THE THEORY OF GENERALIZED DIRICHLET FORMS AND ITS APPLICATIONS IN ANALYSIS AND STOCHASTICS

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Abstract. We present an introduction (also for non–experts) to a new framework for the analysis of (up to) second order differential operators (with merely measurable coefficients and in possibly infinitely many variables) on L^2–spaces via associated bilinear forms. This new framework, in particular, covers both the elliptic and the parabolic case within one approach. To this end we introduce a new class of bilinear forms, so–called *generalized Dirichlet forms*, which are in general neither symmetric nor coercive, but still generate associated C_0–semigroups. Particular examples of generalized Dirichlet forms are symmetric and coercive Dirichlet forms (cf. [FOT], [MR1]) as well as time dependent Dirichlet forms (cf. [O1]). We discuss many applications to differential operators that can be treated within the new framework only, e.g. parabolic differential operators with unbounded drifts satisfying no L^p–conditions, singular and fractional diffusion operators. Subsequently, we analyze the probabilistic counterpart. More precisely, we identify necessary and sufficient analytic properties of a generalized Dirichlet form on an arbitrary topological state space to be associated with a nice strong Markov process, e.g. a diffusion. These results extend previous results on the existence of associated processes in the elliptic as well as the parabolic case.

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Introduction

The main subject of this work is the analysis of differential operators with measurable coefficients up to second order in finite and infinitely many variables on L^2–spaces with the help of bilinear forms. In general these operators will be non–symmetric and non–elliptic. We are in particular interested in the case of parabolic differential operators where the second order part may be degenerate and at the same time the first order part may be unbounded satisfying no L^p–condition. Concerning the L^2-analysis, e.g. whether a properly chosen extension generates a C_0–semigroup, we modify the approach by J.L. Lions and E. Magenes in [LMa] and [L] and adapt their methods so that we can handle operators of particular interest in probability theory. The main difference to the known analysis is that the measure of the underlying L^2–space is not necessarily Lebesgue measure, but a certain measure properly chosen for the operator. As a new input we provide criteria to decide whether the corresponding C_0–semigroup is positivity preserving or sub–Markovian. For the further (pointwise) analysis of the operators we also use probabilistic methods, more precisely the associated stochastic processes. The existence of such processes, as well as an explicit construction, is proved in this paper for the first time.

To this end we extend the known framework of Dirichlet forms to the class of so–called *generalized Dirichlet forms*. (The reader, however, is not assumed to have any preknowledge of Dirichlet forms.) A generalized Dirichlet form is a bilinear form defined on some L^2–space which is in general neither symmetric nor coercive, but still has an associated C_0–semigroup. As motivation let us explain the main ideas and underlying concepts for a rather simple but typical example in finite dimensions: consider the following differential operator on $\mathbb{R}^{d+1}$ in divergence form

$$(0.1)\qquad Lu(t,x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}(a_{ij}(t,x)\frac{\partial u}{\partial x_i}(t,x)) + \sum_{i=1}^{d} b_i(t,x)\frac{\partial u}{\partial x_i}(t,x) + \frac{\partial u}{\partial t}(t,x)$$

where $a(t,x) = (a_{ij}(t,x))_{1\le i,j\le d}$ is measurable in t, x, symmetric and uniformly strictly elliptic, b_i is Lipschitz continuous on $\mathbb{R}^{d+1}$, $1 \le i \le d$, such that $|b_i(t,x)| \le C(|x|+1)$ for some constant C and $\sum_{i=1}^{d} \frac{\partial b_i}{\partial x_i}(t,x) \ge 0$ for all $(t,x) \in \mathbb{R}^{d+1}$. Of course, (0.1) is only an informal expression which can be defined only in the sense of Schwartz distributions for $u \in C_0^\infty(\mathbb{R}^{d+1})$ (= the space of all smooth functions on $\mathbb{R}^{d+1}$ with compact support). Our first problem is to identify a differential operator generating a C_0–semigroup on $\mathcal{H} := L^2(\mathbb{R}^{d+1}, dy)$ which is assigned to (0.1) in a reasonable sense. Here dy denotes the Lebesgue measure on $\mathbb{R}^{d+1}$. The natural way is to look at the associated bilinear form

$$(0.2)\qquad \mathcal{E}(u,v) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^{d+1}} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\, dy - \sum_{i=1}^{d} \int_{\mathbb{R}^{d+1}} b_i \frac{\partial u}{\partial x_i} v\, dy - \int_{\mathbb{R}^{d+1}} \frac{\partial u}{\partial t} v\, dy$$

which can be defined for $u, v \in C_0^\infty(\mathbb{R}^{d+1})$. $\mathcal{E}$ is, of course, not symmetric and satisfies no (weak) sector condition in $\mathcal{H}$. However, it is one of the results of this work that also in this case it is still possible to define some suitable extension of the bilinear form $\mathcal{E}$, that (similar to the symmetric case) *uniquely* determines a linear operator $(L, D(L))$ generating a C_0–semigroup $(T_t)_{t\geq 0}$. $(L, D(L))$ is a reasonable L^2–realization of (0.1). Note that no continuity of a_{ij} in t is assumed and, moreover, that for any $t \in \mathbb{R}$ the bilinear form $\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} a_{ij}(t, \cdot)\, dx - \sum_{i=1}^{d} \int_{\mathbb{R}^d} b_i(t, \cdot) \frac{\partial u}{\partial x_i} v\, dx\,;\, u, v \in C_0^\infty(\mathbb{R}^d)$, is not coercive in general, so that standard existence results of the theory of abstract parabolic equations (cf. e.g. [Ta, Chapter 5]) cannot be applied to construct the operator $(L, D(L))$.

For a pointwise analysis of $(L, D(L))$ we are interested in the existence of associated Markov processes. It is a further result of this work now, that similar to the standard theory of Dirichlet forms, *only* using analytic properties of (the suitable extension of) the bilinear form $\mathcal{E}$, a space–time process $\mathbb{M}$ on $\mathbb{R}^{d+1}$ which is associated with $(L, D(L))$ in the sense that $p_t u$ is a dy–version of $T_t u$ for all $u \in \mathcal{H}$, $t \geq 0$, can be constructed. Here $(p_t)_{t\geq 0}$ denotes the transition semigroup of $\mathbb{M}$. We would like to emphasize that, unless no additional assumptions on $(a_{ij})_{1\leq i,j\leq d}$ are made, standard existence results in stochastic analysis, e.g. existence of solutions of the martingale problem corresponding to $\sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}(a_{ij}(t,x) \frac{\partial u}{\partial x_i}(x)) + \sum_{i=1}^{d} b_i(t,x) \frac{\partial u}{\partial x_i}(x)$, $u \in C_0^\infty(\mathbb{R}^d)$, (cf. [StrV]), cannot be applied to obtain the process $\mathbb{M}$.

To describe our approach more precisely we start with noting that $\mathcal{E}$ can be split up into two parts, where the first part

$$\mathcal{A}(u, v) = \sum_{i,j=1}^{d} \int_{\mathbb{R}^{d+1}} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} a_{ij}\, dy\,;\, u, v \in C_0^\infty(\mathbb{R}^{d+1})\ , \tag{0.3}$$

is a symmetric closable Dirichlet form on $\mathcal{H}$. Let $(\mathcal{A}, \mathcal{V})$ be the corresponding closure, in particular $\mathcal{V}$ is a Hilbert space w.r.t. the inner product $\mathcal{A}_1(u, v) = \mathcal{A}(u, v) + (u, v)_{\mathcal{H}}$, where $(\cdot, \cdot)_{\mathcal{H}}$ denotes the inner product in $\mathcal{H}$. The second part is the bilinear form $(-\Lambda u, v)_{\mathcal{H}}$; $u, v \in C_0^\infty(\mathbb{R}^{d+1})$, induced by the operator

$$\Lambda u = \sum_{i=1}^{d} b_i \frac{\partial u}{\partial x_i} + \frac{\partial u}{\partial t}, u \in C_0^\infty(\mathbb{R}^{d+1})\ . \tag{0.4}$$

The closure $(\Lambda, D(\Lambda))$ generates a C_0–semigroup that can be restricted to a C_0–semigroup on $\mathcal{V}$ (cf. Subsection II.2 (c)). Only $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda))$ are then used to construct the associated semigroup $(T_t)_{t\geq 0}$. We would like to emphasize that the corresponding bilinear form $\mathcal{E}$, defined as the sum of $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda))$ is a *first order object* in contrast to its generator L and thus easier to handle. Moreover, $\mathcal{E}$ is given *explicitly* whereas its generator is not. For example, it is much easier to use the bilinear form $\mathcal{E}$ in order to check that the associated semigroup $(T_t)_{t\geq 0}$ is *sub–Markovian* (i.e., $0 \leq T_t f \leq 1$ if $0 \leq f \leq 1$ for all $t \geq 0$) than using the operator L directly. Moreover, it turns out that the sub–Markov property of $(T_t)_{t\geq 0}$ follows as a *general fact*, only using that $(\mathcal{A}, \mathcal{V})$ is a Dirichlet form and that $(\Lambda, D(\Lambda))$ a Dirichlet operator (cf. Proposition I.4.7).

In the *typical example* above a general framework is inherent for the construction and analysis of differential operators and associated Markov processes on more general, even infinite dimensional, state spaces which we shall deduce now.

Let $(E, \mathcal{B}, m)$ be a σ–finite measure space and $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on the corresponding L^2–space $\mathcal{H} := L^2(E, m)$ and $(\Lambda, D(\Lambda))$ be a linear operator on $\mathcal{H}$ such that

D1 (i) $(\Lambda, D(\Lambda))$ generates a C_0–semigroup of contractions $(U_t)_{t\geq 0}$ on $\mathcal{H}$.

(ii) $(U_t)_{t\geq 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$.

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we obtain that $\mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'$ densely and continuously. Let $(\Lambda, \mathcal{F})$ be the closure of $\Lambda : D(\Lambda) \cap \mathcal{V} \to \mathcal{V}'$ and $(\hat{\Lambda}, \hat{\mathcal{F}})$ be the dual operator. Then the bilinear form associated with $\mathcal{A}$ and Λ is given as follows:

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{if } u \in \mathcal{F}, v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle & \text{if } u \in \mathcal{V}, v \in \hat{\mathcal{F}}\ , \end{cases} \tag{0.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the dualization of $\mathcal{V}'$ with $\mathcal{V}$. Forms of this type have been first studied in [L] (cf. also [LMa]).

We call a bilinear form $\mathcal{E}$ of type (0.5) a *generalized Dirichlet form* if the associated semigroup $(T_t)_{t\geq 0}$ is sub–Markovian. Hence the class of generalized Dirichlet forms includes in particular symmetric and coercive Dirichlet forms (cf. [MR1],[FOT]) but also time dependent Dirichlet forms as considered in [O1] (cf. also [Pi1],[Pi2]). The latter are given as follows: Let $(\mathcal{E}^{(t)}, V)_{t\in\mathbb{R}}$ be a family of coercive Dirichlet forms with uniform sector constant K and common domain V in some L^2–space such that $t \mapsto \mathcal{E}^{(t)}(u,v)$ is Borel–measurable for all $u, v \in V$ and $c\mathcal{E}_1^{(0)}(u,u) \leq \mathcal{E}_1^{(t)}(u,u) \leq c^{-1}\mathcal{E}_1^{(0)}(u,u)$ for some constant c. Then the corresponding *time dependent Dirichlet form*

$$\mathcal{E}(u,v) := \begin{cases} \int_{\mathbb{R}} \mathcal{E}^{(t)}(u(t),v(t))\,dt - \int_{\mathbb{R}} \langle \frac{du}{dt}(t), v(t)\rangle\,dt & \text{if } u \in L^2(\mathbb{R},V)\cap \\ & H^{1,2}(\mathbb{R},V'), v \in L^2(\mathbb{R},V) \\ \int_{\mathbb{R}} \mathcal{E}^{(t)}(u(t),v(t))\,dt + \int_{\mathbb{R}} \langle \frac{dv}{dt}(t), u(t)\rangle\,dt & \text{if } u \in L^2(\mathbb{R},V), v \in \\ & L^2(\mathbb{R},V)\cap H^{1,2}(\mathbb{R},V'), \end{cases}$$

is a generalized Dirichlet form in the above sense. However, if we replace the time derivative in this example by the time derivative $(\frac{d}{dt})^{\alpha}$ of power α, $\alpha \in (0,1)$, we still have a generalized Dirichlet form (cf. Subsection II.5 (a)). Moreover, the *h–transform* of $\mathcal{E}$ (cf. Subsection II.5 (b)) is again a generalized Dirichlet form, while it is, of course, not a time dependent Dirichlet form. In particular, the forward space–time process corresponding to the well–known *Brownian Bridge* is associated with a generalized Dirichlet form (cf. Example II.5.2). Dirichlet forms with time dependent Lipschitz continuous drift, in particular, time dependent linear drift, belong to this class as demonstrated in the introductory example (0.2) (cf. also Section II.2).

Another example, which is also covered by our framework, is given by the bilinear form

(0.6)

$$\mathcal{E}(u,v)=\sum_{i=1}^{d}\int_{\mathbb{R}^d_+}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}x_i\,dx-\sum_{i=1}^{d}\int_{\mathbb{R}^d_+}(Q^Tx)_i\frac{\partial u}{\partial x_i}v\,dx\ ;u,v\in C_0^\infty(\mathbb{R}^d_+)\ ,$$

where $\mathbb{R}^d_+ := \{x|x=(x_1,\dots,x_d)\in\mathbb{R}^d, x_1\geq 0,\dots,x_d\geq 0\}$, $C_0^\infty(\mathbb{R}^d_+)$ denotes the space of all restrictions of smooth functions in $C_0^\infty(\mathbb{R}^d)$ on $\mathbb{R}^d_+$, and Q is the generator of a semigroup of stochastic matrices. Note that the second order part degenerates at the boundary whereas at the same time the first order part is unbounded. However, we will prove in Section II.3 that, nevertheless, there exists a generalized Dirichlet form $\mathcal{E}$ in $L^2(\mathbb{R}^d_+,dx)$ extending the bilinear form (0.6). The associated generator is an extension of the differential operator

$$(0.7)\qquad Lu(x)=\sum_{i=1}^{d}x_i\frac{\partial^2 u}{\partial x_i^2}(x)+\sum_{i=1}^{d}\frac{\partial u}{\partial x_i}(x)+\sum_{i=1}^{d}(Q^Tx)_i\frac{\partial u}{\partial x_i}(x)\ ,u\in C_0^\infty(\mathbb{R}^d_+)\ .$$

$\mathcal{E}$ is associated with a continuous state branching process with immigration, more precisely, the *Dawson–Watanabe process* with constant immigration corresponding to Q (cf. Subsection II.3 (a)).

After having mentioned all these examples which cannot be treated within the classical theory of Dirichlet forms, let us now give a systematic summary of our results. In Chapter I we develop the necessary functional analytic background of bilinear forms of type (0.5) complementing the results in [L]. In particular, we will prove in Section I.3 (cf. Proposition I.3.6) the existence of an associated C_0–resolvent of contractions $(G_\alpha)_{\alpha>0}$ such that $G_\alpha(\mathcal{H})\subset\mathcal{F}$ for all $\alpha>0$ and $\mathcal{E}(G_\alpha u,v)+\alpha(G_\alpha u,v)_\mathcal{H}=(u,v)_\mathcal{H}$ for all $\alpha>0$, $u\in\mathcal{H}$, $v\in\mathcal{V}$. The mere existence of the operator G_α can be deduced from a general result due to J.L. Lions (cf. [L, Théorème 3.1.1]). We will present a slightly different proof providing a more explicit construction (cf. Proposition I.3.2). By the Hille–Yosida–Phillips theorem we then obtain the existence of an associated C_0–semigroup of contractions $(T_t)_{t\geq 0}$. We would like to emphasize that in contrast to the symmetric case the associated resolvent has the particular advantage of being more smoothing than the corresponding semigroup.

In Section I.4 we study contraction properties of $\mathcal{E}$ and obtain in Proposition I.4.6 a complete characterization of the sub–Markov property of the associated resolvent $(G_\alpha)_{\alpha>0}$ only in terms of the bilinear form $\mathcal{E}$: the associated resolvent $(G_\alpha)_{\alpha>0}$ is sub–Markovian if and only if

D2 $u\in\mathcal{F}\Rightarrow u^+\wedge 1\in\mathcal{V}$ and $\mathcal{E}(u,u-u^+\wedge 1)\geq 0$.

D2 is an extension of the second Beurling–Deny criterion (cf. [BeDe], [ReSi3]) to the framework of generalized Dirichlet forms. This characterization allows to deduce a sufficient condition for the sub–Markov property of $(G_\alpha)_{\alpha>0}$ which is easy to check in applications (cf. Proposition I.4.7). Note that the sub–Markov property of $(G_\alpha)_{\alpha>0}$ is equivalent to the sub–Markov property of the associated semigroup

$(T_t)_{t\geq 0}$ (cf. e.g. [MR1, Proposition I.4.3]). We also extend the first Beurling–Deny criterion to the framework of generalized Dirichlet forms (cf. Proposition I.4.4).

In Chapter II we discuss large classes of examples of generalized Dirichlet forms on finite and infinite dimensional state spaces including the examples mentioned above.

The third chapter is a generalization of the classical work on parabolic potential theory in an L^2–setting as developed by M. Pierre in [Pi1], [Pi2], [Pi3] and its extensions to the case of time dependent Dirichlet spaces by Y. Oshima in [O1] to the framework of generalized Dirichlet forms. Suppose from now on that E is a Hausdorff topological space and m some σ–finite measure on the Borel subsets $\mathcal{B}(E)$ of E.

We introduce the notion of *1–excessive functions* and give an equivalent characterization of 1–excessive functions only in terms of the bilinear form $\mathcal{E}$ (cf. Section III.1). We introduce the *1–reduced function* e_h of some element $h \in \mathcal{H}$ which is given by the smallest 1–excessive element dominating h. In Proposition III.1.7 we show, in particular, that for $h \in \mathcal{F}$ the element e_h is a solution of the following variational problem:

Find $u \in \mathcal{V}$, $u \geq h$, such that $\mathcal{A}_1(u,v) - \langle \widehat{\Lambda} v, u\rangle \geq \mathcal{A}_1(u,u)$ for all $v \in \widehat{\mathcal{F}}$, $v \geq h$.

The proof is based on methods used in the theory of parabolic variational inequalities (cf. [MiPu]).

In Section III.2 we define *$\mathcal{E}$–nests* and *$\mathcal{E}$–exceptional sets* with the help of 1–reduced functions, hence only in terms of the bilinear form $\mathcal{E}$, and construct *Choquet capacities* whose zero sets are exactly the $\mathcal{E}$–exceptional sets. This will be an essential tool for the construction of associated processes. In the context of time dependent Dirichlet forms these capacities have not been considered before. They have two particular advantages: first they can be defined also in an infinite dimensional setting in contrast to the usual parabolic capacity (cf. [Pi1], [O1]), and second they are strongly subadditive which is not the case for that kind of capacities that have been introduced in [Pi3].

In Section III.3 we define the notion of *$\mathcal{E}$–quasi continuous ($\mathcal{E}$–q.c.)* functions. Using the capacities as defined in Section III.2 we prove, in particular, that convergence of a sequence of $\mathcal{E}$–q.c. functions in $\mathcal{F}$ implies $\mathcal{E}$–quasi uniform convergence of some subsequence (cf. Corollary III.3.8).

The fourth chapter is an extension of the main result obtained by S. Albeverio, Z.–M. Ma and M. Röckner in [AMR], which is a complete analytic characterization of all Dirichlet forms associated with nice Markov processes, to the class of generalized Dirichlet forms. To this end we introduce the notion of *quasi–regularity* for generalized Dirichlet forms analogously to [AMR] and [MR1]. We call a generalized Dirichlet form $\mathcal{E}$ in $L^2(E,m)$ *quasi–regular* if:

QR (i) There exists an $\mathcal{E}$–nest $(E_k)_{k\geq 1}$ consisting of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$–q.c. m–versions.

(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having $\mathcal{E}$–q.c. m–versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$–exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Note that QR (i) is motivated by the observation that under weak assumptions on the space E it follows that any E–valued stochastic process $(X_t)_{t \geq 0}$ with life time ζ, such that $t \mapsto X_t$ is right continuous on $[0, \infty)$ and has left limits for all $t \in (0, \zeta)$ P–a.s. w.r.t. some probability measure P, lives on a countable union of compact metrizable subsets of E (cf. [MR1, Theorem IV.1.15]).

Let $\mathcal{E}$ be quasi–regular and suppose that the following technical assumption is satisfied:

D3 There exists a linear subspace $\mathcal{Y} \subset \mathcal{H} \cap L^\infty(E, m)$ such that $\mathcal{Y} \cap \mathcal{F}$ is dense in $\mathcal{F}$, $\lim_{\alpha \to \infty} e_{\alpha G_\alpha u - u} = 0$ in $\mathcal{H}$ for all $u \in \mathcal{Y}$ and for the closure $\overline{\mathcal{Y}}$ of $\mathcal{Y}$ in $L^\infty(E, m)$ it follows that $u \wedge \alpha \in \overline{\mathcal{Y}}$ for $u \in \mathcal{Y}$ and $\alpha \geq 0$.

We give a sufficient criterion for D3 to hold which is easy to check in applications (cf. Proposition IV.2.1). Note that D3 is always satisfied in the special case where $\mathcal{E}$ is a Dirichlet form or a time dependent Dirichlet form, but also in all of the examples mentioned above.

Under these assumptions in Section IV.2 we will explicitly construct an associated m–tight special standard process $\mathbb{M}$ (cf. Definition IV.1.6) that is *properly associated in the resolvent sense* with $\mathcal{E}$, i.e., $R_\alpha f$ is an $\mathcal{E}$–q.c. m–version of $G_\alpha f$ for all $f \in \mathcal{B}_b(E) \cap \mathcal{H}$, $\alpha > 0$ (cf. Theorem IV.2.2). Here $(R_\alpha)_{\alpha > 0}$ denotes the resolvent of $\mathbb{M}$ and $\mathcal{B}_b(E)$ the space of all bounded $\mathcal{B}(E)$–measurable functions on E. It is easy to see that this implies in particular that $p_t f$ is an m–version of $T_t f$ for all $f \in \mathcal{B}_b(E) \cap \mathcal{H}$, $t \geq 0$, but in general $p_t f$ will not be $\mathcal{E}$–q.c. The construction of $\mathbb{M}$ extends previous constructions of associated processes in the elliptic and parabolic case.

Conversely, given a generalized Dirichlet form $\mathcal{E}$ and an m–tight m–special standard process $\mathbb{M}$ associated in the resolvent sense with $\mathcal{E}$ we will prove in Section IV.3 (cf. Theorem IV.3.1) that $\mathcal{E}$ is quasi–regular and, moreover, $\mathbb{M}$ is in the resolvent sense properly associated with $\mathcal{E}$.

In Section IV.4 we discuss the quasi–regularity property and assumption D3 in detail in the case of all examples mentioned above. Above all, we will give an easy criterion for quasi–regularity of a time dependent Dirichlet form. Applying then Theorem IV.2.2 we obtain a large class of examples for space–time Markov processes on infinite dimensional state spaces associated with time dependent Dirichlet forms which could not been constructed before. In particular, these examples are not covered by the results of Y. Oshima in [O1] since infinite dimensional state spaces are not locally compact.

We want to emphasize that *any* linear operator L generating a sub–Markovian C_0–semigroup of contractions on some L^2–space $L^2(E, m)$ induces a generalized Dirichlet form given by $\mathcal{E}(u, v) = -(Lu, v)_{L^2(E,m)}$, $u \in D(L)$, $v \in L^2(E, m)$, and $\mathcal{E}(u, v) = -(u, \hat{L}v)_{L^2(E,m)}$, $u \in L^2(E, m)$, $v \in D(\hat{L})$. Hence our results imply, in particular, necessary and sufficient analytic conditions on L to be associated with an m–tight special standard process, and Theorem IV.2.2 can be viewed as the analogue of the well–known existence result on Hunt processes associated with Feller semigroups (cf. [BlG]). For example, Theorem IV.2.2 has been used in [St4] to construct diffusion processes properly associated in the resolvent sense with (particular extensions of non–symmetric) differential operators of type $Lu(x) = \sum_{i,j=1}^d \frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u}{\partial x_i}(x)) + \sum_{i=1}^d b_i(x)\frac{\partial u}{\partial x_i}(x)$ on an arbitrary open subset of $\mathbb{R}^d$, where the b_i, $1 \leq i \leq d$, are no longer Lipschitz continuous but only satisfy some mild integrability assumption. This way the theory of generalized Dirichlet forms provides a new approach to the construction of non–reversible diffusions with singular drifts. The existence of such processes is of interest in the theory of stochastic mechanics, where they are often called Nelson processes (cf. the monograph by M. Nagasawa [Nag] and references therein).

Summarizing, the framework of generalized Dirichlet forms contains many new examples for differential operators that can only be treated by an L^2–approach for a properly chosen reference measure and, in addition, cannot be treated in other frameworks such as e.g. Dirichlet forms. It is the subject of future study whether this (so far most general) framework to apply the method of bilinear forms to stochastic processes can be further perfectionized. We want to emphasize, however, that already in the present state our framework can be used to extend the stochastic analysis of additive functionals of sectorial Markov processes (in particular, the *Fukushima decomposition*) to stochastic processes associated with *generalized Dirichlet forms*. Therefore, applications to non–homogeneous stochastic partial differential equations as well as to infinite–dimensional differential geometry are at hand. Furthermore, we can use our framework to obtain uniqueness results for differential operators associated with generalized Dirichlet forms, including well–posedness of the corresponding martingale problem. In order not to overload the present work, which is meant as a streamlined introduction to the subject, we shall discuss such applications in detail in forthcoming papers.

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I. Functional Analytic Background

I.1 Preliminaries

Let us first make some comments on our notational conventions. $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of natural, rational and real numbers. $\mathbb{R}^d$, $d \in \mathbb{N}$, denotes the d–dimensional Euclidean space, $\langle \cdot, \cdot \rangle$ the inner product, $|\cdot|$ the corresponding norm and $B_r(x) := \{y | y \in \mathbb{R}^d, |y-x| < r\}$ the open ball of radius $r > 0$ and center $x \in \mathbb{R}^d$.

