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LOW-DIMENSIONAL LINEAR REPRESENTATIONS OF $\operatorname{Aut} F_n,\ n \geq 3$

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Abstract. We classify all complex representations of $\operatorname{Aut} F_n$, the automorphism group of the free group F_n $(n \geq 3)$, of dimension $\leq 2n-2$. Among those representations is a new representation of dimension $n+1$ which doesn't vanish on the group of inner automorphisms.

Introduction

This paper continues the study of low-dimensional linear representations of $\Gamma_n = \operatorname{Aut} F_n$ $(n \geq 3)$, the automorphism group of the free group, begun in [R1] (low-dimensional representations of $\operatorname{Aut} F_2$ were analyzed in [DP]). It was shown in [R1] (cf. also Theorem 1.2 below) that any n-dimensional representation of $\operatorname{Aut} F_n$ factors through the the canonical homomorphism $f\colon \Gamma_n \to GL_n(\mathbf{Z})$. In this paper we will show that in dimension higher than n, the group Γ_n acquires new representations. Namely, we will establish the existence of a homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$ which "lifts" f and gives rise to an $(n+1)$-dimensional representation. The main result of the paper (Theorem 3.1) claims that this representation accounts in fact for all representations of Γ_n of dimension $\leq 2n-2$ (in particular, any such representation factors through g). We also use the homomorphism g to construct an infinite family of pairwise inequivalent representations of Γ_n (Proposition 2.4).

1. Notations and preliminaries.

Throughout the paper, E_m will denote the identity $m \times m$-matrix; for a partition $m = m_1 + \cdots + m_k$, $G(m_1, \ldots, m_k)$ will denote the product

$$GL_{m_1}(\mathbf{C}) \times \cdots \times GL_{m_k}(\mathbf{C})$$

diagonally embedded in $GL_m(\mathbf{C})$, with $p_i(m_1, \ldots, m_k)\colon G(m_1, \ldots, m_k) \to GL_{m_i}(\mathbf{C})$ being the corresponding projection; S_n is the symmetric group on $\{1, \ldots, n\}$. To comply with the classical tradition, Γ_n and S_n will be treated as groups of *right* transformations (in other words, the result of application of f to a will be written as $(a)f$, with the law of composition given by: $(a)(fg) = ((a)f)g)$.

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We will use the following elements of $\Gamma_n = \mathrm{Aut}\, F_n$:

$$\rho_{ij}\colon \begin{cases} x_i & \to \quad x_i x_j, \\ x_k & \to \quad x_k, \quad k \neq i \end{cases} \qquad \lambda_{ij}\colon \begin{cases} x_i & \to \quad x_j x_i, \\ x_k & \to \quad x_k, \quad k \neq i \end{cases}$$

$$\varepsilon_i\colon \begin{cases} x_i & \to \quad x_i^{-1}, \\ x_k & \to \quad x_k, \quad k \neq i \end{cases} \qquad \begin{array}{l} \varepsilon_{ij} = \varepsilon_i \varepsilon_j; \\ \varepsilon = \varepsilon_1 \dots \varepsilon_n \colon x_i \to x_i^{-1} \quad \forall i, \end{array}$$

$$\forall \pi \in S_n, \quad \sigma_\pi \colon x_i \to x_{(i)\pi}.$$

It is known that these elements generate Γ_n (cf. [MKS]). One easily checks the following identities:

$$\varepsilon_i^2 = \mathrm{id}_{F_n}, \qquad \varepsilon_i \varepsilon_j = \varepsilon_j \varepsilon_i, \qquad \sigma_\pi^{-1} \varepsilon_i \sigma_\pi = \varepsilon_{(i)\pi}$$

(1)

$$\varepsilon_i^{-1} \rho_{ij} \varepsilon_i = \lambda_{ij}^{-1}, \qquad \varepsilon_{ij}^{-1} \rho_{ij} \varepsilon_{ij} = \varepsilon^{-1} \rho_{ij} \varepsilon = \lambda_{ij} \ \ \forall i, j, \ \ i \neq j.$$

It follows that H_n, the subgroup generated by all ε_i, is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$; all σ_π, $\pi \in S_n$, form a subgroup Σ_n which is a replica of S_n; and the subgroup Ω_n, generated by all ε_i and σ_π together is the semi-direct product $\Sigma_n \ltimes H_n$.

A special feature that distinguishes the case $n \geq 3$ from the case $n = 2$ is the existence of the following commutator relations:

$$[\lambda_{ij}, \lambda_{jk}] = \lambda_{ik} \quad , \quad [\rho_{ij}, \rho_{jk}] = \rho_{ik} \tag{2}$$

(i, j, and k are all distinct) where $[x, y] = xyx^{-1}y^{-1}$.

Passing to abelianization, we obtain a (surjective) homomorphism

$$f\colon \Gamma_n \to \mathrm{Aut}(F_n/[F_n, F_n]) = GL_n(\mathbf{Z})$$

(matrix presentation is taken with respect to the basis of $F_n/[F_n, F_n] \simeq \mathbf{Z}^n$ made up of the images of $x_1, \dots, x_n$; for consistency, $\mathbf{Z}^n$ is treated as *n-rows* of integers, with *right* action of $GL_n(\mathbf{Z})$). Let $N = \mathrm{Ker}\, f$. Our argument makes essential use of the fact (cf. [LS], p.28) that N as a normal subgroup of Γ_n is generated by the element

$$\beta = \lambda_{n,n-1}^{-1} \rho_{n,n-1}.$$

Another element to be frequently used is

$$\gamma = \lambda_{n,n-1} \rho_{n,n-1}.$$

Lemma 1.1. *Let $\varphi\colon \Gamma_n \to G$ be a homomorphism of Γ_n to some group G. If the restriction of φ to Ω_n is not injective, then φ factors through f.*

Proof. First, assume that the restriction $\varphi \mid H_n$ is not injective. It is easy to see that H_n has only two proper subgroups normalized by Σ_n : $\langle \varepsilon \rangle$ and $H' = \{h \in H_n \mid \det f(h) = 1\}$. If $\mathrm{Ker}\, f = \langle \varepsilon \rangle$, then (cf. (1))

$$\varphi(\lambda_{ij}) = \varphi(\varepsilon \rho_{ij} \varepsilon^{-1}) = \varphi(\rho_{ij})$$

implying that $\varphi(\beta) = 1_G$ which in view of the fact quoted above implies that $\varphi(N) = \{1_G\}$, hence our claim. If $\mathrm{Ker}(\varphi \mid H_n) \supset H'$, we observe that $\varepsilon_{ij} \in H'$ $(i \neq j)$, so the same argument applies with ε replaced by ε_{ij}.

