Geometrical Bijections in Discrete Lattices

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1 Introduction

When solving Tarski's circle squaring problem Laczkovich introduced the notion of a "uniformly spread set". Taking a closer look at the combinatorial ideas behind his approach we consider mappings with a "wobbling property". A mapping $f : X \to Y$ (X, Y subsets of a metric space) is a wobbling mapping if no point in X is moved too far, formally if

$$\sup_{x \in X} d(x, f(x)) < \infty .$$

Wobbling mappings occur in many real world situations. Rounding in numerical analysis, image processing, distortion of crystals, but also earthquakes are typical.
We call subsets of a metric space "equivalent" if there exists a wobbling bijection between them. In particular uniformly spraed sets are those which a equivalent to a lattice $\mathbb{Z}^n$ considered as Euclidean space.
The paper is organized as follows: After a short technical introduction giving general useful information also on continued fractions which will play a crucial role in the effective construction of certain wobbling bijections.
In a chapter on geometrical constructions we shall give a constructive proof for the fact that any linear mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ with determinant 1 transforms the lattice $\mathbb{Z}^n$ into a uniformly spread set. It

suffices to consider the Jordan normal form with the typical matrices

- $\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ shearing
- $\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ rotation
- $\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}$ rectangular lattice

In fact shearings will play an important role for an engineering approach to such problems leading to computer visualization. In particular this method already yields a uniform upper bound in terms of a matrix norm (theorem 4.1) for the wobbling distances, which was not obvious from the beginning.

Also by means of continued fractions and the Sturmian sequence from billiards theory we will indicate another efficient constructive method and explain this in particular for rectangular lattices. As a side result we shall investigate the discrepancy of a billards problem. We shall conclude the paper with results on recursive wobblings and an outlook on paradoxical situations [1].

2 General definitions

By $\mathbb{N} = \{1, 2, 3, \dots\}$ we denote the natural numbers. $\mathbb{N}_0$ stands for the set of nonnegative integers and $\mathbb{R}^+$ for the set of nonnegative reals. For a rational or real $x \in \mathbb{R}$ let $[x]$ be the nearest integer, where ties are broken by rounding to the smaller integer. Let (M, d) be a metric space and $X, Y \subseteq M$. A mapping $f : X \to Y$ is a wobbling mapping if $\sup_{x \in X} d(x, f(x)) < \infty$ and the supremun is called the wobbling distance of f. The composition of two wobbling mappings has the same quality and this also holds for the inverse of a wobbling bijection. X and Y are called "wobbling equivalent" or "equivalent" ($X \equiv Y$) if there exists a wobbling bijection from X onto Y. The classical Cantor-Bernstein argument shows that X and Y are equivalent iff there exist wobbling injections $f : X \to Y$ and $g : Y \to X$. As we shall consider lattices in the Euclidean plane with the metric given by the Euclidean norm $\|.\|$ it is noteworthy that for countable discrete metric spaces one can use a graph theoretic approach to wobbling mappings using the Hall-Rado theorem:
For $k \in \mathbb{R}^+$ and $X, Y \subseteq M$ let $C_k(X, Y)$ be the bipartite $k-$neighborhood graph with parts X, Y and edges defined by $d(x, y) \le k$. Let X, Y be subsets of a countable discrete metric space. Then the following statements are equivalent:

(i) There exists an injective wobbling mapping $f : X \to Y$.

(ii) There exists k, such that $C_k(X, Y)$ satisfies the Hall condition: For every finite subset $X^{'}$ of X the cardinality $|N_k(x) \bigcap Y|$ of the $k-$neighborhood of $X^{'}$ in Y satifies $|N_k(x) \bigcap Y| \ge |X^{'}|$.

The continued fraction expansion $[a_0, a_1, ...]$, $a_i \in 0, 1, 2...$ of a real $1/\alpha > 0$ is defined by $1/\alpha = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$. Conversely it is easy to exhibit the continued fraction expansion of any $1/\alpha$ by the generalized eulidean algorithm for the pair $(1, \alpha)$. Let

$$\begin{aligned} 1 &= \alpha a_0 + r_0 \\ \alpha &= r_0 a_1 + r_1 \\ r_0 &= r_1 a_2 + r_2 \\ r_i &= r_{i+1} a_{i+2} + r_{i+2} \text{ for } i \in \mathbb{N}. \end{aligned} \tag{2.1}$$

Then $1/\alpha = [a_0, a_1, ...]$ and if $\alpha > 1$ we have $a_0 = 0$ and $a_i \in \mathbb{N}$ for $i \geq 1$, yielding $\alpha = [a_1, a_2, a_3, ...]$. This definition possibly differs from standard notation by a shift as we start the euclidean algorithm with $(1, \alpha)$ instead of $(\alpha, 1)$. This seems to be more convenient for our purposes.

3 Uniformly spread sets

In his epochal paper [2] Laczkovich introduced the notion of uniformly spread sets and used it for solving the Tarski circle squaring problem. He immediately established the connection of uniformly spread sets to discrepancies and measure theory. The crucial step seems to have been to consider the 2-dimensional case. The generalization to $\mathbb{R}^d$ then relies on some technical details. For simplicity's sake we only consider wobbling equivalenes to the standard lattice $\mathbb{Z}^d$. The more general case of $(a\mathbb{Z})^d$ just needs some rescaling.

Definition 3.1 $X \subseteq \mathbb{R}^d$ is *uniformly spread* if X is wobbling equivalent to Z^d.

Laczkowich's idea may most easily be explained in dimension 2. Let $X \subset \mathbb{R}^2$. In order to test whether X is uniformly spread choose a collection of 'wellbehaved' test regions C with boundary ∂C.

Then the discrepancy $|X \cap C| - |\mathbb{Z}^2 \cap C|$ should be small for all possible test sets C for otherwise no wobbling bijection $X \leftrightarrow \mathbb{Z}^2$ can exist. Now note that a wobbling bijection cannot map points across the boundary ∂C, which are too far away from ∂C implying that the above discrepancy should be small compared to the boundary as measured by a suitable measure.

$|\mathbb{Z}^2 \cap C|$ is approximately the 'area' $\mu_2(C)$ of C.Thus, hopefully, the discrepancy $\delta_C(X, \mathbb{Z}^2) = |X \cap C| - \mu_2(C)|$ is small compared to the boundary. For measuring the latter Laczkowich suggests various possibilities

- Let $\mu_1(\partial C)$ be the perimeter of C measured in the one-dimensional measure.
- Let $\mu_2(N_1(\partial C))$ be the area of the 1- neighborhood of $\partial(C)$.

Theorem 3.1 (Laczkowich) *For every discrete set $X \subset R^d$ the following statements are equivalent*

(i) there exists a positive constant K such that

$$\delta_C(X, \mathbb{Z}^2) \leq K\mu_d(N_1(\partial C)$$

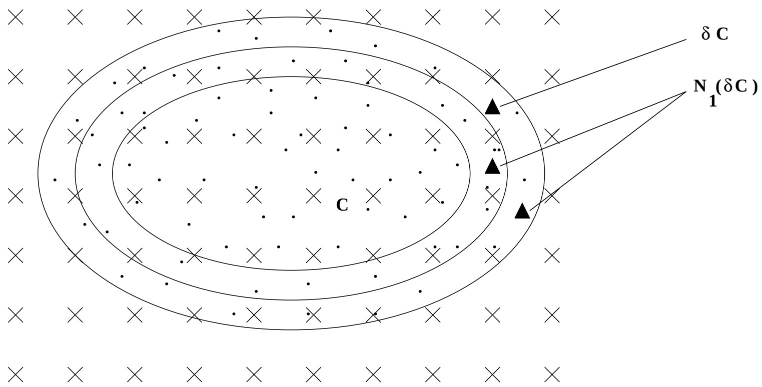
for every bounded Hausdorff measurable set $C \subset \mathbb{R}^d$

(ii) there exists a positive constant K such that

$$\delta_C(X, \mathbb{Z}^2) \leq K \circ \mu_{d-1}(\partial C)$$

for every C which is a finite union of unit cubes.

