TEMPERED SUBGROUPS AND REPRESENTATIONS WITH MINIMAL DECAY OF MATRIX COEFFICIENTS

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ABSTRACT. We present a function F for each simple real linear Lie group G with real rank at least 2 such that F bounds from above all the K-matrix coefficients of non-trivial irreducible spherical unitary representations of G where K is a maximal compact subgroup of G. This enables us to determine when a closed subgroup H is a (G, K)- tempered subgroup of G: for example, if the restriction $F|_H$ of F to H lies in $L^{1-\epsilon}(H)$. We also prove that this function F is the best possible for G a real-split group of type A_n or C_n and as a consequence, we obtain that if H is semisimple, then H is a (G, K)- tempered subgroup of G if and only if $F|_H$ lies in $L^1(H)$.

1. INTRODUCTION

Let G be a connected semisimple linear Lie group and K a maximal compact subgroup of G. We say that a unitary representation of G is *spherical* if it has a K-invariant vector. For a unitary spherical representation ρ, we will use the term "K-matrix coefficients of ρ" to refer to its matrix coefficients with respect to K-invariant unit vectors.

In this paper we are interested in the asymptotic behavior of the K-matrix coefficients of spherical unitary representations of G when restricted to a closed subgroup H of G. One motivation comes from the notion "(G, K)-tempered subgroups" of G defined by Margulis [8]. That is, a closed subgroup H of G is called (G, K)-*tempered* if there exists a (positive) function $q \in L^1(H)$ such that for any non-trivial irreducible spherical unitary representation ρ of G, $|\langle \rho(h)v, w\rangle| \leq q(h)\|v\|\|w\|$ for all $h \in H$ and any K- fixed vectors v and w. Note that any compact subgroup of G is a (G, K)- tempered subgroup for a trivial reason. Margulis also showed in [8] that if a closed subgroup H is a (G, K)-tempered subgroup, then for any non-compact subgroup F of H, the quotient G/F does not allow a compact quotient by a discrete subgroup of G (see [5] for a survey on the general problem).

We denote by $\hat{G}$ (resp. $\hat{G}_K$) the set of equivalence classes of non-trivial irreducible unitary (resp. spherical) representations.

In this paper we first present a "good upper bound function" for K-matrix coefficients for all representations in $\hat{G}_K$ for a simple real linear Lie group G with real rank at least 2. Secondly we show that in simple real-split linear Lie group of type A_n or C_n this function is in fact the best possible by exhibiting a spherical representation of G in $\hat{G}_K$ whose

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K-matrix coefficients are bounded below by this function. We now formulate the main results.

The notation $[x]$ denotes the largest integer which is not greater than x.

Theorem A. *Let G be a connected simple real linear Lie group with real rank $n \geq 2$, K a maximal compact subgroup, B a minimal parabolic subgroup, $A \subset B$ a maximal $\mathbb{R}$-split torus, $A^+ \subset A$ the positive Weyl chamber given by the choice of B. Denote by Φ' the set of all non-multipliable roots in the relative root system $\Phi_{\mathbb{R}}(A, G)$ and by $\alpha_1, ..., \alpha_n$ the basis of Φ' whose subscripts are determined by the highest weight given in section 2.1.*

Then for any $\epsilon > 0$, there exists a constant C (depending on ϵ) such that for any $\rho \in \hat{G}_K$ and f_0 a K-invariant unit vector of ρ, the K-matrix coefficient $\langle \rho(g) f_0, f_0 \rangle$ satisfies

$$|\langle \rho(g) f_0, f_0 \rangle| \leqq C \cdot F(g)^{1-\epsilon} \qquad \text{for any } g \in G$$

where F is the K-bi-invariant function defined on A^+ as follows according to the type of Φ':

Φ'	F
$A_n, n \geq 2$	$\begin{cases} (\prod_{i=1}^{n-1/2} \alpha_i{}^{-i/2})(\prod_{i=n+1/2}^{n} \alpha_i{}^{-(n-i+1)/2}) & \text{for } n \text{ odd} \\ (\prod_{i=1}^{n/2} \alpha_i{}^{-i/2})\alpha_{n/2+1}{}^{-n/4}(\prod_{i=n/2+2}^{n} \alpha_i{}^{-(n-i+1)/2}) & \text{for } n \text{ even} \end{cases}$
$B_n, n \geq 2$	$\left(\prod_{i=1}^{[n/2]} \alpha_i{}^{-i}\right)\left(\prod_{i=[n/2+1]}^{n} \alpha_i{}^{-n/2}\right)$
$C_n, n \geq 2$	$\left(\prod_{i=1}^{n-1} \alpha_i{}^{-i}\right) \alpha_n{}^{-n/2}$
$D_n, n \geq 4$	$\left(\prod_{i=1}^{[n/2]} \alpha_i{}^{-i}\right)\left(\prod_{i=[n/2+1]}^{n-2} \alpha_i{}^{-n/2}\right)(\alpha_{n-1}\alpha_n)^{-n/4}$
E_6	$(\alpha_1 \alpha_2{}^2 \alpha_3^2 \alpha_4^3 \alpha_5^2 \alpha_6)^{-1}$
E_7	$(\alpha_1{}^2 \alpha_2^{7/2} \alpha_3^4 \alpha_4^6 \alpha_5^{9/2} \alpha_6^3 \alpha_7^{3/2})^{-1}$
E_8	$(\alpha_1^2 \alpha_2^4 \alpha_3^5 \alpha_4^8 \alpha_5^7 \alpha_6^5 \alpha_7^3 \alpha_8)^{-1}$
F_4	$(\alpha_1^2 \alpha_2^3 \alpha_3^4 \alpha_4^2)^{-1}$
G_2	$(\alpha_1^2 \alpha_2)^{-1}$

Corollary B. *Let G, K, B, A, A^+ and F be as in Theorem A and let H be a closed subgroup of G. If the restriction $F|_H$ of F to H is in $L^{1-\epsilon}(H)$ for some $\epsilon > 0$, then H is a (G, K)-tempered subgroup of G.*

Remark 1. Suppose further that H is a connected semisimple Lie subgroup of G such that $A \cap H$ is a maximal $\mathbb{R}$-split torus of H and $B \cap H$ is a minimal parabolic subgroup of H. Let δ_H denote the modular function of $B \cap H$, that is, the product of all positive roots including the multiplicity. Let $\lambda_1, \cdots, \lambda_m$ be the fundamental weights of the Lie algebra of H corresponding to $A \cap H$ and $B \cap H$. For any two weights α and β of the Lie algebra of H, we write $\alpha < \beta$ if $(\alpha, \lambda_j) < (\beta, \lambda_j)$ for all $1 \leq j \leq m$.

Then the condition $F|_H \in L^{1-\epsilon}(H)$ is equivalent to

$$-\log F|_H > \log \delta_H;$$

which is again equivalent to the condition $F|_H \in L^1(H)$.

2. If the restriction $F|_H$ is $L^{k-\epsilon}(H)$-integrable for some $\epsilon > 0$ and some positive integer k, then the diagonal embedding $\delta_k(H)$ of H into the group $\prod_{i=1}^k G_i$ is a $(\prod_{i=1}^k G_i, \prod_{i=1}^k K_i)$-tempered subgroup of $\prod_{i=1}^k G_i$ where $G_i = G$ and $K_i = K$ for all $1 \leq i \leq k$. To see this, it is enough to note that for any non-trivial irreducible spherical representation ρ of $\prod_{i=1}^k G_i$, the restrictions of the K-matrix coefficients of ρ to $\delta_k(H)$ are bounded by $(F|_H)^{k(1-\epsilon)}$.