If A is an arbitrary set $\mathbb{1}_A$ denotes the indicator function of A. If $(E, \mathcal{B})$ is a measurable space $\mathcal{B}_b$, $\mathcal{B}^+$, $\mathcal{B}_b^+$ denote the spaces of all bounded, non–negative, bounded and non–negative $\mathcal{B}$–measurable functions. If $(E, \mathcal{B}, m)$ is a measure space and $p \in [1, \infty)$ (resp. $p = \infty$) $L^p(E, m)$ denotes the space of all (equivalence classes of) p–integrable functions w.r.t. m (resp. m–essentially bounded functions). $\|\cdot\|_p$, $p \in [1, \infty]$, denotes the corresponding norm. If W is a subset of $L^p(E, m)$ $\mathcal{B}_b \cap W$, $\mathcal{B}^+ \cap W$ and $\mathcal{B}_b^+ \cap W$ denote the sets of all $\mathcal{B}$–measurable functions in $\mathcal{B}_b$, $\mathcal{B}^+$ and $\mathcal{B}_b^+$ that are m–versions of elements in W. If V is a (real) Banach space with corresponding norm $\|\cdot\|_V$ $L^p(E; V, m)$, $p \in [1, \infty]$, denotes the space of all (equivalence classes of) m–measurable V–valued functions f with $\|f\|_V \in L^p(E, m)$.

If E is a Hausdorff topological space, $\mathcal{B}(E)$ denotes its Borel σ–algebra and $\mathcal{B}^*(E)$ the σ–algebra of universally measurable subsets of E. $\mathcal{M}(E)$ (resp. $\mathcal{P}(E)$) denotes the set of all finite measures (resp. probability measures) on the measurable space $(E, \mathcal{B}(E))$. $C(E)$, $C_b(E)$ denote the spaces of all continuous, continuous and bounded functions on E. If E is locally compact $C_0(E)$ denotes the space of all continuous functions with compact support and $C_\infty(E)$ the space of all continuous functions vanishing at infinity.

Finally, if $U \subset \mathbb{R}^d$, U open, and $m \in \mathbb{N} \cup \{+\infty\}$ $C^m(U)$, $C_b^m(U)$, $C_0^m(U)$ denote the spaces of all m–times continuously differentiable functions on U, m–times continuously differentiable functions on U with f and all partial derivatives up to order m bounded, resp. m–times continuously differentiable functions on U with compact support. Let $H^{1,2}(U) := \{u \in L^2(U, dx) | \frac{\partial u}{\partial x_i} \in L^2(U, dx), 1 \le i \le d\}$ be the $(1,2)$–Sobolev space on U with Neumann–boundary conditions and $H_0^{1,2}(U)$ be the closure of $C_0^\infty(U)$ in $H^{1,2}(U)$, i.e., $H_0^{1,2}(U)$ is the $(1,2)$–Sobolev space on U with Dirichlet–boundary conditions.

Linear operators on Banach and Hilbert spaces

For the rest of this section let E be a real Banach space. Denote by $\|\cdot\|_E$ the corresponding norm, E' the topological dual and $\langle f, e \rangle :=_{E'} \langle f, e \rangle_E := f(e)$ the *dualization* of f with e. Let F be another real Banach space and T be a linear operator from E into F. $D(T)$ denotes the *domain* of T and $\Gamma(T) := \{(e, Te) | e \in D(T)\} \subset E \times F$ the *graph* of T. The *graph norm* $\|\cdot\|_T$ of T is defined on $D(T)$ by $\|e\|_T^2 := \|e\|_E^2 + \|Te\|_F^2$.

T is *closed* if $\Gamma(T) \subset E \times F$ is closed (w.r.t. the norm $\|(e, f)\|_{E \times F}^2 := \|e\|_E^2 + \|f\|_F^2$) and *closable* if there exists a *closed extension* of T, i.e., a closed operator S satisfying

$\Gamma(T) \subset \Gamma(S)$. If T is closable the closure of $\Gamma(T)$ in $E \times F$ is the graph of a closed linear operator $\overline{T}$. $\overline{T}$ is called the *closure* of T.

Let $L(E, F)$ be the linear space of all continuous linear operators T from E into F with domain $D(T) = E$ and $\|T\|_{L(E,F)}$ be the corresponding operator norm. Let $L(E) := L(E, E)$. A linear operator $T \in L(E)$ with $\|T\|_{L(E)} \leq 1$ is called a *contraction*.

Let H be a real Hilbert space. $(\cdot, \cdot)_H$ always denotes the corresponding scalar product, and $J_H : H \to H'$ the isometric isomorphism of the Riesz Theorem, i.e., $J_H(h)(g) = (h, g)_H$. J_H will be sometimes called the *Riesz isometry*.

A subset $D \subset H$ is called *total* if for any $h \in H \setminus \{0\}$ there exists an element $u \in D$ such that $(h, u)_H \neq 0$. The *linear span* $\operatorname{span} D$ of a total set is obviously a dense subspace of H.

Remark 1.1. A subset $D \subset H'$ is total if and only if:

(1.1) For any $h \in H \setminus \{0\}$ there exists an element $f \in D$ such that $f(h) \neq 0$.

Let V be another real Hilbert space and T be a densely defined linear operator from H into V. The *dual operator* T' of T is defined by

$$D(T') = \{u \in V' | h \mapsto {}_{V'}\langle u, Th\rangle_V \text{ is continuous on } D(T) \text{ w.r.t. the } H\text{–norm}\} ,$$

and ${}_{V'}\langle u, Th\rangle_V = {}_{H'}\langle T'u, h\rangle_H$, $h \in D(T)$, $u \in D(T')$. Similarly, the *adjoint operator* $\widehat{T}$ of T is defined by

$$D(\widehat{T}) = \{u \in V | h \mapsto (u, Th)_V \text{ is continuous on } D(T) \text{ w.r.t. the } H\text{–norm}\} ,$$

and $(u, Th)_V = (\widehat{T}u, h)_H$, $h \in D(T)$, $u \in D(\widehat{T})$. In particular, $D(\widehat{T}) = J_V^{-1}(D(T'))$ and $\widehat{T} = J_H^{-1} T' J_V$.

Semigroups of linear operators

A family $(T_t)_{t \geq 0} \subset L(E)$ of bounded linear operators on E is called a *semigroup* if $T_0 = \mathrm{id}_E$ and $T_t T_s = T_{t+s}$ for all $t, s \geq 0$. A semigroup $(T_t)_{t \geq 0}$ is called *strongly continuous* or a *C_0–semigroup* if $\lim_{t \to 0} T_t u = u$ for all $u \in E$ and a *semigroup of contractions* if $\|T_t\|_{L(E)} \leq 1$ for all $t \geq 0$.

Let $(T_t)_{t \geq 0}$ be a C_0–semigroup. The linear operator $Lu := \lim_{t \to 0} \frac{1}{t}(T_t u - u)$ with domain

$$D(L) = \{u \in E | \exists \lim_{t \to 0} \frac{1}{t}(T_t u - u) \text{ in } E\}$$

is called the *infinitesimal generator* or simply the *generator* of $(T_t)_{t \geq 0}$. L is densely defined and closed (cf. [Pa, Corollary 1.2.5]).

Let $\omega \geq 0$. A family $(G_\alpha)_{\alpha > \omega} \subset L(E)$ of bounded linear operators on E is called a *resolvent* if $G_\alpha - G_\beta = (\beta - \alpha) G_\alpha G_\beta$ for all $\alpha, \beta > \omega$. A resolvent $(G_\alpha)_{\alpha > \omega}$ is called *strongly continuous* or a C_0–*resolvent* if $\lim_{\alpha\to\infty} \alpha G_\alpha u = u$ for all $u \in E$ and a *resolvent of contractions* if $\omega = 0$ and $\|\alpha G_\alpha\|_{L(E)} \le 1$ for all $\alpha > 0$.

Let $(G_\alpha)_{\alpha>\omega}$ be a C_0–resolvent. Then G_α, $\alpha > \omega$, is one–to–one and $L := \alpha - G_\alpha^{-1}$, $\alpha > \omega$, (the definition is independent of α) defines a densely defined closed linear operator which is called the *generator* of $(G_\alpha)_{\alpha>\omega}$ (cf. [Pa, Corollary 1.9.5]).

Let L be a linear operator on E. Then $\rho(L)$ denotes the *resolvent set* of L, i.e.,

$$\rho(L) := \{\alpha \in \mathbb{R} | \alpha - L : D(L) \to E \text{ is bijective and } (\alpha - L)^{-1} \text{ continuous}\} .$$

Note that a densely defined operator L with $\rho(L) \neq \emptyset$ is already closed (cf. [MR1, Lemma I.1.2]).

Due to Hille, Yosida and Phillips there is a complete characterization of all linear operators L that *generate C_0–semigroups*, i.e., that are infinitesimal generators of C_0–semigroups (cf. [Pa, Theorem 1.5.3]).

Theorem 1.2. *(Hille - Yosida - Phillips) Let L be a densely defined linear operator on E. Then the following are equivalent:*

(i) L generates a C_0–semigroup $(T_t)_{t\ge0}$.

(ii) $\exists\, \omega \ge 0$ with $(\omega,\infty) \subset \rho(L)$ and
$\exists\, M \ge 0$ with $\|(\alpha - L)^{-n}\|_{L(E)} \le \frac{M}{(\alpha-\omega)^n}$ for all $\alpha \in (\omega,\infty)$, $n \ge 1$.
In this case $\|T_t\|_{L(E)} \le Me^{\omega t}$ for all $t \ge 0$.

Clearly, if L generates a C_0–semigroup, we obtain that the family $((\alpha - L)^{-1})_{\alpha>\omega}$ is a C_0–resolvent with generator L.

Finally, there is a one–to–one correspondence between the set of all densely defined linear operators L satisfying $(0,\infty) \subset \rho(L)$ and $\|\alpha(\alpha - L)^{-1}\|_{L(E)} \le 1$ for all $\alpha > 0$, the set of C_0–semigroups of contractions and the set of C_0–resolvents of contractions (cf. [MR1, Section I.1]).

Suppose that F is a linear subspace of E such that F itself is a real Banach space which is densely and continuously embedded into E. If L is a linear operator on E the *part of L in F* is the linear operator $\widetilde{L}$ on F with domain $D(\widetilde{L}) := \{u \in D(L) \cap F | Lu \in F\}$ and $\widetilde{L}u := Lu$ for $u \in D(\widetilde{L})$.

Definition 1.3. Let L be the infinitesimal generator of a C_0–semigroup $(T_t)_{t\ge0}$ on E. F is called *L–admissible* if $(T_t)_{t\ge0}$ can be restricted to a C_0–semigroup on the Banach space F.

If L is the generator of a C_0–semigroup $(T_t)_{t\ge0}$ on E and F is L–admissible it follows from [Pa, Theorem 4.5.5] that the generator of the restriction of $(T_t)_{t\ge0}$ on F is the part of L in F.

A densely defined linear operator L on the real Hilbert space H is called *bounded from above* if there exists a constant $c \geq 0$ such that $(Lu, u)_H \leq c(u, u)_H$ for all $u \in D(L)$ and *negative definite* if $(Lu, u)_H \leq 0$ for all $u \in D(L)$. Clearly, the infinitesimal generators of C_0–semigroups of contractions on H are negative definite.

Coercive closed forms

Let $\mathcal{H}$ be a real Hilbert space and $(\mathcal{A}, D(\mathcal{A}))$ a positive definite bilinear form, i.e., $D(\mathcal{A}) \subset \mathcal{H}$ is a linear subspace and $\mathcal{A} : D(\mathcal{A}) \times D(\mathcal{A}) \to \mathbb{R}$ a positive definite bilinear map. Since we will only consider *densely defined* bilinear forms we always assume that $D(\mathcal{A})$ is a dense subset of $\mathcal{H}$. $(\tilde{\mathcal{A}}, D(\mathcal{A}))$ will always denote the *symmetric part* of $(\mathcal{A}, D(\mathcal{A}))$, i.e., $\tilde{\mathcal{A}}(u, v) := \frac{1}{2}(\mathcal{A}(u, v) + \mathcal{A}(v, u))$, $(\check{\mathcal{A}}, D(\mathcal{A}))$ the *antisymmetric part* of $(\mathcal{A}, D(\mathcal{A}))$, i.e., $\check{\mathcal{A}}(u, v) := \frac{1}{2}(\mathcal{A}(u, v) - \mathcal{A}(v, u))$, and $(\mathcal{A}_\alpha, D(\mathcal{A}))$, $\alpha > 0$, the form $\mathcal{A}_\alpha(u, v) := \mathcal{A}(u, v) + \alpha\,(u, v)_{\mathcal{H}}$.

$(\tilde{\mathcal{A}}, D(\mathcal{A}))$ is called a *closed form* if $(D(\mathcal{A}), \tilde{\mathcal{A}}_\alpha)$ is a Hilbert space for one (hence all) $\alpha > 0$. $(\tilde{\mathcal{A}}, D(\mathcal{A}))$ is called *closable* if there exists at least one *closed extension* of $(\tilde{\mathcal{A}}, D(\mathcal{A}))$, i.e., a symmetric closed form $(\tilde{\mathcal{A}}', D(\tilde{\mathcal{A}}'))$ with $D(\mathcal{A}) \subset D(\tilde{\mathcal{A}}')$ and $\tilde{\mathcal{A}}'(u, v) = \tilde{\mathcal{A}}(u, v)$ for all $u, v \in D(\mathcal{A})$.

If $(\tilde{\mathcal{A}}, D(\mathcal{A}))$ is closable there exists a minimal closed extension $(\tilde{\mathcal{A}}, D(\bar{\mathcal{A}}))$ of $(\tilde{\mathcal{A}}, D(\mathcal{A}))$. $D(\mathcal{A})$ is dense in $(D(\bar{\mathcal{A}}), \tilde{\mathcal{A}}_\alpha)$ and $(\tilde{\mathcal{A}}, D(\bar{\mathcal{A}}))$ is given as the uniquely determined continuous extension of $(\tilde{\mathcal{A}}, D(\mathcal{A}))$ to $D(\bar{\mathcal{A}})$.

In general, the form $(\mathcal{A}, D(\mathcal{A}))$ cannot be continuously extended to $D(\bar{\mathcal{A}})$. However, if $(\mathcal{A}, D(\mathcal{A}))$ satisfies the *weak sector condition* with a *sector constant* K, i.e.,

$$|\mathcal{A}_1(u, v)| \leq K \mathcal{A}_1(u, u)^{1/2} \mathcal{A}_1(v, v)^{1/2} \text{ for all } u, v \in D(\mathcal{A}) \,, \tag{1.2}$$

then $(\mathcal{A}, D(\mathcal{A}))$ admits a uniquely determined continuous extension to $D(\bar{\mathcal{A}})$. The extension is a coercive closed form in the sense of the following definition.

Definition 1.4. A positive definite bilinear form $(\mathcal{A}, D(\mathcal{A}))$ is called a *coercive closed form* if $D(\mathcal{A})$ is a dense linear subspace of $\mathcal{H}$, the symmetric part $(\tilde{\mathcal{A}}, D(\mathcal{A}))$ is a closed form and $(\mathcal{A}, D(\mathcal{A}))$ satisfies the weak sector condition (1.2).

It is well–known that $(\mathcal{A}, D(\mathcal{A}))$ uniquely determines a closed linear negative definite operator L, called the *generator of* $(\mathcal{A}, D(\mathcal{A}))$, with domain

$$D(L) = \{u \in D(\mathcal{A}) | \exists Lu \in \mathcal{H} \text{ such that } (-Lu, v)_{\mathcal{H}} = \mathcal{A}(u, v) \text{ for all } v \in D(\mathcal{A})\} \,.$$

L generates a C_0–semigroup of contractions on $\mathcal{H}$ (cf. [MR1, Section I.2]). The adjoint operator $\widehat{L}$ of L is called the *cogenerator of* $(\mathcal{A}, D(\mathcal{A}))$.

I.2 Linear perturbations of coercive closed forms

Let $\mathcal{H}$ be a separable real Hilbert space and $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on $\mathcal{H}$ with sector constant $K \geq 1$. Then $\mathcal{V}$ is a real Hilbert space w.r.t. $\tilde{\mathcal{A}}_1(\cdot,\cdot)$. We will always denote the corresponding norm by $\|\cdot\|_{\mathcal{V}}$.

Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we obtain a mapping $i : \mathcal{V} \hookrightarrow \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'$. i maps an element $u \in \mathcal{V}$ onto the linear functional $v \mapsto (v, u)_{\mathcal{H}},\ v \in \mathcal{V}$. Note that i is continuous and different from the Riesz isometry $J_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}'$. Since $\langle i(u), u\rangle = \|u\|^2_{\mathcal{H}} = 0$ if and only if $u = 0$ we obtain that the image $i(\mathcal{V})$ is a total set of $\mathcal{V}'$ by Remark 1.1 and therefore dense.

Because of the inclusions i and $j : \mathcal{H} \cong \mathcal{H}' \hookrightarrow \mathcal{V}'$ we may consider $\mathcal{V}$ and $\mathcal{H}$ as subspaces of $\mathcal{V}'$. Hence we will omit the notations i and j from now on. Then $\langle\cdot,\cdot\rangle_{\mathcal{H}\times\mathcal{V}} = (\cdot,\cdot)_{\mathcal{H}}$.

Remark 2.1. $\mathcal{V}$ is separable.

Proof. Since $\mathcal{H}$ is separable, densely and continuously embedded into $\mathcal{V}'$, we obtain that $\mathcal{V}'$ is separable. By the Riesz isometry $J_{\mathcal{V}}$ the space $\mathcal{V}$ is separable too. □

Lemma 2.2. *Let $(U_t)_{t\geq 0}$ be a C_0–semigroup on $\mathcal{H}$. Suppose that for all $t \geq 0$ U_t can be restricted to a bounded operator on $\mathcal{V}$, again denoted by U_t. Assume there exist positive constants M, ω with $\|U_t\|_{L(\mathcal{V})} \leq Me^{\omega t}, t \geq 0$. Then $(U_t)_{t\geq 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$.*

Proof. It is clear that the family of restrictions $(U_t)_{t\geq 0}$ is a semigroup of bounded operators on $\mathcal{V}$. To prove strong continuity it is now sufficient to show that the restricted semigroup is weakly continuous on $\mathcal{V}$ (cf. [Y, Theorem, p. 233]). To this end let $f \in \mathcal{V}$, $(t_n)_{n\geq 1} \subset [0, \infty)$ be a sequence with $\lim_{n\to\infty} t_n = t$. Since $\lim_{n\to\infty} U_{t_n} f = U_t f$ in $\mathcal{H}$ and $\sup_{n\geq 1} \|U_{t_n} f\|_{\mathcal{V}} < \infty$ we conclude that $\lim_{n\to\infty} U_{t_n} f = U_t f$ weakly in $\mathcal{V}$ (cf. [MR1, Lemma I.2.12]). □

For a linear operator Λ defined on a linear subspace D of one of the Hilbert spaces $\mathcal{V}$, $\mathcal{H}$ or $\mathcal{V}'$ we will use from now on the notation (Λ, D) in order to distinguish clearly between certain restrictions and extensions of Λ.

Let $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying the following assumptions:

D1(i) $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a C_0–semigroup of contractions $(U_t)_{t\geq 0}$.

(ii) $\mathcal{V}$ is Λ–admissible.

Since $\mathcal{V}$ is Λ–admissible $(U_t)_{t\geq 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$. Denote the infinitesimal generator of the restricted semigroup by $(\Lambda, D(\Lambda, \mathcal{V}))$. In particular, we obtain that $D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ is a dense subset of $\mathcal{V}$.

Lemma 2.3. *Let $(\Lambda, D(\Lambda, \mathcal{H}))$ satisfy D1. Then $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$ is closable (as an operator from $\mathcal{V}$ into $\mathcal{V}'$).*

Proof. Let $(u_n)_{n\geq 1} \subset D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ and $f \in \mathcal{V}'$ such that $\lim_{n\to\infty} u_n = 0$ in $\mathcal{V}$ and $\lim_{n\to\infty} \Lambda u_n = f$ in $\mathcal{V}'$. Since $(\Lambda, D(\Lambda, \mathcal{H}))$ is negative definite it follows that

$$\langle \Lambda v - f, v\rangle = \lim_{n\to\infty} \langle \Lambda v - \Lambda u_n, v - u_n\rangle = \lim_{n\to\infty} (\Lambda(v - u_n), v - u_n)_{\mathcal{H}} \leq 0$$

for all $v \in D(\Lambda, \mathcal{H}) \cap \mathcal{V}$. In particular, $\lambda^2 \langle \Lambda v, v\rangle \leq \lambda\langle f, v\rangle$ for all $\lambda \geq 0$, which implies that $\langle f, v\rangle \geq 0$ for all $v \in D(\Lambda, \mathcal{H}) \cap \mathcal{V}$. We conclude that $\langle f, v\rangle = 0$ for all $v \in D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ and subsequently for all $v \in \mathcal{V}$, since $D(\Lambda, \mathcal{H}) \cap \mathcal{V} \subset \mathcal{V}$ dense. It follows that $f = 0$. $\square$

Let $(\Lambda, \mathcal{F})$ be the closure of $\Lambda : D(\Lambda, \mathcal{H}) \cap \mathcal{V} \to \mathcal{V}'$. Then $\mathcal{F}$ is a real Hilbert space with corresponding norm

$$\|u\|^2_{\mathcal{F}} := \|u\|^2_{\mathcal{V}} + \|\Lambda u\|^2_{\mathcal{V}'}$$

(cf. [W, Theorem 5.1]). Let $(V_\alpha)_{\alpha>0}$ be the C_0–resolvent corresponding to $(\Lambda, D(\Lambda, \mathcal{H}))$. Since $\mathcal{V}$ is Λ–admissible it follows from [Pa, Corollary 4.5.6] that there exists a constant $\omega \geq 0$ such that $(V_\alpha)_{\alpha>\omega}$ can be restricted to a C_0–resolvent on $\mathcal{V}$. Furthermore, there exists a constant $M \geq 0$ with $\|V_\alpha\|_{L(\mathcal{V})} \leq \frac{M}{\alpha-\omega}$, $\alpha > \omega$. The corresponding generator is just the operator $(\Lambda, D(\Lambda, \mathcal{V}))$.

Lemma 2.4. *The adjoint semigroup $(\widehat{U}_t)_{t\geq 0}$ of $(U_t)_{t\geq 0}$ can be extended to a C_0–semigroup on $\mathcal{V}'$. The corresponding generator $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$.*

Proof. Let $t \geq 0$. Clearly

$$\langle \widehat{U}_t f, v\rangle = (\widehat{U}_t f, v)_{\mathcal{H}} = (f, U_t v)_{\mathcal{H}} = \langle f, U_t v\rangle \leq M e^{\omega t} \|f\|_{\mathcal{V}'} \|v\|_{\mathcal{V}}$$

for all $f \in \mathcal{H}$ and $v \in \mathcal{V}$ which implies $\|\widehat{U}_t f\|_{\mathcal{V}'} \leq M e^{\omega t}\|f\|_{\mathcal{V}'}$. Consequently, $\widehat{U}_t$ can be extended uniquely to a bounded linear operator on $\mathcal{V}'$, again denoted by $\widehat{U}_t$, such that $\|\widehat{U}_t\|_{L(\mathcal{V}')} \leq M e^{\omega t}$. The family $(\widehat{U}_t)_{t\geq 0}$ is a semigroup of bounded operators since the semigroup property already holds on the dense subset $\mathcal{H}$ of $\mathcal{V}'$ and strongly continuous since it is strongly continuous on $\mathcal{H}$ because $\lim_{t\to 0} \widehat{U}_t f = f$ for $f \in \mathcal{H}$ implies $\lim_{t\to 0} \widehat{U}_t f = f$ in $\mathcal{V}'$. It is easy to see that $\widehat{U}_t : \mathcal{V}' \to \mathcal{V}'$ is the dual operator of $U_t : \mathcal{V} \to \mathcal{V}$ and hence $(\widehat{U}_t)_{t\geq 0}$ the dual semigroup of the restriction of $(U_t)_{t\geq 0}$ on $\mathcal{V}$. It follows from [Pa, Corollary 1.10.6] that $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$ is the dual operator of $(\Lambda, D(\Lambda, \mathcal{V}))$. $\square$

Lemma 2.4 implies in particular that $(\widehat{V}_\alpha)_{\alpha>\omega}$ can be extended to a C_0–resolvent on $\mathcal{V}'$ with corresponding generator $(\widehat{\Lambda}, D(\widehat{\Lambda}, \mathcal{V}'))$. The extension of $\widehat{V}_\alpha$ on $\mathcal{V}'$ for $\alpha > \omega$ is the dual operator of the restriction of V_α on $\mathcal{V}$ and therefore, $\|\widehat{V}_\alpha\|_{L(\mathcal{V}')} \leq \frac{M}{\alpha-\omega}$ for all $\alpha > \omega$.

Let $\widehat{\mathcal{F}} := D(\widehat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$. Then $\widehat{\mathcal{F}}$ is a real Hilbert space with corresponding norm

$$\|u\|^2_{\widehat{\mathcal{F}}} := \|u\|^2_{\mathcal{V}} + \|\widehat{\Lambda} u\|^2_{\mathcal{V}'}$$

(cf. [W, Theorem 5.1]).

Lemma 2.5.

(i) $D(\Lambda, \mathcal{V}) \subset \mathcal{F}$ *dense.*

(ii) $\langle \Lambda u, u\rangle \le 0$ *for all* $u \in \mathcal{F}$.

Proof. (i) By the construction of $\mathcal{F}$ we obtain that $D(\Lambda, \mathcal{H}) \cap \mathcal{V} \subset \mathcal{F}$ is dense w.r.t. the $\mathcal{F}$–norm. Hence it suffices to prove that $D(\Lambda, \mathcal{V}) \subset D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ dense w.r.t. the $\mathcal{F}$–norm. But $\alpha V_\alpha u \in D(\Lambda, \mathcal{V})$ for $u \in D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ and $\alpha > \omega$. Moreover, $\lim_{\alpha\to\infty} \Lambda\alpha V_\alpha u = \Lambda u$ in $\mathcal{H}$. Consequently, $\lim_{\alpha\to\infty} \alpha V_\alpha u = u$ in $\mathcal{F}$ which implies the assertion.