(iii) X is uniformly spread.



4 Geometrical constructions of wobbling equivalences in Euclidean lattices

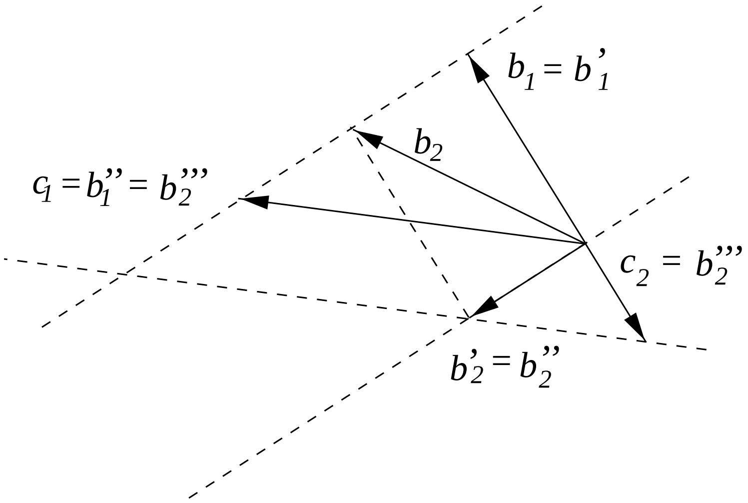
Laczkovich's theorem gives a marvellous abstract characterization for uniformly spread sets. Nevertheless, it depends on the Rado-Hall theorem, which invokes a compactness argument. It therefore does not give explicit wobbling bijections nor reasonably good upper bounds for the wobbling distance. Starting with the 2-dimensional case we shall provide such bijections for elementary geometric situations and will finally indicate the general Euclidean case.

Let τ be a translation of the real line. Then $\mathbb{Z}$ and $\tau\mathbb{Z}$ are wobbling equivalent with wobbling distance $\leq 1/2$. It suffices to consider $\varphi(n) = n + \tau - [\tau]$ to see that $\tau(\mathbb{Z}) \equiv \mathbb{Z}$ and the wobbling distance is $\leq 1/2$.

The next interesting case is the shearing with respect to an abritary basis b_1, b_2; $S = \left(\begin{smallmatrix} 1 & \lambda \\ 0 & 1 \end{smallmatrix}\right)$. Each line $g_a = tb_1 + ab_2$ ($a \in \mathbb{Z}$, fixed) parallel to b_1 is mapped to itself and the restriction of S to g_a is a translation with translation vector $\lambda a b_1$, which of course depends on a. Nothing prevents us to define the wobbling bijection on each line g_a separately by $\varphi(tb_1 + ab_2) = (t + \lambda a - [\lambda a])b_1 + ab_2$. We leave it to the reader to verify that such a φ is bijective when considered on the lattice generated by b_1, b_2. The wobbling distance is as follows: Let $x = x_1 b_1 + x_2 b_2$, where $x_1, x_2 \in \mathbb{Z}$. Then $\parallel \varphi(x) - x \parallel = \parallel (\lambda x_2 - [\lambda x_2])\, b_1 \parallel \leq 1/2 \parallel b_1 \parallel$, which is independent of x.

The next problem is to show that any two 2- dimensional lattices with unit fundamental domain are wobbling equivalent. It suffices to observe that the transformation may be attained as a product of at most 3 shearings: To see this, take the basis b_1, b_2 for the first lattice and c_1, c_2 for the second lattice, oriented in the same way.

- For the first shearing take the fix vector b_1 and shear parallel to it in such a way that b_2 becomes ${b'}_2$ parallel to the line b_1, c_1.
- For the second shearing take ${b'}_2$ as axis and shear parallel to it in such a way that b_1 is mapped to c_1. Note that this is possible by construction of ${b'}_2$ in the first shearing.
- Finally for the third shearing keep c_1 fixed and move ${b'}_2$ into c_2. This is possible as all fundamental domains have unit area rendering the line ${b'}_2, c_2$ parallel to c_1.



As before it is easy to give an upper bound for the wobbling distance of the product of the three shearings by

$$\| \varphi(x) - x \| \leq 1/2 \| b_1 \pm b'_2 \pm c_1 \| \quad . \tag{4.1}$$

For the upper bound one has to take the worst combination of $\pm$ signs. In order to evaluate this we take b_1, b_2 to be the orthonormal basis in $\mathbb{R}^2$ and express all vectors in this basis, obtaining

$$\| \varphi(x) - x \| \leq \frac{1}{2}\left(1 + \sqrt{1 + \left(\frac{c_{11}-1}{c_{12}}\right)^2} + ||c_1||\right) \tag{4.2}$$

$$\leq \frac{1}{2}\left(1 + (1 + \frac{1}{|c_{12}|})\sqrt{(|c_{11}|+1)^2 + c_{12}^2}\right), \tag{4.3}$$

where $c_1 = c_{11} b_1 + c_{12}\ b_2$.

Note that we required $c_{12} \neq 0$ for these calculations. Without loss of generality this may always be achieved by elementary transformations leaving the lattices as point sets unchanged: The transformation mapping b_1, b_2 to $b_2, -b_1$ exchanges the columns in the matrix representation of c_1, c_2, and analogously rows are exchanged by transforming the basis c_1, c_2 to $c_2, -c_1$. As both lattices have unit cells we may assume that at least one coefficient, say c_{12}, does not vanish. We even can do a little better than that.

Fact *Let b_1, b_2 be the standard basis in $\mathbb{R}^2$ and c_1, c_2 be independent vectors with det $(c_1, c_2) = \pm 1$. In the representation $c_1 = c_{11} b_1 + c_{12}\ b_2$, $c_2 = c_{21} b_1 + c_{22}\ b_2$ at least one of the coefficients satisfies $|c_{ij}| \geq \sqrt{2}/2$.*

Proof Otherwise obtain

$$|det(c_1 c_2)| \leq |c_{11}|\ |c_{22}| + |c_{12}|\ |c_{21}| < \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} = 1\,. \tag{4.4}$$

□

Combining this fact with (4.3) obtain the following

Theorem 4.1 *Let b_1, b_2 be the standard orthonomal basis in $\mathbb{R}^2$ and c_1, c_2 be a basis with unit fundamental domain. Then the generated lattices are wobbling equivalent and the wobbling distance is at most*

$$5 \max |c_{ij}|. \tag{4.5}$$

It is remarkable that one can have such a simple universal upper bound for the wobbling distance using only a constant as factor of the max norm. This was not evident to everyone from the beginning.

Remark By induction one verifies that a corresponding theorem holds in higher dimensions, too.

By more carefully inspecting particular cases like rotated and rectangular lattices we can obtain even better results for these:

Rectangular lattices

$$L_a : (c_1, c_2) = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}, \; a \in \mathbb{R}, \; |a| > 1 \tag{4.6}$$

Theorem 4.2 *Let L_1 be the standard lattice and L_a be a rectangular lattice. Then L_1 and L_a are wobbling equivalent and the wobbling distance ist at most $\frac{3+a}{2}$.*

Note that the obvious lower estimate for the wobbling distance is $(a+1)/2$ for irrational a. Moreover Sudmeier [3] showed by using a very involved construction with continued fractions that the upper bound can be improved to $\frac{1+a}{2}$, which he shows to be optimal for irrational α. Anyhow, asymptotically our result is close to optimal.

Proof Because of $c_{12} = 0$ and $c_{21} = 0$ in (4.6) we first take the reflected lattice generated by $\begin{pmatrix} 0 & -a \\ 1/a & 0 \end{pmatrix}$ obtaining $c_{12} = \frac{1}{a}$. Using (4.2) obtain

$$||\varphi(x) - x|| = \frac{1}{2}(1 + \sqrt{1+a^2} + \frac{1}{a}) \leq 1 + \frac{\sqrt{1+a^2}}{2} \leq \frac{3+a}{2}. \tag{4.7}$$

$\square$

Our approach does not discern between rational and irrational a's and gives a universal linear upper bound.

