REALIZATIONS OF FROBENIUS FUNCTIONS

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Abstract. In an earlier paper (see [ALR]) we studied the so-called Frobenius functions on certain translation quivers. Here we show that the classification given there is in some sense complete: every Frobenius length function on the wing $W(n)$ and the tube $T(n)$ is equivalent to the length function on a convex subquiver of the Auslander–Reiten quiver of the module category over some algebra A.

1. Introduction

Let $\Gamma = (\Gamma_0, \Gamma_1, \tau)$ be a translation quiver (without multiple arrows), and let $f : \Gamma_0 \longrightarrow \mathbb{Z}$ be an integral valued function defined on the vertices of Γ. For any $z \in \Gamma_0$ a non-projective vertex we define the defect of the function f at the vertex z (or rather on the mesh ending at z) by $\delta(z) = \delta_f(z) = f(z) + f(\tau z) - \sum\limits_{y \to z} f(y)$. We call the function f a *Frobenius function* if $\delta(z) \neq 0$ implies $\delta(z) > f(z)$ and $\delta(z) > f(\tau z)$. A Frobenius function with positive values only will said to be *positive*. Meshes with non-zero defect are called *incomplete meshes*; otherwise we say that f is additive on the mesh. Two Frobenius functions are said to be *equivalent* provided they have the same set of incomplete meshes. A typical example of a positive Frobenius function is the dimension function on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra over a field k. A *Frobenius length function* is a positive Frobenius function for which $\delta(z) - f(z) = \delta(z) - f(\tau z) = 1$ whenever the mesh ending at the vertex z is incomplete. A typical example of a Frobenius (length) function is the length function ℓ on the stable Auslander–Reiten quiver of a finite dimensional selfinjective algebra over a field k.

In [ALR] we studied in detail positive Frobenius functions on certain translation quivers. In particular we investigated the cases of the wings $W(n)$, the stable tubes $T(n)$ and the related translation quivers $\mathbb{Z}A_\infty$ and $\mathbb{Z}A_\infty^\infty$ and gave a full description of the equivalence classes of Frobenius functions on these quivers by describing geometrically the possible configurations of incomplete meshes.

The aim of this paper is to give algebraic realizations for Frobenius length functions on $W(n)$ and $T(n)$. Namely, we want to prove the following:

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THEOREM 1. *Let f be a positive basic Frobenius length function on the translation quiver $W(n)$. Then there exists a finite dimensional, basic special biserial algebra $\mathcal{C}$ and a translation subquiver Γ of the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ which is isomorphic to (hence it can be identified with) $W(n)$ so that:*

(i) *the length function on Γ coincides with f;*
(ii) *the incomplete meshes of Γ with respect to f are precisely those from which a projective-injective vertex in the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ was removed to obtain Γ.*

We note here that every Frobenius length function on $W(n)$ is equivalent to a basic Frobenius length function (see [ALR]). Thus the above theorem gives algebraic realization up to equivalence for an arbitrary Frobenius length function on $W(n)$.

THEOREM 2. *Let f be a positive Frobenius length function on the translation quiver $T(n)$. Then there exists a finite dimensional special biserial algebra $\mathcal{C}$ and a translation subquiver Γ of the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ which is isomorphic to (hence it can be identified with) $T(n)$ so that:*

(i) *the length function on Γ is equivalent to f;*
(ii) *Γ can be obtained from a component of $\Gamma(\mathcal{C})$ by removing all the projective-injective vertices;*
(iii) *the incomplete meshes of Γ with respect to f are precisely those from which a projective-injective vertex in the Auslander–Reiten quiver $\Gamma(\mathcal{C})$ was removed.*

We would like to mention here that the ideas used in our construction very much rely on ideas contained in the works [HW] and [WW]. In particular we use the explicit description of the Auslander–Reiten sequences for special biserial algebras, given in [WW] (see also [BR]). For the unexplained notation and background we refer to [ALR] and [R].

2. Construction of the algebras

In what follows, k will be an arbitrary field. We shall consider k-algebras (not necessarily finite dimensional) which will be given as path algebras modulo some relations over finite quivers.

Thus let $\mathcal{C}$ be such a k-algebra. The considered $\mathcal{C}$-modules will always be left modules. Accordingly, we shall 'multiply the arrows' of the quiver from the left: if α and β are arrows in the quiver then $\alpha\beta$ will stand for the path with β coming first, followed by α. The category of all left $\mathcal{C}$-modules which are finite dimensional over k will be denoted by $\mathcal{C}$-mod.

Throughout the paper we shall use the following coordinatization for the translation quivers under consideration. The set of vertices of $\mathbb{Z}A_\infty^\infty$ is the the set $\mathbb{Z} \times \mathbb{Z}$ of integral lattice points in the plane. There are arrows $(i,j) \to (i+1,j)$ and $(i,j) \to (i,j+1)$, and the translation is defined by $\tau(i,j) = (i-1,j-1)$ for any $(i,j) \in \mathbb{Z} \times \mathbb{Z}$. We consider $\mathbb{Z}A_\infty$ as the full subquiver of $\mathbb{Z}A_\infty^\infty$ on the set of vertices $\{(i,j) \in \mathbb{Z} \times \mathbb{Z} \mid i \leq j\}$. For $W(n)$ we fix a standard embedding, mapping the projective vertices $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n$ to the points $(1,1), (1,2), \ldots, (1,n)$. Hence, $W(n) = \{(i,j) \mid 1 \leq i \leq j \leq n\}$. Finally, we also consider $\mathbb{Z}A_\infty$ as the universal cover of $T(n)$ so that the vertices on the mouth of $T(n)$ correspond to the boundary $\{(i,i) \mid i \in \mathbb{Z}\}$ of $\mathbb{Z}A_\infty$. The standard coordinatization of $T(n)$ maps the points of $T(n)$ to the strip $\{(i,j) \mid 1 \leq i \leq j,\ i \leq n\}$.

Let us recall from [ALR] that to any Frobenius function f on the wing W_n we can attach a *binary vector* $(f_1, f_2, \ldots, f_n) = \big(f(\mathbf{p}_1), f(\mathbf{p}_2), \ldots, f(\mathbf{p}_n)\big)$, (i.e. a vector where $f_i = f_j$ for $i < j$ implies that there is an index ℓ such that $i < \ell < j$ and $f_\ell < f_i$) and to this a *rooted embedded binary tree* $B(f)$ on the vertex set $\{1, 2, \ldots, n\}$ by the following recursion. If f_r is the (unique) minimal element of the binary vector, then the root of the tree will be the vertex r. Furthermore, we put an arrow going from r to the root of the binary tree assigned to the vector $(f_1 - f_r, \ldots, f_{r-1} - f_r)$ and an arrow going from the root of the binary tree assigned to $(f_{r+1} - f_r, \ldots, f_n - f_r)$ into the vertex r. The first arrow (if it exists) will be 'coloured' by φ, and the second one (if it exists) by ψ. The arrows coloured by φ will be called φ-arrows, those coloured by ψ are the ψ-arrows. A binary vector is called *basic* if the minimal elements of the binary vectors considered in the above recursion are always 1. It is easy to see that if $(f_1, f_2, \ldots, f_n)$ is a basic binary vector, and $B(f)$ the associated binary tree, then f_i is just the number of vertices in the unique path in the underlying unoriented graph of $B(f)$ from the vertex i to the root r. (Note that the rooted embedded binary tree $B(f)$ was denoted by $\vec{B}(f)$ in [ALR], but here we shall use this simplified notation.) A positive Frobenius function on $W(n)$ is called *basic* if the restriction of f to the projective vertices gives a basic binary vector.