(ii) For $u \in \mathcal{F}$ there exists a sequence $(u_n)_{n\ge1} \subset D(\Lambda, \mathcal{H}) \cap \mathcal{V}$ converging to u in $\mathcal{F}$. Since $(\Lambda, D(\Lambda, \mathcal{H}))$ is negative definite we obtain that $\langle \Lambda u_n, u_n\rangle = (\Lambda u_n, u_n)_{\mathcal{H}} \le 0$ and therefore, $\langle \Lambda u, u\rangle = \lim_{n\to\infty} \langle \Lambda u_n, u_n\rangle \le 0$. □

Lemma 2.6. $\langle \widehat{\Lambda} u, u\rangle \le 0$ *for all* $u \in \widehat{\mathcal{F}}$.

Proof.

$$\begin{aligned}\langle \widehat{\Lambda} u, u\rangle &= \lim_{\alpha\to\infty} \langle \alpha \widehat{V}_\alpha \widehat{\Lambda} u, u\rangle = \lim_{\alpha\to\infty} \langle \widehat{\Lambda} \alpha \widehat{V}_\alpha u, u\rangle \\ &= \lim_{\alpha\to\infty} \alpha\langle \alpha \widehat{V}_\alpha u - u, u\rangle = \lim_{\alpha\to\infty} \alpha(\alpha \widehat{V}_\alpha u - u, u)_{\mathcal{H}} \le 0\ . \quad \square\end{aligned}$$

Lemma 2.7. $\langle \Lambda u, v\rangle = \langle \widehat{\Lambda} v, u\rangle$ *for all* $u \in \mathcal{F}, v \in \widehat{\mathcal{F}}$.

Proof. For $u \in \mathcal{F}$ there exists a sequence $(u_n)_{n\ge1} \subset D(\Lambda, \mathcal{V})$ converging to u in $\mathcal{F}$ by Lemma 2.5. By Lemma 2.4 we obtain that $\langle \Lambda u_n, v\rangle = \langle v, \Lambda u_n\rangle = \langle \widehat{\Lambda} v, u_n\rangle$ if $v \in \widehat{\mathcal{F}}$ and consequently,

$$\langle \Lambda u, v\rangle = \lim_{n\to\infty} \langle \Lambda u_n, v\rangle = \lim_{n\to\infty} \langle \widehat{\Lambda} v, u_n\rangle = \langle \widehat{\Lambda} v, u\rangle\ . \quad \square$$

A linear operator L from $\mathcal{V}$ into $\mathcal{V}'$ is called *monotone* if $\langle Lv, v\rangle \ge 0$ for all $v \in D(L)$. L is called *maximal monotone* if there exists no real extension of L which is linear and monotone, i.e., no linear monotone operator $\bar{L}$ from $\mathcal{V}$ into $\mathcal{V}'$ satisfying $D(L) \subsetneqq D(\bar{L})$ and $\bar{L}v = Lv$ for all $v \in D(L)$.

Remark 2.8. It follows from Lemmas 2.4 – 2.7 that $(-\Lambda, \mathcal{F})$ and $(-\widehat{\Lambda}, \widehat{\mathcal{F}})$ are densely defined maximal monotone. Indeed, using [L, Lemma 3.1.1] we have that $(-\Lambda, \mathcal{F})$ and $(-\widehat{\Lambda}, \widehat{\mathcal{F}})$ are maximal monotone if and only if both $\langle -\Lambda u, u\rangle \ge 0$ for all $u \in \mathcal{F}$ and $\langle -\widehat{\Lambda} u, u\rangle \ge 0$ for all $u \in \widehat{\mathcal{F}}$.

In the next step we build a bilinear form $\mathcal{E}$ consisting of the components $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$.

Definition 2.9. Let

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle & \text{if } u \in \mathcal{F}, v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle & \text{if } u \in \mathcal{V}, v \in \widehat{\mathcal{F}}\ , \end{cases} \tag{2.1}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_\mathcal{H}$ for $\alpha > 0$. We call $\mathcal{E}$ the *bilinear form associated with* $(\mathcal{A},\mathcal{V})$ *and* $(\Lambda, D(\Lambda,\mathcal{H}))$.

Note that $\mathcal{E}$ is well–defined by Lemma 2.7 and positive definite by Lemma 2.5 and Lemma 2.6. Obviously $v \mapsto \mathcal{E}(u,v)$ is continuous on $\mathcal{V}$ if $u \in \mathcal{F}$ and $v \mapsto \mathcal{E}(v,u)$ is continuous on $\mathcal{V}$ if $u \in \widehat{\mathcal{F}}$.

Remark 2.10. In the particular case where $(\Lambda, D(\Lambda,\mathcal{H}))$ on $\mathcal{H}$ generates a C_0–semigroup of contractions $(U_t)_{t\geq 0}$ such that both $(U_t)_{t\geq 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$ and at the same time extended to a C_0–semigroup on $\mathcal{V}'$, which is for example satisfied by the time derivative in the case of time dependent Dirichlet forms (cf. Example 4.9 (iii)), it follows that $\mathcal{F} = D(\Lambda,\mathcal{V}') \cap \mathcal{V}$, where $(\Lambda, D(\Lambda,\mathcal{V}'))$ denotes the generator of the extension of the semigroup $(U_t)_{t\geq 0}$ on $\mathcal{V}'$. Indeed, for $u \in D(\Lambda,\mathcal{V}') \cap \mathcal{V}$ it follows that $\lim_{\alpha\to\infty} \alpha V_\alpha u = u$ in $\mathcal{V}$ but also $\lim_{\alpha\to\infty} \Lambda\alpha V_\alpha u = \lim_{\alpha\to\infty} \alpha V_\alpha \Lambda u = \Lambda u$ in $\mathcal{V}'$. Since $\alpha V_\alpha u \in D(\Lambda,\mathcal{V})$ we obtain that $u \in \mathcal{F}$, hence $D(\Lambda,\mathcal{V}') \cap \mathcal{V} \subset \mathcal{F}$. On the other hand $D(\Lambda,\mathcal{V}') \cap \mathcal{V}$ is a closed subspace of $\mathcal{F}$ containing $D(\Lambda,\mathcal{H}) \cap \mathcal{V}$ which implies the assertion.

I.3 The bilinear form $\mathcal{E}$: existence of corresponding resolvents

Throughout the whole section let $\mathcal{H}$ be a separable real Hilbert space, $(\mathcal{A},\mathcal{V})$ a coercive closed form on $\mathcal{H}$ with sector constant $K \geq 1$ and $(\Lambda, D(\Lambda,\mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying assumption D1. Denote by $(V_\alpha)_{\alpha>0}$ the C_0–resolvent corresponding to $(\Lambda, D(\Lambda,\mathcal{H}))$ and let $C, \omega \geq 1$ be constants such that $(V_\alpha)_{\alpha>\omega}$ can be restricted to a C_0–resolvent on $\mathcal{V}$ and $\|V_\alpha\|_{L(\mathcal{V})} \leq C/(\alpha-\omega), \alpha > \omega$. Denote by $\mathcal{E}$ the bilinear form associated with $(\mathcal{A},\mathcal{V})$ and $(\Lambda, D(\Lambda,\mathcal{H}))$. The main purpose of this section is to construct C_0–resolvents $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ associated to the bilinear form $\mathcal{E}$, i.e., C_0–resolvents $(G_\alpha)_{\alpha>0}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ of contractions on $\mathcal{H}$ satisfying $G_\alpha(\mathcal{H}) \subset \mathcal{F}$ (resp. $\widehat{G}_\alpha(\mathcal{H}) \subset \widehat{\mathcal{F}}$) and

$$\mathcal{E}_\alpha(G_\alpha f, g) = (f,g)_\mathcal{H} = \mathcal{E}_\alpha(g, \widehat{G}_\alpha f),\ \ f \in \mathcal{H}, g \in \mathcal{V}\ . \tag{3.1}$$

Let $\alpha > 0$. Then the bilinear form $(\mathcal{A}_\alpha,\mathcal{V})$ induces two continuous, linear mappings $M_\alpha : \mathcal{V} \to \mathcal{V}'$ and $\widehat{M_\alpha} : \mathcal{V} \to \mathcal{V}'$ defined by

$$\langle M_\alpha u, \cdot\rangle := \mathcal{A}_\alpha(u,\cdot) \text{ and } \langle \widehat{M}_\alpha u, \cdot\rangle := \mathcal{A}_\alpha(\cdot, u)\ . \tag{3.2}$$

Note that (3.1) is now equivalent with

$$\langle (M_\alpha - \Lambda) G_\alpha f, g\rangle = \langle f, g\rangle = \langle (\widehat{M}_\alpha - \widehat{\Lambda}) \widehat{G}_\alpha f, g\rangle\ , f \in \mathcal{H}, g \in \mathcal{V}\ .$$

Hence we are searching for solutions $u \in \mathcal{F}$ (resp. $\widehat{u} \in \widehat{\mathcal{F}}$) to the equation

$$M_\alpha u - \Lambda u = f(\text{ resp. } \widehat{M_\alpha}\widehat{u} - \widehat{\Lambda}\widehat{u} = f) \tag{3.3}$$

for $f \in \mathcal{V}'$. The existence of a solution $u \in \mathcal{F}$ (resp. $\widehat{u} \in \widehat{\mathcal{F}}$) for arbitrary $f \in \mathcal{V}'$ follows from [L, Théorème 3.1.1] since $(-\Lambda, \mathcal{F})$ (resp. $(-\widehat{\Lambda}, \widehat{\mathcal{F}})$) is maximal monotone (cf. Remark 2.8).

We will give an alternative proof using the (stronger) assumption D1. Having later applications in mind (cf. Proposition III.1.7), we are also interested in solutions to certain non–linear equations. Hence we consider instead of (3.3) the more general equation

$$Mu - \Lambda u = f$$

for $f \in \mathcal{V}'$, where $M : \mathcal{V} \to \mathcal{V}'$ is some (in general non–linear !) mapping satisfying the following assumptions:

(M.1) M is continuous and bounded, i.e., $M(B)$ is a bounded set whenever B is a bounded set.

(M.2) There exists a constant $\gamma > 0$ such that $\langle Mu - Mv, u - v\rangle \geq \gamma \|u - v\|_{\mathcal{V}}^2$ for all $u, v \in \mathcal{V}$.

(M.3) $\frac{\langle Mv, v\rangle}{\|v\|_{\mathcal{V}}} \to \infty$ if $\|v\|_{\mathcal{V}} \to \infty$.

Lemma 3.1. *Let $M : \mathcal{V} \to \mathcal{V}'$ satisfy assumptions (M.1)–(M.3). Then M is a bijection.*

Proof. Since $\mathcal{V}$ is separable we can apply [L, Théorème 2.2.1] to obtain that M is surjective (cf. also [St1, Lemma I.3.4]). Since (M.2) implies in particular that M is injective, the assertion follows.

Proposition 3.2. *Let M satisfy assumptions (M.1)–(M.3) and $f \in \mathcal{V}'$. Then there exists one and only one solution $u \in \mathcal{F}$ to the equation $Mu - \Lambda u = f$.*

Proof. **Existence:** For $\alpha > 0$ let $\Lambda_\alpha : \mathcal{V} \to \mathcal{V}'$ be defined by $\langle \Lambda_\alpha u, \cdot\rangle := \alpha(\alpha V_\alpha u - u, \cdot)_{\mathcal{H}}$.

1. Step: "Approximation" of the equation $Mu - \Lambda u = f$ through the equation $Mu - \Lambda_\alpha u = f$.

Since $\|\alpha V_\alpha\|_{L(\mathcal{H})} \leq 1$ we obtain that $\langle \Lambda_\alpha u, u\rangle \leq 0$ for all $u \in \mathcal{V}$. Λ_α is linear and bounded (hence continuous) since

$$|\langle \Lambda_\alpha u, v\rangle| \leq 2\alpha \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} \leq 2\alpha \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}}$$

which implies $\|\Lambda_\alpha u\|_{\mathcal{V}'} \leq 2\alpha \|u\|_{\mathcal{V}}$. Therefore, $M - \Lambda_\alpha$ satisfies assumption (M.1). Since

$$\langle (M - \Lambda_\alpha)u - (M - \Lambda_\alpha)v, u - v\rangle \geq \langle Mu - Mv, u - v\rangle$$

(M.2) is obvious and (M.3) is satisfied since $\langle (M-\Lambda_\alpha)u,u\rangle \geq \langle Mu,u\rangle$. By Lemma 3.1 there exists some element $u_\alpha \in \mathcal{V}$ such that $Mu_\alpha - \Lambda_\alpha u_\alpha = f$.

2. Step: Since $\langle Mu_\alpha, u_\alpha\rangle \leq \langle Mu_\alpha - \Lambda_\alpha u_\alpha, u_\alpha\rangle = \langle f, u_\alpha\rangle \leq \|f\|_{\mathcal{V}'}\|u_\alpha\|_{\mathcal{V}}$ we obtain that $\sup_{\alpha>0}\|u_\alpha\|_{\mathcal{V}} < \infty$ by (M.3) and consequently, $\sup_{\alpha>0}\|Mu_\alpha\|_{\mathcal{V}'} < \infty$ by (M.1). Hence for $v \in \mathcal{V}$

$$\begin{aligned}\langle \Lambda_\alpha u_\alpha, v\rangle &= -\langle Mu_\alpha - \Lambda_\alpha u_\alpha, v\rangle + \langle Mu_\alpha, v\rangle \\ &= -\langle f, v\rangle + \langle Mu_\alpha, v\rangle \\ &\leq (\|f\|_{\mathcal{V}'} + \|Mu_\alpha\|_{\mathcal{V}'})\|v\|_{\mathcal{V}}\end{aligned}$$

and therefore, $\sup_{\alpha>0}\|\Lambda_\alpha u_\alpha\|_{\mathcal{V}'} < \infty$.

Hence there exists some subsequence $(\alpha_n)_{n\geq 1} \subset (2\omega, \infty)$ such that $\lim_{n\to\infty}\alpha_n = \infty$, $u_{\alpha_n} \to u$ weakly in $\mathcal{V}$ for some $u \in \mathcal{V}$, $Mu_{\alpha_n} \to h$ weakly in $\mathcal{V}'$ for some $h \in \mathcal{V}'$, and $\Lambda_{\alpha_n}u_{\alpha_n} \to g$ weakly in $\mathcal{V}'$ for some $g \in \mathcal{V}'$.

By the strong continuity of the dual resolvent $(\widehat{V}_\alpha)_{\alpha>\omega}$ in $\mathcal{V}'$ we obtain that

$$\lim_{n\to\infty}\langle v, \alpha_n V_{\alpha_n}u_{\alpha_n}\rangle = \lim_{n\to\infty}\langle \alpha_n \widehat{V}_{\alpha_n}v, u_{\alpha_n}\rangle = \langle v, u\rangle$$

for all $v \in \mathcal{V}'$ and therefore, $\alpha_n V_{\alpha_n}u_{\alpha_n} \to u$ weakly in $\mathcal{V}$.

Since $\|\alpha_n V_{\alpha_n}u_{\alpha_n}\|_{\mathcal{V}} \leq 2C\|u_{\alpha_n}\|_{\mathcal{V}}$ and $\|\Lambda\alpha_n V_{\alpha_n}u_{\alpha_n}\|_{\mathcal{V}'} = \|\Lambda_{\alpha_n}u_{\alpha_n}\|_{\mathcal{V}'}$ we conclude that $\sup_{n\geq 1}\|\alpha_n V_{\alpha_n}u_{\alpha_n}\|_{\mathcal{F}} < \infty$ which implies $u \in \mathcal{F}$ and $\Lambda u = g$.

3. Step: For all $v \in D(\Lambda, \mathcal{V})$ it follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle \Lambda_{\alpha_n}u_{\alpha_n}, u_{\alpha_n}\rangle &= \limsup_{n\to\infty}\langle \Lambda_{\alpha_n}u_{\alpha_n}, u_{\alpha_n} - v\rangle + \langle \Lambda_{\alpha_n}u_{\alpha_n}, v\rangle \\ &\leq \limsup_{n\to\infty}\langle \Lambda_{\alpha_n}v, u_{\alpha_n} - v\rangle + \langle \Lambda_{\alpha_n}u_{\alpha_n}, v\rangle \\ &= \langle \Lambda v, u - v\rangle + \langle \Lambda u, v\rangle\ .\end{aligned}$$

Since $D(\Lambda, \mathcal{V}) \subset \mathcal{F}$ dense this inequality holds for all $v \in \mathcal{F}$, in particular, for $v = u$ which implies $\limsup_{n\to\infty}\langle \Lambda_{\alpha_n}u_{\alpha_n}, u_{\alpha_n}\rangle \leq \langle \Lambda u, u\rangle$. Hence we obtain by (M.2)

$$\begin{aligned}\limsup_{n\to\infty}\gamma\|u_{\alpha_n} - u\|_{\mathcal{V}}^2 &\leq \limsup_{n\to\infty}\langle Mu_{\alpha_n} - Mu, u_{\alpha_n} - u\rangle \\ &= \limsup_{n\to\infty}\langle f - Mu, u_{\alpha_n} - u\rangle + \langle \Lambda_{\alpha_n}u_{\alpha_n}, u_{\alpha_n} - u\rangle \\ &\leq \langle \Lambda u, u\rangle - \langle \Lambda u, u\rangle = 0\ .\end{aligned}$$

Thus $u_{\alpha_n} \to u$ strongly in $\mathcal{V}$ and $Mu_{\alpha_n} \to Mu$ strongly in $\mathcal{V}'$ and consequently, for $v \in \mathcal{V}$

$$\langle Mu - \Lambda u, v\rangle = \lim_{n\to\infty}\langle Mu_{\alpha_n} - \Lambda_{\alpha_n}u_{\alpha_n}, v\rangle = \langle f, v\rangle\ .$$

Uniqueness: $Mu - \Lambda u = Mv - \Lambda v$ implies $0 = \langle Mu - Mv, u - v\rangle - \langle \Lambda(u - v), u - v\rangle \geq \langle Mu - Mv, u - v\rangle$ and hence $u = v$ by (M.2). $\square$

Proposition 3.3. *Let M satisfy assumptions (M.1)–(M.3) and $f \in \mathcal{V}'$. Then there exists one and only one solution $\widehat{u} \in \widehat{\mathcal{F}}$ to the equation $\widehat{M}\widehat{u} - \widehat{\Lambda}\widehat{u} = f$.*

Proof. We proceed in a similar way to the proof of Proposition 3.2.

Existence: For $\alpha > 0$ let $\widehat{\Lambda}_\alpha : \mathcal{V} \to \mathcal{V}'$ be defined by $\langle \widehat{\Lambda}_\alpha u, \cdot \rangle := \alpha(\alpha \widehat{V}_\alpha u - u, \cdot)_{\mathcal{H}}$.

1. Step: "Approximation" of the equation $Mu - \widehat{\Lambda}u = f$ through the equation $Mu - \widehat{\Lambda}_\alpha u = f$.

It is easy to see that the operator $M - \widehat{\Lambda}_\alpha$ satisfies assumptions (M.1)–(M.3) (cf. the 1. Step in the proof of Proposition 3.2). By Lemma 3.1 there exists an element $\widehat{u}_\alpha \in \mathcal{V}$ such that $M\widehat{u}_\alpha - \widehat{\Lambda}\widehat{u}_\alpha = f$.

2. Step: Since $\langle M\widehat{u}_\alpha, \widehat{u}_\alpha \rangle \le \langle M\widehat{u}_\alpha - \widehat{\Lambda}_\alpha \widehat{u}_\alpha, \widehat{u}_\alpha \rangle = \langle f, \widehat{u}_\alpha \rangle \le \|f\|_{\mathcal{V}'} \|\widehat{u}_\alpha\|_{\mathcal{V}}$ we obtain that $\sup_{\alpha > 0} \|\widehat{u}_\alpha\|_{\mathcal{V}} < \infty$ by (M.3) and consequently, $\sup_{\alpha > 0} \|M\widehat{u}_\alpha\|_{\mathcal{V}'} < \infty$ by (M.1).

Hence there exists some subsequence $(\alpha_n)_{n \ge 1} \subset (2\omega, \infty)$ such that $\lim_{n\to\infty} \alpha_n = \infty$, $\widehat{u}_{\alpha_n} \to \widehat{u}$ weakly in $\mathcal{V}$ for some $\widehat{u} \in \mathcal{V}$ and $M\widehat{u}_{\alpha_n} \to h$ weakly in $\mathcal{V}'$ for some $h \in \mathcal{V}'$. In particular, $\widehat{u}_{\alpha_n} \to \widehat{u}$ weakly in $\mathcal{H}$.

Let $v \in D(\Lambda, \mathcal{V})$. Then

$$\begin{aligned}
\langle \widehat{u}, \Lambda v \rangle = (\widehat{u}, \Lambda v)_{\mathcal{H}} &= \lim_{n\to\infty} (\widehat{u}_{\alpha_n}, \alpha_n V_{\alpha_n} \Lambda v)_{\mathcal{H}} \\
&= \lim_{n\to\infty} (\widehat{\Lambda} \alpha_n \widehat{V}_{\alpha_n} \widehat{u}_{\alpha_n}, v)_{\mathcal{H}} \\
&= \lim_{n\to\infty} \langle \widehat{\Lambda}_{\alpha_n} \widehat{u}_{\alpha_n}, v \rangle = \lim_{n\to\infty} \langle M\widehat{u}_{\alpha_n} - f, v \rangle \\
&= \langle h - f, v \rangle \ .
\end{aligned}$$

Hence $v \mapsto \langle \widehat{u}, \Lambda v \rangle$ continuous on $\mathcal{V}$ which implies $\widehat{u} \in D(\widehat{\Lambda}, \mathcal{V}')$ by Lemma 2.4 and

$$\widehat{\Lambda}\widehat{u} = h - f \ . \tag{3.4}$$

3. Step: Since $\widehat{u} \in \widehat{\mathcal{F}}$ we obtain that $\lim_{\alpha\to\infty} \widehat{\Lambda}_\alpha \widehat{u} = \lim_{\alpha\to\infty} \alpha \widehat{V}_\alpha \widehat{\Lambda}\widehat{u} = \widehat{\Lambda}\widehat{u}$ in $\mathcal{V}'$ and therefore,

$$\begin{aligned}
\limsup_{n\to\infty} \gamma \|\widehat{u}_{\alpha_n} - \widehat{u}\|_{\mathcal{V}}^2 &\le \limsup_{n\to\infty} \langle M\widehat{u}_{\alpha_n} - M\widehat{u}, \widehat{u}_{\alpha_n} - \widehat{u} \rangle \\
&= \limsup_{n\to\infty} \langle f - M\widehat{u}, \widehat{u}_{\alpha_n} - \widehat{u} \rangle + \langle \widehat{\Lambda}_{\alpha_n} \widehat{u}_{\alpha_n}, \widehat{u}_{\alpha_n} - \widehat{u} \rangle \\
&\le \limsup_{n\to\infty} \langle f - M\widehat{u}, \widehat{u}_{\alpha_n} - \widehat{u} \rangle + \langle \widehat{\Lambda}_{\alpha_n} \widehat{u}, \widehat{u}_{\alpha_n} - \widehat{u} \rangle \\
&= 0 \ .
\end{aligned}$$

Consequently, $\widehat{u}_{\alpha_n} \to \widehat{u}$ in $\mathcal{V}$ and $M\widehat{u}_{\alpha_n} \to M\widehat{u}$ in $\mathcal{V}'$ by (M.1). Since $h = \lim_{n\to\infty} M\widehat{u}_{\alpha_n}$ weakly in $\mathcal{V}'$ we obtain that $M\widehat{u} = h$ and therefore (3.4) implies $M\widehat{u} - \widehat{\Lambda}\widehat{u} = f$.

Uniqueness: $M\widehat{u} - \widehat{\Lambda}\widehat{u} = M\widehat{v} - \widehat{\Lambda}\widehat{v}$ implies $0 = \langle M\widehat{u} - M\widehat{v}, \widehat{u} - \widehat{v}\rangle - \langle \widehat{\Lambda}(\widehat{u} - \widehat{v}), \widehat{u} - \widehat{v}\rangle \geq \langle M\widehat{u} - M\widehat{v}, \widehat{u} - \widehat{v}\rangle$ and hence $\widehat{u} = \widehat{v}$ by (M.2). □

Using Propositions 3.2 and 3.3 we are now able to construct the C_0–resolvent (resp. C_0–coresolvent) of contractions corresponding to the bilinear form $\mathcal{E}$. Let M_α and $\widehat{M}_\alpha$ be as in (3.2). Obviously M_α and $\widehat{M}_\alpha$ satisfy assumption (M.1). (M.2) holds since

$$\langle M_\alpha u - M_\alpha v, u - v\rangle = \mathcal{A}_\alpha(u - v, u - v) \geq (\alpha \wedge 1)\|u - v\|_{\mathcal{V}}^2 \text{ and}$$

$$\langle \widehat{M}_\alpha u - \widehat{M}_\alpha v, u - v\rangle = \mathcal{A}_\alpha(u - v, u - v) \geq (\alpha \wedge 1)\|u - v\|_{\mathcal{V}}^2 \ ,$$

and (M.3) is satisfied because

$$\langle M_\alpha u, u\rangle = \mathcal{A}_\alpha(u, u) \geq (\alpha \wedge 1)\|u\|_{\mathcal{V}}^2 \text{ and}$$

$$\langle \widehat{M}_\alpha u, u\rangle = \mathcal{A}_\alpha(u, u) \geq (\alpha \wedge 1)\|u\|_{\mathcal{V}}^2 \ .$$

By Propositions 3.2 and 3.3 we obtain the following:

Proposition 3.4. *For all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \to \mathcal{F}$ and $\widehat{W}_\alpha : \mathcal{V}' \to \widehat{\mathcal{F}}$ such that*

$$\mathcal{E}_\alpha(W_\alpha f, v) = \langle f, v\rangle = \mathcal{E}_\alpha(v, \widehat{W}_\alpha f) \textit{ for all } f \in \mathcal{V}', v \in \mathcal{V} \ .$$

$(W_\alpha)_{\alpha>0}$ and $(\widehat{W}_\alpha)_{\alpha>0}$ satisfy the resolvent equation

$$(3.5) \qquad W_\alpha - W_\beta = (\beta - \alpha)W_\alpha W_\beta \textit{ and } \widehat{W}_\alpha - \widehat{W}_\beta = (\beta - \alpha)\widehat{W}_\alpha \widehat{W}_\beta \ , \alpha, \beta > 0 \ .$$

Proof. If $\alpha > 0$ and $f \in \mathcal{V}'$ there exists a unique $W_\alpha f \in \mathcal{F}$ (resp. $\widehat{W}_\alpha f \in \widehat{\mathcal{F}}$) such that

$$M_\alpha(W_\alpha f) - \Lambda(W_\alpha f) = f(\text{ resp. } \widehat{M}_\alpha(\widehat{W}_\alpha f) - \widehat{\Lambda}(\widehat{W}_\alpha f) = f\,)$$

by Proposition 3.2 (resp. 3.3) and therefore,

$$\mathcal{E}_\alpha(W_\alpha f, v) = \langle f, v\rangle = \mathcal{E}_\alpha(v, \widehat{W}_\alpha f) \text{ for all } v \in \mathcal{V} \ .$$

The mapping $f \mapsto W_\alpha f$ (resp. $f \mapsto \widehat{W}_\alpha f$) is linear by the linearity of M and Λ (resp. $\widehat{M}$ and $\widehat{\Lambda}$). W_α (resp. $\widehat{W}_\alpha$) is injective by the uniqueness of the solution $W_\alpha f$ (resp. $\widehat{W}_\alpha f$). On the other hand, if $u \in \mathcal{F}$ (resp. $\widehat{u} \in \widehat{\mathcal{F}}$), the uniqueness in Proposition 3.2 (resp. 3.3) implies that $W_\alpha(M_\alpha u - \Lambda u) = u$ (resp. $\widehat{W}_\alpha(\widehat{M}_\alpha \widehat{u} - \widehat{\Lambda}\widehat{u}) = \widehat{u}$). Consequently, W_α (resp. $\widehat{W}_\alpha$) is surjective. Since $W_\alpha^{-1} = M_\alpha - \Lambda : \mathcal{F} \to \mathcal{V}'$ (resp. $\widehat{W}_\alpha^{-1} = \widehat{M}_\alpha - \widehat{\Lambda} : \widehat{\mathcal{F}} \to \mathcal{V}'$) is continuous and bijective we conclude that W_α (resp.