Rotated lattices

$$L_\alpha : (c_1, c_2) = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{4.8}$$

By symmetry one may assume $0 \leq \alpha \leq \frac{\pi}{4}$.

Theorem 4.3 *The standard lattice $\mathbb{Z}^2$ and the rotated lattice L_α are equivalent and the wobbling distance is at most $\frac{\sqrt{5}}{2}$.*

Proof Using (4.1) obtain with $t = \tan\frac{\alpha}{2}$

$$\begin{aligned} \| \varphi(x) - x \| &\leq \tfrac{1}{2} \left\| \binom{1}{0} \pm \binom{\frac{1-\cos\alpha}{\sin\alpha}}{-1} \pm \binom{\cos\alpha}{\sin\alpha} \right\| \\ &\leq \tfrac{1}{2} \left\| \binom{1 \pm t \pm \frac{1-t^2}{1+t^2}}{\mp 1 \pm \frac{2t}{1+t^2}} \right\| \\ &\leq \tfrac{1}{2} \sqrt{3 + 2(\tfrac{1-t^2}{1+t^2}) + t^2} \leq \tfrac{\sqrt{5}}{2}. \end{aligned} \tag{4.9}$$

□

For irrational trigonometric functions of α the lower bound for the wobbling distance is $\sqrt{2}/2$, as infinitely often the vertices of the rotated lattice L_α are arbitrary close to midpoints of the squares in the standard lattice. Intuitively the upper bound $\frac{\sqrt{5}}{2} = 1.1180...$ seems to be optimal for certain angles: At least for some small angles one may expect that infinitely often the vertices of the rotated lattices are arbitrary close to midpoints of the edges of the standard lattice. Intuitively the rounding along the edges cannot go on indefinitely. Finally one has to jump diagonally with distance close to $\frac{\sqrt{5}}{2}$.

We checked this intuiton numerically for $\alpha = 2.2^o$. The computer showed by the aid of a fast version of Hall's algorithm applied to the neighborhood graph $G_{1.1170}(L_1, L_\alpha)$ that there is no perfect matching covering the $10^3 \times 10^3$ square centered at the origin. This shows that the optimal wobbling distance for $\alpha = 2.2^0$ is between 1.1170 and 1.1180... . We did not care for better bounds.

Remark As rotations are orthogonal mappings one immediately hopes that any rotation is a 3 fold product of shearings along the standard orthogonal basis. Trying to write

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} = \begin{pmatrix} 1 & \lambda_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \lambda_3 \\ 0 & 1 \end{pmatrix} = S_3 \circ S_2 \circ S_1 \tag{4.10}$$

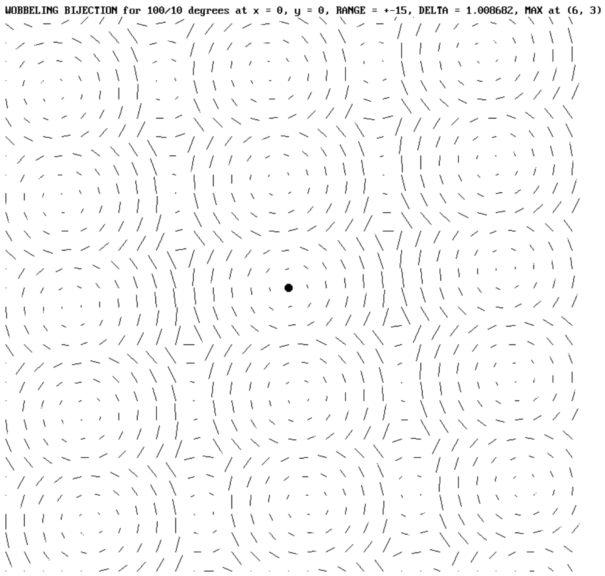
with unknown $\lambda_1, \lambda_2, \lambda_2$ one immediately obtains for $t = \tan\alpha/2$ that

$$\lambda_1 = \lambda_3 = -t \text{ and } \lambda_2 = \sin\alpha. \tag{4.11}$$

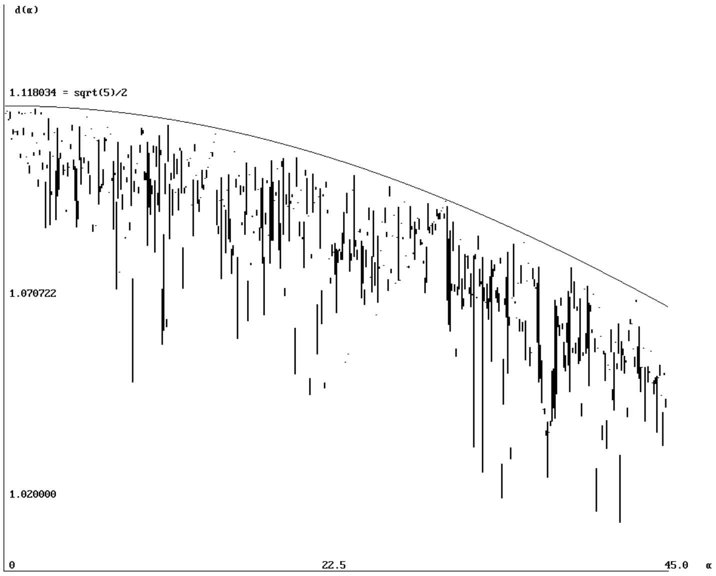
For a matrix A one may introduce the notation $[A]$ for the mapping taking the value $[A]\,x$ as the vector obtained from $A\,x$ by coordinatewise rounding to the nearest integer. Observe that $[A]$ in general is not continuous an mathematically nasty but easily performed on computers. It is certainly not compatible with matrix composition. We use the natural composition as iterated mappings. One obtains the following wobbling bijection between the standard lattice and the rotated lattice L_α

$$\varphi = S_3 \circ S_2 \circ S_1 \circ [S_1^{-1}] \circ [S_2^{-1}] \circ [S_3^{-1}]\ , \tag{4.12}$$

which is easily implemented and may be used for real time visualization. We again obtain $\frac{\sqrt{5}}{2}$ as an upper bound for the wobbling distance. Needless to say that this engineering machinery was the original motivation for the development of the approach to wobbling bijections using shearings.



This picture shows the wobbling φ (4.12) for the angle 10^o.



This figure shows the upper bound $\frac{1}{2}\sqrt{3+2(\frac{1-t^2}{1+t^2})+t^2}$ for the wobbling distances of rotated lattices. Moreover we evaluated the wobbling bijection (4.12) for unit lattices of size 500×500 and 1000×1000 centered at the origin. The vertical bars indicate the achieved wobbling distances and the reasonably good empirical convergence to (4.9) with the usual number theoretic background noise.

5 Applications of Sturmian sequences for wobbling bijections and billiards

5.1 Recursion formulas for some Sturmian sequences

We consider the lattice $\alpha\ \mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}$ (cf. [5]) and give a numbertheoretical construction for a wobbling bijection to a standard lattice. This is motivated by the fact that the continued fractions of α and $1/\alpha$ are essentially the same, a fact wich suggests that there should be some deeper insight than just the results indicated in the previous sections. For related results confer [6].

From billiards theory one knows so called *Sturmian sequences.* Playing billiards on a square table one drives the ball from the lower left corner in the direction $\alpha = tan\ (driving\ angle)$, the ball will successively hit the "horizontal" and "vertical" edges. The sequence $S(\alpha)$ of letters h, v indicating the succession of the hits is the so called Sturmian sequence of the billiard α.

Notation Let A^* be the free monoid over a finite alphabet A with concatenation $*$ of the words. $\emptyset$ is the empty word and a^n ist the n -fold concatenation of $a \in A^*$, where $a^0 = \emptyset$.