Let us fix a basic Frobenius length function on $W(n)$. We shall construct an algebra $\mathcal{C}$ satisfying the requirements of Theorem 1.

Consider first the binary tree $B = B(f)$ corresponding to the basic binary vector $(f_1, f_2, \ldots, f_n)$ as defined above and let $\mathcal{B}$ be the algebra given by the quiver $B = B(f)$ together with the relations $\varphi\psi = 0$ for all arrows of type φ and ψ. Let us denote by r the root of B. (Note that it is well known how to attach to f a basic tilting $A_n(k)$-module with dimension vector $(f_1, f_2, \ldots, f_n)$, where $A_n(k)$ is the quiver algebra over k for the linearly ordered quiver of type A_n; see [HR]. The algebra $\mathcal{B}$ is just the corresponding tilted algebra.)

We define the convex subquiver A^+ of B formed by the endpoints of the φ-arrows in B, and let A^0 be the set of vertices of B not belonging to A^+. Notice that A^+ is just the disjoint union of the maximal φ-paths of B without their starting points. In order to stress that a vertex i belongs to A^+, we sometimes will write i^+ instead of i. Let A^- be an additional copy of the quiver A^+, hence isomorphic to A^+, with the correspondence $i \mapsto i^-$ giving the isomorphism (the inverse isomorphism from A^- to A^+ being denoted by $j \mapsto j^+$).

Finally we define the quiver C on the disjoint union of B and A^- with additional arrows going from A^- to B as follows. If there is a maximal φ-path in B of length at least 1 starting in i and ending in j, then j belongs to A^+ and we add a φ-arrow from j^- to i. If there is a maximal ψ-path starting in i and ending in j, and if j belongs to A^+, then we add an ψ-arrow from j^- to i.

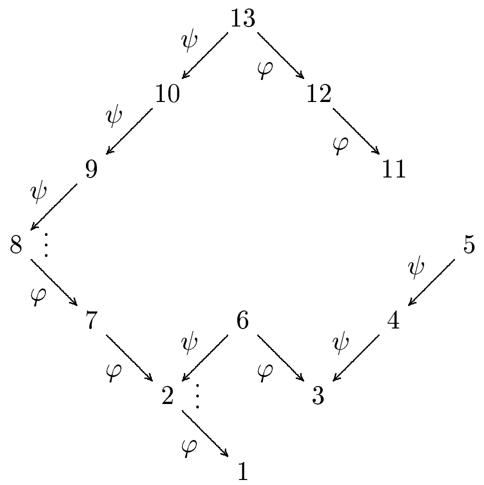
Thus for the sets of vertices we have $C = A^+ \cup A^0 \cup A^-$, furthermore B is the full subquiver of C generated by $A^+ \cup A^0$, and we shall denote by B^- the full subquiver of C generated by $A^0 \cup A^-$. To emphasize a duality between B and B^-, we shall occassionally write $B = B^+$.

We note the following property of C. Every vertex is the starting point of at most one φ-arrow and at most one ψ-arrow, and it is also the end point of at most one φ-arrow and at most one ψ-arrow.

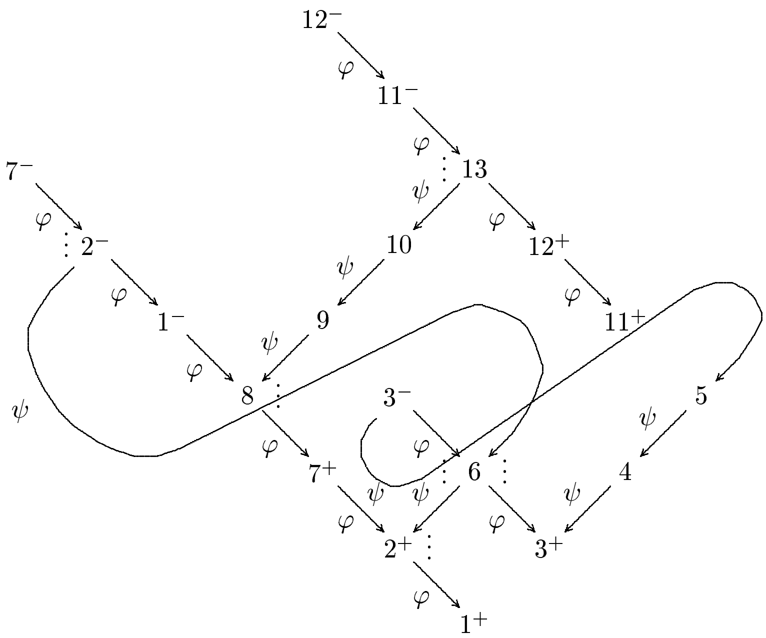
Here is an example. Let us start with the basic binary vector

$$(4, 3, 5, 6, 7, 4, 2, 1, 2, 3, 6, 5, 4).$$

The quiver $B(f)$ has the following shape (the dots indicate where a φ-arrow follows a ψ-arrow). In contrast to a usual convention, the root of the tree (vertex 8) is drawn at the left, not at the bottom of the picture; in this way, all arrows point downwards (this will be helpful for visualizing 'strings').



The subquiver A^+ has three components, namely $7 \to 2 \to 1$, 3 and $12 \to 11$, therefore the quiver of C is as follows (again we have added dots in order to indicate the positions where a φ-arrow and a ψ-arrow follow each other, in any order). Some arrows are drawn curved in order to indicate the local similarity.



Let us define the algebra $\mathcal{C}$ to be the path-algebra of the quiver C modulo the relations $\varphi\psi = 0$ and $\psi\varphi = 0$ for all arrows of type φ and ψ in C, all commutativity relations for the paths going from i^- to i^+ for $i \in A^+$, plus the following zero relations: if $i, j \in A^+$ with an arrow $i \xrightarrow{\varphi} j$ and α is the φ-path from i^- to i^+, then we set the product $\varphi\alpha = 0$.

We should note here that, alternatively, we could have defined $\mathcal{C}$ as the matrix algebra

$$\mathcal{C} = \begin{bmatrix} \mathcal{B} & I^+ \\ 0 & \mathcal{A}^- \end{bmatrix},$$

where $\mathcal{A}^-$ is the path algebra of A^-, the bimodule I^+ is the direct sum of the indecomposable injective $\mathcal{B}$-modules corresponding to the vertices in A^+, with the right action of $\mathcal{A}^-$ on I^+ given by the canonical isomorphism $\mathcal{A}^- \simeq \operatorname{End}_{\mathcal{B}}(I^+)$.

Given a vertex i, we denote by $P_{\mathcal{C}}(i)$ the indecomposable projective $\mathcal{C}$-module corresponding to i and by $I_{\mathcal{C}}(i)$ the indecomposable injective $\mathcal{C}$-module corresponding to i. Similar notations will be used when we consider indecomposable projective or injective $\overline{\mathcal{C}}$-modules (or, later, $\mathcal{D}$-modules and $\overline{\mathcal{D}}$-modules).

It is easy to see that $I_{\mathcal{C}}(i) = P_{\mathcal{C}}(i^-)$ is a projective-injective $\mathcal{C}$-module for every $i \in A^+$. The direct sum of the socle of these projective-injective summands of $\mathcal{C}$ is an ideal J and clearly, it annihilates every other indecomposable module.

Let us define the algebra $\overline{\mathcal{C}} = \mathcal{C}/J$. Thus the quiver of $\overline{\mathcal{C}}$ remains the same as that of $\mathcal{C}$, i.e. it is C, and besides the relations for $\mathcal{C}$, we add the new relations $\alpha = 0$ for every path α going from i^- to i for some $i \in A^+$. These paths will be called *Nakayama paths*. The only difference between the Auslander–Reiten quivers of $\overline{\mathcal{C}}$ and $\mathcal{C}$ is that in the latter we have removed the projective-injective modules $I_{\mathcal{C}}(i)$ with $i \in A^+$, and have broken the τ-orbits at these places.