$\widehat{W}_\alpha$) is continuous by the Open Mapping Theorem. For $f \in \mathcal{V}'$ and $v \in \mathcal{V}$ we obtain that

$$\begin{aligned}\mathcal{E}_\alpha(W_\alpha f - (\beta-\alpha)W_\alpha W_\beta f, v) &= \langle f - (\beta-\alpha)W_\beta f, v\rangle \\ &= \mathcal{E}_\beta(W_\beta f, v) - (\beta-\alpha)(W_\beta f, v)_{\mathcal{H}} \\ &= \mathcal{E}_\alpha(W_\beta f, v)\ .\end{aligned}$$

It follows in particular for $v := W_\alpha f - W_\beta f - (\beta-\alpha)W_\alpha W_\beta f$ that $\mathcal{E}_\alpha(v,v) = 0$ and consequently, $W_\alpha f - W_\beta f = (\beta-\alpha)W_\alpha W_\beta f$. The resolvent equation for $(\widehat{W}_\alpha)_{\alpha>0}$ can be shown in a similar way. □

Let $\alpha > 0$. Restricting W_α (resp. $\widehat{W}_\alpha$) to $\mathcal{H}$ we obtain a linear operator $G_\alpha : \mathcal{H} \to \mathcal{H}$ (resp. $\widehat{G}_\alpha : \mathcal{H} \to \mathcal{H}$) since $\mathcal{F} \subset \mathcal{H}$ (resp. $\widehat{\mathcal{F}} \subset \mathcal{H}$).

Remark 3.5. Since $W_1(\mathcal{V}') = \mathcal{F}$ it follows from the continuity of W_1 that $G_1(\mathcal{H}) = W_1(\mathcal{H})$ is a dense subset of $\mathcal{F}$. Similarly one can show that $\widehat{G}_1(\mathcal{H})$ is a dense subset of $\widehat{\mathcal{F}}$.

Proposition 3.6.

(i) $(G_\alpha)_{\alpha>0}$ *is a* C_0*–resolvent of contractions on* $\mathcal{H}$.

(ii) $(\widehat{G}_\alpha)_{\alpha>0}$ *is the adjoint* C_0*–resolvent of contractions to* $(G_\alpha)_{\alpha>0}$.

Proof. (i) Clearly (3.5) implies that $(G_\alpha)_{\alpha>0}$ satisfies the resolvent equation. Since $\mathcal{E}(u,u) \geq 0$ for all $u \in \mathcal{F}$ we obtain that

$$\alpha\|G_\alpha f\|_{\mathcal{H}}^2 \leq \mathcal{E}_\alpha\left(G_\alpha f, G_\alpha f\right) = (f, G_\alpha f)_{\mathcal{H}} \leq \|f\|_{\mathcal{H}}\|G_\alpha f\|_{\mathcal{H}}$$

and therefore, $\|\alpha G_\alpha\|_{L(\mathcal{H})} \leq 1$. For $f = G_1 h$ we have that

$$\begin{aligned}\alpha\|\alpha G_{\alpha+1}f - f\|_{\mathcal{H}}^2 &\leq \mathcal{E}_{\alpha+1}\left(\alpha G_{\alpha+1}f - f, \alpha G_{\alpha+1}f - f\right) = -\mathcal{E}_1\left(f, \alpha G_{\alpha+1}f - f\right) \\ &= -\left(h, \alpha G_{\alpha+1}f - f\right)_{\mathcal{H}} \\ &\leq \|h\|_{\mathcal{H}}\|\alpha G_{\alpha+1}f - f\|_{\mathcal{H}}\ .\end{aligned}$$

Consequently, $\lim\limits_{\alpha\to\infty}\|\alpha G_\alpha f - f\|_{\mathcal{H}} = 0$. Since $G_1(\mathcal{H})$ is dense in $\mathcal{F}$ by the last remark and $\mathcal{F} \subset \mathcal{H}$ dense the strong continuity of $(G_\alpha)_{\alpha>0}$ now follows by a 3-ε-argument.

(ii) For the proof of (ii) it is now sufficient to show that $\widehat{G}_\alpha$ is the adjoint operator of G_α in $\mathcal{H}$. This follows from $(\widehat{G}_\alpha f, g)_{\mathcal{H}} = \mathcal{E}_\alpha\left(G_\alpha g, \widehat{G}_\alpha f\right) = (f, G_\alpha g)_{\mathcal{H}}$ for all $f, g \in \mathcal{H}$. □

The resolvent $(G_\alpha)_{\alpha>0}$ just constructed is called the *resolvent associated with* $\mathcal{E}$ and $(\widehat{G}_\alpha)_{\alpha>0}$ the *coresolvent associated with* $\mathcal{E}$. The C_0–semigroup of contractions $(T_t)_{t\geq0}$ (resp. $(\widehat{T}_t)_{t\geq0}$) corresponding to $(G_\alpha)_{\alpha>0}$ (resp. $(\widehat{G}_\alpha)_{\alpha>0}$) is called the *semigroup associated with* $\mathcal{E}$ (resp. *cosemigroup associated with* $\mathcal{E}$). The corresponding generator $(L, D(L))$ (resp. $(\widehat{L}, D(\widehat{L}))$) is called the *generator associated with* $\mathcal{E}$ (resp. *cogenerator associated with* $\mathcal{E}$).

Proposition 3.7. *If $v \in \mathcal{V}$ then* $\lim_{\alpha\to\infty} \alpha G_\alpha v = v$ *in* $\mathcal{V}$.

Proof. Let $v \in \mathcal{V}$. Note that $w_\alpha := \alpha G_\alpha v - \alpha V_\alpha v \in \mathcal{F}$ for $\alpha > 2\omega$ and

$$\begin{aligned}\|w_\alpha\|_{\mathcal{V}}^2 &\le \mathcal{E}_\alpha(w_\alpha, w_\alpha)\\ &= \alpha\mathcal{E}_\alpha(G_\alpha v, w_\alpha) - \mathcal{A}(\alpha V_\alpha v, w_\alpha) - \alpha\left(\alpha V_\alpha v - \Lambda V_\alpha v, w_\alpha\right)_{\mathcal{H}}\\ &= -\mathcal{A}\left(\alpha V_\alpha v, w_\alpha\right)\\ &\le 2K\|\alpha V_\alpha v\|_{\mathcal{V}}\|w_\alpha\|_{\mathcal{V}}\\ &\le 4KC\|v\|_{\mathcal{V}}\|w_\alpha\|_{\mathcal{V}}\ .\end{aligned}$$

Hence we obtain for $\alpha > 2\omega$

$$\|\alpha G_\alpha v\|_{\mathcal{V}} \le \|\alpha V_\alpha v\|_{\mathcal{V}} + \|w_\alpha\|_{\mathcal{V}} \le 6KC\|v\|_{\mathcal{V}}\ .$$

For $v = G_1 h$, $h \in \mathcal{H}$, we conclude that

$$\begin{aligned}\|\alpha G_\alpha v - v\|_{\mathcal{V}}^2 &\le \mathcal{E}_1\left(\alpha G_\alpha v - v, \alpha G_\alpha v - v\right)\\ &= \left(\alpha G_\alpha h - h, \alpha G_\alpha v - v\right)_{\mathcal{H}} \to 0 \text{ for } \alpha \to \infty\ .\end{aligned}$$

Since $G_1(\mathcal{H})$ is dense in $\mathcal{V}$ the strong continuity now follows by a 3-ε-argument. □

I.4 Contraction principles of the bilinear form $\mathcal{E}$

In this section we assume that the underlying Hilbert space $\mathcal{H}$ is of L^2–type. Fix some σ–finite measure space $(E, \mathcal{B}, m)$ such that the corresponding L^2–space $L^2(E, m)$ is separable and let $\mathcal{H} := L^2(E, m)$. For functions $u, v : E \to \mathbb{R}$ let

$$\begin{aligned} u \vee v := \sup(u, v), \quad u \wedge v := \inf(u, v)\\ u^+ := u \vee 0 \text{ and } u^- := -(u \vee 0)\,. \end{aligned}$$

For m–equivalence classes u and v $u \vee v$ and $u \wedge v$ are given by their corresponding m–equivalence classes.

Definition 4.1.

(i) A bounded linear operator $G : \mathcal{H} \to \mathcal{H}$ is called *positivity preserving* (resp. *sub–Markovian*) if $Gf \ge 0$ (resp. $0 \le Gf \le 1$) for all $f \in \mathcal{H}$ with $f \ge 0$ (resp. $0 \le f \le 1$).

(ii) A C_0–resolvent $(G_\alpha)_{\alpha>0}$ is called *positivity preserving* (resp. *sub–Markovian*) if αG_α is positivity preserving (resp. sub–Markovian) for all $\alpha > 0$.

(iii) A C_0–semigroup $(T_t)_{t\ge 0}$ is called *positivity preserving* (resp. *sub–Markovian*) if T_t is positivity preserving (resp. sub–Markovian) for all $t \ge 0$.

(iv) A linear operator $(L, D(L))$ on $\mathcal{H}$ is called a *Dirichlet operator* if $(Lu, (u - 1)^+) \le 0$ for all $u \in D(L)$.

Remark 4.2.

(i) Any sub–Markovian operator G is positivity preserving.
Indeed, let $f \in \mathcal{H}$, $f \geq 0$. The continuity of G implies $G(f \wedge n) \to Gf$. Since G is sub–Markovian we obtain that $0 \leq (1/n)G(f \wedge n) \leq 1$ and therefore, $0 \leq G(f \wedge n)$ for all n, which implies the assertion.

(ii) The adjoint operator $\widehat{G}$ of a positivity preserving operator G is again positivity preserving.
Indeed, let $f \in \mathcal{H}$, $f \geq 0$, and $u := \widehat{G}f$. Since $u^- \geq 0$ we obtain that

$$0 \leq (G(u^-), f)_{\mathcal{H}} = (u^-, \widehat{G}f)_{\mathcal{H}} = -\|u^-\|^2_{\mathcal{H}} .$$

It follows that $u^- = 0$ and consequently, $u \geq 0$.

Definition 4.3.

(i) A coercive closed form $(\mathcal{A}, \mathcal{V})$ on $\mathcal{H}$ is called *positivity preserving* if

$$u \in \mathcal{V} \;\Rightarrow\; u^+ \in \mathcal{V} \text{ and } \mathcal{A}(u, u^+) \geq 0 . \tag{4.1}$$

(ii) A coercive closed form $(\mathcal{A}, \mathcal{V})$ on $\mathcal{H}$ is called a *Dirichlet form* if

$$u \in \mathcal{V} \;\Rightarrow\; u^+ \wedge 1 \in \mathcal{V} \text{ and } \mathcal{A}(u + u^+ \wedge 1, u - u^+ \wedge 1) \geq 0 . \tag{4.2}$$

For the rest of this section let $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on $\mathcal{H}$ with sector constant $K \geq 1$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator satisfying D1. Denote by $\mathcal{E}$ the bilinear form associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ and by $(G_\alpha)_{\alpha>0}$ (resp. $(\widehat{G}_\alpha)_{\alpha>0}$) the associated resolvent (resp. coresolvent).

Note that (4.1) (resp. (4.2)) is equivalent to the fact that the resolvent associated with $(\mathcal{A}, \mathcal{V})$ is positivity preserving (resp. sub–Markovian) (cf. [MR1, Theorem I.4.4] resp. [MR2, Theorem 1.5 and Theorem 1.7]). Therefore one can see directly from the form $(\mathcal{A}, \mathcal{V})$ whether the associated resolvent is positivity preserving (resp. sub–Markovian). In the case of the bilinear form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ this fact is also true as is implied by the following two propositions.

Proposition 4.4. *The following statements are equivalent:*

(i) $(G_\alpha)_{\alpha>0}$ *is positivity preserving.*

(ii) $u \in \mathcal{F} \Rightarrow u^+ \in \mathcal{V}$ *and* $\mathcal{E}(u, u^+) \geq 0$.

(iii) $u \in \widehat{\mathcal{F}} \Rightarrow u^+ \in \mathcal{V}$ *and* $\mathcal{E}(u^+, u) \geq 0$.

Proof. (i) $\Rightarrow$ (ii) First consider $u = G_1 h$, $h \in \mathcal{H}$. Since $(G_\alpha)_{\alpha>0}$ is positivity preserving we obtain that

$$\alpha(u^+ - \alpha G_\alpha u^+, u^-)_{\mathcal{H}} \leq 0 \text{ and } \alpha(u^- - \alpha G_\alpha u^-, u^+)_{\mathcal{H}} \leq 0 .$$

Since αG_α is a contraction we have that

$$\alpha(u^- - \alpha G_\alpha u^-, u^-)_{\mathcal{H}} \geq 0\,.$$

Hence

$$\alpha(u^+ - \alpha G_\alpha u^+, u^+)_{\mathcal{H}} \leq \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}} \tag{4.3}$$

and therefore,

$$\begin{aligned}
\mathcal{A}(\alpha G_\alpha u^+, \alpha G_\alpha u^+) &\leq \mathcal{E}(\alpha G_\alpha u^+, \alpha G_\alpha u^+) \\
&= \alpha(u^+ - \alpha G_\alpha u^+, \alpha G_\alpha u^+)_{\mathcal{H}} \\
&\leq \alpha(u^+ - \alpha G_\alpha u^+, u^+)_{\mathcal{H}} \\
&\leq \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}} \\
&= \alpha(G_\alpha h - G_\alpha G_1 h, u)_{\mathcal{H}}\ ,
\end{aligned}$$

which implies

$$\sup_{\alpha>0} \mathcal{A}(\alpha G_\alpha u^+, \alpha G_\alpha u^+) < \infty\ .$$

Consequently, $u^+ \in \mathcal{V}$ by [MR1, Lemma I.2.12] and

$$\begin{aligned}
\mathcal{A}(u^+, u^+) &\leq \liminf_{\alpha\to\infty} \mathcal{A}(\alpha G_\alpha u^+, \alpha G_\alpha u^+) \\
&\leq \limsup_{\alpha\to\infty} \alpha(G_\alpha h - G_\alpha G_1 h, u)_{\mathcal{H}} \\
&= (h - G_1 h, u)_{\mathcal{H}} = \mathcal{E}(u,u) \\
&\leq 3K\|u\|_{\mathcal{F}}\|u\|_{\mathcal{V}}\,.
\end{aligned} \tag{4.4}$$

Moreover,

$$\mathcal{E}(u, u^+) = (h - G_1 h, u^+)_{\mathcal{H}} = \lim_{\alpha\to\infty} \alpha(u - \alpha G_\alpha u, u^+)_{\mathcal{H}} \geq 0\,. \tag{4.5}$$

For arbitrary $u \in \mathcal{F}$ let $(u_n)_{n\geq 1} \subset G_1(\mathcal{H})$ such that $u_n \to u$ in $\mathcal{F}$. Since $u_n^+ \to u^+$ in $\mathcal{H}$ and $\sup_{n\geq 1} \mathcal{A}(u_n^+, u_n^+) < \infty$ by (4.4) we obtain that $u^+ \in \mathcal{V}$ and $u_n^+ \to u^+$ weakly in $\mathcal{V}$ (cf. [MR1, Lemma I.2.12]). It follows from (4.5) that

$$\mathcal{E}(u, u^+) = \lim_{n\to\infty} \mathcal{E}(u_n, u_n^+) \geq 0\,.$$

(ii) $\Rightarrow$ (i) Let $u \in \mathcal{F}$. Then $u^- = (-u)^+$ implies $u^- \in \mathcal{V}$ and $\mathcal{E}(u, u^-) = -\mathcal{E}(-u, (-u)^+) \leq 0$. Let $f \geq 0$ and $u := G_\alpha f$. It follows

$$0 \geq \mathcal{E}(u, u^-) = (f - \alpha u, u^-)_{\mathcal{H}} \geq \alpha\|u^-\|_{\mathcal{H}}^2$$

and therefore, $u^- = 0$, i.e., $u \geq 0$.

(i) $\Rightarrow$ (iii) Let $u \in \widehat{\mathcal{F}}$. Since

$$\alpha(u^+ - \alpha G_\alpha u^+, u^+)_{\mathcal{H}} \leq \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}}$$

(cf. (4.3)) we obtain that

$$\begin{aligned}\mathcal{A}(\alpha G_\alpha u^+, \alpha G_\alpha u^+) &\le \mathcal{E}(\alpha G_\alpha u^+, \alpha G_\alpha u^+)\\ &\le \alpha(u - \alpha G_\alpha u, u)_{\mathcal{H}}\\ &= \mathcal{E}(\alpha G_\alpha u, u)\,.\end{aligned}$$

Since $u \in \widehat{\mathcal{F}}$ and $\lim_{\alpha\to\infty} \alpha G_\alpha u = u$ in $\mathcal{V}$ by Proposition 3.7 we obtain that

$$\limsup_{\alpha\to\infty} \mathcal{A}(\alpha G_\alpha u^+, \alpha G_\alpha u^+) = \mathcal{E}(u,u) < \infty$$

and consequently, $u^+ \in \mathcal{V}$. Moreover,

$$\mathcal{E}(u^+, u) = \lim_{\alpha\to\infty} \mathcal{E}(\alpha G_\alpha u^+, u) = \lim_{\alpha\to\infty} \alpha(u^+ - \alpha G_\alpha u^+, u)_{\mathcal{H}} \ge 0\,,$$

since $\alpha(u^+ - \alpha G_\alpha u^+, u)_{\mathcal{H}} = \alpha(u^+ - \alpha G_\alpha u^+, u^+)_{\mathcal{H}} + \alpha(\alpha G_\alpha u^+, u^-)_{\mathcal{H}} \ge 0$.

(iii) $\Rightarrow$ (i) Let $u \in \widehat{\mathcal{F}}$. Then $u^- = (-u)^+$ implies $u^- \in \mathcal{V}$ and $\mathcal{E}(u^-, u) \le 0$. Let $f \ge 0$ and $u := \widehat{G}_\alpha f$. It follows

$$0 \ge \mathcal{E}(u^-, u) = (f - \alpha u, u^-)_{\mathcal{H}} \ge \alpha \|u^-\|^2_{\mathcal{H}}$$

and therefore, $u^- = 0$, i.e., $u \ge 0$. Hence $(\widehat{G}_\alpha)_{\alpha>0}$ and consequently, $(G_\alpha)_{\alpha>0}$ is positivity preserving. □

Lemma 4.5. *Let $(\mathcal{A}, \mathcal{V})$ be positivity preserving and $(\Lambda u, u^+)_{\mathcal{H}} \le 0$ for all $u \in D(\Lambda, \mathcal{H})$. Then condition (ii) of Proposition 4.4 is satisfied.*

Proof. Since $(\mathcal{A}, \mathcal{V})$ is positivity preserving we obtain that $u^+, u^- \in \mathcal{V}$ if $u \in \mathcal{V}$ and $\mathcal{A}(u^+, u^-) \le 0$ (cf. [MR2, Proposition 1.3]) and therefore,

$$\mathcal{A}(u^+, u^+) \le \mathcal{A}(u^+, u)\ . \tag{4.6}$$

For $u \in \mathcal{F}$ let $(u_n)_{n\ge1} \subset D(\Lambda, \mathcal{V})$ with $u_n \to u$ in $\mathcal{F}$ (cf. Lemma 2.5). Since $u_n^+ \to u^+$ in $\mathcal{H}$ and $\sup_{n\ge1} \mathcal{A}(u_n^+, u_n^+) < \infty$ by (4.6) it follows that $u_n^+ \to u^+$ weakly in $\mathcal{V}$ and thus

$$\langle \Lambda u, u^+\rangle = \lim_{n\to\infty} \langle \Lambda u_n, u_n^+\rangle \le 0\ .$$

Hence $\mathcal{E}(u, u^+) = \mathcal{A}(u, u^+) - \langle \Lambda u, u^+\rangle \ge 0$. □

Proposition 4.6. *The resolvent $(G_\alpha)_{\alpha>0}$ associated with $\mathcal{E}$ is sub–Markovian if and only if*

D2 $\qquad u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V}$ *and* $\mathcal{E}(u, u - u^+ \wedge 1) \ge 0$.

Proof. First suppose that D2 is satisfied. Let $f \in \mathcal{H}$ with $0 \le f \le 1$ and $u := \alpha G_\alpha f$. Then

$$\begin{aligned}0 &\ge -\mathcal{E}(u, u - u^+ \wedge 1) = \alpha(u - f, u - u^+ \wedge 1)_{\mathcal{H}}\\ &= \alpha\|u - u^+ \wedge 1\|^2_{\mathcal{H}} + \alpha(u^+ \wedge 1 - f, u - u^+ \wedge 1)_{\mathcal{H}}\\ &\ge \alpha\|u - u^+ \wedge 1\|^2_{\mathcal{H}}\end{aligned}$$

since $(u^+\wedge 1-f, u-u^+\wedge 1)_{\mathcal{H}} = \int\limits_{\{u>1\}} (1-f)(u-1)dm - \int\limits_{\{u<0\}} fu\,dm \geq 0$. Therefore, $u = u^+ \wedge 1$ which implies $0 \leq u \leq 1$.

On the other hand let $(G_\alpha)_{\alpha>0}$ be sub–Markovian. For $g \in \mathcal{H}$ we have that

$$\begin{aligned}&\big(g^+ \wedge 1 - \alpha G_\alpha(g^+ \wedge 1), g - g^+ \wedge 1\big)_{\mathcal{H}} \\ &\quad = \int_{\{g>1\}} \big(1 - \alpha G_\alpha(g^+ \wedge 1)\big)(g-1)dm - \int_{\{g<0\}} \alpha G_\alpha(g^+ \wedge 1) g\, dm \geq 0 \ ,\end{aligned}$$

since $0 \leq \alpha G_\alpha(g^+ \wedge 1) \leq 1$. It follows

$$\begin{aligned}&\mathcal{A}(\alpha G_\alpha(g - g^+ \wedge 1), \alpha G_\alpha(g - g^+ \wedge 1)) \\ &\qquad \leq \mathcal{E}\left(\alpha G_\alpha\left(g - g^+ \wedge 1\right), \alpha G_\alpha\left(g - g^+ \wedge 1\right)\right) \\ &\qquad \leq \alpha\left(g - g^+ \wedge 1 - \alpha G_\alpha\left(g - g^+ \wedge 1\right), g - g^+ \wedge 1\right)_{\mathcal{H}} \\ &\qquad \leq \alpha\left(g - \alpha G_\alpha g, g - g^+ \wedge 1\right)_{\mathcal{H}} \ .\end{aligned} \tag{4.7}$$

Consequently, for $g = G_1 h$, $h \in \mathcal{H}$, by the resolvent equation

$$\begin{aligned}&\sup_{\alpha>0} \mathcal{A}\left(\alpha G_\alpha\left(g - g^+ \wedge 1\right), \alpha G_\alpha\left(g - g^+ \wedge 1\right)\right) \\ &\qquad \leq \sup_{\alpha>0}\left(\alpha G_\alpha h - \alpha G_\alpha g, g - g^+ \wedge 1\right)_{\mathcal{H}} < +\infty\end{aligned}$$

which implies $g^+ \wedge 1 \in \mathcal{V}$ by [MR1, Lemma I.2.12] and

$$\begin{aligned}\mathcal{A}\left(g - g^+ \wedge 1, g - g^+ \wedge 1\right) &\leq \liminf_{\alpha\to\infty} \mathcal{A}\left(\alpha G_\alpha(g - g^+ \wedge 1), \alpha G_\alpha(g - g^+ \wedge 1)\right) \\ &\leq \limsup_{\alpha\to\infty} \alpha\left(g - \alpha G_\alpha g, g - g^+ \wedge 1\right)_{\mathcal{H}} \\ &= \limsup_{\alpha\to\infty} \alpha(G_\alpha h - G_\alpha G_1 h, g - g^+ \wedge 1)_{\mathcal{H}} \\ &= (h - G_1 h, g - g^+ \wedge 1)_{\mathcal{H}} = \mathcal{E}(g, g - g^+ \wedge 1) \\ &\leq 3K\|g\|_{\mathcal{F}}\|g - g^+ \wedge 1\|_{\mathcal{V}} \ .\end{aligned} \tag{4.8}$$

It follows from (4.7) that

$$\begin{aligned}\mathcal{E}(g, g - g^+ \wedge 1) &= (h - G_1 h, g - g^+ \wedge 1)_{\mathcal{H}} \\ &= \lim_{\alpha\to\infty}\left(\alpha G_\alpha h - \alpha G_\alpha G_1 h, g - g^+ \wedge 1\right)_{\mathcal{H}} \\ &= \lim_{\alpha\to\infty} \alpha\left(g - \alpha G_\alpha g, g - g^+ \wedge 1\right)_{\mathcal{H}} \geq 0 \ .\end{aligned} \tag{4.9}$$

For general $g \in \mathcal{F}$ take a sequence $(g_n)_{n\geq 1} \subset G_1(\mathcal{H})$ such that $g_n \to g$ in $\mathcal{F}$. Since $g_n - g_n^+ \wedge 1 \to g - g^+ \wedge 1$ in $\mathcal{H}$ and $\sup\limits_{n\geq 1} \mathcal{A}(g_n - g_n^+ \wedge 1, g_n - g_n \wedge 1) < \infty$ by (4.8) we obtain that $g - g^+ \wedge 1 \in \mathcal{V}$ hence in particular, $g^+ \wedge 1 \in \mathcal{V}$ and $g_n - g_n^+ \wedge 1 \to g - g^+ \wedge 1$ weakly in $\mathcal{V}$ (cf. [MR1, Lemma I.2.12]). Now (4.9) implies that

$$\mathcal{E}(g, g - g^+ \wedge 1) = \lim_{n\to\infty} \mathcal{E}\left(g_n, g_n - g_n^+ \wedge 1\right) \geq 0 \ . \quad \square$$

The next proposition gives a sufficient criterion for D2 to be satisfied.