Definition 5.1 Let $1/\alpha = [a_0, a_1, a_2, ...]$ be the continued fraction expansion of $1/\alpha$ with $a_0 \in \mathbb{N}_0$, $a_i \in \mathbb{N}$ for $i \geq 1$ and r_i a strictly decreasing sequence with $r_i < a_i$ as defined in section 2. Let

$$b_0 = a_0, \quad b_1 = b_0^{a_1 - 1} * (a_0 + 1), \tag{5.1}$$

$$b_k = b_{k-1}^{a_k - 1} * b_{k-2} * b_{k-1} \quad \text{and} \tag{5.2}$$

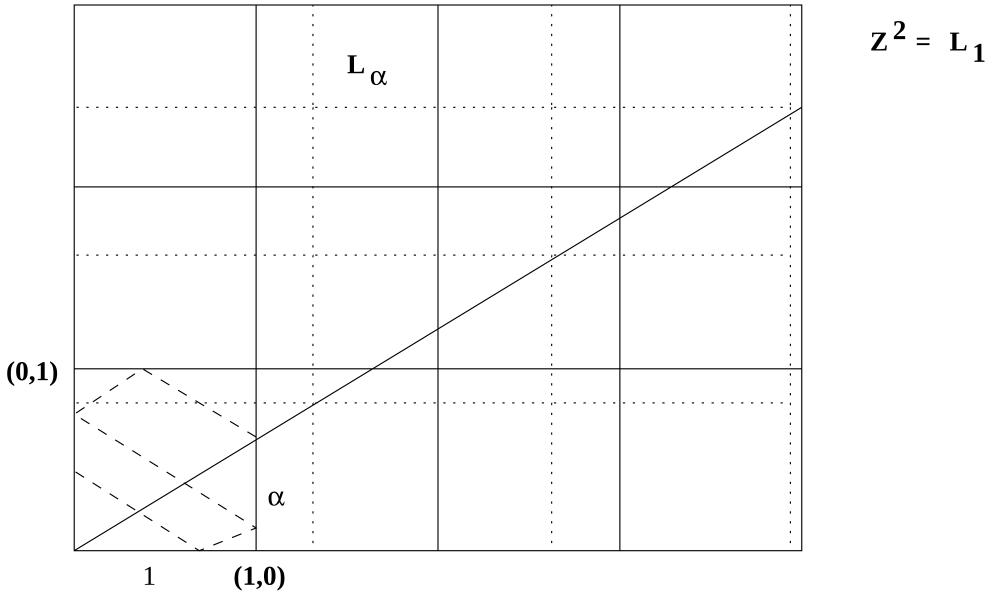
$$B(\alpha) = b_0 * b_1 * b_2 * b_3 * \dots . \tag{5.3}$$

Observe that this yields a method for writing words rather efficiently, in fact exponentially fast. Computer linguists were enthusiastic about the possibility of generating meaningful text exponentially fast. The classical Sturmian sequence for a billiard α (the tangent of the driving angle) $S(\alpha)$ is obtained from $B(\alpha)$ as

$$S(\alpha) = v^{B_0(\alpha)} * h * v^{B_1(\alpha)} * h * v^{B_2(\alpha)} * h * v^{B_3(\alpha)} ..., \tag{5.4}$$

where $B_i(\alpha)$ is the $i-$th letter in $B(\alpha)$. Obviously there is a forgetful functor between $B(\alpha)$ and $S(\alpha)$.

There is a standard technique of unfolding billiards in the plane:



In this way we see that the Sturmian sequence indicates how a straight line with slope α traverses the unit lattice. This and its relation to the unit lattice as below is relevant for the rectangular lattice $\alpha\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}$ where the straight line is generated by one of the diagonals of the basic rectangle.

Definition 5.2 Let $\alpha \in \mathbb{R}$ and $C(\alpha)$ be the sequence $C_k(\alpha) = |\alpha\mathbb{N} \cap (k, k+1]|$.

Let W, W' be two finite segments of $C(\alpha)$ of the same length. Then we have the classical Sturmian property

$$\left| \sum_{x_i \in W} x_i - \sum_{x'_i \in W'} x'_i \right| \leq 1. \tag{5.5}$$

Theorem 5.1 $C(\alpha) = B(\alpha)$.

Theorem 5.1 gives the natural interpretion of the 'Sturmian sequence'. In the following 'Sturmian sequence' refers to any of the three sequences $B(\alpha)$, $C(\alpha)$ or $S(\alpha)$.

The proof of theorem 5.1 will be by induction using the definitions. We shall make use of shifted sequences.

Definition 5.3 For $x \in \mathbb{R}$, $k \in \mathbb{N}$ let

$$b(x, k) = |x + \alpha\mathbb{N} \cap (k, k+1]|. \tag{5.6}$$

$b(x, k)$ tells how many times the arithmetic progression $x + \alpha, x + 2\alpha, ...$ hits the interval $(k, k+1]$. Obviously $b(0, k) = C_k(\alpha)$ for every k, which will be useful for the proof of theorem 5.1.

The following innocent looking shifting lemma will be used frequently.

Lemma 5.1 (Shifting lemma) *Let $x, z \in \mathbb{R}$ and $i, l \in \mathbb{N}_0$, such that $z \le l$ and $z - x + (i - l) \in \alpha\mathbb{N}_0$. Then one has $b(z, l) = b(x, i)$.*

If one wants to evaluate b at some place, one can do so at a more convenient place provided the hypotheses of lemma 5.1 are observed.

Proof of lemma 5.1 Let x, z, i, l satisfy the hypotheses of lemma 5.1 and let $(z - x) + (i - l) = p\alpha$ with $p \in \mathbb{N}_0$. Then one has

$$x + p\alpha = z - l + i \le i. \tag{5.7}$$

Using definition 5.3 and (5.7) one obtains $b(z, l) = |z + \alpha\mathbb{N} \cap (l, l+1]| = |z + p\alpha - z + x + \alpha\mathbb{N} \cap (i, i+1]| = |x + p\alpha + \alpha\mathbb{N} \cap (i, i+1]| = |x + \alpha\mathbb{N} \cap (i, i+1]| = b(x, i)$. □

In the sequel we shall have to verify the hypotheses of lemma 5.1 quite often and it will be the technical routine part of the proofs. The hypothesis $z \le l$ is certainly satisfied if $z \le 0$, a fact we shall use without further mentioning.

Notation Let

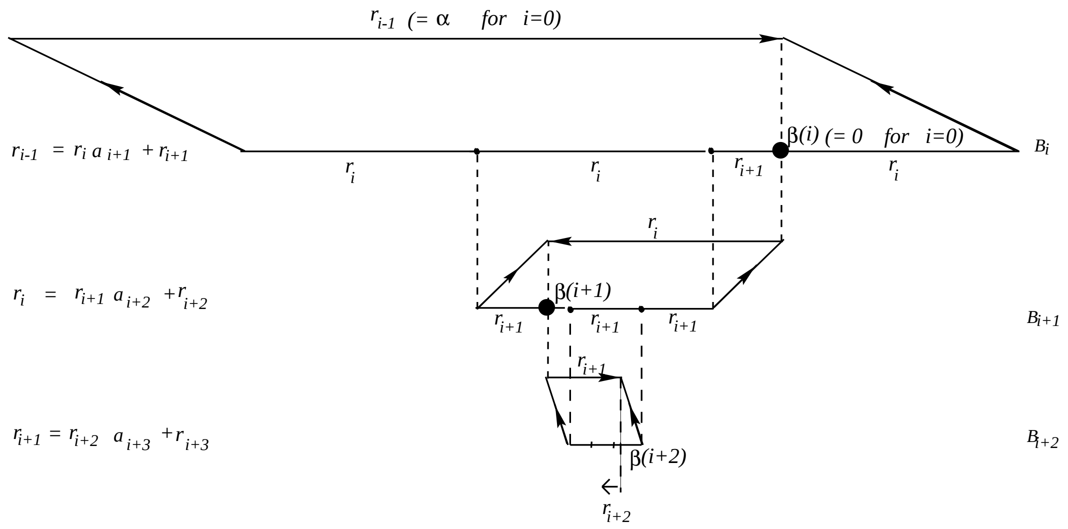
$$\beta(k) = \sum_{j=0}^{k-1} (-1)^{j+1} r_j \quad \text{for } k \in \mathbb{N}_0. \tag{5.8}$$

We need:

Definition 5.4 For $k \in \mathbb{N}_0$ let

$$\begin{aligned} u_k &:= \beta(k) - (-1)^k r_{k+1} - (a_{k+1} - 1)(-1)^k r_k \\ \text{and} \quad v_k &:= \beta(k) + (-1)^k r_k, & (5.9) \\ \text{let} \quad B_k &:= (\min\{u_k, v_k\}, \max\{u_k, v_k\}). & (5.10) \end{aligned}$$

Obviously $\beta(k) \in B_k$ and we shall call $\beta(k)$ the *base point* of B_k. Since the sequence $r_0, r_1 \ldots$ is strictly decreasing all base points satisfy $\beta(k) \le 0$ and for $k > 0$ the inequality is strict. For the same reason $\sup B_k \le 0$ if $k > 0$ and $\sup B_0 = r_0$.