Now, suppose $z = (\sigma_\pi, h) \in \mathrm{Ker}(\varphi \mid \Omega_n)$ with π nontrivial. If i is such that $\pi(i) \neq i$, then the commutator $[z, \varepsilon_i] = \varepsilon_{i\pi(i)} \in \mathrm{Ker}(\varphi \mid H_n)$, reducing the argument to the previous case. □

Lemma 1.1 implies

Theorem 1.2. [R1] *Every n-dimensional representation $\theta\colon \Gamma_n \to GL_n(\mathbf{C})$ factors through the canonical homomorphism $f\colon \Gamma_n \to GL_n(\mathbf{Z})$.*

(For $n = 2$, this result was proved earlier in [DP].)

Proof. It follows from Lemma 1.1 that $\theta(H_n) \subset GL_n(\mathbf{C})$ is isomorphic to $(\mathbf{Z}/2\mathbf{Z})^n$, so after replacing θ by an equivalent representation we may assume that Lemma 1.1 we may assume that $\theta(\Sigma_n)$ contains representatives of all cosets of the Weyl group $\mathcal{W} = \mathcal{N}/\mathcal{D}_n$. Since ε is fixed by Σ_n, and $-E_n$ is the only nontrivial element of $\theta(H_n)$ fixed by $\mathcal{W}$, we conclude that $\theta(\varepsilon) = -E_n$. But then $\theta(\beta) = \theta([\rho_{nn-1}, \varepsilon]) = E_n$, and our claim follows. □

2. The homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$.

To avoid ambiguity, we note that since we are using the *right* action of $GL_n(\mathbf{Z})$ on $\mathbf{Z}^n$, the operation on the semi-direct product $GL_n(\mathbf{Z}) \rtimes \mathbf{Z}^n$ is given by: $(A, u)(B, v) = (AB, uB + v)$.

Let $v_1, \ldots v_n$ be the standard basis of $\mathbf{Z}^n$.

Proposition 2.1. *For every $n \geq 2$, there exists a homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \rtimes \mathbf{Z}^n$ such that*

$$g(\lambda_{ij}) = (f(\lambda_{ij}), v_j)\ ,\ g(\rho_{ij}) = (f(\rho_{ij})), -v_j),$$

$$g(\varepsilon_i) = (f(\varepsilon_i), 0)\ , g(\sigma_\pi) = (f(\sigma_\pi), 0).$$

Proof. For $n \geq 4$, Γ_n is generated by the following four automorphisms:

$$P = \sigma_{(12)}, Q = \sigma_{(12\ldots n)}, S = \varepsilon_1, U = \lambda_{12},$$

and admits the following presentation in terms of these generators (cf. [MKS], [N]):

$$P^2 = S^2 = Q^n = (QP)^{n-1} = 1, \tag{1}$$

$$[S, Q^{-1}PQ] = [S, QP] = [S, Q^{-1}SQ] = [P, Q^{-i}PQ^i] = 1, \quad (i = 2, 3, ..., [n/2]) \tag{2}$$

$$U^{-1}PUPSUSPS = (PSPU)^2 = (PQ^{-1}UQ)^2UQ^{-1}U^{-1}QU^{-1} = 1, \tag{3}$$

$$\begin{aligned}[U, Q^{-2}PQ^2] &= [U, QPQ^{-1}PQ] = [U, Q^{-2}SQ^2] = [U, Q^{-2}UQ^2] = \\ [U, SUS] &= [U, PQ^{-1}SUSQP] = [U, PQ^{-1}PQPUPQ^{-1}PQP] = 1.\end{aligned} \tag{4}$$

Since the GL_n-components of $g(P), g(Q), g(S)$ and $g(U)$ as defined in the statement of the proposition coincide with $f(P)$, $f(Q), f(S)$ and $f(U)$ respectively, for the existence of a homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$ with such images of P, Q, S, and U, we need to verify only the equality of the $\mathbf{Z}^n$-components in the relations (1) - (4). For computations, we need to observe that $f(S) = \mathrm{diag}(-1, 1, \ldots, 1)$, $f(U) = E_{12}$, the elementary matrix with 1 as the (12)-entry, and $f(\sigma_\pi) = (\delta_{i(i)\pi})$, the permutation matrix. Note that since the $\mathbf{Z}^n$-components of $g(P), g(Q)$ and $g(S)$ are trivial, we only need to verify the relations involving U, i.e. (3) and (4).

The triviality of the $\mathbf{Z}^n$-component in all relations (3) is easily verified by direct computation. Namely, the $\mathbf{Z}^n$-component of $U^{-1}PUPSUSPS$ is:

$$(-v_2)f(PUPSUSPS) + (v_2)f(PSUSPS) + (v_2)f(SPS) =$$
$$= -v_2 + (v_1 + v_2) - v_1 = 0.$$

The $\mathbf{Z}^n$-component of $(PSPU)^2$ is:

$$(v_2)f(PSPU) + v_2 = -v_2 + v_2 = 0.$$

Finally, the $\mathbf{Z}^n$-component of $(PQ^{-1}UQ)^2UQ^{-1}U^{-1}QU^{-1}$ is:

$$(2v_2)f(UQ^{-1}U^{-1}QU^{-1}) + (v_2)f(Q^{-1}U^{-1}QU^{-1}) - (v_2)f(QU^{-1}) - v_2 =$$
$$2v_3 + (v_2 - v_3) - v_3 - v_2 = 0.$$

For verification of (4), we note the following commutator identity:

$$[(A, u), (B, v)] = ([A, B], u(B - E_n)A^{-1}B^{-1} + v(A^{-1} - E_n)B^{-1}).$$

It follows that the $\mathbf{Z}^n$-component of $[(A, u), (B, v)]$ is trivial if $uB = u$ and $vA = v$. In the first three commutators, the $\mathbf{Z}^n$-component of the second element is trivial, and it suffices to show that this element fixes v_2 ($= \mathbf{Z}^n$-component of $g(U)$). This is easily done by direct computation, for example:

$$(v_2)Q^{-2}PQ^2 = (v_n)PQ^2 = (v_n)Q^2 = v_2, \qquad \text{etc.}.$$

For triviality of the $\mathbf{Z}^n$-component in the remaining four commutators, we need to show in addition that the $\mathbf{Z}^n$-component of this element is fixed by U, i.e. it doesn't involve v_1. The computations are routine, and we consider below only two out of four cases:

$$(v_2)f(Q^{-2}UQ^2) = (v_n)f(UQ^2) = (v_n)f(Q^2) = v_2,$$

and the $\mathbf{Z}^n$-component of $Q^{-2}UQ^2$ is: $v_2 f(Q^2) = v_4$;

$$(v_2)f(SUS) = (v_2)f(US) = (v_2)f(S) = v_2,$$

and the $\mathbf{Z}^n$-component of SUS is: $v_2 f(Q^2) = v_4$ (in the remaining two commutators the $\mathbf{Z}^n$-components of the second element are v_3 and v_4, respectively).

