$, and let d be defined as in (2.14). Then there exists t_0 such that for all $t \geq t_0$*

$$A_{\otimes}^{t+d} = A_{\otimes}^{t}. \tag{2.15}$$

Proof. Firstly, note that

$$(A_{\otimes}^{d})^{\mathcal{C}} = B^{\mathcal{C}} = (A^{\mathcal{C}})_{\otimes}^{d},$$

since a product of d factors, each of which is less than or equal to 1, is equal to 1 if and only if each factor is 1. As $\frac{(A^{\mathcal{C}})^d}{n^{d-1}} \leq (A^{\mathcal{C}})_{\otimes}^{d} \leq (A^{\mathcal{C}})^d$ and $(A^{\mathcal{C}})^d$ is the direct sum of primitive matrices, then so is $(A^{\mathcal{C}})_{\otimes}^{d} = B^{\mathcal{C}}$. Also $\mu(B) = 1$. If $B = A_{\otimes}^{d}$ is irreducible, then Theorem 2.2 can be applied directly to B. If $B = A_{\otimes}^{d}$ is reducible, then by [13, p.81,Thm.9], B is the direct sum of irreducible matrices, and Theorem 2.2 can be applied to each diagonal block. Hence there is τ such that $B_{\otimes}^{\tau} = B_{\otimes}^{\tau+1}$ and thus for $t_0 = \tau d$, $A_{\otimes}^{t_0+d} = A_{\otimes}^{t_0}$. But then (2.15) holds for all $t \geq t_0$. $\square$

We remark that Theorem 2.2 for A irreducible, is the special case of Theorem 2.3 with $d = 1$. Versions of these theorems under the additional assumption that $A_{\otimes}^{d}$ is irreducible are well known in the max-plus algebra, see, e.g. [1, Thms.3.108,3.112], [6, Th. 4], [7, p.45]. However, our proofs are different and use standard tools from nonnegative matrix theory. We now use our results for a power method algorithm.

3 Power Method in the Max Algebra

In this section we study a power method for calculating eigenvectors of an irreducible matrix A in the max algebra. It also provides a sequence of upper and lower bounds for the eigenvalue $\mu(A)$.

It turns out that in this situation the results are simpler than in the conventional case of the ordinary matrix algebra (see, e.g. [16]), which we nevertheless follow in spirit. In the max algebra, our assumptions are more general than those in [4], and we obtain an eigenvector directly.

In the sequel we make use of the following result.

Lemma 3.1 *Let $A \geq 0$ be irreducible, and $x \geq 0$, $x \neq 0$.*

$$\text{If } \beta \in \mathbf{R}_+ \text{ and } A \otimes x \leq \beta x, \text{ then } \mu(A) \leq \beta. \tag{3.1}$$

$$\text{If } \alpha \in \mathbf{R}_+ \text{ and } \alpha x \leq A \otimes x, \text{ then } \alpha \leq \mu(A). \tag{3.2}$$

Proof. Let $y \geq 0$ be a left eigenvector of A. Then $y^T \otimes A = \mu(A) y^T$ with $y > 0$ by Theorem 1.1. From $A \otimes x \leq \beta x$, we have $\mu(A) y^T \otimes x = y^T \otimes A \otimes x \leq \beta y^T \otimes x$ and thus $\mu(A) \leq \beta$. Similarly we prove the second statement. □

For $x > 0$ we define $q_i(x) = \frac{(A \otimes x)_i}{x_i}$ and

$$\bar{q}(x) = \max_i q_i(x) = \max_i \max_j a_{ij} \frac{x_j}{x_i}$$

$$\underline{q} = \min_i q_i(x) = \min_i \max_j a_{ij} \frac{x_j}{x_i}.$$

Then by Lemma 3.1, as $\underline{q}(x) x \leq A \otimes x \leq \bar{q}(x) x$,

$$\underline{q}(x) \leq \mu(A) \leq \bar{q}(x). \tag{3.3}$$

This inequality can be found for $x = e$ in [3, Cor.4.2] in the max-plus algebra version. It is however obvious that applying the result of [3] to $D^{-1}AD$, where $D = \mathrm{diag}(x_1, x_2, \ldots, x_n)$, and using $\mu(A) = \mu(D^{-1}AD)$, (3.3) is also obtained. The classical analogue of (3.3) was first given in [8].