Notation Let b_k be according to (5.1), (5.2). By $|b_k|$ we denote the length of the word b_k, by $b_{k,i}$ the $i-$th entry and by $\sum(b_k) = \sum_{i<|b_k|} b_{k,i}$ the sum over all entries in b_k.

Theorem 5.1 follows from the following lemmas

Lemma 5.2 *Let $k \in \mathbb{N}_0$. Then $\alpha \sum(b_k) = |b_k| + (-1)^{k+1} r_k$.*

Lemma 5.3 *Let $x \in B_k$ and $i < |b_k|$. Then $b(x,i) = b_{k,i}$.*

Observe that lemma 5.2 gives an indication on the quality of approximation of α in terms of the length of the subwords b_k of $B(\alpha)$ we have $\lim_{k\to\infty} \frac{|b_k|}{\sum(b_k)} = \alpha$; the length of the k^{th} word, divided by the sum of entries tends to α. This approximation is measured by the error term of the continued fraction r_{k+1} divided again the sum of entries $\sum(b_k)$. This obviously relates to the classical approximation via continued fractions.

Proof of Theorem 5.1 We shall evaluate the i^{th} letter of the subword b_k of $B(\alpha)$. Using lemma 5.3 and the shifting lemma obtain

$$B_{i+\sum_{j=0}^{k-1}|b_j|}(\alpha) = b_{k,i} = b(\beta(k), i) = b\left(0, \left(i + \sum_{j=0}^{k-1}|b_j|\right)\right) = C_{i+\sum_{j=0}^{k-1}|b_j|}(\alpha). \tag{5.11}$$

This shows that $B(\alpha) = C(\alpha)$ where the last equality follows from the remark following definition 5.3.

We still have to verify the hypotheses of the shifting lemma. The first one is given by $\beta(k) \le 0$. For the other note that by definition of $\beta(k)$ and lemma 5.2 one has

$$\begin{aligned}(\beta(k) - 0) + \left(i + \sum_{j=0}^{k-1}|b_j| - i\right) &= \beta(k) + \sum_{j=0}^{k-1}|b_j| = \sum_{j=0}^{k-1}(-1)^{j+1}r_j + \sum_{j=0}^{k-1}|b_j| \\ &= \sum_{j=0}^{k-1}\left((-1)^{j+1}r_j + |b_j|\right) = \sum_{j=0}^{k-1}\left(\alpha \sum(b_j)\right) \in \alpha\mathbb{N}_0. \end{aligned} \tag{5.12}$$

□

Proof of lemma 5.2 Proceed by induction on k. For $k = 0$ obtain:

$$\alpha \sum(b_0) = \alpha a_0 = 1 - r_0 = |b_0| + (-1)^{0+1} r_0 \quad . \tag{5.13}$$

For $k = 1$ we use the continued fraction expansion and definition (5.1): $|b_1| + r_1 = (a_1 - 1) + 1 + r_1 = a_1 + r_1 = a_1 + \alpha - r_0 a_1 = a_1(1 - r_0) + \alpha = a_1(a_0\alpha) + \alpha = \alpha(a_1 a_0 - a_0 + a_0 + 1) = \alpha((a_1 - 1)a_0 + (a_0 + 1)) = \alpha(a_1 - 1)\sum(b_0) + (a_0 + 1) = \alpha \sum(b_1)$.

For the induction start with definition 5.3, after some elementary transformations the induction hypothesis may be applied. By another elementary transformation and the definition of b_k using continued fractions the lemma follows:

$$\begin{aligned}\alpha \sum(b_k) &= \alpha\left(a_k \sum(b_{k-1}) + \sum(b_{k-2})\right) = \alpha a_k \sum(b_{k-1}) + \alpha \sum(b_{k-2}) = \\ &= a_k(|b_{k-1}| + (-1)^k r_{k-1}) + (|b_{k-2}| + (-1)^{k-1} r_{k-2} = a_k|b_{k-1}| + |b_{k-2}| + (-1)^{k+1}(r_{k-2} - a_k r_{k-1}) = \\ &= |b_k| + (-1)^{k+1} r_k \end{aligned} \tag{5.14}$$

□

Proof of lemma 5.3 Proceed by induction on k. For $k=0$ we have $b_0 = a_0$. Thus we show that $b(x,0) = a_0$ for $x \in B_0$. By (5.2) and (5.6) we show $|x+\alpha\mathbb{N}\cap(0,1]| = a_0$ for $x \in B_0$. In order to do so let $x \in B_0$. By (5.10) for $k=0$ one has

$$r_0 - \alpha < x < r_0, \tag{5.15}$$

thus $x+\alpha n > r_0 - \alpha + \alpha = r_0 > 0$ for all $n \in \mathbb{N}$. Consider the smallest n_0 such that $x + \alpha n_0 > 1$, i.e. $x + \alpha n_0 \notin (0,1)$, but $x + \alpha(n_0-1) \in (0,1]$ provided $n_0 - 1 \in \mathbb{N}$. By (5.15) obtain $x + \alpha a_0 < r_0 + \alpha a_0 = 1$ and also $x + \alpha(a_0+1) > r_0 - \alpha + \alpha(a_0+1) = r_0 + \alpha a_0 = 1$. So we have $n_0 = a_0 + 1$, which shows that $x + \alpha\mathbb{N}\cap(0,1] = \{x+\alpha, \dots, x + a_0\alpha\}$, thus $b(x,0) = a_0$.

For $k=1$ let $x \in B_1$ and remember that (5.1) states that $b_1 = a_0^{a_1-1} * (a_0+1)$. Thus we consider two cases:

Case I Let $0 \le i < a_i - 1$ and consider the $i-$th letter of b_1. We want to show that $b(x,i) = b_{1,i} = a_0$ for these i's. As $x \in B_1$, (5.10) yields $x - ir_0 \in B_0$. Therefore we can apply lemma 5.3 with $k=0$ and obtain by the shifting lemma $b(x,i) = b(x-ir_0,0) = a_0$ as desired. It remains to check the hypotheses of the shifting lemma:

- For the hypothesis '$z \le l$' observe that $\sup B_1 \le 0$ as noted in the definition of the B_i's.
- For the other hypothesis one has $x - ir_0 - x + i = i - ir_0 = i(1-r_0) = i\alpha a_0$.

Case II For $i = a_1 - 1$ we observe that $x - (a_1-1)r_0 + \alpha \in B_0$; shifting the arguments by $a_1 - 1$ and applying continued fractions obtain

$$\begin{aligned} a_0 + 1 &= b(x-(a_1-1)r_0+\alpha, 0) + 1 = b(x + (a_1-1)(1-r_0) + \alpha, a_1 - 1) + 1 \\ &= b(x + a_0(a_1-1)\alpha + \alpha, a_1-1) + 1 = b(x + a_0(a_1-1)\alpha, a_1-1). = b(x, a_1-1) . \end{aligned} \tag{5.16}$$

So we get $a_0 + 1 = b(x + a_0(a_1-1)\alpha, a_1-1) = b(x, a_1-1)$ by the shifting lemma, since $a_1 \ge 1$ and $x + a_0(a_1-1)\alpha - (a_1-1) < -r_1 + (a_1-1)(a_0\alpha - 1)\ -r_1 - r_0(a_1-1) \le -r_1 \le 0$.