Proposition 4.7. *If $(\mathcal{A},\mathcal{V})$ is a Dirichlet form and $(\Lambda, D(\Lambda,\mathcal{H}))$ a Dirichlet operator then D2 is satisfied.*

Proof. Since $(\mathcal{A},\mathcal{V})$ is a Dirichlet form we obtain that $u^+\wedge 1 \in \mathcal{V}$ if $u \in \mathcal{V}$ and

$$\mathcal{A}\left(u^+\wedge 1, u - u^+\wedge 1\right) \geq 0 \tag{4.10}$$

(cf. [MR1, Theorem I.4.4]). Since $(\Lambda, D(\Lambda,\mathcal{H}))$ is a Dirichlet operator the corresponding semigroup $(U_t)_{t\geq 0}$ is sub–Markovian (cf. [MR1, Proposition I.4.3]) and therefore in particular positivity preserving (cf. Remark 4.2). Consequently, $(\Lambda u, u^+)_{\mathcal{H}} \leq 0$ (cf. [MR2, Theorem 1.7]) and thus $(\Lambda u, u^-)_{\mathcal{H}} = -(\Lambda(-u), (-u)^+)_{\mathcal{H}}$ ≥ 0 for all $u \in D(\Lambda,\mathcal{H})$. Hence we obtain that

$$-\left(\Lambda u, u - u^+\wedge 1\right)_{\mathcal{H}} = \left(\Lambda u, u^-\right)_{\mathcal{H}} - \left(\Lambda u, (u-1)^+\right)_{\mathcal{H}} \geq 0 \text{ for all } u \in D(\Lambda,\mathcal{H})\ .$$

For general $u \in \mathcal{F}$ take a sequence $(u_n)_{n\geq 1} \subset D(\Lambda,\mathcal{V})$ such that $u_n \to u$ in $\mathcal{F}$ (cf. Lemma 2.5). It follows $u_n^+\wedge 1 \to u^+\wedge 1$ weakly in $\mathcal{V}$ since $\sup_{n\geq 1}\mathcal{A}\left(u_n^+\wedge 1, u_n^+\wedge 1\right) < +\infty$ by (4.10) and $u_n^+\wedge 1 \to u^+\wedge 1$ in $\mathcal{H}$. Hence

$$-\langle \Lambda u, u - u^+\wedge 1\rangle = \lim_{n\to\infty} -\langle \Lambda u_n, u_n - u_n^+\wedge 1\rangle \geq 0$$

and therefore, $\mathcal{E}(u, u - u^+\wedge 1) \geq 0$. $\square$

Definition 4.8. The bilinear form $\mathcal{E}$ associated with $(\mathcal{A},\mathcal{V})$ and $(\Lambda, D(\Lambda,\mathcal{H}))$ is called a *generalized positivity preserving form* if one (hence all) of the three equivalent statements in Proposition 4.4 are satisfied and it is called a *generalized Dirichlet form* if D2 is satisfied.

We will give several examples of generalized Dirichlet forms in the next chapter. Before doing so we consider three basic examples of simple structure.

Examples 4.9.

(i) Let $(\mathcal{A},\mathcal{V})$ be a Dirichlet form and $\Lambda = 0$. Clearly $\mathcal{F} = \mathcal{V} = \widehat{\mathcal{F}}$. $\mathcal{E} = \mathcal{A}$ is a generalized Dirichlet form since the resolvent associated with $\mathcal{E}$ is sub–Markovian and therefore D2 satisfied by Proposition 4.6.

(ii) Let $\mathcal{A} = 0$ on $\mathcal{V} := \mathcal{H}$ and $(\Lambda, D(\Lambda))$ be a Dirichlet operator generating a C_0–semigroup of contractions on $\mathcal{H}$. In this case $\mathcal{F} = D(\Lambda), \widehat{\mathcal{F}} = D(\widehat{\Lambda})$ and the corresponding bilinear form $\mathcal{E}(u,v) = -(\Lambda u, v)_{\mathcal{H}}$ if $u \in D(\Lambda)$, $v \in \mathcal{H}$, and $\mathcal{E}(u,v) = -(u, \widehat{\Lambda} v)_{\mathcal{H}}$ if $u \in \mathcal{H}$, $v \in D(\widehat{\Lambda})$, is a generalized Dirichlet form.

(iii) Let $I \subset \mathbb{R}$ be an interval and $\left(\mathcal{E}^{(t)}, V\right)_{t\in I}$ be a family of Dirichlet forms with sector constant K independent of t and common domain V in some separable L^2–space $H = L^2(X,\mu)$ - where V is a Hilbert space with corresponding norm $\|\cdot\|_V$ - satisfying the following conditions:

(a) $t \mapsto \mathcal{E}^{(t)}(u,v)$ is $\mathcal{B}(I)$–measurable for all $u, v \in V$.

(b) There exists a constant c such that

$$c\|u\|_V^2 \le \mathcal{E}_1^{(t)}(u,u) \le c^{-1}\|u\|_V^2 \text{ for all } u \in V, t \in I\ .$$

For $u, v \in \overline{\mathcal{V}} := L^2(I; V, dt)$ let $\bar{\mathcal{A}}(u,v) := \int_I \mathcal{E}^{(t)}(u(t), v(t))\, dt$. Then $(\bar{\mathcal{A}}, \overline{\mathcal{V}})$ is a coercive closed form on $\overline{\mathcal{H}} := L^2(I; H, dt)$ with sector constant K.

For $h \in \overline{\mathcal{H}}$ and $t \ge 0$ let

$$\overline{U}_t h(s) := \begin{cases} h(s+t) & \text{if } s, s+t \in I \\ 0 & \text{otherwise.} \end{cases}$$

The family of operators $(\overline{U}_t)_{t\ge 0}$ defines a C_0–semigroup of contractions on $\overline{\mathcal{H}}$ such that both $(\overline{U}_t)_{t\ge 0}$ and the adjoint semigroup $(\widehat{\overline{U}}_t)_{t\ge 0}$ can be extended to C_0–semigroups of contractions on $\overline{\mathcal{V}}' = L^2(I; V', dt)$. Denote by $(\frac{d}{dt}, D(\frac{d}{dt}, \overline{\mathcal{V}}'))$ and $(\widehat{\frac{d}{dt}}, D(\widehat{\frac{d}{dt}}, \overline{\mathcal{V}}'))$ the generators corresponding to the extended semigroups.

If $I = [0, T]$ it can be shown that

$$D(\frac{d}{dt}, \overline{\mathcal{V}}') = \{u | u \in \overline{\mathcal{V}}', \frac{du}{dt} \in \overline{\mathcal{V}}', u(T) = 0\} \text{ and}$$

$$D(\widehat{\frac{d}{dt}}, \overline{\mathcal{V}}') = \{u | u \in \overline{\mathcal{V}}', \frac{du}{dt} \in \overline{\mathcal{V}}', u(0) = 0\}$$

(cf. [LMa, Subsection 3.4.3]), where $\frac{du}{dt}$ is to be understood in the sense of distributions taking their values in V' and the boundary conditions $u(T) = 0$ and $u(0) = 0$ are well–defined since the mapping $[0, T] \to V'$, $t \mapsto u(t)$, is continuous after a modification on a set of measure zero. Other choices for $(\overline{U}_t)_{t\ge 0}$ lead to time derivatives with different boundary conditions (cf. [LMa, Subsection 3.6.1]). Note that in the particular case $I = \mathbb{R}$ only one choice is possible. Moreover, $D(\widehat{\frac{d}{dt}}, \mathcal{V}') = D(\frac{d}{dt}, \mathcal{V}') = H^{1,2}(\mathbb{R}; V')$, where $H^{1,2}(\mathbb{R}; V') := \{u | u \in \mathcal{V}', \frac{du}{dt} \in \mathcal{V}'\}$, and $\widehat{\frac{d}{dt}} = -\frac{d}{dt}$.

Let $\overline{\mathcal{F}} := D(\frac{d}{dt}, \overline{\mathcal{V}}') \cap \overline{\mathcal{V}}$ and $\widehat{\overline{\mathcal{F}}} := D(\widehat{\frac{d}{dt}}, \overline{\mathcal{V}}') \cap \overline{\mathcal{V}}$. The time dependent Dirichlet form corresponding to $(\mathcal{E}^{(t)}, V)_{t\in I}$ is now given as follows:

$$\overline{\mathcal{E}}(u,v) := \begin{cases} \bar{\mathcal{A}}(u,v) - \langle \frac{du}{dt}, v \rangle & \text{if } u \in \overline{\mathcal{F}}, v \in \overline{\mathcal{V}} \\ \bar{\mathcal{A}}(u,v) + \langle \frac{dv}{dt}, u \rangle & \text{if } u \in \overline{\mathcal{V}}, v \in \widehat{\overline{\mathcal{F}}}\ , \end{cases}$$

(cf. [O1, (2.8)] for the case $I = \mathbb{R}$).

$\overline{\mathcal{E}}$ can be identified with a generalized Dirichlet form as follows. Let $\mathcal{H} := L^2(I \times X\,, dt \otimes d\mu)$ and denote by $T : \mathcal{H} \to \overline{\mathcal{H}}$, $h(t, x) \mapsto (t \mapsto h(t, \cdot))$, the unique isomorhphism between the two spaces (cf. [ReSi2, Theorem II.10]). $(\bar{\mathcal{A}}, \overline{\mathcal{V}})$

can then be identified with a Dirichlet form $(\mathcal{A}, \mathcal{V})$ given by $\mathcal{V} := T^{-1}(\overline{\mathcal{V}})$ and $\mathcal{A}(u,v) := \bar{\mathcal{A}}(T(u), T(v))$. Similarly, $(\overline{U}_t)_{t\geq 0}$ induces a C_0–semigroup $(U_t)_{t\geq 0}$ of contractions on $\mathcal{H}$ if we define $U_t := T^{-1}\overline{U}_t T$, $t \geq 0$. We denote the corresponding generator by $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t}, \mathcal{H}))$. It is easy to see that both $(U_t)_{t\geq 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$ and at the same time be extended to a C_0–semigroup on $\mathcal{V}'$. In particular, $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t}, \mathcal{H}))$ satisfies assumption D1. Since $(U_t)_{t\geq 0}$ is sub–Markovian the bilinear form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t}, \mathcal{H}))$ is a generalized Dirichlet form. In the following we shall always have in mind the generalized Dirichlet form $\mathcal{E}$ when speaking of the *time dependent Dirichlet form* corresponding to $(\mathcal{E}^{(t)}, V)_{t\in I}$.

Remark 4.10. Note that, given the generator $(L, D(L))$ of a Dirichlet form $(\mathcal{A}, \mathcal{V})$, there are two possible ways to construct an associated generalized Dirichlet form. Corresponding to Example 4.9 (i) we can take as our generalized Dirichlet form $\mathcal{E}$ the Dirichlet form $(\mathcal{A}, \mathcal{V})$ itself, thereby taking $\Lambda = 0$. Another possibility corresponds to Example 4.9 (ii). We can take as our generalized Dirichlet form, now denoted by $\mathcal{E}'$, the bilinear form associated with $\mathcal{A} = 0$ on $\mathcal{V} := \mathcal{H}$ and $(\Lambda, D(\Lambda, \mathcal{H})) = (L, D(L))$. Note that the domain of the generalized Dirichlet form $\mathcal{E}'$ is given by the set $D(L) \times \mathcal{H} \cup \mathcal{H} \times D(\widehat{L})$ whereas the domain of the generalized Dirichlet form $\mathcal{E}$ in the first case is given by $\mathcal{V} \times \mathcal{V}$. The $\mathcal{F}'$–norm in the second case, which is the graph norm corresponding to $(L, D(L))$, is stronger in general than the $\mathcal{F}$–norm in the first case. In some sense the generalized Dirichlet form in the first case is a first order object, whereas the generalized Dirichlet form in the second case is a second order object.

II. Examples

In this chapter we will discuss several examples for generalized Dirichlet forms. We will concentrate on examples which are not symmetric nor coercive Dirichlet forms nor semi–Dirichlet forms (cf. [FOT], [MR1], [MOvR]) but also not time dependent Dirichlet forms as treated in [O1]. In the first section we give some examples of Dirichlet forms with time dependent potentials, in the next section we consider first order perturbations of (time dependent) Dirichlet forms on $\mathbb{R}^d$. In the third section we consider some singular diffusion equations connected with finite dimensional continuous state branching processes with immigration, and in the fourth section we give some examples for time dependent Dirichlet forms in infinite dimensions. In the last section we consider two examples for transformations of generalized Dirichlet forms. The first one is the subordination of the perturbation Λ with vaguely continuous convolution semigroups and the second one is the transformation of generalized Dirichlet forms with excessive functions (h–transformation).

II.1 Time dependent potentials

Let $d \geq 1$, and $V : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $V \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d, dt \otimes dx)$, $V \geq 0$, be a time dependent potential. Let $(\mathcal{E}, C_0^\infty(\mathbb{R}^{d+1}))$ be the bilinear form

$$\begin{aligned} \text{(1.1)} \qquad \mathcal{E}(u,v) := &\int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle \, dx \, dt + \int_{\mathbb{R}} \int_{\mathbb{R}^d} uvV \, dx \, dt \\ &- \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\partial u}{\partial t} v \, dx \, dt \, ; \; u, v \in C_0^\infty(\mathbb{R}^{d+1}) \, , \end{aligned}$$

where ∇u means gradient w.r.t. x , i.e. $\nabla u(t,x) = (\frac{\partial u}{\partial x_1}(t,x), \dots, \frac{\partial u}{\partial x_d}(t,x))$. In this section we will construct a generalized Dirichlet form in $\mathcal{H} := L^2(\mathbb{R} \times \mathbb{R}^d, dt \otimes dx)$ extending the bilinear form $\mathcal{E}$. To this end define

$$\mathcal{A}^0(u,v) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle \, dx \, dt \, ; \; u, v \in C_0^\infty(\mathbb{R}^{d+1}) \; ,$$

and

$$\mathcal{A}(u,v) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle \nabla u, \nabla v \rangle \, dx \, dt + \int_{\mathbb{R}} \int_{\mathbb{R}^d} uvV \, dx \, dt \, ; \; u, v \in C_0^\infty(\mathbb{R}^{d+1}) \; .$$

It is easy to see that $(\mathcal{A}^0, C_0^\infty(\mathbb{R}^{d+1}))$ is closable in $\mathcal{H}$.

Lemma 1.1. *$(\mathcal{A}, C_0^\infty(\mathbb{R}^{d+1}))$ is closable in $\mathcal{H}$ and the closure $(\mathcal{A}, \mathcal{V})$ is a symmetric Dirichlet form.*

Proof. Let $(u_n)_{n\geq 1} \subset C_0^\infty(\mathbb{R}^{d+1})$ be an $\mathcal{A}$–Cauchy sequence converging to zero in $\mathcal{H}$. Since $\mathcal{A}^0(u_n - u_m, u_n - u_m) \leq \mathcal{A}(u_n - u_m, u_n - u_m)$ we conclude that $(u_n)_{n\geq 1}$ is $\mathcal{A}^0$–Cauchy and thus $\lim_{n\to\infty} \mathcal{A}^0(u_n, u_n) = 0$ by the closability of $(\mathcal{A}^0, C_0^\infty(\mathbb{R}^{d+1}))$.

Let $(u_{n_k})_{k\geq 1}$ be a subsequence with $u_{n_k} \to 0\,dt\otimes dx$ –a.e., hence $V dt\otimes dx$–a.e.. The Lemma of Fatou implies that

$$\mathcal{A}(u_n,u_n) \leq \mathcal{A}^0(u_n,u_n) + \liminf_{k\to\infty} \int_{\mathbb{R}}\int_{\mathbb{R}^d} (u_n - u_{n_k})^2\, V\, dx\, dt$$

for all n. Since the right hand side converges to zero as $n\to\infty$ we conclude that $\lim_{n\to\infty}\mathcal{A}(u_n,u_n) = 0$ which implies the closability of $(\mathcal{A}, C_0^\infty(\mathbb{R}^{d+1}))$. It is clear that the closure $(\mathcal{A},\mathcal{V})$ is a symmetric Dirichlet form. $\square$

For $u\in C_0^\infty(\mathbb{R}^{d+1})$ let $U_t u(s,x) := u(s+t,x)\,, t\geq 0$. It is easy to see that U_t induces a bounded linear operator on $\mathcal{H}$, again denoted by U_t, such that $(U_t)_{t\geq 0}$ is a sub–Markovian C_0–semigroup of contractions and $U_t(C_0^\infty(\mathbb{R}^{d+1}))\subset C_0^\infty(\mathbb{R}^{d+1})\,, t\geq 0$. The corresponding generator extends the operator $\frac{\partial u}{\partial t}\,, u\in C_0^\infty(\mathbb{R}^{d+1})$, and will be therefore denoted by $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t},\mathcal{H}))$.

Consider now the following assumption on V:

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{\partial u}{\partial t} u\, V\, dx\, dt \leq c\mathcal{A}_1(u,u) \text{ for all } u\in C_0^\infty(\mathbb{R}^{d+1}), \tag{1.2}$$

for some constant $c\geq 0$.

Lemma 1.2. *Let V satisfy (1.2). Then $\mathcal{V}$ is $\frac{\partial}{\partial t}$–admissible.*

Proof. Let $u\in C_0^\infty(\mathbb{R}^{d+1})$. Then the map $[0,\infty)\to\mathcal{V}, t\mapsto U_t u$, is strongly continuous. Indeed, let $(t_n)_{n\geq 1}\subset[0,T]$ with $\lim_{n\to\infty} t_n = t$, where $T>0$ fixed. Note that the set $\{y\in\mathbb{R}^{d+1} | U_t u(y)\neq 0 \text{ for some } t\in[0,T]\}$ is contained in a compact subset K of $\mathbb{R}^{d+1}$. Clearly

$$\sum_{i=1}^d \Big(\frac{\partial}{\partial x_i}(U_{t_n}u - U_t u)\Big)^2(y) + (U_{t_n}u - U_t u)^2(y)\, V(y)$$
$$\leq m(\||\nabla u|\|_\infty^2 \mathbb{1}_K(y) + \|u\|_\infty^2 \mathbb{1}_K(y) V(y))\,, t\in[0,T]\,,$$

for some constant m. Since the right hand side is integrable and the left hand side converges to zero pointwise, we obtain that $\lim_{n\to\infty}\mathcal{A}(U_{t_n}u - U_t u, U_{t_n}u - U_t u) = 0$ using Lebesgue's theorem. By the strong continuity it follows that the Bochner integral $\int_0^t U_s u\, ds$ exists in $\mathcal{V}$.

Since $\frac{\partial u}{\partial t}\in C_0^\infty(\mathbb{R}^{d+1})$ if $u\in C_0^\infty(\mathbb{R}^{d+1})$, we obtain that $U_t u - u = \int_0^t U_s \frac{\partial u}{\partial t}\, ds$ in $\mathcal{V}$, hence the map $[0,\infty)\to\mathcal{V}, t\mapsto U_t u$, is strongly differentiable with derivative $U_s\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} U_s u$.

Since

$$\mathcal{A}_1(\frac{\partial u}{\partial t}, u) = \int_{\mathbb{R}}\int_{\mathbb{R}^d} \frac{\partial u}{\partial t} u\, V\, dx\, dt \leq c\mathcal{A}_1(u,u) \text{ if } u\in C_0^\infty(\mathbb{R}^{d+1})\,,$$

it follows that

$$\begin{aligned}\mathcal{A}_1(U_tu,U_tu) &= \mathcal{A}_1(u,u) + 2\int_0^t \mathcal{A}_1(\frac{\partial}{\partial t}U_su,U_su)\,ds\\ &\le \mathcal{A}_1(u,u) + 2c\int_0^t \mathcal{A}_1(U_su,U_su)\,ds\,, t\ge 0\,,\end{aligned}$$

and therefore $\mathcal{A}_1(U_tu,U_tu) \le e^{2ct}\mathcal{A}_1(u,u)$, using Gronwall's inequality. Hence U_t can be restricted to a bounded operator on $\mathcal{V}$ with $\|U_t\|_{L(\mathcal{V})} \le e^{ct}$, $t\ge 0$. By Lemma I.2.2 $(U_t)_{t\ge 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$. $\square$

Let V satisfy (1.2). The bilinear form associated with $(\mathcal{A},\mathcal{V})$ and $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t},\mathcal{H}))$ extends the bilinear form $\mathcal{E}$ (cf. (1.1)), and is a generalized Dirichlet form by Proposition I.4.7 since $(U_t)_{t\ge 0}$ is sub–Markovian, hence $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t},\mathcal{H}))$ a Dirichlet operator, and $(\mathcal{A},\mathcal{V})$ a Dirichlet form.

Remark 1.3.

A sufficient criterion for (1.2) is given as follows. Let $d\ge 3$ and suppose that $\frac{\partial V}{\partial t}\in L^1_{\text{loc}}(\mathbb{R}\times\mathbb{R}^d, dt\otimes dx)$, where $\frac{\partial}{\partial t}$ is to be understood in the distributional sense. Moreover, assume that $\frac{\partial V}{\partial t}(t,\cdot)\in L^{d/2}(\mathbb{R}^d,dx)\cup L^\infty(\mathbb{R}^d,dx)$ and for some constant $L_0\ge 0$

$$\min\{\|\frac{\partial V}{\partial t}(t,\cdot)\|_{d/2}, \|\frac{\partial V}{\partial t}(t,\cdot)\|_\infty\} \le L_0 \tag{1.3}$$

for all $t\in\mathbb{R}$. Under these assumptions (1.2) holds. Indeed, for $u\in C_0^\infty(\mathbb{R}^{d+1})$ it follows that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d}\frac{\partial u}{\partial t}uV\,dx\,dt = -\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}u^2\frac{\partial V}{\partial t}\,dx\,dt\ .$$

Hence, using Sobolev's inequality, i.e.,

$$\|u\|_q \le \sqrt{2}\lambda(\int_{\mathbb{R}^d}|\nabla u|^2\,dx)^{1/2} \text{ for all } u\in C_0^\infty(\mathbb{R}^d)\,, \tag{1.4}$$

where $\frac{1}{q}+\frac{1}{d}=\frac{1}{2}$ and $\lambda = \frac{2(d-1)}{(d-2)d^{1/2}}$ (cf. [Da2, Theorem 1.7.1]), we obtain that

$$\begin{aligned}\int_{\mathbb{R}^d}u^2(t,x)\frac{\partial V}{\partial t}(t,x)\,dx &\le \|\frac{\partial V}{\partial t}(t,\cdot)\|_{d/2}\|u(t,\cdot)\|_q^2\\ &\le 2\lambda^2L_0\int_{\mathbb{R}^d}|\nabla u|^2(t,x)\,dx\end{aligned}$$

if $t\in\mathbb{R}$ is such that $\|\frac{\partial V}{\partial t}(t,\cdot)\|_{L^{d/2}(\mathbb{R}^d,dx)}\le L_0$ and for all other t

$$\int_{\mathbb{R}^d}u^2(t,x)\frac{\partial V}{\partial t}(t,x)\,dx \le L_0\int_{\mathbb{R}^d}u^2(t,x)\,dx\,.$$

Consequently,

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} u^2 \frac{\partial V}{\partial t}\, dx\, dt \le L_0(\lambda^2+1)\mathcal{A}_1(u,u) \text{ for all } u \in C_0^\infty(\mathbb{R}^{d+1})\,,$$

which implies the assertion.