Once we have established the existence of a homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$ which has the required images on P, Q, S, and U, the standard relations in Γ_n show that the images under this g of *all* elements ε_i, λ_{ij}, ρ_{ij}, and σ_π coincide with those given in the statement of the proposition. Indeed, since the permutations (12) and $(12\ldots n)$ generate S_n, we have

$g(\sigma_\pi) = (f(\sigma_\pi), 0)$ for all $\pi \in S_n$. The identity $\varepsilon_1 \lambda_{12} \varepsilon_1^{-1} = \rho_{12}^{-1}$ implies that $g(\rho_{12}) = (f(\rho_{12}), -v_2)$, and then the identities:

$$\sigma_\pi^{-1} \varepsilon_1 \sigma_\pi = \varepsilon_{(1)\pi} \ , \ \sigma_\pi^{-1} \lambda_{12} \sigma_\pi = \lambda_{(1)\pi(2)\pi} \ , \text{ and } \sigma_\pi^{-1} \rho_{12} \sigma_\pi = \rho_{(1)\pi(2)\pi}$$

easily complete the verification of the fact that g indeed has the prescribed images on all elements.

For the remaining two dimensions: $n = 2$ and 3, one can argue similarly, using the known presentations of Γ_2 and Γ_3 (cf. [MKS]). However, there is a shorter argument based on the "functoriality" of $g = g_n$ with respect to n by which we mean the following: for $m < n$, consider the embedding $\iota_n^m \colon \Gamma_m \to \Gamma_n$ obtained by letting Γ_m act trivially on $x_{n-m+1}, \dots x_n$; then the following diagram is commutative:

$$\begin{array}{ccc} \Gamma_m & \xrightarrow{g_m} & GL_m(\mathbf{Z}) \ltimes \mathbf{Z}^m \\ \iota_n^m \downarrow & & \downarrow \theta_n^m \\ \Gamma_n & \xrightarrow{g_n} & GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n \end{array}$$

where θ_n^m identifies $\mathbf{Z}^m$ with $(\mathbf{Z}^m, 0, \dots, 0) \subset \mathbf{Z}^n$, and embeds $GL_m(\mathbf{Z})$ in the left upper corner of $GL_n(\mathbf{Z})$. In fact, since θ_n^m is an embedding, the images under g_n of the elements $\iota_n^m(\lambda_{ij})$, $\iota_n^m(\rho_{ij})$, $\iota_n^m(\varepsilon_i)$ $(1 \leq i, j \leq m)$ and $\iota_n^m(\sigma_\pi)$ $(\pi \in S_m)$ belong to $\operatorname{Im} \theta_n^m$, and $(\theta_n^m)^{-1}$ applied to these elements yields exactly the elements given in the statement of the proposition for Γ_m, the *existence* of g_n implies the existence of g_m for any $m < n$, completing the proof. $\square$

Corollary 2.2. *For the group of inner automorphisms* $\operatorname{Int} F_n \subset \Gamma_n$, *one has*

$$g(\operatorname{Int} F_n) = (E_n, 2(n-1)\mathbf{Z}^n) \tag{5}$$

In particular, g doesn't vanish on $\operatorname{Int} F_n$.

Proof. Since

$$\operatorname{Int} x_j = \prod_{i \neq j} \lambda_{ij}^{-1} \cdot \prod_{i \neq j} \rho_{ij}^{-1},$$

(5) follows. $\square$

It should be noted that the homomorphism g is not surjective. This plays crucial role in the proof of the following

Corollary 2.3. *The extension*

$$1 \to N^{\mathrm{ab}} \to \Gamma_n/[N, N] \to GL_n(\mathbf{Z}) \to 1, \tag{6}$$

and consequently also the extension

$$1 \to N \to \Gamma_n \to GL_n(\mathbf{Z}) \to 1,$$

doesn't split.

Proof. If (6) were split, so would be the extension

$$1 \to g(N) \to g(\Gamma_n) \to GL_n(\mathbf{Z}) \to 1,$$

in particular, $g(\Gamma_n) \subset GL_n(\mathbf{Z}) \ltimes g(N)$. But $g(\beta) = (2v_n, E_n)$, so $g(N) = 2\mathbf{Z}^n$, and we see that $g(\lambda_{nn-1}) \notin GL_n(\mathbf{Z}) \ltimes g(N)$. □

As yet another application of the existence of the homomorphism g, we will show that (contrary to some expectations) Γ_n does not have finite representation type.

Proposition 2.4. *For any $n \geq 2$, there exists an infinite family $\{\rho_s\}$ of pairwise inequivalent representations $\rho_s : \Gamma_n \to GL_m(\mathbf{C})$ in some dimension $m = m(n)$. Thus, Γ_n has infinite representation type for all $n \geq 2$.*

Proof. Since Γ_2 has the virtually free group $GL_2(\mathbf{Z})$ as a quotient, our assertion is trivial for $n = 2$. So, in what follows $n \geq 3$.

We will say that two representations $\rho_1, \rho_2 : H \to GL_m(\mathbf{C})$ of a certain group H are *strongly inequivalent* if $\mathrm{Hom}_H\left((\mathbf{C}^m, \rho_1), (\mathbf{C}^m, \rho_2)\right) = 0$; in other words, there is no nonzero operator $T : \mathbf{C}^m \to \mathbf{C}^m$ such that $\rho_1(h)T = T\rho_2(h)$ for all $h \in H$. Our construction of the required family $\{\rho_s\}$ (which generalizes the construction described in [R2]) hinges on the existence of an infinite family of pairwise strongly inequivalent representation of the algebraic group $G_n = SL_n(\mathbf{C}) \ltimes \mathbf{C}^n$. Though this should be well-known to the specialists in representation theory, we have been unable to find a reference, and therefore include the formulation and proof of the required fact.

Lemma 2.5. *For any $n \geq 3$, there exists an $m = m(n)$ such that G_n admits an infinite family of pairwise strongly inequivalent representations $\theta_s : G_n \to GL_m(\mathbf{C})$.*

Proof. We will need six pairwise inequivalent irreducible rational representations $\rho_1, \ldots, \rho_6$ of the algebraic group $SL_n(\mathbf{C})$ with the following property: for for every pair (i, j) from the set

$$I = \{(1,4), (2,4), (2,5), (3,5), (3,6), (1,6)\},$$

the tensor product $\hat{\rho}_i \otimes \rho_j$, contains the standard representation of $SL_n(\mathbf{C})$ on $V = \mathbf{C}^n$, where for a representation ρ, $\hat{\rho}$ denotes the contragredient representation defined by $\hat{\rho}(g) = {}^t\rho(g)^{-1}$. Such representations (which exist for any $n \geq 3$) were kindly constructed for us by E.B.Vinberg to whom we acknowledge our thanks. Namely, it follows from the formula for the multiplicities of irreducible components of the tensor products of two irreducible representations (cf. [OV], pp. 290 and Exercise 14 to §9 of Ch. VIII in [Bo]) that the representations with the following highest weights satisfy the required property:

$$\begin{array}{llll} \rho_1 : & 3\varepsilon_1 + \varepsilon_2 + \varepsilon_3, & \rho_4 : & 2\varepsilon_1 + \varepsilon_2 + \varepsilon_3, \\ \rho_2 : & 2\varepsilon_1 + 2\varepsilon_2 + \varepsilon_3, & \rho_5 : & 2\varepsilon_1 + 2\varepsilon_3, \\ \rho_3 : & 3\varepsilon_1 + 2\varepsilon_2, & \rho_6 : & 3\varepsilon_1 + \varepsilon_2. \end{array}$$