Now assume that lemma 5.3 holds for all $i < k+2$ for some $k \ge 0$. Fix $x \in B_{k+2}$ and recall that $b_{k+2} = b_{k+1}^{a_{k+2}} * b_k * b_{k+1}$, thus $|b_{k+2}| = (a_{k+2}-1)|b_{k+1}| + |b_k| + |b_{k+1}|$. For $i < |b_{k+1}|$ one of the following cases applies:

$$\begin{array}{llcll} (i) & 0 \le i & = & |b_{k+1}|j + l, & \text{where} \quad 0 \le j < a_{k+2} - 1 \text{ and } 0 \le l < |b_{k+1}| \\ (ii) & i & = & |b_{k+1}|(a_{k+2}-1) + l, & \text{where} \quad 0 \le l < |b_k| \\ (iii) & i & = & |b_{k+1}|(a_{k+2}-1) + |b_k| + l, & \text{where} \quad 0 \le l < |b_{k+1}|. \end{array} \tag{5.17}$$

By construction of B_k, B_{k+1}, B_{k+2} observe that in each case

$$\begin{array}{llcllcl} (i) & 0 & > & x + (-1)^{k+2} r_{k+1} j & & = & x_1 \in B_{k+1}, \\ (ii) & 0 & > & x + (-1)^{k+2} r_{k+1}(a_{k+2}-1) & & = & x_2 \in B_k \\ (iii) & 0 & > & x + (-1)^{k+2} r_{k+1}(a_{k+2}-1) + (-1)^{k+1} r_k & & = & x_3 \in B_{k+1}. \end{array}$$

Using the induction hypothesis we show

$$\begin{array}{l} (i) \\ (ii) \\ (iii) \end{array} \ b_{k+2,i} = \left\{ \begin{array}{ll} b_{k+1,l} & = b(x_1 l) \\ b_{k,l} & = b(x_2, l) \\ b_{k+1,l} & = b(x_3, l) \end{array} \right\} = b(x,i), \tag{5.18}$$

for each case, respectively. As $x_m < 0 \leq l$ for $m \in \{1,2,3\}$, verifying the first hypothesis of lemma 5.1. The proof is complete provided we can verify the other hypothesis of the shifting lemma:

$$(x_m - x) + (i - l) \in \alpha\mathbb{N}_0 \text{ for } m = 1,2,3. \tag{5.19}$$

We prototypically consider case (iii) and leave the rest to the reader.

$$\begin{aligned}(x_3 - x) + (i - l) &= (-1)^{k+2} r_{k+1}(a_{k+2} - 1) + (-1)^{k+1} r_k + |b_{k+1}|(a_{k+1} - 1) + |b_k| \\ &= (a_{k+2} - 1)((-1)^{k+2} r_{k+1} + |b_{k+1}|) + (-1)^{k+1} r_k + |b_k| \\ &= \alpha(a_{k+2} - 1)\sum(b_{k+1}) + \alpha\sum(b_k)\end{aligned}$$

by lemma 5.2. But the last term is in $\alpha\mathbb{N}_0$ as desired. This completes the proof of lemma 5.3 and thus the proof of the theorem 5.1. $\square$

5.2 Wobbling bijections for rectangular lattices

In this section we shall give an "explicit" wobbling bijection between the rectangular lattice $R = \alpha\mathbb{N} \times 1/\alpha\mathbb{N}$ and $\mathbb{N}^2$ using the Sturmian sequence.
For $\alpha < 1$ let $\alpha = [0, a_0, a_1, a_2, \ldots]$ be the continued fraction expansion. The Sturmian seqences $B(\alpha)$ and $B(1/\alpha)$ are as follows:

$\boldsymbol{\alpha}$			$\mathbf{1/\boldsymbol{\alpha}}$		
b_0	$=$	a_0	b_0'	$=$	0
b_1	$=$	$a_0^{a_1-1} * (a_0+1)$	b_1'	$=$	$0^{a_0-1} * 1$
	$\vdots$			$\vdots$	
b_k	$=$	$b_{k-1}^{a_{k-1}} * b_{k-2} * b_{k-1}$	b_k'	$=$	$b_{k-1}^{'a_{k-1}-1} * b_{k-2}' * b_{k-1}'$
$B(\alpha)$	$=$	$b_0 * b_1 * \ldots$	$B(1/\alpha)$	$=$	$b_0' * b_1' * \ldots$

Note that $B(\alpha)$ is over the alphabet $\{a_0, a_0+1\}$ and $B(1/\alpha$ is over the alphabet $\{0, 1\}$. One easily obtains $B(1/\alpha)$ from $B(\alpha)$ by means of the following *rewriting system*:

$$\begin{aligned} f(a_0) &= 0^{a_0-1} * 1 \\ f(a_0+1) &= 0^{a_0} * 1 \\ \text{Then one obtains}\qquad B(1/\alpha) &= 0 * f(B(\alpha)) \end{aligned} \tag{5.20}$$

Of course $B_i(\alpha)$ is the ith term in the sequence $B(\alpha)$. Define the following infinite $0, \pm 1$ matrix $N = (n_{i,j})_{i,j}$, where $n_{i,j} = B_i(\alpha) - B_j(\alpha)$.

Lemma 5.4 *Along all diagonals of N the elements $+1$ and -1 occur alternately.*

Proof Assume that $n_{i,j}$ and $n_{i+k,j+k}$ are two consecutive nonzero entries (the case $n_{i+k,j-k}$ can be handled similarly). Say $n_{i,j} = +1$. Then by definition

$$\begin{array}{lcl} B_i(\alpha) & \neq & B_j(\alpha) \\ B_{i+1}(\alpha) & = & B_{j+1}(\alpha) \\ \vdots & & \vdots \\ B_{i+k-1}(\alpha) & = & B_{j+k-1}(\alpha) \\ B_{i+k}(\alpha) & = & ? \end{array} \tag{5.21}$$

The words $W = (B_i(\alpha), \ldots, B_{i+k}(\alpha))$ and $W' = (B_j(\alpha), \ldots, B_{j+k}(\alpha))$ are segments of $B(\alpha)$ of the same length k. By the Sturmian property $|\sum(W) - \sum(W')| \leq 1$. Then $B_{i+k}(\alpha) = B_{j+k}(\alpha) - 1$ for otherwise $|\sum(W) - \sum(W')| = 2$ and $n_{i+k,j+k} = -1$, as desired. $\square$

We need the tiling

$$\mathbb{R}^+ \times \mathbb{R}^+ = \left(\bigcup_{j=0}^{\infty} [0,1) \times [j, j+1) \right) \ \dot{\bigcup} \ \left(\bigcup_{i,j \in \mathbb{N}} I_i \times [j, j+1) \right), \tag{5.22}$$

where (I_i) is a sequence of consecutive intervals of length $l\,(I_i) = B_i(\alpha)$.

Lemma 5.5 *For every i, j the tile $I_i \times [j, j+1)$ intersects the lattice R in $B_j(\alpha)$ many points.*

Proof I_i has length $B_i(\alpha)$. By means of the rewriting definition (5.20) of $B(1/\alpha)$ the segment I_i of $B(1/\alpha)$ has exactly 1 entry of value one. Moreover by theorem 5.1 the Sturmian sequence $B_j(\alpha)$ counts $|\alpha\mathbb{N} \cap (j, j+1)|$. $\square$

Now we define a wobbling bijection between R and $\mathbb{N} \times \mathbb{N}$ as follows.