For a unitary representation ρ of G, ρ is said to be strongly L^q if there is a dense subset V in the Hilbert space attached to ρ such that the matrix coefficients of ρ with respect to the vectors in V lie in $L^q(G)$. Let $p(G)$ be the smallest real number such that for any $\rho \in \hat{G}$, ρ is strongly L^q for any $q > p(G)$ (cf. [6]). Similarly let $p_K(G)$ be the smallest real number such that for any $\rho \in \hat{G}_K$, the K-matrix coefficients of ρ are $L^q(G)$-integrable for any $q > p_K(G)$. The estimate of the Harish-Chandra function Ξ of G shows that $p_K(G)$ is at least 2 (cf. [1]) and hence G cannot be a (G, K)-tempered subgroup of itself. The method used in proving Theorem A yields upper bounds for both $p(G)$ and $p_K(G)$.

Corollary C. *Let G, B, A, A^+, Φ, Φ' and $\alpha_1, \cdots, \alpha_n$ be as in Theorem A and let δ be the modular function of B. Define*

$$r(G) = \max\{\frac{\text{the power of } \alpha_i \text{ in } \delta}{\text{the power of } \alpha_i \text{ in } F^{-1}} \mid i = 1, \cdots, n\}.$$

Then $p(G) \leq r(G)$ and $p_K(G) \leq r(G)$.

If G is split over $\mathbb{R}$, $r(G)$ is as follows:

$\Phi = \Phi'$:	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
$r(G)$:	$2n$	$2n$	$2n$	$2(n-1)$	16	18	58	11	6

For $n \geq 3$, Vogan's classification of unitary duals for $GL_n(D)$ yields that for $G = SL_n(D)$, $D = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, $p(G)$ is $2(n-1)$, $2(n-1)$ and $2n-1$ respectively and for $Sp_{2n}(\mathbb{R})$, it follows from Howe's result in [1] that $p(G) = 2n$. The number $p(G)$ in other classical group cases was 0 calculated by Li [6] and was given an upper bound by Li and Zhu [7] in exceptional split group cases. We remark that the numbers $r(G)$ in Corollary C coincide with $p(G)$ calculated in [6] for all classical real split groups except B_n type. For a split group of type E_6, by obtaining $r(G) = 16$, we improve the bound for $p(G)$ in [7].

We state a theorem which yields a necessary and sufficient condition for a closed semisimple subgroup to be a (G, K)-tempered subgroup in a simple real split linear Lie group G of type A_n or C_n. Let G be either $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. The group $Sp_{2n}(\mathbb{R})$ is defined by the bi-linear form $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where $\bar{I}_n$ denotes the skew diagonal $n \times n$-identity matrix. Set $K = SO_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R})$ respectively. Define the parabolic subgroup P of G as follows:

$$\text{for } G = SL_n(\mathbb{R}), \qquad P = \{(g_{ij}) \in G \mid g_{i1} = 0 \text{ if } i \neq 1\},$$

$$\text{for } G = Sp_{2n}(\mathbb{R}), \qquad P = \{(g_{ij}) \in G \mid g_{i1} = 0, g_{nj} = 0 \text{ if } i \neq 1, j \neq n\}.$$

Note that P is the maximal parabolic subgroup which stabilizes the line $\mathbb{R}e_1$. We fix an ordering in the root system of G so that the positive Weyl chamber A^+ is as follows:

$$SL_n(\mathbb{R}), A^+ = \{\mathrm{diag}(a_1, \cdots, a_n) \mid \prod_{i=1}^{n} a_i = 1,\ a_i \geqq a_{i+1} \text{ for all } 1 \leq i \leq n-1\},$$

$$Sp_{2n}(\mathbb{R}), A^+ = \{\mathrm{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1}) \mid a_1 \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq n-1\}.$$

Example. The function F defined in Theorem A is as follows:

$$\text{for } G = SL_n(\mathbb{R}), \qquad F(a) = \begin{cases} \prod_{i=1}^{n/2} {a_i}^{-1} & \text{for n even} \\ \left(\prod_{i=1}^{(n-1)/2} {a_i}^{-1}\right) {a_{(n+1)/2}}^{-1/2} & \text{for n odd} \end{cases}$$

$$\text{for } G = Sp_{2n}(\mathbb{R}), \qquad F(a) = \prod_{i=1}^{n} {a_i}^{-1}$$

where $a \in A^+$.

Theorem D. *Let G be $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$ and P, K and A^+ be as above.*

(1) *For any $\epsilon > 0$ there exist constants C_1 and C_2 such that*

$$C_1 \cdot F(a) \leq |\langle \mathrm{Ind}_P^G(I)(a) f_0, f_0 \rangle| \leq C_2 \cdot F(a)^{1-\epsilon}$$

for any $a \in A^+$ and for any K-invariant unit vector f_0 in $\mathrm{Ind}_P^G(I)$.

(2) *If a closed subgroup H of G is (G, K)-tempered, $F|_H$ is in $L^1(H)$.*

(3) *A closed semisimple subgroup H of G is (G, K)-tempered if and only if $F|_H$ is in $L^1(H)$.*

(4) *$p_K(G) = r(G) = p(G)$.*

Acknowledgment. I am very grateful to Professors Roger Howe and Gregory Margulis for valuable suggestions and for their encouragement. Most of this paper was written during my visit at Bielefeld University. I would like to thank Professor Abels and the Mathematics department there for their warm hospitality.

2. A maximal system of strongly orthogonal roots in each irreducible root system

2.1. Let Φ be an irreducible reduced root system with a fixed ordering. Denote by Φ^+ the set of positive roots and by $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ the set of simple roots of Φ. The subscripts of α_i's are determined by the following choice of the highest root.

Φ	the highest root
A_n	$\alpha_1 + \alpha_2 + \cdots + \alpha_n$
B_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n$
C_n	$2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$
D_n	$\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$
G_2	$3\alpha_1 + 2\alpha_2$

We define the number $N(\Phi)$ as follows:

$$N(\Phi) = \begin{cases} [\frac{n+1}{2}], & \text{for } \Phi = A_n \\ 2[\frac{n}{2}], & \text{for } \Phi = D_n \\ 4, & \text{for } \Phi = E_6 \\ \operatorname{rank}(\Phi)\ , & \text{for } \Phi = B, C, F_4, G_2, E_7, E_8 \end{cases}$$

2.2. Construction of some strongly orthogonal roots. Two roots α and β are called *strongly orthogonal* if neither one of $\alpha \pm \beta$ is a root. Consider the family $\mathcal{S}(\Phi)$ of all subsets of Φ^+ whose elements are pairwise strongly orthogonal. We call an element $\vee$ in $\mathcal{S}(\Phi)$ a *strongly orthogonal system*. Let f be the function on $\mathcal{S}(\Phi)$ given by $f(\mathcal{O}) = \sum_{\alpha \in \mathcal{O}} \alpha$. The aim of this section is to construct an element $\mathcal{Q}(\Phi) = \{\gamma_1, \cdots, \gamma_{N(\Phi)}\}$ in $\mathcal{S}(\Phi)$ on which f attains its maximum. For simplicity, we set $N(\Phi) = N$.