Let $n_i = \dim \rho_i$, $(i = 1, ..., 6)$ and let M_{ij} be the space of complex $(n_i \times n_j)$-matrices with the following action of $SL_n(\mathbf{C})$:

$$A \cdot g = \rho_i(g)^{-1} A \rho_j(g),\ g \in SL_n(\mathbf{C}),$$

where in the right-hand side we have the usual product of matrices. Obviously, this representation of $SL_n(\mathbf{C})$ is equivalent to $\hat{\rho}_i \otimes \rho_j$. Let $m = n_1 + ... + n_6$. This partition of m makes it sensible to talk about the ij-block (of size $n_i \times n_j$) of a matrix from $M_m(\mathbf{C})$, where $i, j = 1, ..., 6$. Let $V_{ij} \subset M_m(\mathbf{C})$ be the subspace of matrices in which all blocks, except ij, equal zero. Then V_{ij} is invariant under $\mathrm{Ad}\rho$ where $\rho = \rho_1 \oplus ... \oplus \rho_6$, and the restriction $\mathrm{Ad}\rho|_{V_{ij}}$ defines an $SL_n(\mathbf{C})$-module isomorphic to M_{ij}, hence to $\hat{\rho}_i \otimes \rho_j$. Thus, according to our choice of ρ_i, for each pair $(i, j) \in I$ there exists an embedding of $SL_n(\mathbf{C})$-modules $\varphi_{ij} : V \to V_{ij}$. (As remarked by E.B.Vinberg, since V does not have multiple weights, it follows from the above mentioned formula for multiplicities that φ_{ij} is actually unique up to a scalar.)

Let $W_{ij} = \varphi_{ij}(V)$. For any two pairs $(i_1, j_1), (i_2, j_2) \in I$ we have $j_1 \neq i_2$ and therefore $W_{i_1,j_1} \cdot W_{i_2,j_2} = 0$ (the product is taken in $M_m(\mathbf{C})$). Since $W_{ij}^2 = 0$ for all $i \neq j$, we see that

$$W = \bigoplus_{(i,j) \in I} W_{ij}$$

satisfies $W^2 = 0$.

Consider the family of linear maps $\varphi_s : V \to W$, $s \in \mathbf{C}$, defined by

$$\varphi_s(v) = s\varphi_{14}(v) + \sum_{\substack{(i,j) \in I \\ (i,j) \neq (1,4)}} \varphi_{ij}(v). \tag{7}$$

One easily checks that $\theta_s : G_n \to GL_m(\mathbf{C})$, given by

$$\theta_s(g, v) = \rho(g)(E_m + \varphi_s(v)),$$

is a family of rational representations of G_n, so it only remains to be shown that θ_s and θ_t are strongly inequivalent for $s, t \in \mathbf{C}$, $s \neq t$.

Suppose $T : \mathbf{C}^m \to \mathbf{C}^m$ intertwines ρ_s and ρ_t. Then T must commute with ρ, and therefore $T = \mathrm{diag}(\alpha_1 E_{n_1}, ..., \alpha_6 E_{n_6})$ for some $\alpha_1, ...\alpha_6 \in \mathbf{C}$, as ρ is the direct sum of the pairwise inequivalent irreducible representations $\rho_1, ..., \rho_6$. In view of (7) the condition $T\varphi_s = \varphi_t T$ implies that $\alpha_i = \alpha_j$ for all i, j such that $(i, j) \in I$ and $(i, j) \neq (1, 4)$, and therefore $\alpha_1 = ... = \alpha_6$. So, if $T \neq 0$ then $T\varphi_s = \varphi_t T$ forces $s = t$, completing the proof. □

We now proceed with the proof of the proposition. Clearly, it suffices to construct an infinite family of pairwise inequivalent representations of $\Lambda_n = g(\Gamma_n)$. Note that $\Phi_n = \Lambda_n \cap (SL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n)$ is a normal subgroup of Λ_n of index 2; on the other hand, being of finite index in $SL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$, it is Zariski dense in G_n. It follows that restricting θ_s to Φ_n (θ_s as in Lemma 2.5) we obtain a family of pairwise strongly inequivalent representations π_s of

Φ_n. Fix $g \in \Lambda_n - \Phi_n$ and let $\tilde{\pi}_s(h) = \pi_s(ghg^{-1})$. We are going to show that the family

$$\rho_s = \mathrm{Ind}_{\Phi_n}^{\Lambda_n}(\pi_s)$$

contains infinitely many inequivalent representations. Since representations $\tilde{\pi}_s$ are pairwise inequivalent, it suffices to establish the implication

$$\rho_s \simeq \rho_t \text{ for } s \neq t \;\Rightarrow\; \pi_s \simeq \tilde{\pi}_t, \tag{8}$$

where $\simeq$ denotes the equivalence of representations.

Obviously, the restriction $\rho_s|_{\Phi_n}$ is equivalent to $\pi_s \oplus \tilde{\pi}_s$, so $\rho_s \simeq \rho_t$ implies the existence of an isomorphism

$$T : \pi_s \oplus \tilde{\pi}_s \to \pi_t \oplus \tilde{\pi}_t.$$

However, since π_s and π_t are strongly inequivalent, the projection of $T \circ \pi_s$ to π_t must be trivial implying that T yields the equivalence of π_s and $\tilde{\pi}_t$, which proves (8) and completes the proof of Proposition 2.4. $\square$

Remark 2.6. Note that for the family ρ_s constructed in Proposition 2.4, $\rho_s(N)$ is abelian, so Theorem 2 in [R1] which claims that Γ_n $(n \geq 3)$ is SS-rigid with respect to the class of representations ρ for which $\rho(N)$ is nilpotent of a fixed nilpotency class, cannot be upgraded to a statement about finiteness of representation type. On the other hand, representations ρ_s are not completely reducible, so the question whether or not Γ_n is SS-rigid, i.e. has only finitely many inequivalent completely reducible representations in every dimension, remains open.