We shall prove that the convex subquiver of the Auslander–Reiten quiver of $\mathcal{C}$ spanned by $P_{\mathcal{C}}(r)$ and $I_{\mathcal{C}}(n)$ with the projective-injective modules $I_{\mathcal{C}}(i)$ with $i \in A^+$ being removed, gives a required representation of the Frobenius function f.

Let us observe first that both $\mathcal{C}$ and $\overline{\mathcal{C}}$ are special biserial algebras, thus we may use the description of the Auslander–Reiten sequences for these algebras given in [WW]. In order to conform with the technical requirements in [WW], we shall give an explicit description of the subquiver between $P_{\mathcal{C}}(r) = P_{\overline{\mathcal{C}}}(r)$ and $I_{\mathcal{C}}(n) = I_{\overline{\mathcal{C}}}$ in $\overline{\mathcal{C}}$-mod rather than in $\mathcal{C}$-mod.

Note that both $P_{\mathcal{C}}(r)$ and $I_{cC}(n)$ are so-called *string modules*, thus in order to describe the corresponding subquiver, we shall first make a few observations concerning strings and string modules over $\mathcal{C}$ and $\overline{\mathcal{C}}$. The proofs will be deferred to the last section.

Let us recall the definition of a string. Suppose $\mathbf{u} = (u_1, u_2, \ldots, u_t)$ is a sequence of vertices in C such that for every $i = 1, \ldots, t-1$ we have either $u_i \xrightarrow{\varphi} u_{i+1}$ or $u_i \xleftarrow{\psi} u_{i+1}$. We shall also assume that no substring is a 0 path in the algebra $\overline{\mathcal{C}}$ (i.e. strings cannot contain Nakayama paths). Such a sequence will be called a *string from u_1 to u_t over $\overline{\mathcal{C}}$*.

Note that defining strings in this way, we give a direction to the walks in the graph C which correspond to non-zero elements in $\overline{\mathcal{C}}$. Hence a string from a to b will be different from a string from b to a for any vertices $a \neq b$ of C. With this definition we have the following uniqueness result for strings.

LEMMA 2.1. *For any $a, b \in C$ we have at most one string from a to b over $\overline{\mathcal{C}}$.*

To each string $\mathbf{u}$ over $\overline{\mathcal{C}}$ we can attach a uniquely defined indecomposable $\overline{\mathcal{C}}$-module $M(\mathbf{u})$, called a *string module* (see for example [WW]). Observe that $\mathbf{u} \neq \mathbf{v}$ implies that $M(\mathbf{u}) \not\simeq M(\mathbf{v})$.

The previous lemma shows that we may introduce the following notation: if $a, b \in \overline{B}$ and there exists a string from a to b then the corresponding unique string module will be denoted by $M(a,b)$.

In order to deal with the explicit structure of the Auslander–Reiten quivers of $\mathcal{C}$ and $\overline{\mathcal{C}}$, respectively, let us first observe a partial symmetry of the graph C.

If x is a vertex of C, write $\varphi(x) = y$ and $\varphi^{-1}(y) = x$ provided that there is a φ-arrow from x to y in C. In particular, $\varphi(x)$ is defined for all vertices in A^- and $\varphi^{-1}(x)$ is defined for all vertices in A^+.

Also, we write $\varphi^{\omega}(x) = y$ provided the longest φ-path (possibly of length 0) starting in x and not including a Nakayama path ends in y. Similarly, we write $\varphi^{-\omega}(y) = x$ provided the longest φ-path ending in y and not including a Nakayama path starts in x. Note that these functions φ^{ω} and $\varphi^{-\omega}$ are defined for all vertices of C. They are not inverse to each other, but they satisfy the relations $\varphi^{\omega}\varphi^{-\omega}\varphi^{\omega} = \varphi^{\omega}$ and $\varphi^{-\omega}\varphi^{\omega}\varphi^{-\omega} = \varphi^{-\omega}$. It is clear that the image of φ^{ω} is the set of vertices of $B = B^+ = A^0 \cup A^+$, that of $\varphi^{-\omega}$ is the set of vertices of $B^- = A^0 \cup A^-$ and it follows that the restriction of φ^{ω} is a bijection from the set of vertices of B^- to the set of vertices of B^+, with the inverse map being $\varphi^{-\omega}$. To simplify the notation, we shall write i^* for $\varphi^{-\omega}(i)$ for any vertex i of B^+. Thus for example, $1^* = r$, where r is the root of the binary tree $B = B^+$.

In a similar way, we deal with the ψ-arrows. If there is a ψ-arrow in C from x to y, then we write $\psi(x) = y$ and $\psi^{-1}(y) = x$. Also, we define ψ^{ω} and $\psi^{-\omega}$ as above: we put $\psi^{\omega}(x) = y$, provided the longest ψ-path starting in x and not including a Nakayama path ends in y, and $\psi^{-\omega}(y) = x$ provided the longest ψ-path ending in y and not including a Nakayama path starts in x.

Note that for each vertex in C there is at most one φ-arrow and one ψ-arrow starting at this point and at most one of each type ending at this point. Thus the above maps and partial maps are well-defined.

Since the vertices of B are indexed by the numbers from 1 to n, they may be considered as being totally ordered. Actually, this ordering may be recovered from the binary tree structure of B. Using the correspondence $\varphi^{-\omega}$ between the vertices of B and B^- we get a similar total ordering on the vertices of B^-. The next lemma shows that this ordering, given by $(1^*, 2^*, \ldots, n^*)$, coincides with the ordering determined by the binary tree structure of B^-, provided we switch the role of the φ- and ψ-arrows.

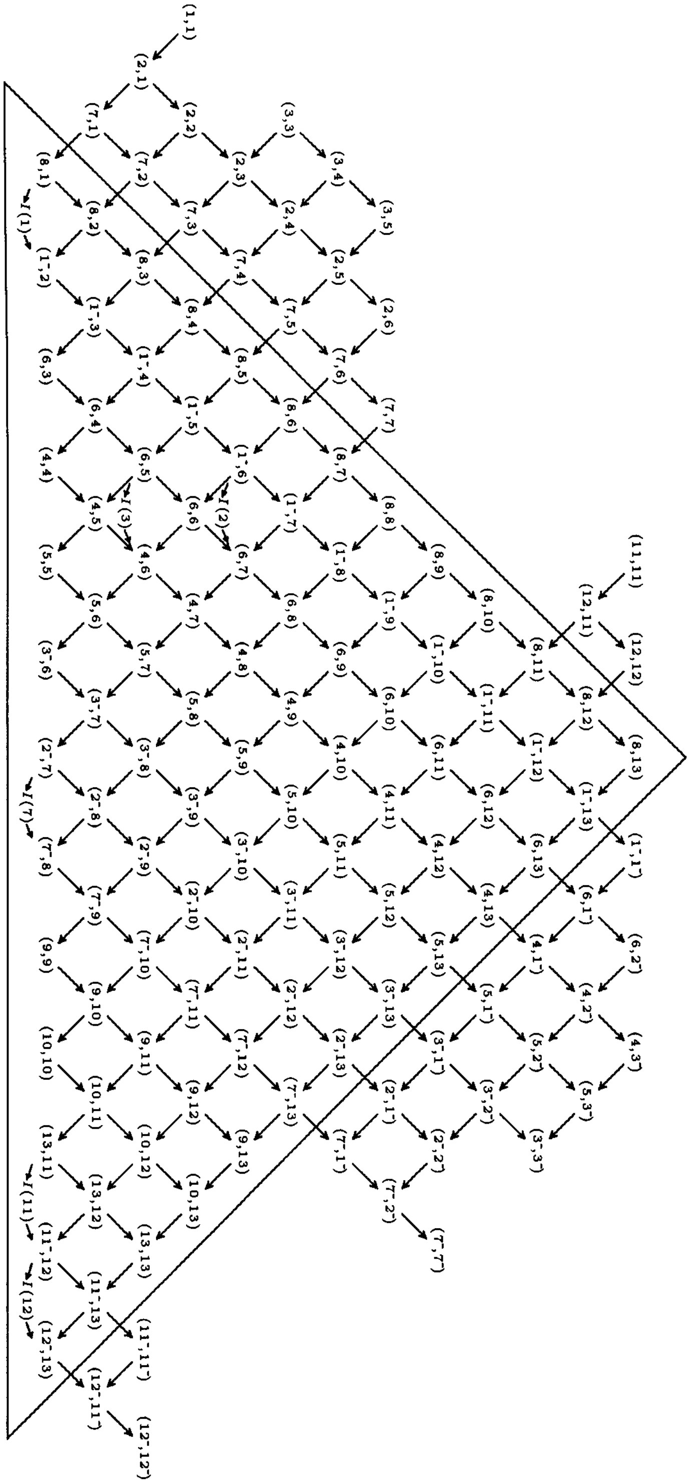
Lemma 2.2. *For $i = 1, \ldots, n-1$ we have:*