We define $\mathcal{Q}(\Phi)$ as follows:

Φ	$\mathcal{Q}(\Phi)$
A_n	$\begin{cases} \gamma_i = \alpha_i + \cdots + \alpha_{n-i+1} \text{ for } i \leq N-1 \\ \gamma_N = \begin{cases} \alpha_N & \text{for } n \text{ odd} \\ \alpha_N + \alpha_{N+1} & \text{for } n \text{ even} \end{cases} \end{cases}$
B_n	$\begin{cases} \gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_n \\ \gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} \text{ for } i \leq [n/2] \\ \gamma_n = \alpha_{(n+1)/2} + \cdots + \alpha_n \text{ for } n \text{ odd} \end{cases}$
C_n	$\begin{cases} \gamma_i = 2\alpha_i + \cdots + 2\alpha_{n-1} + \alpha_n \text{ for } i \leq N-1 \\ \gamma_N = \alpha_n \end{cases}$
D_n	$\begin{cases} \gamma_1 = \alpha_1 + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_{n-2} + \alpha_n \\ \gamma_2 = \alpha_1 + \cdots + \alpha_{n-1} \\ \gamma_{2i-1} = \alpha_i + \cdots + \alpha_{n-i} + 2\alpha_{n-i+1} + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n \\ \gamma_{2i} = \alpha_i + \cdots + \alpha_{n-i} \text{ for } 3 \leq i \leq [2/n] \end{cases}$
E_6	$\begin{cases} \gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\ \gamma_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\ \gamma_3 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_4 = \alpha_2 \end{cases}$
E_7	$\begin{cases} \gamma_1 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_2 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_3 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 \\ \gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 \\ \gamma_5 = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_6 = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 \\ \gamma_7 = \alpha_2 \end{cases}$

$$E_8 \quad \begin{cases} \gamma_1 = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7 + \alpha_8 \\ \gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \\ \gamma_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8 \\ \gamma_4 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8 \\ \gamma_5 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 \\ \gamma_6 = \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 \\ \gamma_7 = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 \\ \gamma_8 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 \end{cases}$$

$$F_4 \quad \begin{cases} \gamma_1 = \alpha_1 + 2\alpha_2 + 4\alpha_3 + 2\alpha_4 \\ \gamma_2 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 \\ \gamma_3 = \alpha_1 + 2\alpha_2 + 2\alpha_3 \\ \gamma_4 = \alpha_1 \end{cases}$$

$$G_2 \quad \begin{cases} \gamma_1 = 3\alpha_1 + 2\alpha_2 \\ \gamma_2 = \alpha_1 \end{cases}$$

The following lemma can be checked.

Lemma. *The set $\mathcal{Q}(\Phi)$ is a strongly orthogonal system.*

2.3. Proposition. *$f(\mathcal{Q}(\Phi)) = \max_{\mathcal{O}\in\mathcal{S}(\Phi)} f(\mathcal{O})$, that is, for any $\mathcal{O} \in \mathcal{S}(\Phi)$, the coefficient of α_i in $f(\mathcal{Q}(\Phi))$ is greater than or equal to that of α_i in $f(\mathcal{O})$ for all $1 \le i \le n$ where n is the rank of Φ.*

Proof. Let $\mathcal{O}$ be any element in $\mathcal{S}(\Phi)$. We prove this proposition by induction. We can easily check that the proposition is true for $n = 2$. Suppose that $n \ge 3$.

For $\Phi = A_n$, take any element in $\mathcal{O}$, say $\alpha = \epsilon_i - \epsilon_j$. Then $\alpha \le \gamma_1$ since γ_1 is the highest root. On the other hand $\mathcal{O} - \{\alpha\}$ is contained in $\{\epsilon_m - \epsilon_l \mid m, l \notin \{i, j\}\}$, which is a root system of type A_{n-2}. Note that $\mathcal{Q}(A_n) \cap \{\epsilon_m - \epsilon_l \mid m, l \notin \{i, j\}\} = \mathcal{Q}(A_{n-2})$. Therefore by the induction assumption, $f(\mathcal{O} - \{\alpha\}) \le f(\mathcal{Q}(A_{n-2}))$. Hence we have $f(\mathcal{O}) \le \gamma_1 + f(\mathcal{Q}(A_{n-2})) \le f(\mathcal{Q}(A_n))$, proving the claim.

For $\Phi = B_n$, it is easy to check that if three positive roots are mutually strongly orthogonal, then the coefficient of α_1 in at least one of them is o. Note that for any $\alpha \in \Phi^+$, we have that the coefficient of α_1 in α is not bigger than 1. Write $\mathcal{O}$ as $\mathcal{O}_1 \cup \mathcal{O}_2$ so that $\beta \in \mathcal{O}_1$ if and only if the coefficient of α_1 in β is 1 and $\mathcal{O}_2 = {\mathcal{O}_1}^c$. Note that $\gamma_1 + \gamma_2 = \sum_{i=1}^n 2\alpha_i$. We can easily check that for any two strongly orthogonal $\beta_1, \beta_2 \in \Phi^+$ such that the coefficient of α_1 in β_i is 1 for both $i = 1, 2$, we have $\beta_1 + \beta_2 \le \sum_{i=1}^n 2\alpha_i$; hence $\sum_{\beta\in\mathcal{O}_1} \beta \le \gamma_1 + \gamma_2$ because $|\mathcal{O}_1| \le 2$. For $\theta \subset \Delta$, the notation $[\theta]$ denotes the set of the roots in Φ which can be expressed as integral combinations of the roots in θ. Since

$\mathcal{O}_2 \subset [\alpha_2, \cdots, \alpha_n]$, $\gamma_3 + \gamma_4 = \sum_{i=2} 2\alpha_i$ and $[\alpha_2, \cdots, \alpha_n]$ is a root system of type B_{n-1}, we can proceed by induction as before.

The argument for D_n is exactly the same; so we omit it.

If Φ is of type C_n, write $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ so that $\beta \in \mathcal{O}_1$ if and only if the coefficient of α_1 in β is positive and $\mathcal{O}_2 = {\mathcal{O}_1}^c$. It is easy to see that $|\mathcal{O}_1| \leq 1$. Therefore $\sum_{\alpha \in \mathcal{O}_1} \alpha \leq \gamma_1$ for γ_1 is the highest root in Φ. Since $\mathcal{O}_2 \subset [\alpha_2, \cdots, \alpha_n]$, it remains to use induction process.

For exceptional root system cases, we can prove the proposition by checking each root system case by case. □

As a corollary of the above proposition, we obtain that $\mathcal{Q}(\Phi)$ is a maximal element in $\mathcal{S}(\Phi)$ with respect to the inclusion ordering.

Remark. I learned from E. Vinberg that this construction of a strongly orthogonal system coincides with the so called Kostant's cascade construction (cf. [3]) if Φ is one of the types A_n, C_n or G_2. But in all cases the cardinalities of the sets in Kostant's cascade construction coincide with the numbers $N(\Phi)$, which are the cardinalities of $\mathcal{Q}(\Phi)$ in our construction. We note that not all maximal strongly orthogonal systems in Φ have the same cardinality. For example, $\{\alpha_2, \alpha_4, 2\alpha_2 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4\}$ is a maximal strongly orthogonal system in the root system of F_4.