3. Statement of the Main Theorem.

We need to introduce some special representations of Γ_m. Let $\mu\colon \Gamma_n \to GL_n(\mathbf{C})$ be the representation obtained by composing $f\colon \Gamma_n \to GL_n(\mathbf{Z})$ with the embedding $GL_n(\mathbf{Z}) \subset GL_n(\mathbf{C})$. Let $\delta(x) = \det \mu(x)$; since for $n \geq 3$, the commutator subgroup $[\Gamma_n, \Gamma_n]$ coincides with $f^{-1}(SL_n(\mathbf{Z}))$ (this easily follows from the commutator relations), δ is the only nontrivial character of Γ_n.

Next, the homomorphism $g\colon \Gamma_n \to GL_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$, $g(x) = (f(x), v(x))$, we have constructed in the previous section, gives rise to the following representation of Γ_n in $GL_{n+1}(\mathbf{Z})$:

$$x \;\mapsto\; \left(\begin{array}{c|c} f(x) & 0 \\ \hline v(x) & 1 \end{array}\right),$$

and we let $\nu\colon \Gamma_n \to GL_{n+1}(\mathbf{C})$ denote the composition of this representation with the embedding $GL_{n+1}(\mathbf{Z}) \subset GL_{n+1}(\mathbf{C})$. (Obviously, $\mathrm{Ker}\,\nu = \mathrm{Ker}\,g$, in particular, ν doesn't vanish on $\mathrm{Int}\,F_n$.)

Theorem 3.1. *Let $n \geq 3$, $\theta\colon \Gamma_n \to GL_m(\mathbf{C})$ be a representation of dimension $m \leq 2n-2$. Then either θ factors through f, or it is equivalent to a direct sum $\theta_1 \oplus \theta_2$ where θ_1 is one of the following $(n+1)$-dimensional representations: $\nu, \delta\nu$, or their contragredient representations, and θ_2 is a* *direct sum of 1-dimensional representations of* Γ_n. *In particular,* θ *always factors through* g *and therefore vanishes on* $[N, N]$.

(Here $\delta\nu$ denotes the δ-twist of ν given by: $\delta\nu(x) = \delta(x)\nu(x)$, and the contragredient representation for $\varphi\colon \Gamma_n \to GL_d(\mathbf{C})$ is $\tau\varphi$ where $\tau\colon g \mapsto {}^t g^{-1}$.)

Corollary 3.2. *Let* $n \geq 3$. *Then* Γ_n *has only finitely many inequivalent representations in any dimension* $\leq 2n-2$.

Indeed, it follows, for example, from the affirmative solution of the congruence subgroup problem for $SL_n(\mathbf{Z})$ (cf. [BMS]), or from Margulis's superrigidity (cf. [Ma]) that $GL_n(\mathbf{Z})$ has finitely many inequivalent representations in every dimension.

4. Restricting low-dimensional representations of Γ_n to Ω_n.

Let $\hat{H}_n$ be the group of characters of H_n, $\hat{\varepsilon}_i$ be defined by: $\hat{\varepsilon}_i(\varepsilon_j) = \delta_{ij}$ (Kronecker's delta). We will need some specific representations of Ω_n. Let $\psi\colon \Sigma_n \to \{\pm 1\}$ be the sign homomorphism, and $\chi_0 = \hat{\varepsilon}_1 + \ldots + \hat{\varepsilon}_n$. Then for $k, l \in \{0, 1\}$ we define a character of Ω_n

$$\eta_{k,l}(\sigma_\pi, h) = \psi(\sigma_\pi)^k \chi_0(h)^l$$

and the corresponding twisted n-dimensional representation

$$\mu_{k,l} = \eta_{k,l} \cdot \mu \mid_{\Omega_n}$$

where $\mu\colon \Gamma_n \to GL_n(\mathbf{C})$ is the standard representation introduced in the previous section.

Proposition 4.1. *Let* $\theta\colon \Gamma_n \to GL_m(\mathbf{C})$ $(n \geq 3)$ *be a representation of dimension* $m \leq 2n-2$. *Then* $\theta(N) \neq \{E_m\}$ *is possible only if the restriction* $\theta \mid \Omega_n$ *is equivalent to a representation of the form:*

$$\mu_{p,q} \bigoplus \Big(\bigoplus_{k,l \in \{0,1\}} (\eta_{k,l})^{\alpha_{k,l}} \Big)$$

for some $p, q \in \{0, 1\}$, *and some integers* $\alpha_{k,l}$ $(k, l \in \{0,1\})$ *such that* $\alpha_{00} + \cdots + \alpha_{11} = m - n$.

Proof. Let $V = \mathbf{C}^m$, and

$$V = V_{\chi_1} \oplus \cdots V_{\chi_r} \tag{1}$$

be the decomposition of V as the direct sum of nonzero eigenspaces of H_n corresponding to characters $\chi_i \in \hat{H}_n$ (observe that $\{\chi_1, \ldots \chi_r\}$ is the union of some orbits of Σ_n acting on $\hat{H}_n$). If $\chi \in \hat{H}_n$ is of the form

$$\chi = \hat{\varepsilon}_{i_1} + \cdots + \hat{\varepsilon}_{i_l},$$

then the orbit $\Sigma_n \cdot \chi$ consists of $\begin{pmatrix} n \\ l \end{pmatrix}$ elements. It follows that if $n \geq 5$, $\mid \Sigma_n \cdot \chi \mid \geq 2n$ unless $l = 0, 1, n-1$, or n, i.e. $\chi = 0, \hat{\varepsilon}_i, \chi^0 - \hat{\varepsilon}_i$, or χ^0 for some i, and therefore only such characters can appear in (1) as $\dim V < 2n$. On the other hand, if $n = 3$, then the characters of such form exhaust all characters, so only the case $n = 4$ will require special consideration. Furthermore, in view of Lemma 1.1, there should be a χ_i in (1) which is different from 0 and χ^0. Then the orbit $\mathcal{O} = \Sigma_n \cdot \chi_i$ consists of n elements, so $W = \oplus_{\chi \in \mathcal{O}} V_\chi$ has dimension $n \cdot \dim V_{\chi_i}$, so $\dim V_{\chi_i} = 1$. The stabilizer $\Sigma_n(\chi_i)$ is isomorphic to S_{n-1}, so its representation on V_{χ_i} is either the trivial or sign representation, which easily implies that the representation of Ω_n on W is equivalent to one of $\mu_{p,q}$. Those V_χ in (1) which are not in W correspond to $\chi = 0$ or χ^0, hence Σ_n-invariant, and have dimension $\leq n-2$. But for $n \neq 4$, S_n has only one nontrivial irreducible representation of dimension $\leq n-2$, viz. the 1-dimensional sign representation (cf. [JK], Theorem 2.4.10), and our assertion follows.