Consider a tile $T = I_i \times [j, j+1)$ with the property $n_{i,j} = +1$ (say). By definition

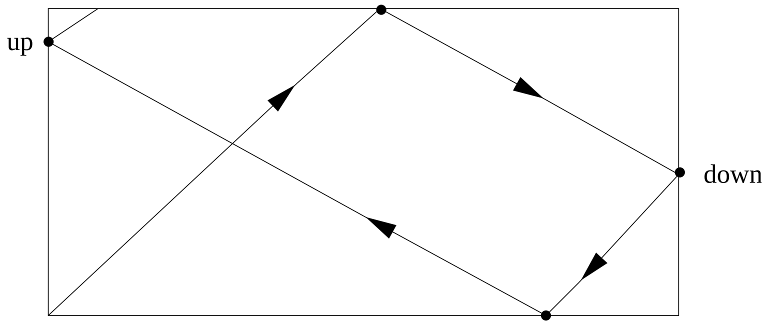
$$1 + |T \cap \mathbb{N}^2| = |T \cap R| \tag{5.23}$$

thus there is one vertex too much for allowing a local bijection between parts of R and $\mathbb{N}^2$ within T. Now successively shift one surplus point along the diagonal into the next tile. After suitably many iterations according to lemma 5.4 this procedure ends in a tile T' with $n_{i,j} = -1$ balancing the process. Finally this local procedure gives a global bijection by patching together the local mappings.

This is a wobbling mapping as one maps from any tile either to itself or an adjoining diagonal neighbouring tile and tiles have size atmost $(a_0+1)\times 1$. For the particular choice of $R = \frac{1}{\alpha}\mathbb{N}\times\alpha\mathbb{N}$ it should be technically possible to calculate reasonable wobbling distances.

5.3 Discrepancies of billiards

Here we explore the Sturmian sequence for a very special billiard with $\alpha = \sqrt{2}$. This can surely be generalized, but this simple case is illustrative. Consider the billiard on a square table hitting the ball in the lower left corner with angle of slope α. Then the ball will alternatively hit the left and right vertical edges. In order to obtain a more sophisticated problem we consider the sequence of “upwards” and “downwards” hits of vertical edges and ask for the discrepancy as follows:



Let $\bigcup_N(\alpha)$ be the number of upward hits among the first N vertical hits and similarly $\bigcap_N(\alpha)$ for the downward hits. The discrepancy will be

$$D_N(\alpha) = \bigcup\nolimits_N(\alpha) - \frac{N}{2} = \frac{\bigcup_N(\alpha) - \bigcap_N(\alpha)}{2} \ . \tag{5.24}$$

We only want to consider this discrepancy for very special N's. Consider the Sturmian sequence and let N be of the form

$$N = \sum(b_0 * \cdots * b_k) = \sum_{i<\mathrm{length}(b_0*\cdots*b_k)} B_i(\alpha) \tag{5.25}$$

for some k. Obviously by (5.3) $k \sim \log N$. Then from (5.4)

$$\begin{array}{rcl} \bigcup_N(\alpha) & = & B_0(\alpha) + B_2(\alpha) + B_4(\alpha) + \dots \\ \bigcap_N(\alpha) & = & B_1(\alpha) + B_3(\alpha) + \dots + \ , \end{array} \tag{5.26}$$

where sums extend over all even (odd) $i < length(b_0 * \cdots * b_k)$. Therefore

$$\bigcup\nolimits_N(\alpha) - \bigcap\nolimits_N(\alpha) = B_0(\alpha) - B_1(\alpha) + B_2(\alpha) - B_3(\alpha) \dots \tag{5.27}$$

In order to simplify the further calculations we take $\alpha = \sqrt{2} = [1, \bar{2}, \dots]$. One has $b_0 = 0$, $b_1 = 1$, $b_2 = 101$ and $b_k = b_{k-1} * b_{k-2} * b_{k-1}$.

Notation Let $x = (x_0 \dots x_l)$ then $\overline{\overline{\sum}}(x) = \sum(-1)^j x_j$.

Lemma 5.6 *For* $\alpha = \sqrt{2}$

$$\bigcup\nolimits_N(\alpha) - \bigcap\nolimits_N(\alpha) = \sum_{j\le k}\left((-1)^j \ \overline{\overline{\sum}}(b_j)\right). \tag{5.28}$$

Proof Proceed by induction on k. For $k = 0, 1$ the statement obviously holds. Assume that it holds for some $k \ge 1$. Then

$$\begin{aligned} &\bigcup\nolimits_{N(k+1)}(\sqrt{2}) - \bigcap\nolimits_{N(k+1)}(\sqrt{2}) \bigcup\nolimits_{N(k)}(\sqrt{2}) - \bigcap\nolimits_{N(k)}(\sqrt{2}) + (-1)^{k+1}\ \overline{\overline{\sum}}(b_{k+1}) = \\ &\qquad = \sum_{j\le k}\left((-1)^j\ \overline{\overline{\sum}}(b_j)\right) + (-1)^{k+1}\ \overline{\overline{\sum}}(b_{k+1}) \sum_{j\le k+1}\left((-1)^j\ \overline{\overline{\sum}}(b_j)\right) \end{aligned} \tag{5.29}$$

□

Lemma 5.7 *For $\alpha = \sqrt{2}$ one has for all $j \in \mathbb{N}_0$*

$$\overline{\overline{\sum}}(b_j) = j \qquad . \tag{5.30}$$

Proof Proceed by induction on j. The cases $j = 0, 1$ are obvious. For $\alpha = \sqrt{2}$ the recursion (5.2) specializes to $b_j = b_{j-1} * b_{j-2} * b_{j-1}$. The inductionstep shows

$$\overline{\overline{\sum}}(b_j) = 2\overline{\overline{\sum}}(b_{j-1}) - \overline{\overline{\sum}}(b_{j-2}) = j \tag{5.31}$$

again using the fact that $|b_j| = 1 \ (mod 2)$ for all j. Combining these facts obtain

$$\bigcup\nolimits_N(\sqrt{2}) - \bigcap\nolimits_N(\sqrt{2}) = \sum_{j \le k}\left((-1)^j \overline{\overline{\sum}}(b_j)\right) = \sum_{j \le k}(-1)^j \cdot j = \begin{cases} +k & \text{for } k \text{ even} \\ -(k+1) & \text{for } k \text{ odd} \end{cases} \tag{5.32}$$

Corollary 5.1 *For the $\sqrt{2}$-billiard the discrepancy $D_N(\sqrt{2})$ is at least $\log N$ for infinitely many N.*

This is a very special case of a deep theorem by Roth telling that discrepancies are not too small. Here we indicated the simple billiard model in order to have an elementary, accessible example.

6 Paradoxical Situations

Following the classical result by Tarski [4] that the unit ball B in $\mathbb{R}$ may be decomposed into two parts B_1, B_2 such that B, B_1, B_2 are pairwise congruent. Paradoxical decompositions were defined for metric spaces with wobbling equivalences [1].

Definition 6.1 Let M be a metric space. M is paradoxical if there is a decomposition into two parts M_1, M_2 such that M, M_1, M_2 are pairwise equivalent.

For discrete countable metric spaces one can define exponential growth via the concept of the 'doubling radius' r :

Definition 6.2 Let M be a countable discrete metric space M has *at least exponential growth* if there exists $r \in \mathbb{R}^+$ such that for every finite subset X of M the $r-$ neighbourhood of X satisfies

$$|N_r(X)| \geq 2|X|. \tag{6.1}$$

Theorem 6.1 *[1] Let M be a discrete countable metric space. Then M is paradoxical iff M has at least exponential growth.*

Let us report that Elek [7] used this theorem in order to characterize the finitely generated amenable groups.

Theorem 6.1 is as effective as the matching theorem of Hall- Rado, which is used for the proof of the Rado- Hall theorem. A recursive version does not hold [8] but paradoxity can be tested by countably many tests on finite sets, in the sense of Hall's theorem applied to a neighborhood graph.

Let X be a recursive subset of $\mathbb{Q}^2$, and d be a positive real. If X has a d-wobbling to $\mathbb{Z}^2$, is there a recursive one? For trivial reasons, the answer is yes if d is less than $\frac{1}{2}$. Let us show, that the answer in general is no if $d = \frac{1}{2}$.