We remark that if Φ is none of A_n, C_n or G_2, the function f attains its maximum in our construction but not in Kostant's cascade construction.

3. An upper bound function for matrix coefficients in simple non-compact linear Lie groups

3.1. In the following let G be a connected simple linear real Lie group with real rank at least 2, B a minimal parabolic subgroup, A a maximal $\mathbb{R}$-split torus contained in B, A^+ the positive Weyl chamber and K a maximal compact subgroup. Consider a Cartan decomposition of $G: G = KA^+K$. Since the K-matrix coefficients of a spherical unitary representation are K-bi-invariant, they are determined by its restriction to the A^+-part.

It is known that the induced representation $\mathrm{Ind}_B^G(I)$ of the trivial representation of B is irreducible and has a unique (up to a sign) K-invariant unit vector, say f_0. The matrix coefficient of $\mathrm{Ind}_B^G(I)$ defined by $g \mapsto \langle \mathrm{Ind}_B^G(I)(g)f_0, f_0\rangle$ is called the *Harish-Chandra function* of G, which we will denote by Ξ_G. When there is no confusion, Ξ_G will simply be denoted by Ξ.

If we fix a Haar measure dg on G, then the modular function δ of B is the product of all the positive roots of A.

Harish-Chandra has shown the following.

Proposition. (cf. [1]) *For any $\epsilon > 0$, there exist constants c_1 and c_2 such that*

$$c_1\delta^{-1/2}(a) < \Xi(a) < c_2\delta^{-1/2+\epsilon}(a)$$

for all $a \in A^+$.

In particular the value of Harish-Chandra function Ξ of $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$ at $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ for $a > 1$ is asymptotically $(\log a)/a$ up to some constant multiple.

3.2. We can write the Haar measure dg of G in terms of the Cartan decomposition KA^+K as follows:

$$dg = \Delta(a)dk_1dadk_2$$

where $\Delta(a)$ is a positive function on A^+ satisfying $d_1(t)\delta(a) \leqq \Delta(a) \leqq d_2\delta(a)$ for all $a \in \{a \in A^+ \mid |\alpha(a)| \geq t \text{ for all } \alpha \in \Phi^+\}$ and for some constants $d_1(t)$ and d_2 if $t > 1$.

For a K-matrix coefficient $\phi(g) = \langle \rho(g)v, w\rangle$, $\rho \in \hat{G_K}$, it is well known that $\phi \in L^p(G)$ if and only if $\int_{A^+} |\phi(a)|^p\delta(a)da < \infty$.

3.3. Proposition. *Let H be $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$. Suppose that for some $k \geq 2$, H acts on $\mathbb{R}^k$ by a rational representation so that the only H-invariant vector is the origin. Let $H \ltimes \mathbb{R}^k$ be the associated semidirect product. Let ρ be a unitary representation of $H \ltimes \mathbb{R}^k$ without any $\mathbb{R}^k$-invariant vectors. Then we have*

$$|\langle \rho|_H(h)v, w\rangle| \leq \Xi_H(h)(\dim\langle K \cdot v\rangle \dim\langle K \cdot w\rangle)^{1/2}$$

where $h \in H$, $K = SO_2(\mathbb{R})$ and v and w are K-finite unit vectors.

In 0 if ρ is spherical, then the K-matrix coefficients of $\rho|_H$ are bounded by Ξ_H.

Proof. By [9, Theorem 7.3.9], the restriction $\rho|_H$ of ρ to H is weakly contained in the infinite sum of the regular representation of H. It is well known (cf. [2, Ch V, Theorem 3.2 1]) that the K-finite (K-fixed) matrix coefficients of the regular representation of H satisfy the above inequality. □

In the spirit of Howe's strategy (see [2]) we state the following proposition.

The notation $\mathfrak{u}_\alpha$ for $\alpha \in \Phi$ denotes the root space in the Lie algebra of G corresponding to α.

3.4. Proposition. *Let G be split over $\mathbb{R}$. Denote by Φ^+ the set of positive roots in $\Phi(G, A)$. Let $\{\beta_1, \cdots, \beta_m\} \subset \Phi^+$ be a strongly orthogonal system. For each $1 \leq i \leq m$,* *let H_i be the connected closed subgroup of G whose Lie algebra is generated by $\mathfrak{u}_{\pm\beta_i}$. Then for any $\rho \in \hat{G}$ we have*

$$|\langle \rho(a)v, w\rangle| \leq C \cdot \prod_{i=1}^{m} \exp\left(-\frac{1}{2}\beta_i(\log a) + \epsilon\right)$$

where $a \in A^+$, v and w are K-finite vectors, and C is a constant depending on v and w.

Proof. Note that for each $1 \leq i \leq m$, (1) H_i is isomorphic to $SL_2(\mathbb{R})$ or $PSL_2(\mathbb{R})$; (2) the subgroups $H_i \cap B$, $H_i \cap A$ and $H_i \cap K$ are a minimal parabolic subgroup, a maximal $\mathbb{R}$-split torus and a maximal compact subgroup of H_i respectively; (3) $A^+(H_i) \subset A^+$.

It is not difficult to see that $A^+ \subset \prod_{i=1}^m A^+(H_i)C_G(\prod_{i=1}^m H_i)$ where $C_G(\prod_{i=1}^m H_i)$ denotes the centralizer of $\prod_{i=1}^m H_i$. Since $\{\beta_1, \cdots, \beta_m\}$ is a strongly orthogonal system, it follows that $x_i x_j = x_j x_i$ for all $i \neq j$, $x_i \in H_i$ and $x_j \in H_j$. By looking at the root system, it is not difficult to see that for each H_i, there exists an abelian unipotent subgroup U_i of G of dimension at least 2 such that H normalizes U_i and $C_G(H_i) \cap U_i$ is trivial. It follows that for each $1 \leq i \leq m$, $H_i \ltimes \mathbb{R}^{k_i}$ is embedded into G where the semidirect product is as described in Proposition 3.3 and $k_i = \dim U_i$.

Let $\rho \in \hat{G}_K$ and v, w be K-fixed vectors. The restriction $\rho|_{\prod_{i=1}^m H_i}$ can be written as a direct integral $\int_X \rho_\alpha d\mu(\alpha)$ where X is some Borel measure space with measure μ, $\alpha \in X$ and ρ_α is an irreducible representation of $\prod_{i=1}^m H_i$ for all $\alpha \in X$. Without loss of generality we may assume that all ρ_α's are non-trivial spherical representations. Any element $a \in A^+$ is uniquely written as $a = a_1 \cdots a_m c$ where $a_i \in A^+(H_i)$ and $c \in C_G(\prod_{i=1}^m H_i)$. Write $\rho(c)v$ and w as $\int v_\alpha d\mu(\alpha)$ and $\int w_\alpha d\mu(\alpha)$ respectively where v_α and w_α are vectors in ρ_α. Since c centralizes H_i, $\rho(c)v$ is $K \cap H_i$-fixed for all $1 \leq i \leq m$. Therefore there is no loss of generality in assuming that for all $\alpha \in X$, v_α and w_α are $K \cap H_i$-fixed for all $1 \leq i \leq m$. Fix $\alpha \in X$. Then $\rho_\alpha| \prod_{i=1}^m H_i = \otimes_{i=1}^m \rho_{\alpha i}$, $v_\alpha = \otimes_{i=1}^m v_{\alpha i}$ and $w_\alpha = \otimes_{i=1}^m w_{\alpha i}$ where $\rho_{\alpha i}$ is a spherical irreducible representation of H_i and $v_{\alpha i}$ and $w_{\alpha i}$ are $K \cap H_i$-fixed vectors for each $1 \leq i \leq m$. By Moore's theorem (cf. [9, Theorem 2.1.9]), $\rho_{\alpha i}$ is non-trivial and $\rho_{\alpha i}$ has no U_i-invariant vector for each $1 \leq i \leq m$.