$$i+1 = \begin{cases} \varphi^{\omega}\psi^{-1}(i) & \text{if } i \in \psi(C), \\ \varphi^{-1}\psi^{\omega}(i) & \text{if } i \notin \psi(B^+); \end{cases}$$

$$(i+1)^* = \begin{cases} \psi^{\omega}\varphi^{-1}(i^*) & \text{if } i^* \in \varphi(C), \\ \psi^{-1}\varphi^{\omega}(i^*) & \text{if } i^* \notin \varphi(B^-). \end{cases}$$

Note that the cases for $i+1$ and for $(i+1)^*$ are not disjoint; if both conditions are satisfied, then the two expressions will coincide.

Using this notation and the duality between B^+ and B^-, we can completely describe the subquiver under consideration.



The Auslander–Reiten quiver of $\mathcal{C}$-mod. *Encircled is* U.

Proposition 2.3. *The string modules $M_{i,j} = M(i^*, j)$ exist for every pair of indices $1 \le i \le j \le n$, and they form a convex subquiver U of the Auslander–Reiten quiver of $\overline{\mathcal{C}}$. This subquiver U is isomorphic to $W(n)$, with $M_{i,j}$ corresponding to the point (i, j) in the standard embedding of $W(n)$. This is also an isomorphism of translation quivers, when the vertices of U are considered as $\mathcal{C}$-modules.*

In order to conclude the proof of Theorem 1, we shall need only the following statement.

Lemma 2.4. *Denote by $\ell(i, j)$ the composition length of the module $M_{i,j}$, defined above. Then $\ell(i, j) = f(i, j)$ for every $(i, j) \in W(n)$.*

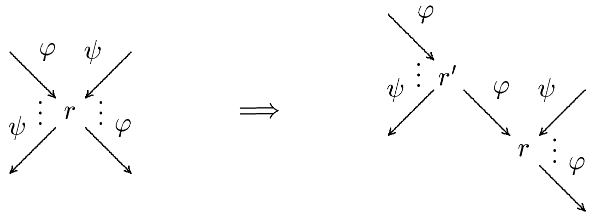
Let us illustrate these results by showing on the Auslander–Reiten quiver of the algebra $\mathcal{C}$ corresponding to our previous example (see the figure on previous page).

Turning now to the case of the stable tube $T(n)$, we should recall first from [ALR] that for every Frobenius function f on $T(n)$ there is a maximal subwing U of $T(n)$, isomorphic to $W(n)$, so that f is additive on the meshes outside U.

Thus let us take a Frobenius length function f on $T(n)$ and choose a maximal wing, denoted by $W(n)$, containing all the incomplete meshes. We may assume that the restriction of f to the wing is basic, since we will anyhow represent our Frobenius function only up to equivalence.

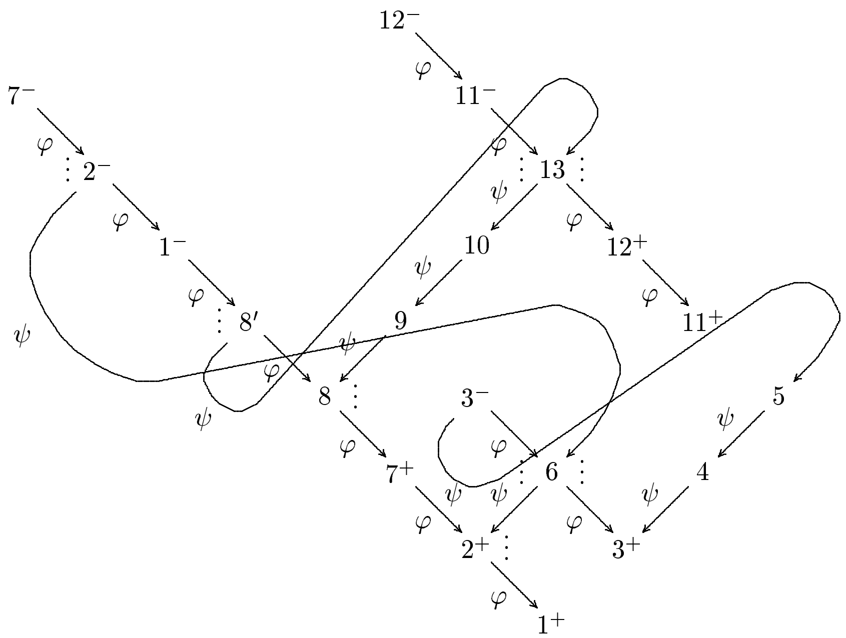
The restriction of f to $W(n)$ will give us, as it was defined above, a tree $B = B(f)$, and we may construct the graph C as before. In order to get a representation of the Frobenius function on the whole of $T(n)$, we have to add one ψ-arrow to C, joining the root r of the tree $B(f)$ to the vertex n and then we take the corresponding path algebra modulo the previously defined relations (including also those new relations of the type $\varphi\psi = 0$ or $\psi\varphi = 0$ which arise from adding the new φ-arrow). Unfortunately, by adding this new ψ-arrow, we get a cycle in the graph, hence the corresponding algebra will be infinite dimensional and in general it will not have almost split sequences.

In order to get a finite dimensional algebra, we can do the following. Let us replace the root r of the binary tree $B(f)$ by two vertices, r and r', and a φ-arrow joining r' to r in the following way. The old ψ-arrow (if any) in C going to r will now still go to r, while the old φ-arrow will go to r'. The old φ-arrow going from r will now still go from r, while the newly added ψ-arrow, joining r to the vertex n will go from r'. The local situation around r and r' is shown on the following diagram.



Let us denote the new graph by D, and the corresponding algebras by $\mathcal{D}$ and $\overline{\mathcal{D}}$, respectively. As before, $\overline{\mathcal{D}}$ is obtained from $\mathcal{D}$ by factoring out the Nakayama paths for every $i \in A^+$.

From our earlier example we will get the following graph D.