We consider now for given $x(0) > 0$ the sequence $x(t),\ t = 1, 2, \ldots$ generated by

$$x(t+1) = A \otimes x(t), \quad t \geq 0, \tag{3.4}$$

and show that the iterates $x(t)$ yield a sequence of increasingly better upper and lower bounds for $\mu(A)$.

Lemma 3.2 *Let $A \geq 0$ be irreducible. Define $x(t),\ t = 0, 1, \ldots$ as above, and*

$$\bar{q}_t = \bar{q}(x(t)), \quad \underline{q}_t = \underline{q}(x(t)).$$

Then

$$\underline{q}_t \leq \underline{q}_{t+1} \leq \mu(A) \leq \bar{q}_{t+1} \leq \bar{q}_t, \quad t = 0, 1, \ldots \tag{3.5}$$

Proof. By the irreducibility of A, all $x(t)$ are strictly positive and hence $\underline{q}_t$, $\bar{q}_t$ are well defined. We show $\bar{q}_{t+1} \leq \bar{q}_t$. Multiply the inequality

$$A \otimes x(t) \leq \bar{q}_t x(t)$$

by A. This leads to

$$A \otimes x(t+1) = A \otimes A \otimes x(t) \leq A \otimes \bar{q}_t x(t) = \bar{q}_t x(t+1),$$

and hence by definition, $\bar{q}_{t+1} \leq \bar{q}_t$. Similarly $\underline{q}_t \leq \underline{q}_{t+1}$. The other inequalities of (3.5) follow from (3.3). $\square$

In the conventional case, i.e. $\tilde{x}(t+1) = A\tilde{x}(t)$, with the standard matrix product, defining $\underline{\tilde{q}}_t = \underline{q}(\tilde{x}(t)), \bar{\tilde{q}}_t = \bar{q}(\tilde{x}(t))$ gives

$$\underline{\tilde{q}}_t \leq \underline{\tilde{q}}_{t+1} \leq \rho(A) \leq \bar{\tilde{q}}_{t+1} \leq \bar{\tilde{q}}_t.$$

Here it is known that for A primitive $\lim_{t\to\infty} \underline{\tilde{q}}_t = \lim_{t\to\infty} \bar{\tilde{q}}_t = \rho(A)$ and that the $x(t)$, suitably normalized, converge to the (suitably normalized) eigenvector of A corresponding to $\rho(A)$.

In the max algebra case we have

$$\mu_1(A) := \lim_{t\to\infty} \underline{q}_t \leq \mu(A) \leq \mu_2(A) := \lim_{t\to\infty} \bar{q}_t \tag{3.6}$$

and the existence of a vector $z > 0$, an accumulation point of the suitably normalized sequence $\{x(t)\}_{t=0,1,\ldots}$ satisfying

$$\mu_1(A) = \underline{q}(A^s_\otimes \otimes z),\ \ \mu_2(A) = \bar{q}(A^s_\otimes \otimes z), \quad s = 0, 1, \ldots \ .$$

The example $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ with $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ gives $x(1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $x(2) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $x(3) = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $z = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mu_1(A) = 1 < \sqrt{2} = \mu(A) < \mu_2(A) = 2$, and shows that primitivity of A alone does not guarantee convergence of either q_t to $\mu(A)$ or $x(t)$ to an eigenvector.

We prove the following result.

Theorem 3.3 *Let $A \geq 0$ be irreducible, and consider the sequence $x(t+1) = A \otimes x(t),\ t = 0, 1, \ldots$ where $x(0) > 0$ is given. If the critical matrix $A^{\mathcal{C}}$ is the direct sum of primitive matrices, then $x(t)$ is an eigenvector of A for some finite t_0 (and hence for all $t \geq t_0$), $\underline{q}_t = \mu(A) = \bar{q}_t$. Otherwise there exist integers $p,\ t$ and a positive number s such that*

$$A^p_\otimes \otimes x(t) = sx(t), \tag{3.7}$$

in which case

$$\mu(A) = s^{\frac{1}{p}} \tag{3.8}$$

and

$$z = x(t) \oplus \frac{1}{\mu(A)} A \otimes x(t) \oplus \cdots \oplus \frac{1}{\mu(A)^{p-1}} A^{p-1}_\otimes \otimes x(t) \tag{3.9}$$

is an eigenvector of A.