By Proposition 3.3, we obtain that for each $1 \leq i \leq m$,

$$|\langle \rho_{\alpha i}(a_i)v_{\alpha i}, w_{\alpha i}\rangle| \leq \Xi_{H_i} \|v_{\alpha i}\| \|w_{\alpha i}\|.$$

Hence

$$\begin{aligned}
|\langle \rho(a)v, w\rangle| &\leq \int_\alpha |\langle \rho_\alpha(\prod_{i=1}^m a_i)v_\alpha, w_\alpha\rangle| d\mu(\alpha) \\
&\leq \int_\alpha \prod_{i=1}^m |\langle \rho_{\alpha i}(a_i)v_{\alpha i}, w_{\alpha i}\rangle| d\mu(\alpha) \\
&\leq \int_\alpha \prod_{i=1}^m \left(\Xi_{H_i}(a_i)\|v_{\alpha i}\| \cdot \|w_{\alpha i}\|\right) d\mu(\alpha) \\
&= \prod_{i=1}^m \Xi_{H_i}(a_i)\|\rho(c)v\| \cdot \|w\| \leq \prod_{i=1}^m \Xi_{H_i}(a_i)\|v\| \cdot \|w\|.
\end{aligned}$$

Note that the modular function δ_{H_i} is equal to $\exp(\beta_i)$. Hence by Proposition 3.1, $\Xi_{H_i}(a_i) = \exp(-\frac{1}{2}\beta_i(\log a) + \epsilon)$ for some $\epsilon > 0$. This proves the proposition when ρ is spherical and v and w are K-fixed. It is not difficult to generalize this proof for non-spherical representation ρ and v, w K-finite vectors. $\square$

3.5. Proof of Theorem A. It is well known that G contains a connected simple closed subgroup G_0 such that G_0 is split over $\mathbb{R}$, rank $G_0 = \mathbb{R}$-rank G and Φ' is isomorphic to $\Phi(G_0, G_0 \cap A)$ where Φ' is the set of all non-multipliable roots in $\Phi(G, A)$. Recall the strongly orthogonal system $\mathcal{Q}(\Phi') = \{\gamma_1, \cdots, \gamma_{N(\Phi')}\}$ for Φ' we constructed in section 2.2.

The notation $\mathfrak{u}_\alpha$ for $\alpha \in \Phi'$ denotes the one-dimensional root subalgebra of $\mathfrak{g}_0 =$ Lie (G_0). Set $N = N(\Phi')$. For each $1 \leq i \leq N$, we define H_i to be the connected closed subgroup of G_0 whose Lie algebra is generated by $\mathfrak{u}_{\pm\gamma_i}$.

The proof of Theorem A in the case when $\Phi' \neq D_{n=2k+1}$ is obtained by simply applying Proposition 3.4 with $\mathcal{Q}(\Phi')$ and H_i's.

Case ($\Phi' \neq D_{n=2k+1}$): We note that the function F in Theorem A is equal to $\prod_{i=1}^{N(\Phi')} \delta_{H_i}{}^{-1/2}$ or equivalently, $F = \prod_{i=1}^{N(\Phi')} \exp(-\frac{1}{2}\gamma_i)$. Then by Proposition 3.4, this proves Theorem A.

Case ($\Phi' = D_{n=2k+1}$): In this case we define H'_N to be the connected closed subgroup of G_0 whose Lie algebra is generated by $\mathfrak{u}_{\pm(\alpha_{k+1}+2(\alpha_{k+2}+\cdots+\alpha_{n-2})+\alpha_{n-1}+\alpha_n)}$, $\mathfrak{u}_{\pm\alpha_k}$ and $\mathfrak{u}_{\pm\alpha_{k+1}}$. Note that the Lie algebra of H'_N is isomorphic to that of $SO(3,3)$. We have that $\delta_{H'_N}^{1/4} = \exp(\alpha_k)\left(\prod_{i=k+1}^{n-2}\exp(\frac{3}{2}\alpha_i)\right)\exp(\frac{3}{4}\alpha_{n-1})\exp(\frac{3}{4}\alpha_n)$.

By [6, Lemma 4.1], the restriction $\rho|_{H'_N}$ is strongly $L^{4+\epsilon}$ for any $\rho \in \hat{G}$. This implies (see [1, Corollary 7.2]) that the restrictions to H'_N of the K-finite matrix coefficients of ρ are bounded by $\Xi_{H'_N}{}^{1/2}$. It is not difficult to see from the proof of Proposition 3.4 that when we replace H_N by $H_N{}'$, a statement similar to Proposition 3.4 holds, that is, the K-matrix coefficients of ρ for any $\rho \in \hat{G}_K$ are bounded above by $(\prod_{i=1}^{N-1} \delta_{H_i}{}^{-1/2})\delta_{H'_N}{}^{-1/4}$. Therefore it remains to observe that the function F in Theorem

A coincides with $\left(\prod_{i=1}^{n-2} \exp(-\frac{1}{2}\gamma_i)\right) \exp(\alpha_k) \left(\prod_{i=k+1}^{n-2} \exp(\frac{3}{2}\alpha_i)\right) \exp(\frac{3}{4}\alpha_{n-1}) \exp(\frac{3}{4}\alpha_n)$, which is equal to $(\prod_{i=1}^{n-2} \delta_{H_i}{}^{-1/2})\delta_{H'_N}{}^{-1/4}$ to complete the proof.

Remark. The results in section 2.2 show that the function F is the best possible upper bound for K-matrix coefficients, which can be obtained using Proposition 3.4 when $\Phi' \neq D_{n=2k+1}$. Note that when $\Phi' = D_{n=2k+1}$, we improved F by replacing one $SL_2(\mathbb{R})$ by $SO(3,3)$.

3.6. Corollary. *With the same notation as in Proposition 3.4, if* $\vee = \{\beta_1, \cdots, \beta_t\}$ *is a strongly orthogonal system of* Φ' *and* $r \cdot \left(\sum_{i=1}^{t} \beta_i\right) > 2\log(\delta_G)$ *for some number* r*, then we have*

$$p(G) \leq r \text{ and } p_K(G) \leq r.$$

3.7. Proof of Corollary C. Corollary C easily follows from the remark 1 after Corollary B.

The modular function δ for a Borel subgroup B in each simple real-split Lie group G is given as below, from which the remark after Corollary C follows.