Theorem 6.2 *There is a recursive set $X \subseteq \mathbb{Q}^2$ having a $\frac{1}{2}$ − wobbling to $\mathbb{Z}^2$, but no recursive one.*

Proof The proof will be a classical diagonalization argument. Every index e_i is associated with the vertical line L_i through $(i,0)$. The aim is to construct X in such a way, that φ_{e_i}, the algorithm having index e_i, working wrongly on L_i, cannot be a $\frac{1}{2}$ − wobbling of X to $\mathbb{Z}^2$. The set X is defined inductively by stages:

$$X_0 \subseteq X_1 \subseteq \cdots \subseteq X_n \subseteq \ldots \qquad \text{s.t.} \bigcup_n X_n = X \tag{6.2}$$

For bookkeeping purposes we use marker and lists Λ_n of "inactive" indices. Let $e_1, e_2, e_3, \ldots$ be a recursive enumeration of indices of all partial recursive functions on $\mathbb{Q}^2$ to $\mathbb{Z}^2$.

$$\begin{aligned} \text{stage}\,0: \qquad X_0 \;&:=\; \{(a,b) \mid a,b \in \mathbb{Z} \wedge a \le 0\} \\ &\;\cup\; \left\{(i,\tfrac{1}{2}) \mid i \in \mathbb{N} \setminus \{0\}\right\} \\ &\;\cup\; \left\{(i,-\tfrac{1}{2}) \mid i \in \mathbb{N} \setminus \{0\}\right\}, \\[1em] \Lambda_0 \;&:=\; \emptyset. \end{aligned}$$

X_0, Λ_0 are recursive sets.

$$\text{stage}\ n{+}1: \qquad \begin{array}{l} \text{We consider} \\ \{e_1, \ldots, e_{n+1}\} \setminus \Lambda_n =: A_{n+1}, \end{array}$$

the set of indices active in stage $n+1$.

Let e_i be an element of A_{n+1}. We compute $n+1$ steps of φ_{e_i} for argument $(i, \frac{1}{2})$.

case $n+1\,.\,1$: $\varphi_{e_i}^{n+1}$ gives output $(i,1)$
We put a marker on $(i, -(n+1)+\frac{1}{2})$

case $n+1\,.\,2$: $\varphi_{e_i}^{n+1}$ gives output $(i,0)$
We put a marker on $(i, n+1+\frac{1}{2})$

case $n+1\,.\,3$: $\varphi_{e_i}^{n+1}$ gives no output or does something else
We place no marker.

We define
$$X_{n+1} := X_n \cup \left\{ \begin{array}{l} (a, b+\tfrac{1}{2}) \mid a \in \mathbb{N} \text{ and } 1 \le a \le n+1 \\ \text{and } b \in \mathbb{Z} \text{ and } -(n+1) \le b \le n+1 \\ \text{and } (a, b+\tfrac{1}{2}) \text{ unmarked} \end{array} \right\} \tag{6.3}$$

Obviously X_{n+1} is recursive. $\Lambda_{n+1} := \Lambda_n \cup \{e_i \in A_{n+1} \mid \varphi_{e_i}^{n+1} \text{ gives an output}\}$.
Clearly $X = \bigcup_n X_n$ is recursive.

Let us show, that there is a $\frac{1}{2}$ – wobbling φ of X to $\mathbb{Z}^2$. We consider the vertical lines L_{-i} for $i \in \mathbb{N}$. On these lines φ is the identity. We consider the vertical lines L_i, $1 \leq i$. On these lines φ works as follows. If there is no marker on line L_i we define $\varphi(i, z + \frac{1}{2}) = (i, z)\,,\ z \in \mathbb{Z}$.

If a marker is on $(i, -(n+1) + \frac{1}{2})$, we define

$$\begin{aligned} \varphi(i, z + \tfrac{1}{2}) &= (i, z)\,, \quad z \geq -n \\ \text{and} \qquad \varphi(i, z + \tfrac{1}{2}) &= (i, z+1)\,, \quad z < -n \end{aligned} \tag{6.4}$$

If a marker is on $(i, n + 1 + \frac{1}{2})$, we define

$$\begin{aligned} \varphi(i, z + \tfrac{1}{2}) &= (i, z+1)\,, \quad z \leq n \\ \text{and} \qquad \varphi(i, z + \tfrac{1}{2}) &= (i, z)\,, \quad z > n \end{aligned} \tag{6.5}$$

Obviously $\varphi(X) = \mathbb{Z}^2$, and φ is injective. So φ is a $\frac{1}{2}$ – wobbling of X onto $\mathbb{Z}^2$.
Now we prove that no φ_{e_i} is a $\frac{1}{2}$ – wobbling of X to $\mathbb{Z}^2$. Suppose that φ_{e_i} is a $\frac{1}{2}$ – wobbling. Then φ_{e_i} is eventually defined on $(i, \frac{1}{2})$, say in step $n+1$. Any $\frac{1}{2}$ – wobbling of X to $\mathbb{Z}^2$ moves only at vertical lines. If $\varphi_{e_i}^{n+1}(i, \frac{1}{2}) = (i, 1)$, a marker is on $\big(i, -(n+1) + \frac{1}{2}\big)$. Hence $\big(i, -(n+1) + \frac{1}{2}\big)$ is not in X_{n+1} and a fortiori not in X. To close the gap at $\big(i, -(n+1) + \frac{1}{2}\big)$ any $\frac{1}{2}$ – wobbling is forced to work like φ above. So φ_{e_i} moves $(i, \frac{1}{2})$ in the wrong direction.
If $\varphi_{e_i}^{n+1}(i, \frac{1}{2}) = (i, 0)$ the same argument applies. Hence there is no recursive $\frac{1}{2}$ – wobbling of X to $\mathbb{Z}^2$.$\square$

Remark The above argument is prototypical and may be adapted to show that for every $d \geq 1/2$ the analogue of theorem 6.2 holds. A general method to construct counterexamples as the one above is given in [9].

References

[1] Deuber, W.A., Simonovits, M., Sós, V.T., *A note on paradoxical metric spaces*, Studia Scientiarum Mathematicarum Hungarica **30** (1995), 17-23

[2] Laczkovich, M., *Equidecomposability and discrepancy; a solution of Tarski's circle-squaring problem*, J. Reine Angew. Math. **404** (1990), 77-117. MR **91b**:51034

[3] Sudmeier, R., *Beitrag zum Studium distanzbeschränkter Bijektionen zwischen Laczkovich-äquivalenten Mengen unter besonderer Berücksichtigung von Rechteckgittern*, Diplomarbeit, Bielefeld (1997)

[4] Banach, S., Tarski, A., *Sur la decomposition des ensembles de points en parties respectivement congruents*, Fund. Math. **6** (1924), 244-277

[5] Koppenrade, E., *Beiträge zum Studium effektiver distanzbeschränkter Abbildungen zwischen Gittern mit besonderer Berücksichtigung rotierter Gitter unter Verwendung der Kettenbruchdarstellung*, Dissertation, Bielefeld (1996)

[6] Brown, T., C., *Description of the characteristic sequence of an irrational*, Canad. Math. Bull., Vol. 36 (1) 1993, 15-21

[7] Elek, G., *The K-theory of Gromov's translation algebras and the amenability of discrete groups*, Preprint (1996), to appear int the Proceedings of the AMS

[8] Manaster, A., Rosenstein, J., *Effective matchmaking (recursion theoretic aspects of a theorem of Philip Hall)*, Proc. London Math. Soc. (3) 25 (1972), 615–654

[9] Carstens, H.-G., Päppinghaus, P., *Abstract Construction of Counterexamples in Recursive Graph Theory*, in M. M. Richter, E. Börger, W. Oberschelp, B. Schinzel, and W. Thomas (ed.): Computation and Proof Theory, Lecture Notes in Mathematics 1104, Springer 1984