II.2 First order perturbations of Dirichlet forms and time dependent Dirichlet forms on $\mathbb{R}^d$

(a) A result from the theory of ordinary differential equations

Let $B = (b_1, \dots, b_d) : \mathbb{R}^d \to \mathbb{R}^d$ be a smooth vector field. Suppose that for some continuous, monotone increasing function $f : [0,\infty) \to [1,\infty)$ with $\int_0^\infty \frac{dr}{f(r)} = \infty$ we have that

$$(2.1) \qquad |B(x)| \le f(|x|)\ , x \in \mathbb{R}^d\ .$$

By [Am, Theorem 7.6] there exists for all $x \in \mathbb{R}^d$ a unique maximal solution $\xi_x : (t^-,t^+) \to \mathbb{R}^d$ of the initial value problem $y' = B(y), y(0) = x$. Let $F(t) := \int_0^t \frac{dr}{f(r)}$, $t \ge 0$. Using (2.1) we obtain that $|\xi_x(s)| \le |x| + |\int_0^s f(|\xi_x(r)|)\, dr|$ for all $s \in (t^-,t^+)$ and therefore,

$$(2.2) \qquad F(|\xi_x(s)|) \le F(|x| + |\int_0^s f(|\xi_x(r)|)\, dr|) \le F(|x|) + |s|$$

for all $s \in (t^-,t^+)$. Let $T > 0$ be arbitrary. By (2.2) $\{\xi_x(s) | s \in (t^- \vee (-T), 0]\}$ and $\{\xi_x(s) | s \in [0, t^+ \wedge T)\}$ are contained in a compact subset of $\mathbb{R}^d$. By [Am, Corollary 7.7] we conclude that $t^- \le -T$ and $t^+ \ge T$, hence $t^- = -\infty$ and $t^+ = +\infty$ since T was arbitrary. Consequently, ξ_x is a global solution to the corresponding initial value problem. The map $\xi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d\,, (t,x) \mapsto \xi_x(t)$, is a smooth map with $\xi(0,\cdot) = \mathrm{id}_{\mathbb{R}^d}$, and $\xi(s,\xi(t,x)) = \xi(s+t,x)$ for all $x \in \mathbb{R}^d\,, s,t \in \mathbb{R}$ (cf. [Am, Theorem 10.3]). In particular, $\xi(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is a C^∞–diffeomorphism of $\mathbb{R}^d$ for all $t \in \mathbb{R}$ and the Jacobian matrix $D\xi(t,\cdot)(x)$ is the unique solution to the linear initial value problem

$$(2.3) \qquad y' = DB\, y,\ \ y(0) = \mathrm{id}_{\mathbb{R}^d}\ ,$$

where $DB(x)$ denotes the Jacobi matrix of the vector field B at the point x (cf. [Am, Theorem 9.2]). For the Jacobian determinant we have the formula

$$(2.4) \qquad \det D\xi(t,\cdot)(x) = \exp(\int_0^t \sum_{i=1}^d \frac{\partial b_i}{\partial x_i}(\xi(s,x))\, ds)\,, t \ge 0,$$

(cf. [CoLe, Theorem 1.7.2]).

(b) First order perturbations of Dirichlet forms on $\mathbb{R}^d$

Let $d \geq 1$ and $b_i \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$, $1 \leq i \leq d$. Consider the bilinear form

$$\mathcal{E}(u,v) := \sum_{i=1}^{d} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx - \sum_{i=1}^{d} \int_{\mathbb{R}^d} b_i \frac{\partial u}{\partial x_i} v\, dx\,;\, u, v \in C_0^\infty(\mathbb{R}^d)\ .$$

Let $B := (b_1, \dots, b_d)$ and assume that there exist constants $c, L \geq 0$ such that

$$\int_{\mathbb{R}^d} \langle B, \nabla u\rangle\, dx \leq 2c\|u\|_1 \text{ for all } u \in C_0^\infty(\mathbb{R}^d), u \geq 0, \tag{2.5}$$

$$-\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} b_i \frac{\partial u}{\partial x_j}\, dx\, h_i h_j \leq L\|u\|_1 |h|^2, \tag{2.6}$$

$$\text{for all } u \in C_0^\infty(\mathbb{R}^d), u \geq 0, h \in \mathbb{R}^d, \text{ and}$$

for some continuous, monotone increasing function $f : [0,\infty) \to [1,\infty)$ with $\int_0^\infty \frac{dr}{f(r)} = \infty$ we have that

$$|B(x)| \leq f(|x|)\ , x \in \mathbb{R}^d\ . \tag{2.7}$$

Clearly $(\mathcal{E}_c, C_0^\infty(\mathbb{R}^d))$ is a positive definite bilinear form by (2.5). Note that $(\mathcal{E}_c, C_0^\infty(\mathbb{R}^d))$ does not satisfy the weak sector condition in $\mathcal{H} := L^2(\mathbb{R}^d, dx)$ since no integrability condition on B is assumed. However, we will construct in this subsection a generalized Dirichlet form in $\mathcal{H}$ extending the bilinear form $\mathcal{E}_c$.

Remark 2.1. (i) Assumptions (2.5)–(2.7) are fulfilled in the case where B is Lipschitz continuous. For if

$$|B(x) - B(y)| \leq L|x - y| \text{ for all } x, y \in \mathbb{R}^d,$$

for some constant L we obtain for $u \in C_0^\infty(\mathbb{R}^d)$

$$\left|\int b_i(x) \frac{\partial u}{\partial x_j}(x)\, dx\right| \leq L\|u\|_1\, , 1 \leq i, j \leq d\ .$$

Consequently,

$$\int_{\mathbb{R}^d} \langle B, \nabla u\rangle\, dx \leq dL\|u\|_1\ ,$$

and for $h \in \mathbb{R}^d$

$$\begin{aligned}-\sum_{i,j=1}^{d} \int_{\mathbb{R}^d} b_i(x) \frac{\partial u}{\partial x_j}(x)\, dx\, h_i h_j &\leq L\|u\|_1 \sum_{i,j=1}^{d} |h_i h_j| \\ &\leq dL\|u\|_1 |h|^2\ .\end{aligned}$$

(ii) If B is continuously differentiable then (2.6) is equivalent to the fact that the matrix $(\frac{\partial b_i}{\partial x_j}(x) - L\delta_{ij})_{1\le i,j\le d}$ is negative definite for all $x \in \mathbb{R}^d$. Consequently, if B is of the form $\nabla\varphi$ for some function $\varphi : \mathbb{R}^d \to \mathbb{R}$ which is two times differentiable we obtain that (2.6) is satisfied if and only if the function $\frac{L}{2}|x|^2 - \varphi(x) : \mathbb{R}^d \to \mathbb{R}$ is convex.

Let $(D, H^{1,2}(\mathbb{R}^d))$ be the symmetric Dirichlet form

$$D(u,v) := \int_{\mathbb{R}^d} \langle \nabla u, \nabla v\rangle \, dx; u, v \in H^{1,2}(\mathbb{R}^d) \ ,$$

and $\Lambda u := \langle B, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$. Note that $(\Lambda, C_0^\infty(\mathbb{R}^d))$ is a densely defined operator in $\mathcal{H}$ which is bounded from above by (2.5).

Proposition 2.2. *Assume that B satisfies (2.5)–(2.7). Then $(\Lambda, C_0^\infty(\mathbb{R}^d))$ is closable and the closure $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a sub–Markovian C_0–semigroup $(U_t)_{t\ge 0}$ with $\|U_t\|_{L(\mathcal{H})} \le \exp(ct)$, $t \ge 0$. $(U_t)_{t\ge 0}$ can be restricted to a C_0–semigroup on $H^{1,2}(\mathbb{R}^d)$ with $\|U_t\|_{L(H^{1,2}(\mathbb{R}^d))} \le \exp((c+L)t)$, $t \ge 0$.*

For the proof of the proposition we will need two lemmas. Assume first that B is a smooth vector field satisfying only (2.5) and (2.7). Denote by ξ the corresponding flow on $\mathbb{R}^d$ (cf. Subsection (a)). Let

$$U_t f := f(\xi(t,\cdot)),\ f \in \mathcal{B}_b(\mathbb{R}^d) \cap \mathcal{H}\,, t \ge 0 \ .$$

Lemma 2.3. *U_t induces a bounded linear operator on $\mathcal{H}$, again denoted by U_t, such that $\|U_t\|_{L(\mathcal{H})} \le e^{ct}$, $t \ge 0$. The family $(U_t)_{t\ge 0}$ is a sub–Markovian C_0–semigroup on $\mathcal{H}$ with $U_t(C_0^\infty(\mathbb{R}^d)) \subset C_0^\infty(\mathbb{R}^d)$, $t \ge 0$. The corresponding generator extends $(\Lambda, C_0^\infty(\mathbb{R}^d))$.*

Proof. Since $\sum_{i=1}^d \frac{\partial b_i}{\partial x_i}$ is continuous it is easy to see that (2.5) implies that $\sum_{i=1}^d \frac{\partial b_i}{\partial x_i} \ge -2c$. Using (2.4) we conclude that

$$\det D\xi(t,\cdot)(x) = \exp(\int_0^t \sum_{i=1}^d \frac{\partial b_i}{\partial x_i}(\xi(s,x))\, ds) \ge \exp(-2ct) \ , \tag{2.8}$$

and consequently, for $f \in \mathcal{B}_b(\mathbb{R}^d) \cap \mathcal{H}$ by the change of variables formula for the Lebesgue measure

$$\begin{aligned}\int_{\mathbb{R}^d} f^2(\xi(t,x))\, dx &\le \exp(2ct) \int_{\mathbb{R}^d} f^2(\xi(t,x)) \det D\xi(t,\cdot)(x)\, dx \\ &= \exp(2ct)\|f\|_{\mathcal{H}}^2 \ ,\end{aligned} \tag{2.9}$$

which implies that U_t induces a bounded linear operator on $\mathcal{H}$, again denoted by U_t, such that $\|U_t\|_{L(\mathcal{H})} \le e^{ct}$ and $U_t(C_0^\infty(\mathbb{R}^d)) \subset C_0^\infty(\mathbb{R}^d)$, $t \ge 0$. The family $(U_t)_{t\ge 0}$ is a sub–Markovian semigroup on $\mathcal{H}$. Note that for $f \in C_0(\mathbb{R}^d)$, $f \geq 0$,

$$\begin{aligned}\int_{\mathbb{R}^d}(U_tf-f)^2\,dx &\leq (e^{2ct}+1)\int_{\mathbb{R}^d} f^2\,dx - 2\int_{\mathbb{R}^d} fU_tf\,dx\\ &= (e^{2ct}+1)\int_{\mathbb{R}^d} f^2\,dx - 2\int_{\mathbb{R}^d} f(U_tf-f)^+\,dx\\ &\qquad -2\int_{\mathbb{R}^d} f(f\wedge U_tf)\,dx\\ &\leq (e^{2ct}+1)\int_{\mathbb{R}^d} f^2\,dx - 2\int_{\mathbb{R}^d} f(f\wedge U_tf)\,dx\end{aligned}$$

and the right hand side converges to zero as $t \to 0$ using Lebesgue's theorem. It follows that $\lim_{t\downarrow 0} U_tf = f$ in $\mathcal{H}$ for all $f \in C_0(\mathbb{R}^d)$ and hence for all $f \in \mathcal{H}$, which implies the strong continuity of $(U_t)_{t\geq 0}$. To prove that the corresponding generator extends $(\Lambda, C_0^\infty(\mathbb{R}^d))$ let $u \in C_0^\infty(\mathbb{R}^d)$ and note that

$$\begin{aligned}U_tu(x)-u(x) &= u(\xi(t,x))-u(x)\\ &= \int_0^t \langle \nabla u(\xi(s,x)), B(\xi(s,x))\rangle\,ds\\ &= \int_0^t U_s(\langle B,\nabla u\rangle)(x)\,ds \text{ for all } x\in\mathbb{R}^d\,, t\geq 0\,.\end{aligned}$$

Consequently, $\lim_{t\downarrow 0}\frac{1}{t}(U_tu-u) = \langle B,\nabla u\rangle$ in $\mathcal{H}$ by the strong continuity of $(U_t)_{t\geq 0}$. □

If in addition to (2.5) and (2.7) B satisfies (2.6), it is easy to see that (2.6) implies

$$\langle DB(x)h,h\rangle \leq L|h|^2\,, h\in\mathbb{R}^d\,, x\in\mathbb{R}^d\ .$$

Since

$$D\xi(t,\cdot)(x) = \mathrm{id}_{\mathbb{R}^d} + \int_0^t DB(\xi(s,x))D\xi(s,\cdot)(x)\,ds$$

we conclude that

$$\begin{aligned}|D\xi(t,\cdot)(x)h|^2 &= |h|^2 + 2\int_0^t \langle DB(\xi(s,x))D\xi(s,\cdot)(x)h, D\xi(s,\cdot)(x)h\rangle\,ds\\ &\leq |h|^2 + 2L\int_0^t |D\xi(s,\cdot)(x)h|^2\,ds\,, h\in\mathbb{R}^d\,, x\in\mathbb{R}^d\ ,\end{aligned}$$

for $t \geq 0$ and consequently,

$$|D\xi(t,\cdot)(x)h|^2 \leq \exp(2Lt)|h|^2 \tag{2.10}$$

by Gronwall's inequality. (2.10) implies that for $u \in C_0^\infty(\mathbb{R}^d)$

$$|\nabla U_tu(x)| \leq \exp(Lt)|\nabla u(\xi(t,x))|\,, x\in\mathbb{R}^d\ , \tag{2.11}$$

and therefore, using (2.9), we see that

$$\int_{\mathbb{R}^d} |\nabla U_tu|^2\,dx \leq \exp(2(L+c)t)\,D(u,u)\,, u\in C_0^\infty(\mathbb{R}^d)\ . \tag{2.12}$$

Lemma 2.4. *The C_0–semigroup $(U_t)_{t\geq 0}$ can be restricted to a C_0–semigroup on $H^{1,2}(\mathbb{R}^d)$ with $\|U_t\|_{L(H^{1,2}(\mathbb{R}^d))} \leq \exp((L+c)t)$, $t \geq 0$.*

Proof. By (2.12) U_t can be restricted to a bounded operator on $H^{1,2}(\mathbb{R}^d)$ with $\|U_t\|_{L(H^{1,2}(\mathbb{R}^d))} \leq \exp((L+c)t)$, $t \geq 0$. The assertion now follows from Lemma I.2.2. $\square$

Proof of Proposition 2.2.

Existence: By (2.5) the operator $(\Lambda - c, C_0^\infty(\mathbb{R}^d))$ is negative definite on $\mathcal{H}$. Hence there exists at least one closed extension $(\Lambda - c, D(\Lambda))$ of $(\Lambda - c, C_0^\infty(\mathbb{R}^d))$ on $\mathcal{H}$ generating a C_0–semigroup (cf. [P], [Da1, Section 6.1]). Then $(\Lambda, D(\Lambda))$ is an extension of $(\Lambda, C_0^\infty(\mathbb{R}^d))$ generating a C_0–semigroup.

Uniqueness: To prove uniqueness let $(\Lambda, D(\Lambda,\mathcal{H}))$ be an arbitrary extension of $(\Lambda, C_0^\infty(\mathbb{R}^d))$ generating a C_0–semigroup $(U_t)_{t\geq 0}$. We approximate B with the help of smooth vector fields $B^\varepsilon = (b_1^\varepsilon, \dots, b_d^\varepsilon)$, $\varepsilon > 0$, which are given as the convolution $B * \delta_\varepsilon$ of B with δ_ε, where $\delta_\varepsilon(x) := \prod_{i=1}^d \psi_\varepsilon(x_i)$, $\psi_\varepsilon(t) := (1/\varepsilon)\psi(t/\varepsilon)$ for some $\psi \in C_0^\infty((-1,1))$, $\psi \geq 0$, with $\int_{\mathbb{R}} \psi\, dt = 1$. It is easy to see that B^ε, $\varepsilon > 0$, satisfies (2.5) and (2.6) with the same constants c and L. By (2.7) we have that

$$\begin{aligned} |B^\varepsilon(x)| &\leq \int_{\mathbb{R}^d} |B(x-y)|\delta_\varepsilon(y)\, dy \\ &\leq \int_{\mathbb{R}^d} f(|x-y|)\delta_\varepsilon(y)\, dy \\ &\leq f(|x|+\varepsilon)\,, x \in \mathbb{R}^d\ , \end{aligned} \tag{2.13}$$

which implies that there exists a (global) smooth flow ξ^ε corresponding to B^ε (cf. Subsection 2.(a)). Let

$$U_t^\varepsilon f := f(\xi^\varepsilon(t,\cdot))\,, f \in \mathcal{B}_b(\mathbb{R}^d) \cap \mathcal{H}\,, t \geq 0\ .$$

By Lemma 2.3 U_t^ε induces a bounded linear operator U_t^ε on $\mathcal{H}$ such that $(U_t^\varepsilon)_{t\geq 0}$ is a sub–Markovian C_0–semigroup on $\mathcal{H}$ with $\|U_t^\varepsilon\|_{L(\mathcal{H})} \leq e^{ct}$, $t \geq 0$. Denote the corresponding generator, which is an extension of $\Lambda^\varepsilon u = \langle B^\varepsilon, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, by $(\Lambda^\varepsilon, D(\Lambda^\varepsilon,\mathcal{H}))$. By Lemma 2.4 $(U_t^\varepsilon)_{t\geq 0}$ can be restricted to a C_0–semigroup on $H^{1,2}(\mathbb{R}^d)$ with $\|U_t^\varepsilon\|_{L(H^{1,2}(\mathbb{R}^d)} \leq \exp((L+c)t)$, $t \geq 0$. Moreover, $U_t^\varepsilon(C_0^\infty(\mathbb{R}^d)) \subset C_0^\infty(\mathbb{R}^d)$ and for $u \in C_0^\infty(\mathbb{R}^d)$

$$|\nabla U_t^\varepsilon u(x)| \leq e^{Lt}|\nabla u(\xi^\varepsilon(t,x))|\,, x \in \mathbb{R}^d\,, t \geq 0, \tag{2.14}$$

(cf. (2.11)). Fix $T \geq 0$ and $u \in C_0^\infty(\mathbb{R}^d)$ with $\mathrm{supp}\,(u) \subset B_R(0)$ for some $R > 0$. Let $F(t) := \int_0^t \frac{dr}{f(r)}$, $t \geq 0$. Since by (2.2) and (2.13)

$$F(|\xi^\varepsilon(-t,x)|) \leq F(|x|+\varepsilon) + t\,, t \geq 0, x \in \mathbb{R}^d, \varepsilon > 0\,,$$

it follows that

$$F(|x|) = F(|\xi^\varepsilon(-t,\xi^\varepsilon(t,x))|) \leq F(|\xi^\varepsilon(t,x)|+\varepsilon) + t\ .$$

Since $\lim_{t\to\infty} F(t) = \infty$ there exist $R' \geq 0$ such that $F(R') = F(R+1) + T$. Then $|x| \geq R'$ implies that $F(|\xi^\varepsilon(t,x)| + \varepsilon) \geq F(R+1)$ if $t \in [0,T]$. Consequently, $|\xi^\varepsilon(t,x)| + \varepsilon \geq R+1$ and $U_t^\varepsilon u(x) = 0$ if $t \in [0,T], \varepsilon \in (0,1)$. Therefore, the set

$$\{x | \exists t \in [0,T]\,, \exists \varepsilon \in (0,1) \text{ such that } U_t^\varepsilon u(x) \neq 0\}$$

is contained in a compact subset K of $\mathbb{R}^d$.

Since $b_i \in L^2_{\mathrm{loc}}(\mathbb{R}^d, dx)$, $1 \leq i \leq d$, we obtain that $|B^\varepsilon - B|\mathbb{1}_K \to 0$ in $\mathcal{H}$ if $\varepsilon \to 0$ and by (2.14) it follows that

$$\begin{aligned}
\lim_{\varepsilon \downarrow 0} \|U_T u - U_T^\varepsilon u\|_{\mathcal{H}} &\leq \limsup_{\varepsilon \downarrow 0} \int_0^T \|U_{T-t}(\Lambda^\varepsilon - \Lambda) U_t^\varepsilon u\|_{\mathcal{H}}\, dt \\
&\leq \limsup_{\varepsilon \downarrow 0} \int_0^T \|U_{T-t}\|_{L(\mathcal{H})} \||B^\varepsilon - B|\mathbb{1}_K\|_{\mathcal{H}} \exp(Lt) \||\nabla u|\|_\infty\, dt \\
&= 0\,,
\end{aligned}$$

because $\|U_t\|_{L(\mathcal{H})}$ is bounded on compact sets by the strong continuity of $(U_t)_{t\geq 0}$.

Thus $U_t^\varepsilon u \to U_t u$ for all $u \in C_0^\infty(\mathbb{R}^d)$ and hence, using a 3–ε–argument, $U_t^\varepsilon f \to U_t f$ for all $f \in \mathcal{H}$. In particular, $(U_t)_{t\geq 0}$ is a sub–Markovian C_0–semigroup with $\|U_t\|_{L(\mathcal{H})} \leq \exp(ct)$, $t \geq 0$.

Consequently, $(\Lambda, D(\Lambda, \mathcal{H}))$ is the only closed extension of $(\Lambda, C_0^\infty(\mathbb{R}^d))$ generating a C_0–semigroup which implies $C_0^\infty(\mathbb{R}^d) \subset D(\Lambda, \mathcal{H})$ dense w.r.t. the graph norm (cf. [Na, Theorem A-II, 1.33]).

Since $U_t^\varepsilon f \in H^{1,2}(\mathbb{R}^d)$ if $f \in H^{1,2}(\mathbb{R}^d)$, $\varepsilon > 0$, and $\sup_{\varepsilon > 0} \|U_t^\varepsilon f\|_{H^{1,2}(\mathbb{R}^d)} < \infty$ by Lemma 2.4 it follows from [MR1, Lemma I.2.12] that $U_t f \in H^{1,2}(\mathbb{R}^d)$ and $\|U_t f\|_{H^{1,2}(\mathbb{R}^d)} \leq \exp((L+c)t)\|f\|_{H^{1,2}(\mathbb{R}^d)}$, $t \geq 0$. By Lemma I.2.2 $(U_t)_{t\geq 0}$ can be restricted to a C_0–semigroup on $H^{1,2}(\mathbb{R}^d)$. $\square$

Remark 2.5. In general the adjoint semigroup $(\widehat{U}_t)_{t\geq 0}$ of the semigroup $(U_t)_{t\geq 0}$ corresponding to the closure of $(\Lambda, C_0^\infty(\mathbb{R}^d))$ cannot be restricted to a C_0–semigroup on $H^{1,2}(\mathbb{R}^d)$. Hence our framework is more general than the framework considered in [St2] and [St3]. The reason why $(\widehat{U}_t)_{t\geq 0}$ cannot be restricted to $H^{1,2}(\mathbb{R}^d)$ in general is given as follows. Assume that B satisfies (2.5) - (2.7) and is continuously differentiable. Denote by ξ the corresponding flow of continuously differentiable diffeomorphisms of $\mathbb{R}^d$ (cf. [Am, Theorem 10.3]). Then $U_t f = f(\xi(t,\cdot))$, $f \in \mathcal{B}_b(\mathbb{R}^d) \cap \mathcal{H}$, and the adjoint semigroup $(\widehat{U}_t)_{t\geq 0}$ is given by
$\widehat{U}_t f = \exp(-\int_{-t}^0 \sum_{i=1}^d \frac{\partial b_i}{\partial x_i}(\xi(s,\cdot))\, ds) f(\xi(-t,\cdot))$, $f \in \mathcal{B}_b(\mathbb{R}^d) \cap \mathcal{H}$, since

$$\begin{aligned}
\int_{\mathbb{R}^d} \widehat{U}_t f(x) g(x)\, dx &= \int_{\mathbb{R}^d} f(\xi(-t,x)) g(\xi(t,\xi(-t,x))) \det D\xi(-t,\cdot)(x)\, dx \\
&= \int_{\mathbb{R}^d} f(x) g(\xi(t,x))\, dx \text{ for all } f,g \in \mathcal{B}_b(\mathbb{R}^d) \cap \mathcal{H}\ ,
\end{aligned}$$

using (2.3) and the change of variables formula for the Lebesgue measure. If $\widehat{U}_t$ can be restricted to a bounded operator on $H^{1,2}(\mathbb{R}^d)$ it follows that $\exp(-\int_{-t}^0 \sum_{i=1}^d \frac{\partial b_i}{\partial x_i}(\xi(s,\cdot))\,ds) \in H^{1,2}_{\mathrm{loc}}(\mathbb{R}^d)$, i.e., $\exp(-\int_{-t}^0 \sum_{i=1}^d \frac{\partial b_i}{\partial x_i}(\xi(s,\cdot))\,ds) \in H^{1,2}(U)$ for all $U \subset \mathbb{R}^d$ open and relatively compact.
As a typical example consider the vector field $B(x) = x^+$ on $\mathbb{R}^1$ which is, of course, only weakly differentiable. In this case $U_t f(x) = f(e^t x)$, $x \geq 0$, and $U_t f(x) = f(x)$, $x < 0$. Hence $\widehat{U}_t f(x) = e^{-t} f(e^{-t}x)$, $x \geq 0$, and $\widehat{U}_t f(x) = f(x)$, $x < 0$. It follows for $f \in C_0^\infty(\mathbb{R}^1)$ with $f(0) \neq 0$ that $\widehat{U}_t f \notin H^{1,2}(\mathbb{R}^1)$ since $\widehat{U}_t f$ has no continuous dx–version.

Let $B := (b_1, \ldots, b_d)$ be a vector field satisfying (2.5)–(2.7). Denote by $(\Lambda, D(\Lambda, \mathcal{H}))$ the closure of $\Lambda u := \langle B, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d)$, in $\mathcal{H}$ and by $(U_t)_{t\geq 0}$ the corresponding semigroup. By Proposition 2.2 we obtain that $H^{1,2}(\mathbb{R}^d)$ is Λ–admissible. Since $(U_t)_{t\geq 0}$ is sub–Markovian the semigroup corresponding to $(\Lambda - c, D(\Lambda, \mathcal{H}))$ is sub–Markovian too and therefore, $(\Lambda - c, D(\Lambda, \mathcal{H}))$ a Dirichlet operator (cf. [MR1, Proposition I.4.3]). Since $(D, H^{1,2}(\mathbb{R}^d))$ is a Dirichlet form Propositon I.4.7 now implies that the bilinear form $\mathcal{E}$ associated with $(D, H^{1,2}(\mathbb{R}^d))$ and $(\Lambda - c, D(\Lambda, \mathcal{H}))$ is a generalized Dirichlet form.