Φ	δ
A_n	$\prod_{i=1}^{n} \alpha_i{}^{i(n-i+1)}$
B_n	$\left(\prod_{i=1}^{n-1} \alpha_i{}^{2ni-i^2}\right) \alpha_n{}^{n^2}$
C_n	$\left(\prod_{i=1}^{n} \alpha_i^{2ni-i^2+i}\right) \alpha_n^{n(n+1)/2}$
D_n	$\left(\prod_{i=1}^{n-2} \alpha_i{}^{2ni-i^2-i}\right) (\alpha_{n-1}\alpha_n)^{n(n-1)/2}$
E_6	$\alpha_1^{16}\alpha_2^{22}\alpha_3^{30}\alpha_4^{42}\alpha_5^{30}\alpha_6^{16}$
E_7	$\alpha_1^{34}\alpha_2^{49}\alpha_3^{66}\alpha_4^{96}\alpha_5^{75}\alpha_6^{52}\alpha_7^{27}$
E_8	$\alpha_1^{92}\alpha_2^{136}\alpha_3^{182}\alpha_4^{270}\alpha_5^{220}\alpha_6^{168}\alpha_7^{114}\alpha_8^{58}$
F_4	$\alpha_1^{16}\alpha_2^{30}\alpha_3^{42}\alpha_4^{22}$
G_2	$\alpha_1^{10}\alpha_2^{6}$

4. Spherical unitary representations with minimal decay in $SL_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R})$

4.1. In this section we will show that the upper bound function F we obtained in Theorem A is the best possible when $G = SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. This will be proved by showing that there exists a spherical unitary representation of G whose K-matrix coefficient is bounded from below by a constant multiple of F. Those representations are the induced representations $\mathrm{Ind}_P^G(I)$ of the trivial representation where P is the maximal parabolic subgroup which stabilizes the line $\mathbb{R}e_1$.

4.2. For the rest of section 4, let G be either $SL_n(\mathbb{R})$ or $Sp_{2n}(\mathbb{R})$. The group $Sp_{2n}(\mathbb{R})$ is defined by the bi-linear form $\begin{pmatrix} 0 & \bar{I}_n \\ -\bar{I}_n & 0 \end{pmatrix}$ where $\bar{I}_n$ denotes the skew diagonal $n \times n$-identity matrix. Set $K = SO_n(\mathbb{R})$ and $Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R})$ respectively. Define the maximal parabolic subgroup P of G as follows:

$$\text{for } G = SL_n(\mathbb{R}), \qquad P = \{(g_{ij}) \in G \mid g_{i1} = 0 \text{ if } i \neq 1\};$$

$$\text{for } G = Sp_{2n}(\mathbb{R}), \qquad P = \{(g_{ij}) \in G \mid g_{i1} = 0, g_{nj} = 0 \text{ if } i \neq 1, j \neq n\}.$$

We fix an ordering in the root system of G so that the positive Weyl chamber A^+ is as follows:

$$SL_n(\mathbb{R}), A^+ = \{\mathrm{diag}(a_1, \cdots, a_n) \mid \prod_{i=1}^{n} a_i = 1,\ a_i \geqq a_{i+1} \text{ for all } 1 \leq i \leq n-1\};$$

$$Sp_{2n}(\mathbb{R}), A^+ = \{\mathrm{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1{}^{-1}) \mid a_1 \geq a_{i+1} \geq 1 \text{ for all } 1 \leq i \leq n-1\}.$$

4.3. We recall the formula for the matrix coefficients of the induced representation $\mathrm{Ind}_P^G(I)$. Consider the Langlands decomposition of P: $P = MA_PN$. Denote by $\bar{N}$ the unipotent radical of the opposite parabolic subgroup to P with the common Levi subgroup MA_P.

If g decomposes under the Iwasawa decomposition $G = KMA_PN$ we denote by $\exp H(g)$ the A_P-component of g. It is well known that the representation space of $\mathrm{Ind}_P^G(I)$ of the trivial representation I of P can be realized as $L^2(\bar{N}, d\bar{n})$. If g decomposes under $\bar{N}MA_PN$ as

$$g = \bar{n}(g)m(g)a(g)n(g),$$

then the action is given by

$$\mathrm{Ind}_P^G(I)(g)f(\bar{n}) = e^{-\delta_0(\log a(g^{-1}\bar{n}))} f(\bar{n}(g^{-1}\bar{n})) \text{ for any } f \in L^2(\bar{N}, d\bar{n})$$

where δ_0 is the half sum of positive N-roots (cf. [4]).

Lemma. *Define the vector f_0 of* $\mathrm{Ind}_P^G(I)$ *as follows:*

$$f_0(\bar{n}) = e^{-\delta_0(H(\bar{n}))}.$$

Then f_0 is K-fixed.

Proof. We have to show that $e^{-\delta_0(\log a(k^{-1}\bar{n}))}e^{-\delta_0(H(\bar{n}(k^{-1}\bar{n})))} = e^{-\delta_0(H(\bar{n}))}$ for all $k \in K$.

Let $\bar{n} = k_1 m_1 a_1 n_1$ be the decomposition of $\bar{n}$ under KMA_PN and $k^{-1}\bar{n} = \bar{n}_2 m_2 a_2 n_2$ the decomposition of $k^{-1}\bar{n}$ under $\bar{N}MA_PN$. Then $a_1 = \|\bar{n}.e_1\|$ and $a_2 = \|\bar{n}.e_1\|/\|\bar{n}_2.e_1\|$. If $\bar{n}_2 = k_3 m_3 a_3 n_3$ is the Cartan decomposition of $\bar{n}_2$, we have $a_3 = \|\bar{n}_2.e_1\|$. Hence $a_1 = a_2 a_3$. Since $\log(a_1) = H(\bar{n})$, $\log(a_3) = H(\bar{n}(k^{-1}\bar{n}))$ and $\log(a_2) = \log(a(k^{-1}\bar{n}))$, this proves the lemma. □

The matrix coefficient of $\mathrm{Ind}_P^G(I)$ with respect to f_0 is as follows:

$$\langle \mathrm{Ind}_P^G(I)(g)f_0, f_0\rangle = \int_{\bar{N}} e^{-\delta_0(\log a(g^{-1}\bar{n}))}e^{-\delta_0(H(\bar{n}(g^{-1}\bar{n})))}e^{-\delta_0(H(\bar{n}))}d\bar{n}.$$

4.4. Theorem D obviously follows from the following proposition and Theorem A.

Proposition. *For any $\epsilon > 0$, there exists a constant C such that*

$$C \cdot F(a) \leq |\langle \mathrm{Ind}_P^G(I)(a)f_0, f_0\rangle|$$

where $a \in A^+$ and F is as in Theorem A

In the following two sections we will prove Proposition 4.4. We will sometimes omit the exponential notation.

Proof of Proposition 4.4.

Case: $G = SL_n(\mathbb{R}), n \geq 3$

Denote by $\tilde{a}$ the matrix $\mathrm{diag}(a_1, \cdots, a_n)$ where $a_i \geqq a_{i+1}$ for all $1 \leq i \leq n-1$ and $\prod_{i=1}^n a_i = 1$ and by $\bar{n}$ the matrix in $\bar{N}$ whose first column is $(1, x_2, \cdots, x_n)$, that is $\bar{n}.e_1 = (1, x_2, \cdots, x_n)$. To simplify notation, set $x_1 = 1$.