For $n = 4$, there is one orbit of Σ_4 on $\hat{H}_4$ consisting of 6 elements, viz. $\mathcal{O} = \Sigma_4 \cdot \chi$, $\chi = \hat{\varepsilon}_1 + \hat{\varepsilon}_2$, so we need to eliminate the possibility of this χ to occur in (1) when $\dim V = 6$. In this case $\mathcal{O} = \{\hat{\varepsilon}_i + \hat{\varepsilon}_j \mid i \neq j\}$ which restricts to three distinct characters of the subgroup $\tilde{H} = \langle \varepsilon_1, \varepsilon_2 \rangle$, each of the corresponding eigenspaces being 2-dimensional. Let $\tilde{\Gamma}$ be the group $\mathrm{Aut}\, F\langle x_3, x_4\rangle$ isomorphically embedded in Γ_4 as acting identically on x_1 and x_2. Then $\tilde{H}$ and $\tilde{\Gamma}$ commute, so each of the eigenspace is $\tilde{\Gamma}$-invariant. Since any 2-dimensional representation of $\mathrm{Aut}\, F_2$ factors through the canonical homomorphism $\mathrm{Aut} F_2 \to GL_2(\mathbf{Z})$ (Theorem 1.2), this implies that $\theta(\beta) = E_m$, hence $\theta(N) = \{E_m\}$ - a contradiction.

It remains to be shown that if V_χ from (1) doesn't occur in W (so $\chi = 0$ or χ^0, and V_χ is Σ_4-invariant), then the representation of Σ_4 on V_χ cannot be a 2-dimensional irreducible representation of Σ_4. It is known (cf. [FuH], p.19) that S_4 has only one 2-dimensional irreducible representation τ which is obtained by composing the standard representation of S_3 with the isomorphism

$$S_4/\{1, (12)(34), (13)(24), (14)(23)\} \simeq S_3.$$

We need only the fact that $\det \tau(t) = -1$ for any transvection $t \in S_4$. So, let us suppose that

$$\theta \mid \Omega_n \simeq \mu_{p,q} \bigoplus \tau_l$$

where $\tau_l(\sigma_\pi, h) = \chi^0(h)^l \tau(\pi)$, $l \in \{0, 1\}$. Let $x = (\sigma_{(12)}, \varepsilon_1)$. Then

$$\det \theta(x) = \det f(x) \det \tau((1,2)) = -1.$$

On the other hand, since for $n \geq 3$, $[\Gamma_n, \Gamma_n] = f^{-1}(SL_n(\mathbf{Z}))$, we have $x \in [\Gamma_n, \Gamma_n]$, and consequently $\det \theta(x) = 1$ - a contradiction. $\square$

5. Proof of Theorem 3.1.

Our proof of Theorem 3.1 is based on the description of the restriction of a representation $\theta\colon \Gamma_n \to GL_m(\mathbf{C})$ to Ω_n provided by Proposition 4.1, and information about how elements of Ω_n interact with ρ_{ij}, λ_{ij}. We may (and we will) assume that

$$\theta \mid \Omega_n = \mu_{p,q} \bigoplus \bar{\theta}$$

where $\bar{\theta} = \oplus_{k,l \in \{0,1\}} (\eta_{k,l})^{\alpha_{k,l}}$ (notations as in §4) has dimension $l = m - n$, $0 \leq l \leq n-2$. We collect in the following statement some properties of $\mu_{p,q}$ and $\bar{\theta}$ that are immediate consequences of their description and will be used in the sequence.

Lemma 5.1. *(i)* $\mu_{p,q}(\varepsilon_{ij}) = \mathrm{diag}(\alpha_1, \dots, \alpha_n)$ *where* $\alpha_i = \alpha_j = -1$ *and* $\alpha_k = 1$ *for* $k \neq i, j$*;*

(ii) $\bar{\theta}(\varepsilon_i) = \bar{\theta}(\varepsilon_j)$ *for all* i, j*, so* $\bar{\theta}(\varepsilon_{ij}) = E_l$ *for all* $i \neq j$*;*

(iii) let $g \in \Gamma_n$ *be such that* $\theta(g) = \mathrm{diag}(A, B) \in G(n, l)$*, and suppose that either* $n = 3$*, or* g *commutes with* ε_i *and* $\sigma_{(jk)}$ *for some* i *and some* $j \neq k$*; then* B *commutes with* $\bar{\theta}(\Omega_n)$*.*

(Assertion (iii) is immediate if $n = 3$ since then $\bar{\theta}$ is at most one-dimensional; otherwise one needs to use (ii) and the fact that for any two transposition, the corresponding automorphisms have the same image under $\bar{\theta}$.)

Let Γ' be the subgroup of Γ_n generated by ρ_{ij}, λ_{ij}, ε_i for $i, j \in \{n-1, n\}$, $i \neq j$, and $\sigma_{(n-1 n)}$, $\Delta = \langle \Gamma', H_n \rangle$.

Lemma 5.2. *If either* $n \geq 4$*, or* $n = 3$ *but* θ *doesn't factor through* f*, then* $\theta(\Delta) \subset \mathcal{D}_{n-2} \times GL_{l+2}(\mathbf{C})$ *where* $\mathcal{D}_{n-2} \subset GL_{n-2}(\mathbf{C})$ *is the diagonal torus.*

PROOF. Let $H_{n-2} \subset H_n$ be the subgroup generated by ε_i for $i \leq n-2$. Then Δ commutes element-wise with H_{n-2} implying that $\theta(\Delta)$ is contained in the centralizer $C = C_{GL_m(\mathbf{C})}(\theta(H_{n-2}))$. First, we assume that $n \geq 4$. Since $\theta(\varepsilon_{ij}) = \mathrm{diag}(\alpha_1, \dots, \alpha_m)$ with $\alpha_i = \alpha_j = -1$ and all other α's equal to 1, we see that $C \subset G(n-2, l+2)$. Furthermore, since for $i \leq n-2$, we have $p_1(n-2, l+2)(\theta(\varepsilon_i)) = \pm\mathrm{diag}(\beta_1, \dots, \beta_{n-2})$ where $\beta_i = -1$ and all other β's equal to 1, we get our claim.

If $n = 3$, we only need to consider the case $m = 4$. We have: $\theta(\varepsilon_1) = \pm\mathrm{diag}(-1, 1, 1, \alpha)$ where α can be 1 or -1. If $\alpha = 1$, our claim is immediate. Otherwise, the subspaces $V_1, V_2 \subset \mathbf{C}^4$ spanned by 1^{st} and 4^{th}, and 2^{nd} and 3^{rd} basic vectors respectively, are invariant under $\theta(\Delta)$. Since $\Gamma' \simeq \mathrm{Aut}\, F_2$, we conclude from Theorem 1.2 that the restriction of $\theta(\Gamma')$ to each of them factors through the canonical homomorphism $f' \colon \Gamma' \to GL_2(\mathbf{Z})$. This implies that $\theta(\beta) = E_4$, and therefore θ factors through f.