We can replace now $(D, H^{1,2}(\mathbb{R}^d))$ by any other Dirichlet form $(\mathcal{A}, H^{1,2}(\mathbb{R}^d))$ provided that the $\mathcal{A}_1$–norm is equivalent to the D_1–norm. In particular, if $d \geq 3$ we may consider the bilinear form

$$
\begin{aligned}
\mathcal{E}(u,v) = &\sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\,dx + \sum_{i=1}^d \int_{\mathbb{R}^d} b_i^1 \frac{\partial u}{\partial x_i} v\,dx \\
&+ \sum_{i=1}^d \int_{\mathbb{R}^d} d_i \frac{\partial v}{\partial x_i} u\,dx + \int_{\mathbb{R}^d} uvc^1\,dx \\
&- \sum_{i=1}^d \int_{\mathbb{R}^d} b_i \frac{\partial u}{\partial x_i} v\,dx + c\int_{\mathbb{R}^d} uv\,dx \ ; u,v \in C_0^\infty(\mathbb{R}^d)\ ,
\end{aligned} \tag{2.15}
$$

where $a_{ij}, b_i^1, d_i, c^1 \in L^1_{\mathrm{loc}}(\mathbb{R}^d, dx), 1 \leq i,j \leq d$, satisfy the following assumptions: There exist positive constants ν and C_0 such that

$$
\begin{aligned}
\nu^{-1}|\xi|^2 \leq \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \leq \nu|\xi|^2 \text{ for all} \\
\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d\ dx - a.e.,
\end{aligned} \tag{2.16}
$$

$$
\check{a}_{ij} \in L^\infty(\mathbb{R}^d, dx) \text{ where } \check{a}_{ij} := (1/2)(a_{ij} - a_{ji})\,, 1 \leq i,j \leq d, \tag{2.17}
$$

(2.18)

$$
0 \leq \int_{\mathbb{R}^d} u^2c^1\,dx + \sum_{i=1}^d \int_{\mathbb{R}^d} (b_i^1 + d_i)\frac{\partial u}{\partial x_i} u\,dx \leq C_0 D_1(u,u) \text{ and}
$$

$$
0 \leq \int_{\mathbb{R}^d} u^2c^1\,dx + 2\sum_{i=1}^d \int_{\mathbb{R}^d} d_i \frac{\partial u}{\partial x_i} u\,dx \text{ if } u \in C_0^\infty(\mathbb{R}^d)\ . \tag{2.19}
$$

Moreover, we assume that

$$b_i^1 - d_i \in L^d(\mathbb{R}^d, dx) \cup L^\infty(\mathbb{R}^d, dx)\,, 1 \le i \le d\ . \tag{2.20}$$

Let

(2.21)

$$\begin{aligned}\mathcal{A}(u,v) := \sum_{i,j=1}^d \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\, dx + \sum_{i=1}^d \int_{\mathbb{R}^d} b_i^1 \frac{\partial u}{\partial x_i} v\, dx \\ + \sum_{i=1}^d \int_{\mathbb{R}^d} d_i \frac{\partial v}{\partial x_i} u\, dx + \int_{\mathbb{R}^d} uvc^1\, dx\ ; u,v \in C_0^\infty(\mathbb{R}^d)\ .\end{aligned}$$

By (2.16) and (2.18) we obtain that

$$\nu^{-1} D(u,u) \le \mathcal{A}(u,u) \le \nu D(u,u) + C_0 D_1(u,u)\,, u \in C_0^\infty(\mathbb{R}^d)\ . \tag{2.22}$$

By (2.20) we obtain for $u, v \in C_0^\infty(\mathbb{R}^d)$ in the case $b_i^1 - d_i \in L^\infty(\mathbb{R}^d, dx)$

$$\begin{aligned}&\left|\int_{\mathbb{R}^d} (b_i^1 - d_i)(\frac{\partial u}{\partial x_i} v - \frac{\partial v}{\partial x_i} u)\, dx\right| \\ &\qquad \le 2\|b_i^1 - d_i\|_\infty D_1(u,u)^{1/2} D_1(v,v)^{1/2}\ ,\end{aligned} \tag{2.23}$$

and in the case $b_i^1 - d_i \in L^d(\mathbb{R}^d, dx)$ using Sobolev's inequality (cf. (1.4))

$$\begin{aligned}\left|\int_{\mathbb{R}^d} (b_i^1 - d_i)(\frac{\partial u}{\partial x_i} v - \frac{\partial v}{\partial x_i} u)\, dx\right| &\le D(u,u)^{1/2} (\int_{\mathbb{R}^d} (b_i^1 - d_i)^2 v^2\, dx)^{1/2} \\ &\quad + D(v,v)^{1/2} (\int_{\mathbb{R}^d} (b_i^1 - d_i)^2 u^2\, dx)^{1/2} \\ &\le 2^{3/2}\lambda \|b_i^1 - d_i\|_d D(u,u)^{1/2} D(v,v)^{1/2}\ ,\end{aligned} \tag{2.24}$$

where $\lambda := \frac{2(d-1)}{(d-2)d^{1/2}}$. Hence, by (2.17), (2.23) and (2.24) there exists a constant K with

$$|\check{\mathcal{A}}(u,v)| \le K D_1(u,u)^{1/2} D_1(v,v)^{1/2} \text{ for all } u,v \in C_0^\infty(\mathbb{R}^d)\ .$$

It is now easy to see that $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$ satisfies the weak sector condition and hence can be extended to a coercive closed form $\mathcal{A}$ on $H^{1,2}(\mathbb{R}^d)$ by (2.22) with

$$(\nu+1)^{-1} D_1(u,u) \le \mathcal{A}_1(u,u) \le (\nu + 1 + C_0) D_1(u,u)\ , u \in H^{1,2}(\mathbb{R}^d)\ .$$

By (2.19) we obtain that $(\mathcal{A}, H^{1,2}(\mathbb{R}^d))$ is a Dirichlet form in the sense of Definition I.4.3 by the results in [MR1, Subsection II.2.d)]. Proposition I.4.7 now implies that the bilinear form associated with $(\mathcal{A}, H^{1,2}(\mathbb{R}^d))$ and $(\Lambda - c, D(\Lambda, \mathcal{H}))$ is a generalized Dirichlet form. Hence we have proved the following

Proposition 2.6. *Let $d \geq 3$. Assume that (2.16)–(2.20) hold and let B be a vector field satisfying (2.5)–(2.7). Then there exists a generalized Dirichlet form $\mathcal{E}'$ extending the bilinear form $\mathcal{E}_c$ (cf. (2.15)). $\mathcal{E}'$ is given as the bilinear form associated with the closure of $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$ in $\mathcal{H}$ (cf. (2.21)) and the closure of $\langle B, \nabla u\rangle - cu$, $u \in C_0^\infty(\mathbb{R}^d)$, in $\mathcal{H}$.*

(c) First order perturbations of time dependent Dirichlet forms on $\mathbb{R}^d$

Let $d \geq 1$ and $b_i \in L^1_{\text{loc}}(\mathbb{R} \times \mathbb{R}^d, dt \otimes dx)$, $1 \leq i \leq d$, and $b_i(t, \cdot) \in L^1_{\text{loc}}(\mathbb{R}^d, dx)$ for all $t \in \mathbb{R}$, $1 \leq i \leq d$. Consider the bilinear form

$$\begin{aligned}\mathcal{E}(u,v) = &\sum_{i=1}^d \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}\, dx\, dt - \sum_{i=1}^d \int_{\mathbb{R}} \int_{\mathbb{R}^d} b_i \frac{\partial u}{\partial x_i} v\, dx\, dt \\ &- \int_{\mathbb{R}} \int_{\mathbb{R}^d} \frac{\partial u}{\partial t} v\, dx\, dt\ ; u, v \in C_0^\infty(\mathbb{R}^{d+1})\ .\end{aligned} \tag{2.25}$$

Let $B := (b_1, \dots, b_d)$ and assume that there exists a constant $c \geq 0$ such that

$$\int_{\mathbb{R}^{d+1}} \langle B, \nabla u\rangle\, dy \leq 2c\|u\|_1 \text{ for all } u \in C_0^\infty(\mathbb{R}^{d+1}), u \geq 0. \tag{2.26}$$

We suppose furthermore that there exist functions $L = L_2 + L_\infty$ and $M = M_2 + M_\infty$ with $L_2, M_2 \in \mathcal{B}^+(\mathbb{R}) \cap L^2(\mathbb{R}, dt)$ and $L_\infty, M_\infty \in \mathcal{B}^+(\mathbb{R}) \cap L^\infty(\mathbb{R}, dt)$ such that

$$|B(t,x) - B(t,y)| \leq L(t)|x - y| \text{ for all } x, y \in \mathbb{R}^d, t \in \mathbb{R} \tag{2.27}$$

$$\text{and } |B(t,x)| \leq M(t)(|x| + 1)\ ,\ x \in \mathbb{R}^d, t \in \mathbb{R}\ . \tag{2.28}$$

Note that $(\mathcal{E}_c, C_0^\infty(\mathbb{R}^{d+1}))$ is a positive definite bilinear form by (2.26) that satisfies no weak sector condition in $\mathcal{H} := L^2(\mathbb{R} \times \mathbb{R}^d, dt \otimes dx)$. However, we will construct in this subsection a generalized Dirichlet form extending the bilinear form $\mathcal{E}_c$. To this end let $(\mathcal{A}, \mathcal{V})$ be the symmetric Dirichlet form given by the closure of

$$\mathcal{A}(u,v) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle \nabla u, \nabla v\rangle\, dx\, dt\ ;\ u, v \in C_0^\infty(\mathbb{R}^{d+1})\ ,$$

in $\mathcal{H}$. Similar to Section 1 ∇u means gradient w.r.t. x, i.e., $\nabla u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d})$.

Let $\Lambda u := \langle B, \nabla u\rangle + \frac{\partial u}{\partial t}$, $u \in C_0^\infty(\mathbb{R}^{d+1})$, and note that $(\Lambda, C_0^\infty(\mathbb{R}^{d+1}))$ is a densely defined operator in $\mathcal{H}$ which is bounded from above by (2.26).

Remark 2.7. In the particular case where $B(t,x) = A_t x$, where $(A_t)_{t \in \mathbb{R}}$ is a family of (non–symmetric) $d \times d$–matrices $A_t = (a_{ij}(t))_{1 \leq i,j \leq d}$ with trace$(A_t) \geq -2c$, $t \in \mathbb{R}, t \mapsto a_{ij}(t)$ measurable, $1 \leq i,j \leq d$, and $\|A_t\|_{L(\mathbb{R}^d)}$ can be represented as the sum $L_2(t) + L_\infty(t)$ of two functions, where $L_2 \in \mathcal{B}^+(\mathbb{R}) \cap L^2(\mathbb{R}, dt)$ and $L_\infty \in \mathcal{B}^+(\mathbb{R}) \cap L^\infty(\mathbb{R}, dt)$ assumptions (2.26) - (2.28) are fulfilled (cf. [St3, Example II.11.vi]).

Proposition 2.8. *$(\Lambda, C_0^\infty(\mathbb{R}^{d+1}))$ is closable and the closure $(\Lambda, D(\Lambda, \mathcal{H}))$ generates a sub–Markovian C_0–semigroup $(U_t)_{t\geq 0}$ with $\|U_t\|_{L(\mathcal{H})} \leq \exp(ct)$, $t \geq 0$. $(U_t)_{t\geq 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$ with $\|U_t\|_{L(\mathcal{V})} \leq \exp(t(c + d\|L_\infty\|_\infty) + t^{1/2} d\|L_2\|_2)$, $t \geq 0$.*

Similar to the proof of Proposition 2.2 we approximate B with smooth vector fields $B^\varepsilon = (b_1^\varepsilon, \dots, b_d^\varepsilon)$, $\varepsilon > 0$, given by the convolution $B * \delta_\varepsilon$ of B with δ_ε, where $\delta_\varepsilon(t,x) := \psi_\varepsilon(t) \prod_{i=1}^d \psi_\varepsilon(x_i)$, $\psi(t) := (1/\varepsilon)\psi(t/\varepsilon)$ for some $\psi \in C_0^\infty((-1,1))$, $\psi \geq 0$, such that $\int_{\mathbb{R}} \psi \, dt = 1$. By (2.28) we obtain that

$$
\begin{aligned}
|B^\varepsilon(t,x)| &\leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} |B(t-s, x-y)| \delta_\varepsilon(s,y) \, dy \, ds \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} M(t-s)(|x-y|+1) \delta_\varepsilon(s,y) \, dy \, ds \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} M(t-s)(|x|+1+\varepsilon) \delta_\varepsilon(s,y) \, dy \, ds \\
&= M^\varepsilon(t)(|x|+1+\varepsilon) \ ,
\end{aligned}
\tag{2.29}
$$

where $M^\varepsilon := M * \psi_\varepsilon$. In particular, the operator $\Lambda^\varepsilon u := \langle B^\varepsilon, \nabla u \rangle + \frac{\partial u}{\partial t}$, $u \in C_0^\infty(\mathbb{R}^{d+1})$, is a densely defined operator on $\mathcal{H}$.

Lemma 2.9. *Let $\varepsilon > 0$. Then there exists a closed extension $(\Lambda^\varepsilon, D(\Lambda^\varepsilon, \mathcal{H}))$ of $(\Lambda^\varepsilon, C_0^\infty(\mathbb{R}^{d+1}))$ generating a sub–Markovian C_0–semigroup $(U_t^\varepsilon)_{t\geq 0}$ with $\|U_t^\varepsilon\|_{L(\mathcal{H})} \leq \exp(ct)$ and $U_t^\varepsilon(C_0^\infty(\mathbb{R}^{d+1})) \subset C_0^\infty(\mathbb{R}^{d+1})$, $t \geq 0$.*

Proof. By (2.29) we obtain for the smooth vector field $\overline{B}^\varepsilon := (1, B^\varepsilon)$ on $\mathbb{R}^{d+1}$ that

$$
|\overline{B}^\varepsilon(t,x)| \leq 1 + |B^\varepsilon(t,x)| \leq 1 + (\|M_2\|_2 \|\psi_\varepsilon\|_2 + \|M_\infty\|_\infty)(|x|+1+\varepsilon) \ .
$$

Consequently, there exists a (global) smooth flow $\overline{\xi}^\varepsilon$ corresponding to $\overline{B}^\varepsilon$ (cf. Subsection (a)). Note that $\overline{\xi}^\varepsilon$ can be written in the form $\overline{\xi}^\varepsilon(t,(s,x)) = (s+t, \xi^\varepsilon(t,(s,x)))$, $(s,x) \in \mathbb{R} \times \mathbb{R}^d$, for some smooth mapping $\xi^\varepsilon = (\xi_1^\varepsilon, \dots, \xi_d^\varepsilon) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$. By (2.26) we obtain that

$$
\begin{aligned}
\sum_{i=1}^d \frac{\partial b_i^\varepsilon}{\partial x_i}(t,x) &= \sum_{i=1}^d \frac{\partial}{\partial x_i} \int_{\mathbb{R}} \int_{\mathbb{R}^d} b_i(s,y) \delta_\varepsilon(t-s, x-y) \, dy \, ds \\
&= -\sum_{i=1}^d \int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle B(s,y), \nabla \delta_\varepsilon(t-s, x-y) \rangle \, dy \, ds \geq -2c,
\end{aligned}
\tag{2.30}
$$

and consequently, for the Jacobian determinant $\det D\overline{\xi^\varepsilon}(t,\cdot)(y)$ of the diffeomorphism $\overline{\xi}^\varepsilon(t,\cdot)$ in $y \in \mathbb{R}^{d+1}$ by (2.4)

$$
\det D\overline{\xi^\varepsilon}(t,\cdot)(y) = \exp\Big(\int_0^t \sum_{i=1}^d \frac{\partial b_i^\varepsilon}{\partial x_i}(\overline{\xi}^\varepsilon(s,y)) \, ds\Big) \geq \exp(-2ct) \, , t \geq 0 \, .
$$

By the change of variables formula for the Lebesgue measure we obtain for $f \in \mathcal{B}_b(\mathbb{R}^{d+1}) \cap \mathcal{H}$

(2.31)
$$\begin{aligned}\int_{\mathbb{R}^{d+1}} f^2(\overline{\xi}^{\varepsilon}(t,y))\,dy &\le \exp(2ct)\int_{\mathbb{R}^{d+1}} f^2(\overline{\xi}^{\varepsilon}(t,y))\det D\overline{\xi^{\varepsilon}}(t,\cdot)(y)\,dy\\ &= \exp(2ct)\|f\|_{\mathcal{H}}^2\ .\end{aligned}$$

Hence $U_t^\varepsilon f := f(\overline{\xi}^{\varepsilon}(t,\cdot))$, $f \in \mathcal{B}_b(\mathbb{R}^{d+1}) \cap \mathcal{H}$, induces a bounded linear operator U_t^ε on $\mathcal{H}$ with $\|U_t^\varepsilon\|_{L(\mathcal{H})} \le \exp(ct)$, $t \ge 0$. The family $(U_t^\varepsilon)_{t\ge 0}$ is a sub–Markovian semigroup of operators with $U_t^\varepsilon(C_0^\infty(\mathbb{R}^{d+1})) \subset C_0^\infty(\mathbb{R}^{d+1})$, $t \ge 0$. Similar to the proof of the corresponding statements of Lemma 2.3 it can be shown that $(U_t^\varepsilon)_{t\ge 0}$ is strongly continuous and that the corresponding generator $(\Lambda^\varepsilon, D(\Lambda^\varepsilon,\mathcal{H}))$ extends $(\Lambda^\varepsilon, C_0^\infty(\mathbb{R}^{d+1}))$. $\square$

Lemma 2.10. *Let $\varepsilon > 0$. Then $(U_t^\varepsilon)_{t\ge 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$ with $\|U_t^\varepsilon\|_{L(\mathcal{V})} \le \exp(t(c + d\|L_\infty\|_\infty) + t^{1/2}d\|L_2\|_2)$, $t \ge 0$.*

Proof. For $DB^\varepsilon := (\frac{\partial b_i^\varepsilon}{\partial x_j})_{1\le i,j\le d}$ and $h \in \mathbb{R}^d$ we obtain by (2.27)

$$\begin{aligned}\langle DB^\varepsilon(t,x)h,h\rangle &= \sum_{i,j=1}^d \frac{\partial b_i^\varepsilon}{\partial x_j}(t,x)h_ih_j\\ &= -\sum_{i.j=1}^d \int_{\mathbb{R}}\int_{\mathbb{R}^d} b_i(s,y)\frac{\partial \delta_\varepsilon}{\partial y_j}(t-s,x-y)\,dy\,ds\,h_ih_j \qquad (2.32)\\ &\le d\int_{\mathbb{R}} L(s)\psi_\varepsilon(t-s)\,ds|h|^2 = dL^\varepsilon(t)|h|^2\ ,\end{aligned}$$

where $L^\varepsilon := L * \psi_\varepsilon$.

For the Jacobian matrix $D\xi^\varepsilon(t,\cdot)(y) = (\frac{\partial \xi_i^\varepsilon}{\partial x_j}(t,y))_{1\le i,j\le d}$ and $y = (s,x) \in \mathbb{R}\times\mathbb{R}^d$ it follows from (2.3) that

$$D\xi^\varepsilon(t,\cdot)(y) = \mathrm{id}_{\mathbb{R}^d} + \int_0^t DB^\varepsilon(s+r,\xi^\varepsilon(r,y))D\xi^\varepsilon(r,\cdot)(y)\,dr$$

and by (2.32) we obtain for $h \in \mathbb{R}^d$

$$\begin{aligned}|D\xi^\varepsilon(t,\cdot)(y)h|^2 = |h|^2 + 2\int_0^t &\langle DB^\varepsilon(s+r,\xi^\varepsilon(r,y))D\xi^\varepsilon(r,\cdot)(y)h,\\ &D\xi^\varepsilon(r,\cdot)(y)h\rangle\,dr\\ \le |h|^2 + 2d&\int_0^t L^\varepsilon(s+r)|D\xi^\varepsilon(r,\cdot)(y)h|^2\,dr\ .\end{aligned}$$

By Gronwall's inequality we obtain that

$$
\begin{aligned}
|D\xi^\varepsilon(t,\cdot)(y)h|^2 &\le |h|^2 \exp(2d\int_0^t L^\varepsilon(s+r)\,dr) \\
&\le |h|^2 \exp(2td\|L_\infty\|_\infty + 2t^{1/2}d\|L_2\|_2)\ .
\end{aligned}
$$

Therefore, if $u \in C_0^\infty(\mathbb{R}^{d+1})$ we obtain that

$$
|\nabla U_t^\varepsilon u(y)| \le \exp(td\|L_\infty\|_\infty + t^{1/2}d\|L_2\|_2)|\nabla u(\overline{\xi}^\varepsilon(t,y))|\,, y \in \mathbb{R}^{d+1}\ , \tag{2.33}
$$

and

$$
\int_{\mathbb{R}^{d+1}} |\nabla U_t^\varepsilon u(y)|^2\,dy \le \exp(2t(c+d\|L_\infty\|_\infty) + 2t^{1/2}d\|L_2\|_2)\mathcal{A}(u,u)\ .
$$

Hence U_t^ε can be restricted to a bounded operator on $\mathcal{V}$ with $\|U_t^\varepsilon\|_{L(\mathcal{V})} \le \exp(t(c + d\|L_\infty\|_\infty) + t^{1/2}d\|L_2\|_2), t \ge 0$. Lemma I.2.2 now implies that the family of restrictions on $\mathcal{V}$ is a C_0–semigroup on $\mathcal{V}$. □

Proof of Proposition 2.8.

Existence: The existence of a closed extension of $(\Lambda, C_0^\infty(\mathbb{R}^d))$ generating a C_0–semigroup can be proved in the same way as the corresponding statement in Proposition 2.2.

Uniqueness: Let $(\Lambda, D(\Lambda,\mathcal{H}))$ be an arbitrary extension of $(\Lambda, C_0^\infty(\mathbb{R}^{d+1}))$ generating a C_0–semigroup $(U_t)_{t\ge 0}$.

Let $T \ge 0$ and $u \in C_0^\infty(\mathbb{R}^{d+1})$. Since

$$
-\xi^\varepsilon(-t,y) = -y + \int_{-t}^0 B^\varepsilon(s+r,\xi^\varepsilon(r,y))\,dr\ , t \ge 0\,, y=(s,x) \in \mathbb{R}^{d+1}\ ,
$$

(2.29) and Gronwall's inequality imply that for $\varepsilon \in (0,1)$

$$
\begin{aligned}
|\xi^\varepsilon(-t,y)| &\le (|y| + 2\int_{-t}^0 M^\varepsilon(s+r)\,dr)\exp(\int_{-t}^0 M^\varepsilon(s+r)\,dr) \\
&\le (|y| + 2t\|M_\infty\|_\infty + 2t^{1/2}\|M_2\|_2) \\
&\qquad \exp(t\|M_\infty\|_\infty + t^{1/2}\|M_2\|_2)\ , t \ge 0\,.
\end{aligned} \tag{2.34}
$$

Hence we obtain for $t \in [0,T]$, $\varepsilon \in (0,1)$, that

$$
\begin{aligned}
|y| = |\xi^\varepsilon(-t,\xi^\varepsilon(t,y))| &\le (|\xi^\varepsilon(t,y)| + 2T\|M_\infty\|_\infty + 2T^{1/2}\|M_2\|_2) \\
&\qquad \exp(T\|M_\infty\|_\infty + T^{1/2}\|M_2\|_2)\,, y \in \mathbb{R}^{d+1}\ .
\end{aligned}
$$

Similar to the proof of the corresponding statement in Proposition 2.2 it follows that the set

$$\{y | \exists t \in [0,T]\,, \exists \varepsilon \in (0,1) \text{ such that } U_t^\varepsilon u(y) \neq 0\}$$

is contained in a compact subset K of $\mathbb{R}^{d+1}$.

Since $b_i \in L^2_{\mathrm{loc}}(\mathbb{R}^{d+1}, dy)$, $1 \le i \le d$, we conclude that $|B^\varepsilon - B|\mathbb{1}_K \to 0$ in $\mathcal{H}$ if $\varepsilon \to 0$ and by (2.33)

$$\begin{aligned}
&\lim_{\varepsilon \downarrow 0} \|U_T u - U_T^\varepsilon u\|_{\mathcal{H}} \le \limsup_{\varepsilon \downarrow 0} \int_0^T \|U_{T-t}(\Lambda^\varepsilon - \Lambda)U_t^\varepsilon u\|_{\mathcal{H}}\, dt \\
&\le \limsup_{\varepsilon \downarrow 0} \int_0^T \|U_{T-t}\|_{L(\mathcal{H})} \| |B^\varepsilon - B|\mathbb{1}_K\|_{\mathcal{H}} \\
&\qquad\qquad \exp(td\|L_\infty\|_\infty + t^{1/2} d\|L_2\|_2) \| |\nabla u| \|_\infty \, dt \\
&= 0\ ,
\end{aligned}$$

because $\|U_t\|_{L(\mathcal{H})}$ is bounded on compact sets by the strong continuity of $(U_t)_{t\ge0}$. Hence $U_T^\varepsilon u \to U_T u$ for all $u \in C_0^\infty(\mathbb{R}^{d+1})$ and subsequently for all $u \in \mathcal{H}$. In particular, we obtain that $(U_t)_{t\ge0}$ is a sub–Markovian C_0–semigroup with $\|U_t\|_{L(\mathcal{H})} \le \exp(ct)$, $t \ge 0$.

Therefore, $(\Lambda, D(\Lambda, \mathcal{H}))$ is the only closed extension of $(\Lambda, C_0^\infty(\mathbb{R}^{d+1}))$ generating a C_0–semigroup and consequently, $C_0^\infty(\mathbb{R}^{d+1}) \subset D(\Lambda, \mathcal{H})$ dense w.r.t. the graph norm (cf. [Na, Theorem A-II, 1.33]).