The decomposition of $a^{-1}\bar{n}$ under $\bar{N}MA_PN$ is as follows:

$$a(\tilde{a}^{-1}\bar{n}) = \mathrm{diag}(a_1, {a_1}^{-1/(n-1)}, \cdots, {a_1}^{-1/(n-1)}) \text{ and } \bar{n}(\tilde{a}^{-1}\bar{n}).e_1 = \left(1, \frac{a_1}{a_2}x_2, \cdots, \frac{a_1}{a_n}x_n\right).$$

Then $\delta_0(a(\tilde{a}^{-1}\bar{n})) = {a_1}^{-n/2}$ and $H(\bar{n}) = \|\bar{n}.e_1\| = \sqrt{\sum_{i=1}^n {x_i}^2}$.

Therefore

$$\begin{aligned}\langle \mathrm{Ind}_P^G(I)(\tilde{a})f_0, f_0\rangle &= \int_{\bar{N}} {a_1}^{n/2}\|\bar{n}(\tilde{a}^{-1}\bar{n}).e_1\|^{-n/2} \cdot \|\bar{n}.e_1\|^{-n/2}d\bar{n} \\ &= \int_{\mathbb{R}^{n-1}} \left(\sum_{i=1}^n \left(\frac{1}{a_i}\right)^2 x_i^2\right)^{-n/4} \left(\sum_{i=1}^n x_i^2\right)^{-n/4} dm\end{aligned}$$

where dm is the standard measure in $\mathbb{R}^{n-1}$.

Set $k = [(n+1)/2]$ and let T be the set

$$\{(x_2, \cdots, x_n) \mid 0 \leq x_i \leq 1 \text{ for } 2 \leq i \leq k-1, 0 \leq x_k \leq 2, x_i \leq \frac{a_i}{a_k} x_k \text{ for } k+1 \leq i \leq n\}.$$

Note that if $(x_2, \cdots, x_n) \in T$, then for all $1 \leq i \leq n$, we have

$$x_i \leq 2 \quad \text{and} \quad \frac{x_i}{a_i} \leq \frac{x_k}{a_k}.$$

Thus for $(x_2, \cdots, x_n) \in T$, we have

$$\left(\sum_{i=1}^{n} \left(\frac{1}{a_i} \right)^2 x_i^2 \right)^{-n/4} \left(\sum_{i=1}^{n} x_i^2 \right)^{-n/4} \geq C \cdot \left(\frac{x_k}{a_k} \right)^{-n/2}$$

for some constant $C > 0$.

Therefore

$$|\langle \mathrm{Ind}_P^G(I)(\tilde{a}) f_0, f_0 \rangle| \geq C \cdot \int_T \left(\frac{x_k}{a_k} \right)^{-n/2} dm \geq C \cdot {a_k}^{n/2} \prod_{i=k+1}^{n} \left(\frac{a_i}{a_k} \right) \geq C \cdot F(a).$$

Case: $G = Sp_{2n}(\mathbb{R}), n \geq 2$

For $\tilde{a} = \mathrm{diag}(a_1, \cdots, a_n, a_n^{-1}, \cdots, a_1^{-1})$ where $a_i \geq a_{i+1} \geq 1$ for all $1 \leq i \leq n-1$, we have

$$\langle \mathrm{Ind}_P^G(I)(\tilde{a}) f_0, f_0 \rangle = \int_{\mathbb{R}^{2n-1}} \left(\sum_{i=1}^{n} \left(\frac{x_i}{a_i} \right)^2 + \sum_{i=1}^{n} (a_i y_i)^2 \right)^{-n/2} \left(\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 \right)^{-n/2} dm$$

where $x_1 = 1$. Let T be the set

$$\{(x_1, \cdots, x_n, y_n, \cdots, y_1) \mid 1 \leq y_n \leq 2, y_i \leq \frac{a_n}{a_i} y_n, 0 \leq x_i \leq 1, 0 \leq y_i \leq 2 \text{ for } 1 \leq i \leq n\}.$$

Note that if $(x_1, \cdots, x_n, y_n, \cdots, y_1) \in T$, then

$$x_i \leq a_i a_n y_n$$

since $a_i \geq 1$ for all $1 \leq i \leq n$. Therefore

$$|\langle \mathrm{Ind}_P^G(I)(\tilde{a}) f_0, f_0 \rangle| \geq C \cdot \int_T (a_n y_n)^{-n} dm \geq C \cdot F(a)$$

where C is some positive constant.

5. (G, K)-TEMPERED SUBGROUPS AND FINITE DIMENSIONAL REPRESENTATIONS

5.1. Let H be a connected semisimple Lie group and π a 0 from H to $SL_n(\mathbb{R})$. Let B be the Borel subgroup of $SL_n(\mathbb{R})$ consisting of the upper triangular matrices, $K = SO_n(\mathbb{R})$ and A^+ the positive Weyl chamber such that

$$A^+ = \{\mathrm{diag}(a_1, \cdots, a_n) \mid a_i \geqq a_{i+1} \text{ for all } 1 \leq i \leq n-1\}.$$

Consider a minimal parabolic subgroup B_H of H and a Cartan decomposition of H: $H = K_H {A_H}^+ K_H$ where K_H is a maximal compact subgroup of H, A_H a maximal $\mathbb{R}$-split torus and A_H^+ is a positive Weyl chamber for a suitably chosen ordering so that $\pi(A_H^+) \subset A^+$. We assume that $\pi(K_H) \subset K$ and $\pi(B_H) \subset B$. We denote by δ_H the modular function of B_H. Let $\alpha_1, \cdots, \alpha_k$ be the simple roots and $\lambda_1, \ldots, \lambda_k$ the fundamental weights of the Lie algebra of H with respect to A_H^+. For any weights γ_1 and γ_2 of the Lie algebra of H, we write $\gamma_1 > \gamma_2$ if $(\gamma_1, \lambda_i) > (\gamma_2, \lambda_i)$ for all $1 \leq j \leq k$. If the weights $\beta_1, \ldots, \beta_n$ of π are such that $\beta_i(a)$ is the (i, i)-entry of $\pi(a)$ and $\beta_i \geq \beta_{i+1}$ for all $i = 1, \ldots, n-1$, then the following corollary follows from the remark (2) after Corollary B and Theorem D.

5.2. Corollary. *$\pi(H)$ is an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup if and only if the following holds:*

$$\begin{cases} \beta_1 + \cdots + \beta_{n/2} > \log(\delta_H) & \text{for } n \text{ even} \\ \beta_1 + \cdots + \beta_{(n-1)/2} + \frac{1}{2}\beta_{(n+1)/2} > \log(\delta_H) & \text{for } n \text{ odd.} \end{cases}$$

5.3. Let π be irreducible and λ the highest weight π. We denote by the lowest weight of π by $\Lambda(\lambda)$. Note that $(\Lambda(\lambda), \lambda_j) = -(\lambda, i(\lambda_j))$ where i is the opposition involution.

The following lemma is well known.

Lemma. *Let H and π be as in section 5.1. The representation π is self-dual if and only if $\lambda = -\Lambda(\lambda)$.*

In such cases, π is orthogonal (resp. symplectic) if $\sum_{j=1}^{k} \frac{1}{2}(\log \delta_H, \lambda_j)(\lambda, \lambda_j)$ is even (resp. odd).