Let us note first that the maps φ^{ω}, $\varphi^{-\omega}$, ψ^{ω} and $\psi^{-\omega}$ are still well defined, and we shall also keep the notation i^* to denote $\varphi^{-\omega}(i)$ for $1 \le i \le n$. Observe, however, that now $1^* = r'$.

If we consider the strings over $\overline{\mathcal{D}}$, analogously defined as for the case of $\overline{\mathcal{C}}$, it is obvious that the uniqueness of strings stated in Lemma 2.1 does not hold any more under the new circumstances. For example, there are many strings from r to n: apart from the "straightforward" ψ^{-1}-path between them, one may get another path by adding a step from n to r' on the inverse of the newly added ψ-arrow, then continuing to r on a φ-arrow and finally repeating the walk along the ψ^{-1}-path from r to n. By repeating the procedure we get infinitely many such paths.

A crucial fact in this example is that in order to get new paths, we have to go through the inverse of the new arrow $r' \xrightarrow{\psi} n$ at least once. Thus for a string $\mathbf{u}$, let us denote by $w(\mathbf{u})$ the number of occurences of this arrow in $\mathbf{u}$; we shall call it the *winding number of* $\mathbf{u}$. Now we may formualate the following uniqueness statement.

LEMMA 2.5. *For any $a, b \in D$ and any natural number $t \ge 0$, we have at most one string from a to b over $\overline{\mathcal{D}}$ with winding number equal to t.*

This allows us to introduce the following notation: $M_t(a, b)$ will denote the unique string module corresponding to a string from a to b with winding number t, provided such a string exists. With this notation we have the following proposition.

PROPOSITION 2.6. *The string modules $M_t(i^*, j)$ exist for every pair of indices $1 \le i \le j \le n$, and $t = 0$ and for every $1 \le i, j \le n$ and $t > 0$. They form a component S of the Auslander–Reiten quiver of $\overline{\mathcal{D}}$, which is isomorphic to $T(n)$, with $M_t(i, j)$ corresponding to the point $(i, t \cdot n + j)$ in the standard coordinatization of $T(n)$. This is also an isomorphism of translation quivers, when the vertices of S are considered as $\mathcal{D}$-modules.*

Let us denote by U the maximal wing $\{(i,j) \mid 1 \le i \le j \le n\}$ in S. Thus U consists of those modules in S whose underlying string has winding number 0.

Note that by "doubling" the vertex r we will not get on U the original Frobenius function f but a Frobenius function which is equivalent to f.

LEMMA 2.7. *Let us denote by $\ell(i,j)$ the length of the module corresponding to the vertex (i,j) in S. Then we have $\ell(1,j) = f(1,j) + 1$ for $1 \le j \le n$. Thus ℓ and f are equivalent Frobenius length functions on U, hence they have the same set of incomplete meshes on S.*

In this way we get a realization of f on $T(n)$ up to an equivalence, as stated in Theorem 2. If we would like to represent the function f itself (assuming that f is basic on a maximal wing, containing the incomplete meshes), we can go back to the construction of D, but keep the root "together", as originally existed in $B(f)$ and C (i. e. not replacing it with $r' \xrightarrow{\varphi} r$). Let us denote by $\mathcal{E}$ the corresponding algebra over this quiver with the usual relations. The new arrow $r \xrightarrow{\psi} n$ makes the algebra $\mathcal{E}$ infinite dimensional. Note that the full category of (left) $\mathcal{E}$-modules in general will not have Auslander–Reiten sequences. For instance, in our standard example there is no almost split sequence starting with the simple module corresponding to the vertex 12.

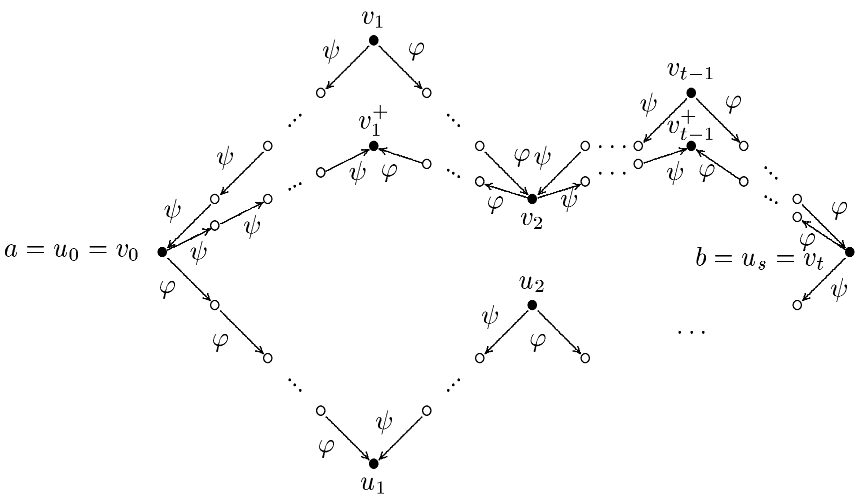
On the other hand, it is clear that the category of (left) $\mathcal{E}$-modules can be identified with the full subcategory of (left) modules M over $\mathcal{D}$ for which the multiplication by the arrow $r' \xrightarrow{\varphi} r$ is an isomorphism of the corresponding vector spaces $M_{r'}$ and M_r. This subcategory, as indicated, will not have Auslander–Reiten sequences in general. However, it contains the full component S, defined in Proposition 2.6, of the Auslander–Reiten quiver of $\mathcal{D}$, and it is clear that S will remain to be a full component of the "partial" Auslander-Reiten quiver within this subcategory. Furthermore, if we consider the elements of this component as $\mathcal{E}$-modules, the length function on S will coincide with the original Frobenius function f, thus giving a realization of the function itself (not only up to equivalence).

3. The proofs

Proof of Lemma 2.1. Suppose the statement is false and let us take a minimal counterexample, i.e. let us consider points $a, b \in C$ such that there are two different strings from a to b and we assume that the sum of the lengths of the two strings is minimal. Let $(a = u_0, \dots, u_1, \dots \; \dots, u_s = b)$ and $(a = v_0, \dots, v_1, \dots \; \dots, v_t = b)$ be two such strings, with $(u_i, \dots, u_{i+1})$ (for $0 \le i \le s-1$) and $(v_j, \dots, v_{j+1})$ (for $0 \le j \le t-1$) the maximal oriented subpaths of the two strings. As a consequence of the minimality assumption, we may assume that the first string starts with a φ-arrow, while the second with ψ^{-1}.

Since $\psi(C) \subseteq B$, we get that $a \in B$. Now, it follows by induction that $u_{2i-1} \in A^+$ and $u_{2i} \in A^0$ for $i \ge 1$, and it is easy to see that the first string completely belongs to B.

The minimality of the strings clearly implies that if the first string would end with a φ-arrow, then the second one would have to end in a ψ^{-1}-step. This would lead to a contradiction since then we would have $b \in A^+$, but ψ is not defined on vertices of A^+. Hence s and t are even, and $b \in A^0$, furthermore the last arrow of the second string is a φ-arrow. Now we can use induction once more to show that $v_{t-(2i+1)} \in A^-$ and $v_{t-2i} \in A^0$ for every $i \geq 0$. In particular we get that $a \in A^0$.