So, in terms of proving Theorem 3, we may (and we will) assume hereinafter the inclusion given in Lemma 5.2. Since $\beta = [\varepsilon_{nn-1}, \lambda_{nn-1}]$ and $\gamma = [\lambda_{nn-1}, \varepsilon_n]$, we obtain that

$$p_1(n-2, l+2)(\theta(\beta)) = p_1(n-2, l+2)(\theta(\gamma)) = E_{n-2}.$$

We let θ' denote the representation of Δ obtained by composing θ with the projection $p_2(n-2, l+2)$. Then

$$\theta'(\varepsilon_{1,n}) = \mathrm{diag}(1, -1, E_l) \quad \text{and} \quad \theta'(\varepsilon_{n,n-1}) = \mathrm{diag}(-1, -1, E_l),$$

so the fact that γ and $\varepsilon_{n,n-1}$ commute (which immediately follows from (1) in §1) implies that

$$\theta'(\gamma) = \mathrm{diag}(A, B)$$

with $A \in GL_2(\mathbf{C})$, and $B \in GL_l(\mathbf{C})$. Moreover, using the identity

$$(\gamma\varepsilon_{1,n})^2 = 1$$

(which is again an immediate consequence of (1) in §1) we conclude that

$$(\mathrm{diag}(1, -1)A)^2 = E_2 \quad \text{and} \quad B^2 = E_m. \tag{1}$$

If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then we obtain from (1) that

$$a^2 - bc = d^2 - bc = 1 \quad \text{and} \quad (a-d)b = (a-d)c = 0. \tag{2}$$

Case 1. $b \neq 0$ *and* $c \neq 0$. We claim that in this case A cannot possibly have eigenvalues ± 1. Indeed, we obtain from (2) that $a = d$ and $\det A = 1$. This means that if one of the eigenvalues is ± 1, the other is the same. So, $\mathrm{tr}\, A = 2a = \pm 2$ which implies that $bc = 0$ - a contradiction. Since the eigenvalues of B are ± 1, the fact that γ and λ_{nn-1} commute implies that $\theta'(\lambda_{nn-1}) \in G(2, l) \subset GL_{2+l}(\mathbf{C})$, hence

$$\theta'(\beta) = \theta'([\varepsilon_{nn-1}, \lambda_{nn-1}]) = E_{l+2}.$$

So, $\theta(\beta) = E_m$, and θ factors through f.

Case 2. $b = c = 0$, i.e. A is diagonal. We will show that in this case θ factors through f as well. We claim that in addition to the obvious identity $\theta'(\gamma)^2 = E_{l+2}$, we also have

$$\theta'(\varepsilon_n \sigma_{(nn-1)}\gamma)^4 = E_{l+2}. \tag{3}$$

Indeed, the matrices $\theta'(\gamma)$, $\theta'(\sigma_{(nn-1)})$ and $\theta'(\varepsilon_n)$ belong to $G(2, l)$. Then (3) for their 2×2-blocks follows from the fact that these have the following shape:

$$\mathrm{diag}(\pm 1, \pm 1), \quad \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathrm{diag}(\pm 1, \pm 1).$$

To analyze their $l \times l$-blocks, we observe that since γ commutes with $\sigma_{(12)}$ and ε_1 if $n \geq 4$, it follows from Lemma 5.1 that the $m \times m$-block of γ commutes with those of $\sigma_{(nn-1)}$ and ε_n, and again (3) follows.

Now, the identity

$$(\sigma_{(nn-1)}\lambda_{nn-1})^{-1}\lambda_{nn-1}(\sigma_{(nn-1)}\lambda_{nn-1}) = \varepsilon_n \sigma_{(nn-1)}\gamma$$

implies that $\theta'(\lambda_{nn-1})^4 = E_{l+2}$, so

$$\theta'(\beta)^2 = \theta'(\lambda_{nn-1})^{-4}\theta'(\gamma)^2 = E_{l+2}.$$

Thus, $\theta'(\varepsilon_{nn-1}\beta\varepsilon_{nn-1}^{-1}) = \theta'(\beta)^{-1} = \theta'(\beta)$, and since $\theta'(\varepsilon_{nn-1}) = \mathrm{diag}(-E_2, E_l)$, we obtain that

$$\theta'(\beta) = \mathrm{diag}(B_1, B_2) \in G(2, l).$$

Moreover, since β commutes with ε_n, and ε_n has $\pm\mathrm{diag}(1, -1)$ as its 2×2-block, we see that B_1 is a diagonal matrix.

Next, we need the following easily verifiable identity:

$$\lambda_{n-1n}^{-1}\beta\lambda_{n-1n} = (\sigma_{(nn-1)}\beta)^2 \tag{4}$$

Since $\theta'(\beta)$ and $\theta'(\sigma_{(nn-1)})$ belong to $G(2, l)$, and the same argument as above shows that their $m \times m$-blocks commute, (4) implies that $\beta = \mathrm{diag}(B_1, B_2)$ is conjugate to

$$\mathrm{diag}((\pm B_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})^2, B_2^2).$$

The eigenvalues of β are ± 1, and it follows from the conjugacy of these matrices that the multiplicity of -1 is ≤ 2. Since $\det \theta(\beta) = 1$ (β is a commutator), it can only be 0 or 2. If it is 0 or $B_1 = \pm E_2$, we obtain that $\theta(\beta) = E_m$. So, it remains to eliminate the possibility of –1 occurring as an eigenvalue in each of B_1 and B_2 with multiplicity one.

Assume that this is the case. Then the matrix B_1 can be either $\mathrm{diag}(1, -1)$ or $\mathrm{diag}(-1, 1)$, however by switching x_{n-1} and x_n (or alternatively by passing to the element $\beta' = \sigma_{(n-1n)}^{-1}\beta\sigma_{(n-1n)}$ which equally generates N), we may assume that $B_1 = \mathrm{diag}(1, -1)$. Furthermore, it follows from Lemma5.1 that B_2 commutes with $\bar{\theta}(\Omega_n)$, and therefore there exists a matrix $C \in GL_l(\mathbf{C})$ that conjugates B_2 to $\mathrm{diag}(-1, 1, \ldots, 1)$ and preserves the shape of $\bar{\theta}$ (i.e. $C^{-1}\bar{\theta}C$ remains the direct sum of 1-dimensional representations of Ω_n). So, by passing to an equivalent representation whose restriction to Ω_n has the same structure, we may assume that $\theta(\beta) = \mathrm{diag}(E_{n-1}, -E_2, E_{l-1})$.