If H is none of the type A_n, D_{2k+1} or E_6, then all finite dimensional irreducible representations of H are self-dual.

5.4. Corollary.

(1) *Let H be a connected simple non-compact real linear Lie group and π a finite dimensional irreducible representation of H with the highest weight λ. Suppose that*

$$\lambda - \Lambda(\lambda) > 2 \log \delta_H.$$

Then $\pi(H)$ is an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup where $n = \dim \pi$.

(2) *Furthermore assume that*

$$\lambda = -\Lambda(\lambda) \text{ and } \lambda > \log \delta_H.$$

Then if $\sum_{j=1}^{k} \frac{1}{2}(\log \delta_H, \lambda_j)(\lambda, \lambda_j)$ is even (resp. odd), $\pi(H)$ is an $(SO(n,n')(\mathbb{R}), SO(n,n')(\mathbb{R}) \cap SO_{n+n'}(\mathbb{R}))$-tempered subgroup (resp. $(Sp_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R}))$-tempered subgroup) where $n = [(\dim\pi)/2]$ and $n' = \dim\pi - n$.

Proof. (1) follows from a simple observation that $\beta_1 + \cdots + \beta_{[n/2]} \geqq (\beta_1 - \beta_n)/2$ since $\sum_{i=1}^{n} \beta_i = 0$. (2) follows from Lemma 5.3 and the inequality $\sum_{i=1}^{[(\dim\pi)/4]} \beta_i \geq \beta_1$, since $\beta_i \geq 0$ for all $1 \leq i \leq [(\dim\pi)/2]$. $\square$

Remark. It is easy to see that if $\lambda > \log \delta_H$, then $\lambda - \Lambda(\lambda) > 2 \log \delta_H$. Therefore it is enough to assume that $\lambda > \log \delta_H$ in (1).

Corollary 5.5. *Let π be a finite dimensional irreducible representation of $SL_{k+1}(\mathbb{R})$ with the highest weight $\lambda = c_1\alpha_1 + \cdots + c_k\alpha_k$.*

(1) *If $c_i + c_{k+1-i} > 2i(k+1-i)$ for each $i = 1, \cdots, k$, then $\pi(SL_{k+1}(\mathbb{R}))$ is an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup where $n = \dim \pi$.*

(2) *If $c_i = c_{k+1-i}$, $\frac{1}{2}\sum_{i=1}^{k} i(k+1-i)c_i$ is even (resp. odd) and $c_i > 2i(k+1-i) + \epsilon$ for each $i = 1, \cdots, [n/2]$, then $\pi(SL_{k+1}(\mathbb{R}))$ is an $(SO(n,n')(\mathbb{R}), SO(n,n')(\mathbb{R}) \cap SO_{n+n'}(\mathbb{R}))$-tempered subgroup (resp. $(Sp_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}) \cap SO_{2n}(\mathbb{R}))$-tempered subgroup) where $n = [(\dim\ \pi)/2]$ and $n' = \dim\ \pi - n$.*

Proof. This follows from the facts that the modular function of B is $\sum_{i=1}^{k} i(k+1-i)\alpha_i$ and $i(\alpha_j) = \alpha_{k-j+1}$ and the previous proposition. $\square$

5.6. From the well known theory of finite dimensional irreducible representations of $SL_2(\mathbb{R})$, we have the following corollary.

Corollary. *If π_n is an n-dimensional irreducible representation of $SL_2(\mathbb{R})$, $\pi_n(SL_2(\mathbb{R}))$ is an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup if and only if $n \geqq 4$.*

5.7. For matrices A of order s and B of order t, the Kronecker product $A \otimes B$ of A and B is the matrix of order st such that the (ij)-matrix block of $A \otimes B$ is $a_{ij}B$ where a_{ij} is the (ij)-entry of A.

Corollary.

(1) *The embedding of $SL_k(\mathbb{R})$ as the first k by k diagonal block matrix in $SL_n(\mathbb{R})$ is not an $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$-tempered subgroup for any positive integers k and n.*

(2) *The subgroup* $SL_m(\mathbb{R})\otimes I_k = \{A\otimes I_k \in SL_{mk}(\mathbb{R}) \mid A \in SL_m(\mathbb{R})\}$ *is an* $(SL_{mk}(\mathbb{R}), SO_{mk}(\mathbb{R}))$*-tempered subgroup if and only if* $k > m$.

5.8. Unipotent tempered subgroups. Lastly we give an example of a unipotent tempered subgroup of $SL_n(\mathbb{R})$. For a non-semisimple subgroup H, to apply Theorem D we need to know how each element in H decomposes under the Cartan decomposition of G.

Consider the decomposition of the element $v_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$ as $k_1 a k_2$ under the Cartan decomposition of $SL_2(\mathbb{R})$ with $K = SO_2(\mathbb{R})$.

Since $v_s(v_s)^t = k_1 a^2 k_1^{-1}$, the eigenvalues of a^2 coincide with those of $v_s(v_s)^t$. If $a = \mathrm{diag}(b, b^{-1})$, then $\{b^2, b^{-2}\} = \{\frac{1}{2}(2 + s^2 + s\sqrt{s^2+4}), \frac{1}{2}(2 + s^2 - s\sqrt{s^2+4})\}$.

Consider the one parameter unipotent subgroup U_{ij} of $SL_n(\mathbb{R})$ consisting of the elements $u_{ij}(s) = I + sE_{ij}$, $s \in \mathbb{R}$ where $i \neq j$ and E_{ij} is the elementary matrix whose non-zero entry is 1 only at (i,j)-th place. We keep the same notation as in section 5.1. Then the A^+-component of $u_{ij}(s)$ under the Cartan decomposition of $SL_n(\mathbb{R})$ is $\mathrm{diag}(a, 1, \cdots, 1, a^{-1})$ where $a = \sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2+4})}$.

Therefore $F(u_{ij}(s))$ is asymptotically equal to $\sqrt{\frac{1}{2}(2 + s^2 + s\sqrt{s^2+4})}$.

Proposition. *Let* $n \geq 2$ *and* $i \neq j$.

(1) *For any* $\epsilon > 0$, *the restriction* $F|_{U_{ij}}$ *is* $L^{1+\epsilon}(U_{ij})$*-integrable.*

(2) *The diagonal embedding* $\delta(U_{ij}) = \{(g,g) \in SL_n(\mathbb{R}) \times SL_n(\mathbb{R}) \mid g \in U_{ij}\}$ *is an* $(SL_n(\mathbb{R}) \times SL_n(\mathbb{R}), SO_n(\mathbb{R}) \times SO_n(\mathbb{R}))$*-tempered subgroup.*

The part (1) is clear. For the second claim, see the remark following Corollary B.

Now consider the unipotent one-parameter subgroup U of $SL_4(\mathbb{R})$ consisting of the elements $U(s) = \begin{pmatrix} 1 & s & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & s \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $s \in \mathbb{R}$. It is easy to see that the following proposition holds.

5.9. Proposition. *The subgroup* U *is an* $(SL_4(\mathbb{R}), SO_4(\mathbb{R}))$*-tempered subgroup.*

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