Since $v_{2i+1} \in A^-$, we can take the corresponding elements $w_{2i+1} = v_{2i+1}^+ \in A^+$, i.e. we have $v_{2i+1} = w_{2i+1}^-$ for $0 \leq i \leq t/2$. The construction of C shows that the existence of an oriented φ-path in B from $c \in B$ to $d \in A^+$ is equivalent to the existence of an oriented φ-path in C from d^- to c. Similarly for the ψ-paths. This will give us a nontrivial string from a to a through the elements $a = u_0, u_1, \dots, u_s = v_t, w_{t-1}, v_{t-2}, w_{t-3}, \dots, v_0 = a$ with all the intermediate steps inside B. But B is a tree, so we get a contradiction. $\square$

Proof of Lemma 2.2. Let us observe first that the natural ordering of the tree B gives that the following holds for $1 \leq i \leq n-1$:

$$i+1 = \begin{cases} \varphi^\omega\psi^{-1}(i) & \text{if } i \in \psi(B) \quad \text{(i.e. this is the first element of the } \psi \text{ branch of the subtree whose root is } i \text{ — if there is still such a branch);} \\ \varphi^{-1}\psi^\omega(i) & \text{if } i \notin \psi(B) \quad \text{(i.e. this is the root of the tree whose } \varphi \text{ branch we have just finished).} \end{cases}$$

This settles the second formula for $i+1$. We still have to show that the first formula is also valid if $i \in \psi(C) \setminus \psi(B) = \psi(A^-)$. So let $\psi^{-1}(i) = j^-$; then we get that $\varphi^\omega\psi^{-1}(i) = \varphi^\omega(j^-) = \varphi^{-1}\big(\varphi\varphi^\omega(j^-)\big) = \varphi^{-1}(j) = \varphi^{-1}\big(\psi^\omega\psi(j^-)\big) = \varphi^{-1}\psi^\omega(i)$, as required.

Turning now to the formulas for $(i+1)^*$, if $i^* \notin \varphi(B^-)$ then we have that $i = \varphi^\omega(i^*) \in A^0$. Since $i < n$, we get from here that $i \in \psi(C)$, hence we may use the first formula for $i+1$ to get that $i+1 = \varphi^\omega\psi^{-1}(i)$. Since $\psi^{-1}(i) \in B^-$, we get that $(i+1)^* = \varphi^{-\omega}\varphi^\omega\psi^{-1}(i) = \psi^{-1}(i) = \psi^{-1}\varphi^\omega(i^*)$. Thus the second case of the formula holds.

As for the first case, note that the condition $i^* \in \varphi(C)$ is equivalent to saying that $i^* = \varphi(i^-)$, where $i \in A^+$. Now, if $i \in \psi(C)$ (and equivalently, $i^- \in \psi^{-1}(C)$), then using the first case of the formula for $i+1$, we get $i+1 = \varphi^\omega\psi^{-1}(i)$, and as in the previous case, $(i+1)^* = \varphi^{-\omega}\varphi^\omega\psi^{-1}(i) = \psi^{-1}(i) = \psi^\omega(i^-) = \psi^\omega\varphi^{-1}(i^*)$, as required. Finally, if $i \notin \psi(C)$ then $i \in A^+$ implies that $i^- \notin \psi^{-1}(C)$, thus, in particular, $\psi^\omega(i^-) = i^-$. Note that $i \in A^+$ also implies that $i \notin \psi^{-1}(C)$, i.e. $\psi^\omega(i) = i$. Hence by the second case of the formula for $i+1$ we get that $(i+1)^* = \varphi^{-\omega}(i+1) = \varphi^{-\omega}\big(\varphi^{-1}\psi^\omega(i)\big) = \varphi^{-1}\varphi^{-\omega}(i) = i^- = \psi^\omega(i^-) = \psi^\omega\varphi^{-1}(i^*)$. Thus the first case of the formula for $(i+1)^*$ is fully proved. □

Proof of Proposition 2.3. The string from $1^* = r$ to 1 is just the φ-path from the root of B to the vertex 1. Thus $M_{1,1} \simeq P_{\overline{C}}(r)$. We shall use the characterization of the irreducible maps given in [WW] to show inductively that the other string modules also exist, with the irreducible maps going from $M_{i,j}$ to $M_{i+1,j}$ for $1 \le i < j \le n$ and to $M_{i,j+1}$ for $1 \le i \le j < n$.

According to [WW], for a string $(a,\dots,b)$ there are at most two irreducible morphisms going from the module $M(a,b)$: one to a string module $M(a,c)$ and one to a string module $M(d,b)$. We get the corresponding strings by the following algorithm.

(1) If $\big(a,\dots,b,\psi^{-1}(b)\big)$ is a string, then we complete it by φ-arrows to a string $\big(a,\dots,b,\dots,\varphi^\omega\psi^{-1}(b) = c\big)$. Otherwise let v be the first element in $(a,\dots,b)$ such that the string $(v,\dots,b)$ contains only ψ^{-1}-arrows. Then either $a \ne v$, in which case we choose $c = \varphi^{-1}(v)$, this way obtaining $(a,\dots,c)$ as a substring of $(a,\dots,b) = \big(a,\dots,c,\dots,\psi^{-\omega}\varphi(c) = b\big)$ or $a = v$ and then this type of morphisms does not exist.
(2) If $\big(\varphi^{-1}(a),a,\dots,b\big)$ is a string, then we complete it by ψ^{-1}-arrows to the string $\big(d = \psi^\omega\varphi^{-1}(a),\dots,a,\dots,b\big)$. Otherwise let u be the last element in $(a,\dots,b)$ such that the string $(a,\dots,u)$ contains only φ-arrows. Then either $u \ne b$, in which case we choose $d = \psi^{-1}(u)$, this way obtaining $(d,\dots,b)$ as a substring of $(a,\dots,b) = \big(a = \varphi^{-\omega}\psi(d),\dots,d,\dots,b\big)$ or $u = b$ and then this type of morphisms does not exist.

So we shall consider strings of the form $(a,\dots,u,\dots,v,\dots,b)$ with u being the last element of the string so that the substring from a to u is a (possibly empty) φ-path, while v is the first element of the string so that the substring from v to b is a (possibly empty) ψ^{-1}-path.

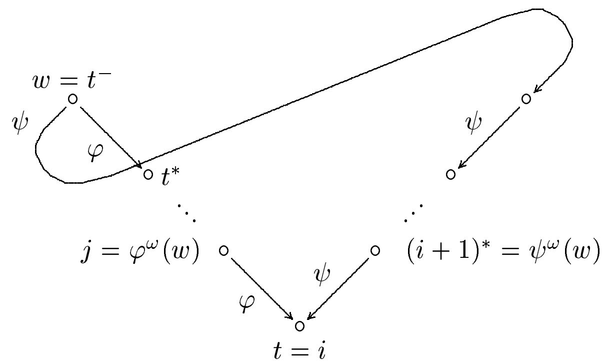
We are going to prove the existence of a morphism of type (1) from $M_{i,j}$ to $M_{i,j+1}$ when $1 \le i \le j < n$. If $j \in \psi(C)$ and $(v,\dots,j,\psi^{-1}(j))$ is a string, then the first case of the formula for $j+1$ in Lemma 2.2 (and the uniqueness of strings) shows that we have a string from i^* to $j+1$ of the specified form. Otherwise $j = \psi^{-\omega}(v)$. Consider the directed ψ^{-1}-path $\mathbf{p} = (\psi^\omega(j),\dots,v,\dots,j)$. Since $j \in B$, we have either $\psi^\omega(j) \in A^+$ or $\psi^\omega(j) = r$. But $\psi^\omega(j) = r$ would imply $j = \psi^{-\omega}(v) = \psi^{-\omega}(r) = n$, contradicting $j < n$. Thus $\psi^\omega(j) \in A^+$. If $\mathbf{p}$ is of length 0 then clearly $\psi^\omega(j) = v$. Otherwise $\big(\psi^\omega(j),\dots,v,\dots,j,\psi^{-1}(j)\big)$ is a Nakayama path, and since $\big(v,\dots,j,\psi^{-1}(j)\big)$ is not a string, $v = \psi^\omega(j)$ must hold in this case, too. Now, $v = \psi^\omega(j) \in A^+$ implies that $i^* \ne v$, hence $\big(i^*,\dots,\varphi^{-1}(v)\big)$ is a substring of $(i^*,\dots,j)$, and here $\varphi^{-1}(v) = \varphi^{-1}\psi^\omega(j) = j+1$ by Lemma 2.2. This proves the second case of (1).