Then since λ_{nn-1} commutes with β, $\theta'(\lambda_{nn-1})$ must be of the form:

$$\begin{pmatrix} s & 0 & 0 & u \\ 0 & x & y & 0 \\ 0 & z & t & 0 \\ v & 0 & 0 & W \end{pmatrix}$$

where $s \in \mathbf{C}^*$, $u \in \mathbf{C}^{1\times(l-1)}$, $v \in \mathbf{C}^{(l-1)\times 1}$, and $W \in M_{l-1}(\mathbf{C})$. Now, using the identity

$$\varepsilon_{nn-1}^{-1}\lambda_{nn-1}\varepsilon_{nn-1} = \lambda_{nn-1}\beta \tag{5}$$

and the fact that $\theta'(\varepsilon_{nn-1}) = \mathrm{diag}(-E_2, E_m)$, one easily obtains that $u = 0$ and $v = 0$. In particular, $\theta(\lambda_{nn-1}) \in G(n+1, l-1)$. Since also $\theta(\Omega_n) \subset G(n+1, l-1)$, and Ω_n and λ_{nn-1} together generate Γ_n, we see that $\theta(\Gamma_n) \subset$

$G(n+1, l-1)$. Let $\theta^!\colon \Gamma_n \to GL_{n+1}$ be the composition of θ with the projection to $p_1(n+1, l-1)$.

Obviously, $\theta(\lambda_{nn-1}) \in G(n-2, 3, l-1)$, and letting $p = p_2(n-2, 3, l-1)$, we will have

$$p(\theta(\lambda_{nn-1})) = \begin{pmatrix} s & 0 & 0 \\ 0 & x & y \\ 0 & z & t \end{pmatrix}.$$

Since $p(\theta(\varepsilon_{nn-1})) = \mathrm{diag}(-1, 1, 1)$, $p(\theta(\beta)) = \mathrm{diag}(1, -1, -1)$, the identity (5) implies that $x = t = 0$. In particular, $\theta^!(\lambda_{nn-1})$ belongs to the group of monomial matrices $\mathcal{M}_{n+1} \subset GL_{n+1}$. Since $\theta^!(\Omega_n) \subset \mathcal{M}_{n+1}$, we obtain that $\theta^!(\Gamma_n) \subset \mathcal{M}_{n+1}$. If $\mathcal{M}_{n+1} \xrightarrow{\phi} S_{n+1}$ is the canonical homomorphism, then $\phi(\lambda_{nn-1}) = (n, n+1)$. Using $\pi = (n-1, n, n-2) \in S_n$, we obtain

$$\phi(\lambda_{n-2,n}) = \phi(\sigma_\pi^{-1} \lambda_{nn-1} \sigma_\pi) = (n-2, n+1).$$

Then the commutator identity (2) §1 implies that $\phi(\lambda_{n-2,n-1}) = (n-2, n+1, n)$; on the other hand, $\phi(\lambda_{n-2,n-1}) = (n-2, n+1)$ – a contradiction.

Case 3. $b = 0, c \neq 0$. Then

$$A^2 = \begin{pmatrix} 1 & 0 \\ 2c & 1 \end{pmatrix}.$$

Since λ_{nn-1} and γ commute, we have

$$\theta'(\lambda_{nn-1}) = \left(\begin{array}{cc|c} x & 0 & 0 \\ y & x & u \\ \hline v & 0 & W \end{array}\right),$$

for some $x, y \in \mathbf{C}$, $u \in \mathbf{C}^{1\times m}$, $v \in \mathbf{C}^{m\times 1}$, $W \in M_m(\mathbf{C})$. The relation $\gamma = \varepsilon_{nn-1}^{-1} \lambda_{nn-1} \varepsilon_{nn-1} \lambda_{nn-1}$ implies that

$$x^2 = 1 \quad \text{and} \quad xu - uW = 0,$$

while the identity $\varepsilon_{1n-1}^{-1} \lambda_{nn-1} \varepsilon_{1n-1} = \lambda_{nn-1}^{-1}$ yields

$$xu + uW = 0.$$

So, $u = 0$.

Next, since λ_{nn-1} commutes with $\sigma_{(12)}$, W commutes with $p_2(n,l)(\theta(\sigma))$ for any $\sigma \in \Sigma_n$, to the effect that $\theta(\lambda_{ij})$ has the structure

$$\left(\begin{array}{c|c} * & 0 \\ \hline * & W \end{array}\right),$$

with the *same* W for *all* $i \neq j$. Then the identity (2) in §1 implies that actually $W = E_l$. Since the elements λ_{ij}, ε_i, and σ_π generate Γ_n, θ has the following block structure:

$$\theta(g) = \begin{pmatrix} \omega(g) & 0 \\ * & \eta(g) \end{pmatrix}, \quad g \in \Gamma_n.$$

where ω and η have dimension n and l, respectively. It follows from [R1] that ω coincides either with μ or with $(\det \mu)\mu$ (it cannot be $\tau\mu$ or $(\det \mu)\tau\mu$ as $\omega(\lambda_{nn-1})$ is a lower triangular matrix). Replacing θ by $(\det \mu)\theta$ if necessary, we may assume that $\omega = \mu$; in particular, $\omega(\lambda_{ij}) = E_{ij}$. We see that $\theta(\lambda_{nn-1})$ has the following shape:

$$\left(\begin{array}{c|cc|c} E_{n-2} & 0 & & 0 \\ \hline 0 & 1 & 0 & 0 \\ & 1 & 1 & \\ \hline 0 & v & 0 & E_l \end{array}\right).$$

The fact that λ_{nn-1} commutes with ε_1 and $\sigma_{(12)}$ easily implies that v is fixed by $\eta(\Gamma_n)$, and therefore after conjugation by a suitable matrix of the form $\mathrm{diag}(E_n, D)$, $D \in GL_l(\mathbf{C})$, which preserves the shape of η, we may assume that $v = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then, in particular, $\theta(\lambda_{nn-1}) \in G(n+1, l-1)$, and therefore $\theta(\Gamma_n) \subset G(n+1, l-1)$ as the images under θ of the elements ε_i and σ_π, $\pi \in S_n$, also belong to $G(n+1, l-1)$, and together with λ_{nn-1} these elements generate Γ_n. Moreover, on all these elements (including λ_{nn-1}) $p_1(n+1, l-1)(\theta)$ coincides ν. On the other hand, $p_2(n+1, l-1)(\theta)$ is E_{l-1} on λ_{ij} and is the direct some of 1-dimensional representations of Ω_n. So, we finally are able to conclude that θ is equivalent to a representation of the form: $\nu \oplus \varkappa$ where $\varkappa$ is a direct sum of 1-dimensional representations of Γ_n, as required.

Case 4. $b \neq 0, c = 0$. This case is immediately reduced to Case 3 by applying the automorphism $g \overset{\tau}{\mapsto} {}^t g^{-1}$ of $GL_n(\mathbf{C})$ (observe that $\theta(\Omega_n)$ lies in the orthogonal group, hence is fixed by τ).

The proof of Theorem 3.1 is now complete.

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