Proof. Using the result of Theorem 2.2, if $A^{\mathcal{C}}$ is the direct sum of primitive matrices, then

$$\frac{1}{\mu(A)^t} A_\otimes^t = A_\infty$$

for $t = t_0$, where $A \otimes A_\infty = \mu(A) A_\infty$. Hence $x(t) = A_\otimes^t \otimes x(0) = \mu(A)^t A_\infty \otimes x(0)$ showing that $x(t)$ is an eigenvector. Hence $\underline{q}_t = \mu(A) = \bar{q}_t$ and $\mu_1(A) = \mu(A) = \mu_2(A)$. In general, if A is only irreducible, we have by Theorem 2.3 that $\frac{1}{\mu(A)} A$ is cyclic, so there exist t_0 and a minimal integer $p \le d$, where d is given by (2.14), such that

$$\frac{1}{\mu(A)^{t+p}} A_\otimes^{t+p} = \frac{1}{\mu(A)^t} A_\otimes^t \quad \text{for all } t \ge t_0.$$

Hence

$$A_\otimes^p \otimes x(t) = x(t+p) = \mu(A)^p x(t) \quad \text{for all } t \ge t_0$$

and we see that (3.7) holds with $s = \mu(A)^p$. It is verified in a straight forward manner that z given by (3.9) satisfies $A \otimes z = \mu(A) z$. $\square$

Observe that $p = 1$ gives the case of convergence. So we have shown that the following algorithm will stop after finitely many steps, giving a vector z and number $\mu(A)$ satisfying $A \otimes z = \mu(A) z$.

Algorithm 3.4 *Let $A \ge 0$, irreducible, $x(0) > 0$.*

For $t = 0, 1, \ldots$

1. $x(t+1) = A \otimes x(t)$,
2. *For* $r = 1, 2, \ldots, t+1$
   If $x(t+1) = s x(t+1-r)$ *for some* $s > 0$*, then*
   $\mu(A) = s^{\frac{1}{r}}$
   $z = x(t+1-r) \oplus \frac{1}{\mu(A)} x(t+2-r) \oplus \cdots \oplus \frac{1}{\mu(A)^{r-1}} x(t)$
   $p = r$
   stop.

We remark that the number p calculated by this algorithm and in Theorem 3.3 is in general only a lower bound for d, the cyclicity of A, see (2.14). If by chance $x(0)$ is an eigenvector, p will be 1, regardless of the cyclicity of A. If $\mathcal{G}(A^{\mathcal{C}})$ has r strongly connected components, then r linearly independent (in the max sense) eigenvectors exist [2, Thm.5]. These can be found by starting with different $x(0)$. If $x(0) \ge 0$, $x(0) \ne 0$, but after some iterations

$x(k) > 0$, then the previous results apply. Note also that (3.9) gives an eigenvector without any further calculation, in contrast to the power algorithm in [4, Alg.3.1].

Example 3.5 Let $A = \begin{bmatrix} 0 & e^3 & 0 \\ e^2 & 0 & 1 \\ e & e^2 & e^2 \end{bmatrix}$, cf [4, Ex.5.2]. Then A is irreducible, $A^{\mathcal{C}}$ is the leading 2×2 principal minor of A and is irreducible but not primitive. Take $x(0) = (1, 0, 0)^T$, then $x(1) = (0, e^2, e)^T$, $x(2) = (e^5, e, e^4)^T$, $x(3) = (e^4, e^7, e^6)^T$, $x(4) = e^5 x(2)$, giving $\mu_2(A) = e^{-1}$, $\mu_1(A) = e^6$ and $A^2_{\otimes} \otimes x(2) = e^5 x(2)$. Thus $\mu(A) = e^{\frac{5}{2}}$, with eigenvector $z = x(2) \oplus e^{-\frac{5}{2}} A \otimes x(2) = (e^5, e^{\frac{9}{2}}, e^4)$.