Now let us assume that $1 \le i < j \le n$; we shall prove the existence of a morphism of type (2) from $M_{i,j}$ to $M_{i+1,j}$. If $i^* \in \varphi(C)$ and $(\varphi^{-1}(i^*),i,\dots,u)$ is a string, then the first case of the formula for $(i+1)^*$ in Lemma 2.2 shows that there is a string from $(i+1)^*$ to j of the specified form. Otherwise $i^* = \varphi^{-\omega}(u)$. Observe that $i \neq j$ implies that $\varphi^{-\omega}(u) = i^* \neq j^* = \varphi^{-\omega}(j)$, so $u \neq j$. Hence by the choice of u we get that $u \in \psi(C)$, giving that $u \in B$. But then $u = \varphi^{\omega}\varphi^{-\omega}(u) = \varphi^{\omega}(i^*) = i$ and the new string will go from $\psi^{-1}(u)$ to j, and here $\psi^{-1}(u) = \psi^{-1}\varphi^{\omega}(i^*) = (i+1)^*$ by Lemma 2.2. This proves the second case of (2).

Thus we have proved that the modules $M_{i,j}$ $(1 \leq i \leq j \leq n)$ form a subquiver U of the Auslander-Reiten quiver of $\overline{\mathcal{C}}$ isomorphic to the wing $W(n)$. We still have to show that this subquiver is convex. Note that for the modules $M_{i,i}$ $(i = 1, \ldots, n)$ the morphisms of type (2) do not exist, since the string $(i^*, \ldots, i)$ is a maximal φ-path, so it can neither be completed nor shortened in the specified way. It is also worth mentioning that there is no morphism of type (1) going into these modules, since these morphisms cannot go to modules corresponding to maximal φ-strings. The only way to leave the wing is by a morphism of type (1) from a module $M_{i,n}$. But a morphism of type (1) will map a module $M(a,b)$ with $b \in \{n\} \cup A^-$ into a module $M(a,c)$ with $c \in A^-$, since $b \notin \psi(C)$ and in the case when $b = \psi^{-\omega}\varphi(c)$, we get that $\varphi(c) \notin B^+$, thus $c \in A^-$. On the other hand, morphisms of type (2) do not change the end vertex of the string, so once we have left the wing, we cannot get back again by a sequence of irreducible morphisms.

It is clear that the knowledge of the irreducible maps determines the translation structure of U for non-projective vertices. Thus to finish the proof, we still have to show that those projective vertices in U which are not projective in $W(n)$ are the endterms of incomplete meshes from which a projective-injective $\mathcal{C}$-module has been removed. More precisely, we shall prove that a string module $M_{i+1,j}$ from U for some $1 \leq i < j \leq n$ is projective in $\overline{\mathcal{C}}$-mod if and only if there is an arrow $j \xrightarrow{\varphi} i$ in B; moreover we will also show that in this case $M_{i+1,j} \simeq P_{\overline{\mathcal{C}}}(i^-)$ in $\overline{\mathcal{C}}$-mod, thus the mesh ending at this vertex will be completed with the projective-injective module $I_{\mathcal{C}}(i) = P_{\mathcal{C}}(i^-)$ in $\mathcal{C}$-mod.

Assume that $M_{i+1,j}$ is projective as a $\overline{\mathcal{C}}$-module for some $1 \leq i < j \leq n$. Then the string from $(i+1)^*$ to j must be of the form $\big((i+1)^*, \ldots, w, \ldots, j\big)$ with $\psi^{\omega}(w) = (i+1)^*$ and $\varphi^{\omega}(w) = j$. Clearly, we must have $M_{i+1,j} \simeq P_{\overline{\mathcal{C}}}(w) \in \overline{\mathcal{C}}$-mod. Observe first that $(i+1)^* \neq 1^* = r$, so $\psi^{\omega}(w) = (i+1)^* \notin A^+$ implies that $w \in A^-$. Thus $w = t^-$ for $t = \varphi\varphi^{\omega}(t^-) = \varphi\varphi^{\omega}(w) = \varphi(j) \in A^+$. But then $t^* = \varphi(w)$, therefore Lemma 2.2 implies that $(t+1)^* = \psi^{\omega}(w) = (i+1)^*$. So $t = i$, hence $M_{i+1,j} \simeq P_{\overline{\mathcal{C}}}(i^-)$ and we also have an arrow $j \xrightarrow{\varphi} i$ in B.



On the other hand, if there is an arrow $j \xrightarrow{\varphi} i$ in B, then $i \in A^+$ implies that there is an arrow $i^- \xrightarrow{\varphi} i^*$ in C, hence $i^* \in \varphi(C)$. Thus by the first case in the formula of Lemma 2.2 for $(i+1)^*$, we get that $(i+1)^* = \psi^\omega \varphi^{-1}(i^*) = \psi^\omega(i^-)$, and we also have that $j = \varphi^\omega(i^-)$. Since the string from $(i+1)^*$ to j is unique, we get that $M_{i+1,j} \simeq P_{\overline{C}}(i^-)$, as required.

This finishes the proof. $\square$

We should mention that instead of proving the existence of irreducible maps of type (2) between modules $M_{i,j}$ and $M_{i+1,j}$, for $1 \leq i < j \leq n$, we could first show the existence of these morphisms for the modules $M_{i,i+1}$ and $M_{i+1,i+1}$ — this is a simple special case of the general argument — and then by induction on the difference $j-i$ show that $\tau M_{i+1,j+1} = M_{i,j}$ for $1 \leq i \leq j < n$. This would also imply the existence of all required morphisms of type (2).

Proof of Lemma 2.4. Proposition 2.3 gives us that the length function $\ell(i,j)$ defined on $W(n)$ is a Frobenius length function with the same set of incomplete meshes as the original Frobenius function f. What remains to be shown is that the two functions are actually equal. To this end it is enough to show that the two functions agree on the projective vertices of the wing $W(n)$, since the extension of a basic binary vector to a Frobenius length function on $W(n)$ is unique (cf. [ALR], Lemma 5.1).

The projective vertices of $W(n)$ correspond to string modules of the form $M_{1,j} = M(1^*, j)$ for $1 \leq j \leq n$, and the composition length of these modules coincides with the number of vertices in the unique path (string) from the root $1^* = r$ to the vertex j in the binary tree $B = B(f)$. On the other hand, since the given Frobenius function was basic, this is also the value of $f_j = f(\mathbf{p}_j)$.

Thus the composition length of the modules $M_{i,j}$ for $1 \leq i \leq j \leq n$ is $f(i,j)$, as required. $\square$

Proof of Theorem 1. By Lemma 2.4 and Proposition 2.3 the algebra C together with the subquiver U gives the required realization. $\square$

Proof of Lemma 2.5. It is enough to prove that for any $a, b \in D$ there is at most one string from a to b with winding number being equal to 0. Namely, every string $\mathbf{u}$ with winding number equal to $k > 0$ can be partitioned uniquely into substrings as follows: $\mathbf{u} = (u_1, \ldots u_{i_1}, u_{i_1+1}, \ldots, u_{i_2}, \ldots u_{i_k}, u_{i_k+1}, \ldots, u_t)$, where $u_{i_j} = n$, $u_{i_j+1} = r'$ for $1 \leq j \leq k$, furthermore the winding number of the substrings $(u_1, \ldots, u_{i_1})$, $(u_{i_1+1}, \ldots, u_{i_2})$,..., $(u_{i_k+1}, \ldots, u_t)$ is 0. Then from the uniqueness of the strings with winding number equal 0 we shall get that the knowledge of the endterms of the string and of the winding number uniquely determines the string.

Thus we can work with strings over the graph D' obtained from D by removing the arrow $r' \xrightarrow{\psi} n$. Furthermore, by collapsing the arrow $r' \xrightarrow{\varphi} r$ to one vertex r, any string over D' gives a string over C. Two distinct strings over D' from a to b would go to distinct strings over C, since clearly, a given string over D' from a to b can be recovered from the collapsed one and the knowledge of the endpoints a and b, by replacing $\xrightarrow{\varphi} r$ by $\xrightarrow{\varphi} r' \xrightarrow{\varphi} r$, if necessary." Thus the uniqueness result of Lemma 2.1 gives the required statement. $\square$

Proof of Proposition 2.6. We shall again use the classification of irreducible morphisms by [WW], described earlier in the proof of Proposition 2.3.

First we have to show that for $1 \leq i, j \leq n$ the following sequences of irreducible morphisms of type (1) exist: $M_0(i^*, i) \to M_0(i^*, i+1) \to \cdots \to M_0(i^*, n) \to M_1(i^*, 1) \to M_1(i^*, 2) \to \cdots \to M_1(i^*, n) \to M_2(i^*, 1) \to \cdots$.

Similarly, for $1 \leq i, j \leq n$ the following sequences of irreducible morphisms of type (2) exist: $\cdots \to M_2((n-1)^*, j) \to M_2(n^*, j) \to M_1(1^*, j) \to M_1(2^*, j) \to \cdots \to M_1(n^*, j) \to M_0(1^*, j) \to \ldots \to M_0((j-1)^*, j) \to M_0(j^*, j)$.

Let us observe that the existence of the string modules of the type $M_0(i^*, j)$ for $1 \leq i \leq j \leq n$ and of the specified irreducible morphisms between them follows from the proof of Proposition 2.3. Note first that, having winding number 0, these strings are not effected by the existence of the new arrow $r' \xrightarrow{\psi} n$. Secondly, none of the strings under consideration will start with r or end with r'. Thus, the needed strings over $\overline{\mathcal{D}}$ can be obtained from the corresponding strings over $\overline{\mathcal{C}}$ by replacing the steps of type $\xrightarrow{\varphi} r$ in the interior of the string with $\xrightarrow{\varphi} r' \xrightarrow{\varphi} r$; if the string should start with r', take the string over $\overline{\mathcal{C}}$ starting with r and add $r' \xrightarrow{\varphi} r$. The morphisms will not be effected because the formulas of Lemma 2.2 are still valid in this new graph D.

Next, the existence of the modules $M_t(i^*, j)$ for $t > 0$ and $1 \leq i, j \leq n$ and of the morphisms $M_t(i^*, j) \to M_t(i^*, j+1)$ for $1 \leq j < n$ of type (1) follows from the existence of the modules $M_0(1^*, j)$ and of morphisms $M_0(1^*, j) \to M_0(1^*, j+1)$. This can be seen by writing the corresponding string from i^* to j with winding number t as the concatenation of a string from i^* to n, then of $t-1$ copies of the string from $1^* = r'$ to n and finally of the string from 1^* to j. (The various pieces are joined by the inverse of the arrow $r' \xrightarrow{\psi} n$. Similar considerations show the existence of the specified morphisms $M_t(i^*, j) \to M_t\big((i+1)^*, j$ with $1 \leq i < n$ and $t > 0$ of type (2) between modules with winding number t.

Finally, let us consider the "connecting morphisms" $M_t(i^*, n) \to M_{t+1}(i^*, 1)$ of type (1) for $1 \leq i \leq n$ and $t \geq 0$. As in the previous case, the corresponding string of $M_t(i^*, n)$ can be written as the concatenation of a sring from i^* to n and of t copies of the string from 1^* to n. Since by adding at the end the vertex $\psi^{-1}(n) = r'$ gives a string (no zero relation may be involved), the algorithm gives us, that the irreducible morphism must go from $M_t(i^*, n)$ to $M_{t+1}\big(i^*, \varphi^{\omega}(r')\big) = M_{t+1}(i^*, 1)$, as required. — Similarly, the string modules of the form $M_t(n^*, j)$ for $1 \leq j \leq n$ and $t > 0$ will be mapped to $M_{t-1}(1^*, j)$ by a morphism of type (2), since $n^* \notin \varphi(D)$, hence one has to cut off the whole substring from n^* to r', thus decreasing the winding number by one.

The fact that S is a full component of the Auslander–Reiten quiver of $\overline{\mathcal{D}}$ can be shown by observing (as in the proof of Proposition 2.3) that there is no irreducible morphism of type (1) going into the modules $M_0(i^*, i)$, and there is no irreducible morphism of type (2) leaving from these modules, since these modules correspond to maximal φ-strings. Otherwise all the other vertices of S have two incoming and two outgoing arrows.

To finish the proof, we only have to note that the proof given for the position of the projective vertices over $\overline{C}$, given in the proof of Proposition 2.3 goes through almost verbatim for this situation. The only thing we still have to check is that neither the modules outside U nor the modules $M_0(1^*, j)$ for $1 \leq j \leq n$ cannot be projective over $\overline{\mathcal{D}}$. Thus note first that the all these modules contain a simple composition factor of type corresponding to $r' = 1^*$ in their top. Thus we have to show only that $P_{\overline{\mathcal{D}}}(1^*)$ cannot occur among the modules in S. But $P_{\overline{\mathcal{D}}}(1^*) \simeq M_1(r,1)$, and since $r \notin \varphi^{-\omega}(D)$, we are done.

In particular, observe that we do not have incomplete meshes outside U. □

We should remark here that the decomposition of the strings with higher winding number as a concatenation of maximal substrings with winding number equal to 0 corresponds to filtrations of these modules with quotients belonging to U. In particular, the module $M_t(i^*,j)$ for $t > 0$ will have a filtration where the first quotient (which is a sumodule) of $M_t(i^*,j))$ is isomorphic to $M_0(i^*,n)$, then $t-1$ quotients follow, each isomorphic to $M_0(1^*,n)$, and finally the top quotient is isomorphic to $M_0(1^*,j)$.

Proof of Lemma 2.7. The length of the string from 1^* to j for $1 \le j \le n$ is always greater by 1 than that of the string from r to j (since we start with the arrow $1^* = r' \xrightarrow{\varphi} r$). On the other hand, since we started with a Frobenius fucntion which was basic on U, the values $f(1,j)$ for $1 \le j \le n$ are equal to the length of the string from the root r of the tree $B = B(f)$ to j. Thus indeed, $\ell(1,j) = f(1,j) + 1$ for $1 \le j \le n$. This implies, in particular, that the sequences $\big(f(1,1),\dots,f(1,n)\big)$ and $\big(\ell(1,1),\dots,\ell(1,n)\big)$ give rise to the same binary tree, hence by Theorem 1.3 of [ALR], the Frobenius functions ℓ and f are equivalent, i. e. they have the same set of incomplete meshes on $S = T(n)$. □

Proof of Theorem 2. Lemma 2.7 and Proposition 2.6 show that the algebra $\mathcal{D}$ and the subquiver S of the Auslander–Reiten quiver of $\mathcal{D}$ gives the required realization. □

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