4 Asymptotic formulas

For $A \in M_n = \{n \times n \text{ complex matrices}\}$, it is well known that for any matrix norm

$$\lim_{k \to \infty} ||A^k||^{\frac{1}{k}} = \rho(A), \tag{4.1}$$

see, e.g., [15, Cor. 5.6.14]. The *Hadamard power* of A is defined via the Hadamard (componentwise) product as $A^k_{\odot} = [a^k_{ij}]$, and it is clear from the definition of μ for A nonnegative that

$$\mu(A^k_{\odot}) = (\mu(A))^k, \tag{4.2}$$

see, e.g., [14, p. 140]. For A nonnegative and irreducible, then $A \otimes x = \mu(A)x$ with $x > 0$ by Theorem 1.1, thus

$$\mu(A^k_{\otimes}) = (\mu(A))^k. \tag{4.3}$$

By continuity this holds also for reducible $A \geq 0$. It is also well known (see [10, (6.11)], [14, p. 138]) that, for an $n \times n$ nonnegative matrix A,

$$\mu(A) \leq \rho(A) \leq n\mu(A), \tag{4.4}$$

which implies that

$$\lim_{k \to \infty} [\rho(A^k_{\odot})]^{\frac{1}{k}} = \mu(A), \tag{4.5}$$

as given in [10, (6.1)], [12, Th. 1], [14, p. 140]. Also from (4.4) and the fact that $\rho(A^k) = (\rho(A))^k$, we have the result of [2, Th.14], namely

$$\lim_{k\to\infty}[\mu(A^k)]^{\frac{1}{k}} = \rho(A). \tag{4.6}$$

We now give corresponding asymptotic formulas when the standard and Hadamard products are replaced by the max product.

Lemma 4.1 *If* $A \geq 0$ *then*

$$\lim_{k\to\infty}||A^k_{\otimes}||^{\frac{1}{k}} = \mu(A). \tag{4.7}$$

Proof. Since for an operator norm $||A|| \geq \rho(A)$ for any $A \in M_n$ [15, Th. 5.6.9],

$$||A^k_{\otimes}|| \geq \rho(A^k_{\otimes}) \geq \mu(A^k_{\otimes}) = (\mu(A))^k,$$

by (4.4) and (4.2). So if the required limit exists, it must be at least $\mu(A)$. Assume for the other direction that $\mu(A) = 1$, and A is scaled as in Theorem 2.2 so that $a_{ij} \leq 1$ for all i, j. Then $A^k_{\otimes} \leq ee^T$, so the maximum row sum norm of $A^k_{\otimes}$ is at most n. Since the limit does not depend on the choice of norm, its value is at most $1 = \mu(A)$. Hence the limit exists and has value $\mu(A)$. $\square$

By (4.3) and (4.4) it is obvious that

$$\lim_{k\to\infty}[\mu(A^k_{\otimes})]^{\frac{1}{k}} = \lim_{k\to\infty}[\rho(A^k_{\otimes})]^{\frac{1}{k}} = \mu(A). \tag{4.8}$$

As the final relation, we have that

$$\lim_{k\to\infty}||A^k_{\odot}||^{\frac{1}{k}} = \max_{i,j}|a_{i,j}| \tag{4.9}$$

for $A \in M_n$. This follows by taking the norm $||A|| = n\max_{i,j}|a_{ij}|$; see [15, p. 292].

We summarize the above results for $A \geq 0$ in the following table where $\lim_{k\to\infty}[\alpha(A^k_{\beta})]^{\frac{1}{k}}$ is given in row α with $\alpha \in \{\mu,\ \rho,\ ||\cdot||\}$, and column β where β denotes the matrix product used (standard, Hadamard or max product).

$\alpha\backslash\beta$	standard	Hadamard $\odot$	max $\otimes$
μ	$\rho(A)$	$\mu(A)$	$\mu(A)$
ρ	$\rho(A)$	$\mu(A)$	$\mu(A)$
$\|\|\cdot\|\|$	$\rho(A)$	$\max_{i,j}\|a_{ij}\|$	$\mu(A)$

Note that the 3 entries not involving μ hold for all $A \in M_n$.

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