Since $U_t^\varepsilon f \in \mathcal{V}$ if $f \in \mathcal{V}$ and $\varepsilon > 0$ and moreover, $\sup_{\varepsilon>0} \|U_t^\varepsilon f\|_{\mathcal{V}} < \infty$ by Lemma 2.10 we obtain that $U_t f \in \mathcal{V}$ with $\|U_t f\|_{\mathcal{V}} \le \exp(t(c + d\|L_\infty\|_\infty) + t^{1/2} d\|L_2\|_2)\|f\|_{\mathcal{V}}$ by [MR1, Lemma I.2.12]. Lemma I.2.2 implies that $(U_t)_{t\ge0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$. $\square$

Remark 2.11. For later purposes (cf. Section IV.4) we state the following inequality. If $u, v \in C_0^\infty(\mathbb{R}^{d+1})$ then

$$\begin{aligned}
\Big|\int_{\mathbb{R}} \int_{\mathbb{R}^d} \langle B, \nabla(uv)\rangle \, dx\, dt\Big| \le (d^2\|L_\infty\|_\infty + 2c)\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}} \\
+ d^2\|L_2\|_2 \sup_{t\in\mathbb{R}} \|v(t,\cdot)\|_2^2 \|u\|_{\mathcal{H}}\ .
\end{aligned}$$

Proof. Since for $\varepsilon > 0$

$$-2c \le \sum_{i=1}^d \frac{\partial b_i^\varepsilon}{\partial x_i}(t,x) = \sum_{i=1}^d \langle DB^\varepsilon(t,x)e_i, e_i\rangle \le d^2 L^\varepsilon(t)$$

by (2.30) and (2.32) we obtain that

$$
\begin{aligned}
|\int_{\mathbb{R}}\int_{\mathbb{R}^d} \langle B, \nabla(uv)\rangle \, dx\, dt| &= \lim_{\varepsilon\downarrow 0} |\int_{\mathbb{R}}\int_{\mathbb{R}^d} \langle B^\varepsilon, \nabla(uv)\rangle \, dx\, dt| \\
&\le \limsup_{\varepsilon\downarrow 0} \int_{\mathbb{R}} (d^2 L^\varepsilon(t) + 2c) \int_{\mathbb{R}^d} |u(t,x)v(t,x)| \, dx\, dt \\
&\le \limsup_{\varepsilon\downarrow 0} (d^2\|L_\infty\|_\infty + 2c)\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}} \\
&\qquad + d^2(\int_{\mathbb{R}} L_2^\varepsilon(t)^2\, dt)^{1/2}(\int_{\mathbb{R}} \|u(t,\cdot)\|_2^2 \|v(t,\cdot)\|_2^2\, dt)^{1/2} \\
&= (d^2\|L_\infty\|_\infty + 2c)\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}} \\
&\qquad + d^2\|L_2\|_2 \sup_{t\in\mathbb{R}} \|v(t,\cdot)\|_2^2 \|u\|_{\mathcal{H}}\ . \quad \square
\end{aligned}
$$

Let $B := (b_1, \dots, b_d) : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be a vector field satisfying (2.26)–(2.28). Denote by $(\Lambda, D(\Lambda, \mathcal{H}))$ the closure of $\Lambda u := \langle B, \nabla u\rangle + \frac{\partial u}{\partial t}$, $u \in C_0^\infty(\mathbb{R}^{d+1})$, in $\mathcal{H}$ and by $(U_t)_{t\ge 0}$ the corresponding semigroup. By the last Proposition we obtain that $\mathcal{V}$ is Λ–admissible. Since $(U_t)_{t\ge 0}$ is sub–Markovian the semigroup corresponding to the operator $(\Lambda - c, D(\Lambda, \mathcal{H}))$ is sub–Markovian too and therefore, $(\Lambda - c, D(\Lambda, \mathcal{H}))$ a Dirichlet operator (cf. [MR1, Proposition I.4.3]). Proposition I.4.7 now implies that the bilinear form $\mathcal{E}'$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda - c, D(\Lambda, \mathcal{H}))$ is a generalized Dirichlet form. Note that $\mathcal{E}'$ extends $\mathcal{E}_c$ where $\mathcal{E}$ is given as in (2.25). We can replace now $(\mathcal{A}, \mathcal{V})$ by any other Dirichlet form $(\mathcal{A}', \mathcal{V})$ provided that the $\mathcal{A}'_1$–norm is equivalent with the $\mathcal{A}_1$–norm. In particular, if $d \ge 3$ we may consider the bilinear form

$$
\begin{aligned}
\mathcal{E}(u,v) = &\sum_{i,j=1}^d \int_{\mathbb{R}^{d+1}} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\, dy + \sum_{i=1}^d \int_{\mathbb{R}^{d+1}} b_i^1 \frac{\partial u}{\partial x_i} v\, dy \\
&+ \sum_{i=1}^d \int_{\mathbb{R}^{d+1}} d_i \frac{\partial v}{\partial x_i} u\, dy + \int_{\mathbb{R}^{d+1}} uvc^1\, dy \\
&- \sum_{i=1}^d \int_{\mathbb{R}^{d+1}} b_i \frac{\partial u}{\partial x_i} v\, dy + c\int_{\mathbb{R}^{d+1}} uv\, dy \\
&- \int_{\mathbb{R}^{d+1}} \frac{\partial u}{\partial t} v\, dy\ ;\ u, v \in C_0^\infty(\mathbb{R}^{d+1})\ ,
\end{aligned} \tag{2.35}
$$

where $a_{ij}, b_i^1, d_i, c^1 \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1}, dy), 1 \le i,j \le d$, satisfy the following assumptions: There exist constants ν and C_0 such that

$$
\begin{aligned}
\nu^{-1}|\xi|^2 \le \sum_{i,j=1}^d a_{ij}(y)\xi_i\xi_j \le \nu|\xi|^2 \text{ for all} \\
\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d\ dy - a.e.\,,
\end{aligned} \tag{2.36}
$$

$$(2.37) \qquad \check{a}_{ij} \in L^\infty(\mathbb{R}^{d+1}, dy) \text{ where } \check{a}_{ij} := (1/2)(a_{ij} - a_{ji})\,, 1 \le i, j \le d\;.$$

(2.38)

$$0 \le \int_{\mathbb{R}^{d+1}} u^2 c^1\, dy + \sum_{i=1}^d \int_{\mathbb{R}^{d+1}} (b_i^1 + d_i) \frac{\partial u}{\partial x_i} u\, dy \le C_0 \mathcal{A}_1(u,u) \text{ and}$$

$$(2.39) \qquad 0 \le \int_{\mathbb{R}^{d+1}} u^2 c^1\, dy + 2\sum_{i=1}^d \int_{\mathbb{R}^{d+1}} d_i \frac{\partial u}{\partial x_i} u\, dy \text{ if } u \in C_0^\infty(\mathbb{R}^{d+1})\;.$$

Moreover, we assume that

$$(b_i^1 - d_i)(t, \cdot) \in L^d(\mathbb{R}^d, dx) \cup L^\infty(\mathbb{R}^d, dx)\,, 1 \le i \le d, \text{ for all } t \in \mathbb{R}$$

and for some constant L_0

$$(2.40) \qquad \min\{\|(b_i^1 - d_i)(t,\cdot)\|_d, \|(b_i^1 - d_i)(t,\cdot)\|_\infty\} \le L_0\,,$$

$1 \le i \le d\,, t \in \mathbb{R}$. Let

(2.41)

$$\mathcal{A}'(u,v) := \sum_{i,j=1}^d \int_{\mathbb{R}^{d+1}} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\, dy + \sum_{i=1}^d \int_{\mathbb{R}^{d+1}} b_i^1 \frac{\partial u}{\partial x_i} v\, dy$$
$$+ \sum_{i=1}^d \int_{\mathbb{R}^{d+1}} d_i \frac{\partial v}{\partial x_i} u\, dy + \int_{\mathbb{R}^{d+1}} uvc^1\, dy\;; u, v \in C_0^\infty(\mathbb{R}^{d+1})\;.$$

Similar to the proof of the corresponding statement in Subsection (b) one can show that $(\mathcal{A}', C_0^\infty(\mathbb{R}^{d+1}))$ can be extended to a coercive closed form $(\mathcal{A}', \mathcal{V})$ on $\mathcal{V}$ such that the $\mathcal{A}'_1$–norm is equivalent to the $\mathcal{A}_1$–norm.

By (2.39) we obtain that $(\mathcal{A}', \mathcal{V})$ is a Dirichlet form in the sense of Definition I.4.3 by the results in [MR1, Subsection II.2.d)]. Proposition I.4.7 now implies that the bilinear form associated with $(\mathcal{A}', \mathcal{V})$ and $(\Lambda - c, D(\Lambda, \mathcal{H}))$ is a generalized Dirichlet form. Hence we have proved the following

Proposition 2.12. *Let $d \ge 3$. Assume that (2.36)–(2.40) hold and let $B = (b_1, \dots, b_d) : \mathbb{R}^{d+1} \to \mathbb{R}^d$ be a vector field satisfying (2.26)–(2.28). Then there exists a generalized Dirichlet form $\mathcal{E}'$ extending the bilinear form $\mathcal{E}_c$ (cf. (2.35)). $\mathcal{E}'$ is given as the bilinear form associated with the closure of $(\mathcal{A}', C_0^\infty(\mathbb{R}^{d+1}))$ in $\mathcal{H}$ (cf. (2.41)) and the closure of $\langle B, \nabla u\rangle + \frac{\partial u}{\partial t} - cu$, $u \in C_0^\infty(\mathbb{R}^{d+1})$, in $\mathcal{H}$.*

II.3 Singular diffusion equations connected with generalized Dirichlet forms

Let $d \geq 1$, $\mathbb{R}^d_+ := \{x | x = (x_1, \ldots, x_d) \in \mathbb{R}^d, x_1 \geq 0, \ldots, x_d \geq 0\}$ and $\overset{\circ}{\mathbb{R}^d_+}$ be the open interior of $\mathbb{R}^d_+$. Consider the bilinear form

(3.1)

$$\mathcal{E}(u,v) = \sum_{i=1}^{d} \int_{\mathbb{R}^d_+} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} x_i \, dx - \sum_{i=1}^{d} \int_{\mathbb{R}^d_+} b_i \frac{\partial u}{\partial x_i} v \, dx \; ; u, v \in C_0^\infty(\overset{\circ}{\mathbb{R}^d_+}) \; ,$$

where x_i denotes the projection on the i–th coordinate and $B = (b_1, \ldots, b_d) : \mathbb{R}^d \to \mathbb{R}^d$ is a smooth vector field on $\mathbb{R}^d$, such that there exist positive constants c, L with

(3.2) $|B(x) - B(y)| \leq L|x - y|$ for all $x, y \in \mathbb{R}^d$.

(3.3) $b_i \geq 0$ on $\{x_i = 0\} \cap \mathbb{R}^d_+$, $1 \leq i \leq d$.

(3.4) $\sum_{j=1}^d x_j |\frac{\partial b_i}{\partial x_j}(x)| \leq b_i(x) + c x_i$ for all $x \in \mathbb{R}^d_+$, $1 \leq i \leq d$.

Note that by (3.2) $(\mathcal{E}_{d(L/2)}, C_0^\infty(\overset{\circ}{\mathbb{R}^d_+}))$ is a positive definite bilinear form which satisfies in general no weak sector condition in $\mathcal{H} := L^2(\mathbb{R}^d_+, dx)$. However, there exists a generalized Dirichlet form in $\mathcal{H}$ extending the bilinear form $\mathcal{E}_{d(L/2)}$. In order to construct this particular extension let

$$\text{(3.5)} \qquad \mathcal{A}(u,v) := \sum_{i=1}^{d} \int_{\mathbb{R}^d_+} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} x_i \, dx \, ; u, v \in C_0^\infty(\overset{\circ}{\mathbb{R}^d_+}) \; .$$

Using [MR1, Proposition I.3.3] it is easy to see that $(\mathcal{A}, C_0^\infty(\overset{\circ}{\mathbb{R}^d_+}))$ is closable in $\mathcal{H}$. Its closure $(\mathcal{A}, \mathcal{V})$ in $\mathcal{H}$ is a symmetric Dirichlet form.

Let us introduce another class of smooth functions. Let $C_0^\infty(\mathbb{R}^d_+)$ be the set of all functions on $\mathbb{R}^d_+$ which are restrictions of functions in $C_0^\infty(\mathbb{R}^d)$, i.e.,

$$C_0^\infty(\mathbb{R}^d_+) = \{f \in C(\mathbb{R}^d_+) | \exists g \in C_0^\infty(\mathbb{R}^d) \text{ such that } g_{|\mathbb{R}^d_+} = f\} \; .$$

Lemma 3.1. *Let $u \in C_0^\infty(\mathbb{R}^d_+)$. Then $u \in \mathcal{V}$ and*

$$\mathcal{A}(u,u) = \sum_{i=1}^{d} \int_{\mathbb{R}^d_+} (\frac{\partial u}{\partial x_i})^2 x_i \, dx \; .$$

Proof. Let $R > 0$ such that $u = 0$ on $\bigcup_{i=1}^d \{x_i \geq R\}$. Let $\delta > 0$ and $\psi \in C^\infty(\mathbb{R})$, $0 \leq \psi \leq 1$, such that $\psi(t) = 0$ if $t \leq \delta$, $\psi(t) = 1$ if $t \geq 1$, and $\|\psi'\|_\infty \leq 2$. Let

$\varphi_n(x) := \prod_{i=1}^d \psi(nx_i)$ and $u_n(x) := \varphi_n(x)u(x)$, $x \in \mathbb{R}^d_+$. Clearly $u_n \in C_0^\infty(\overset{\circ}{\mathbb{R}^d_+})$ and $\lim_{n\to\infty} u_n = u$ in $\mathcal{H}$. Since

$$|\frac{\partial u_n}{\partial x_i}(x)| \leq 2n|u(x)|\mathbb{1}_{\{x_i \leq 1/n\}} + |\frac{\partial u}{\partial x_i}(x)|$$

for all $x \in \mathbb{R}^d_+$ we conclude that

$$\int_{\mathbb{R}^d_+} (\frac{\partial u}{\partial x_i})^2 x_i\, dx \leq 8n^2 \int_{\mathbb{R}^d_+} u^2 \mathbb{1}_{\{x_i \leq 1/n\}} x_i\, dx + 2\int_{\mathbb{R}^d_+} (\frac{\partial u}{\partial x_i})^2 x_i\, dx$$
$$\leq 4\|u\|_\infty^2 R^{d-1} + 2\int_{\mathbb{R}^d_+} (\frac{\partial u}{\partial x_i})^2 x_i\, dx\ .$$

Consequently, $\sup_{n\geq 1} \mathcal{A}(u_n, u_n) < \infty$ which implies that $u \in \mathcal{V}$ and $\lim_{n\to\infty} u_n = u$ weakly in $\mathcal{V}$ (cf. [MR1, Lemma I.2.12]). Moreover, the Cesaro mean $w_n := (1/n)\sum_{k=1}^n u_{n_k}$ of some subsequence $(u_{n_k})_{k\geq 1}$ converges to u in $\mathcal{V}$. In particular, $(1/n)\sum_{k=1}^n x_i(\frac{\partial u_{n_k}}{\partial x_i})^2$ converges in $L^1(\mathbb{R}^d_+, dx)$ to $x_i(\frac{\partial u}{\partial x_i})^2$ since $\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i}$ dx–a.e., $1 \leq i \leq d$. Hence

$$\mathcal{A}(u,u) = \lim_{n\to\infty} \mathcal{A}(w_n, w_n) = \sum_{i=1}^d \int_{\mathbb{R}^d_+} (\frac{\partial u}{\partial x_i})^2 x_i\, dx\ . \quad \square$$

Define the linear operator $\Lambda u := \langle B, \nabla u\rangle$, $u \in C_0^\infty(\mathbb{R}^d_+)$.

Lemma 3.2. $(\Lambda, C_0^\infty(\mathbb{R}^d_+))$ *is closable. The closure generates a sub–Markovian* C_0–*semigroup* $(U_t)_{t\geq 0}$ *on* $\mathcal{H}$ *such that* $\|U_t\|_{L(\mathcal{H})} \leq \exp(d(L/2)t)$ *and* $U_t(C_0^\infty(\mathbb{R}^d_+)) \subset C_0^\infty(\mathbb{R}^d_+)$, $t \geq 0$.

Proof. Denote by ξ the smooth flow corresponding to B (cf. Subsection II.2 (a)). Applying [Am, Theorem 16.5] we obtain that $\mathbb{R}^d_+$ is a positively invariant set for ξ, i.e., $\xi(t,x) \in \mathbb{R}^d_+$ if $x \in \mathbb{R}^d_+$, $t \geq 0$. Indeed, (3.3) implies that $x + tB(x) \in \mathbb{R}^d_+$ if $x \in \mathbb{R}^d_+$ and $t > 0$ small. Hence $\operatorname{dist}(x + tB(x), \mathbb{R}^d_+) = 0$ if $x \in \mathbb{R}^d_+$ and $t > 0$ small. Thus we can define

$$U_t f(x) := f(\xi(t,x))\,, f \in \mathcal{B}_b(\mathbb{R}^d_+) \cap \mathcal{H}\,, x \in \mathbb{R}^d_+\,, t \geq 0\ .$$

Since $\xi(t,x) \in \mathbb{R}^d_+$ for $x \in \mathbb{R}^d_+$, $t \geq 0$, we conclude, using (2.4),

$$\det D\xi(t,\cdot)(x) = \exp(\int_0^t \sum_{i=1}^d \frac{\partial b_i}{\partial x_i}(\xi(s,x))\, ds) \geq \exp(-dLt)$$

if $x \in \mathbb{R}^d_+$ and hence

$$\int_{\mathbb{R}^d_+} (U_t f)^2(x)\, dx \leq e^{dLt} \int_{\mathbb{R}^d} f^2(\xi(t,x)) \mathbb{1}_{\mathbb{R}^d_+}(x) \det D\xi(t,x)\, dx$$
$$= e^{dLt} \int_{\mathbb{R}^d} f^2(x) \mathbb{1}_{\mathbb{R}^d_+}(\xi(-t,x))\, dx$$
$$\leq e^{dLt} \int_{\mathbb{R}^d_+} f^2(x)\, dx\ ,$$

because $1\!\!1_{\mathbb{R}^d_+}(\xi(-t,x)) \le 1\!\!1_{\mathbb{R}^d_+}(x)$, $x \in \mathbb{R}^d$.

Thus U_t induces a bounded linear operator on $\mathcal{H}$, again denoted by U_t, with $\|U_t\|_{L(\mathcal{H})} \le \exp(d(L/2)t)$, $t \ge 0$. The family $(U_t)_{t\ge 0}$ is obviously a sub–Markovian semigroup of operators on $\mathcal{H}$. Similar to the proof of Lemma 2.3 one can show that $(U_t)_{t\ge 0}$ is strongly continuous and that the corresponding generator $(\Lambda, D(\Lambda, \mathcal{H}))$ extends $(\Lambda, C_0^\infty(\mathbb{R}^d_+))$. Clearly $U_t(C_0^\infty(\mathbb{R}^d_+)) \subset C_0^\infty(\mathbb{R}^d_+)$ for all $t \ge 0$ which implies that $C_0^\infty(\mathbb{R}^d_+) \subset D(\Lambda, \mathcal{H})$ dense w.r.t. graph norm (cf. [ReSi2, Theorem X.49]). $\square$

We will show in the next step that $\mathcal{V}$ is Λ–admissible.

Lemma 3.3. *Let $u \in C_0^\infty(\mathbb{R}^d_+)$. Then the map $[0,\infty) \to \mathcal{V}$, $t \mapsto U_t u$, is strongly continuous.*

Proof. Since $t \mapsto U_t u$ is strongly continuous on $\mathcal{H}$ it suffices to prove that $(t_n)_{n\ge 1} \subset [0,T]$ with $\lim_{n\to\infty} t_n = t$, where $T > 0$ fixed, implies

$$\lim_{n\to\infty} x_i\Big(\frac{\partial}{\partial x_i}(U_{t_n}u - U_t u)\Big)^2 = 0 \text{ in } L^1(\mathbb{R}^d_+, dx)\,, 1 \le i \le d\,. \tag{3.6}$$

Since by (3.2) and Gronwall's inequality it follows that $|x| = |\xi(-t,\xi(t,x))| \le (|\xi(t,x)| + Mt)e^{Mt}$, $t \ge 0$, for some positive constant M we conclude that the set

$$\{x \in \mathbb{R}^d_+ \,|\, \exists t \in [0,T] \text{ such that } U_t u(x) \ne 0\}$$

is contained in a compact subset K of $\mathbb{R}^d_+$.

Since $\frac{\partial U_t u}{\partial x_i}(x) = \sum_{j=1}^d \frac{\partial u}{\partial x_j}(\xi(t,x))\frac{\partial \xi^j}{\partial x_i}(t,x)\,, 1 \le i \le d$, and $\frac{\partial \xi^j}{\partial x_i} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is locally bounded it follows that

$$x_i\Big(\frac{\partial}{\partial x_i}(U_{t_n}u - U_t u)\Big)^2 \le M x_i \||\nabla u|\|_\infty^2 1\!\!1_K\,, 1 \le i \le d\,,$$

for some positive constant M. Since the right hand side is integrable and the left hand side converges to zero pointwise we obtain (3.6) using Lebesgue's theorem. $\square$

Lemma 3.4. *$\mathcal{V}$ is Λ–admissible.*

Proof. Let $u \in C_0^\infty(\mathbb{R}^d_+)$. Then

$$\begin{aligned}\sum_{i,j=1}^d \int_{\mathbb{R}^d_+} \frac{\partial}{\partial x_i}\Big(b_j\frac{\partial u}{\partial x_j}\Big)\frac{\partial u}{\partial x_i} x_i\,dx &= \sum_{i,j=1}^d \int_{\mathbb{R}^d_+} \frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i}\frac{\partial b_j}{\partial x_i} x_i\,dx \\ &+ \sum_{i,j=1}^d \int_{\mathbb{R}^d_+} \frac{\partial^2 u}{\partial x_i \partial x_j}\frac{\partial u}{\partial x_i} b_j x_i\,dx\,.\end{aligned} \tag{3.7}$$

By (3.3) it follows that

$$\int_{\mathbb{R}^d_+} \frac{\partial^2 u}{\partial x_i \partial x_j} \frac{\partial u}{\partial x_i} b_j x_i \, dx = \frac{1}{2} \int_{\mathbb{R}^d_+} \frac{\partial}{\partial x_j} ((\frac{\partial u}{\partial x_i})^2) b_j x_i \, dx$$
$$\leq -\frac{1}{2} \int_{\mathbb{R}^d_+} (\frac{\partial u}{\partial x_i})^2 \frac{\partial}{\partial x_j}(b_j x_i) \, dx \ .$$

Hence (3.7) is dominated by

$$\begin{aligned} &\frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d_+} ((\frac{\partial u}{\partial x_j})^2 + (\frac{\partial u}{\partial x_i})^2) |\frac{\partial b_j}{\partial x_i}| x_i \, dx \\ &\quad - \frac{1}{2} \sum_{i=1}^d \int_{\mathbb{R}^d_+} (\frac{\partial u}{\partial x_i})^2 b_i \, dx - \frac{1}{2} \sum_{i,j=1}^d \int_{\mathbb{R}^d_+} (\frac{\partial u}{\partial x_i})^2 \frac{\partial b_j}{\partial x_j} x_i \, dx \\ &\quad \leq (c/2 + dL) \mathcal{A}(u,u) \, . \end{aligned} \tag{3.8}$$

Since $\Lambda u \in C_0^\infty(\mathbb{R}^d_+)$ if $u \in C_0^\infty(\mathbb{R}^d_+)$ we obtain that $U_t u - u = \int_0^t U_s \Lambda u \, ds$ in $\mathcal{V}$ by Lemma 3.3, hence the map $[0,\infty) \to \mathcal{V}$, $t \mapsto U_t u$, is strongly differentiable with derivative $U_t \Lambda u = \Lambda U_t u$. Therefore,

$$\mathcal{A}_1(U_t u, U_t u) = \mathcal{A}_1(u,u) + 2 \int_0^t \mathcal{A}_1(\Lambda U_s u, U_s u) \, ds$$
$$\leq \mathcal{A}_1(u,u) + (c + 2dL) \int_0^t \mathcal{A}_1(U_s u, U_s u) \, ds \, , t \geq 0 \ ,$$

using (3.8) and the fact that $(\Lambda v, v)_{\mathcal{H}} \leq d(L/2) \|v\|^2_{\mathcal{H}}$ for all $v \in C_0^\infty(\mathbb{R}^d_+)$. It follows that $\mathcal{A}_1(U_t u, U_t u) \leq \exp((c+2dL)t) \mathcal{A}_1(u,u)$. Hence the operator U_t can be restricted to a bounded operator on $\mathcal{V}$ with $\|U_t\|^2_{L(\mathcal{V})} \leq \exp((c+2dL)t)$, $t \geq 0$. By Lemma I.2.2 $(U_t)_{t \geq 0}$ can be restricted to a C_0–semigroup on $\mathcal{V}$. $\square$

The bilinear form $\mathcal{E}'$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda - d(L/2), D(\Lambda, \mathcal{H}))$ is a generalized Dirichlet form by Proposition I.4.7. $\mathcal{E}'$ extends the bilinear form $\mathcal{E}_{d(L/2)}$ (cf. (3.1)). If $(L, D(L))$ denotes the generator associated with $\mathcal{E}'$ it is easy to see that $C_0^\infty(\mathbb{R}^d_+) \subset D(L)$ and

$$Lu = \sum_{i=1}^d x_i \frac{\partial^2 u}{\partial x_i^2} + \sum_{i=1}^d \frac{\partial u}{\partial x_i} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} - d(L/2) u \, , u \in C_0^\infty(\mathbb{R}^d_+) \ . \tag{3.9}$$

(a) Associated continuous state branching processes

Let $E := \{e_1, \ldots, e_d\}$ be a discrete set of d points and Q be the generator of a Markov chain on E. Let $\Phi(\lambda) = -\lambda^2$, $\lambda \in \mathbb{R}$. A (Q,Φ)–*superprocess* (or Dawson–Watanabe process corresponding to (Q,Φ)) is an $\mathcal{M}(E)$–valued Markov process $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t \geq 0}, (P_\mu)_{\mu \in \mathcal{M}(E)})$ characterized by its Laplace functional

$$E_\mu[\exp(-\int \lambda \, dX_t)] = \exp(-\int \psi(t,\lambda) \, d\mu) \, , t \geq 0, \mu \in \mathcal{M}(E) \, , \tag{3.10}$$

where λ is a non–negative function on E and $\psi(t,\lambda)$ is the unique solution of the (non–linear) equation

$$\frac{\partial\psi}{\partial t}(t,\lambda) = Q\psi(t,\lambda) - \psi^2(t,\lambda)\,, t\geq 0\,, \psi(0,\lambda)=\lambda$$

(cf. [Daw, Chapter 4]). Clearly Q can be identified with a matrix $Q=(q_{ij})_{1\leq i,j\leq d}$ satisfying $q_{ij}\geq 0$, $1\leq i,j\leq d$, $i\neq j$, and $Q\mathbb{1}=0$, where $\mathbb{1}=(1,\ldots,1)\in\mathbb{R}^d_+$. $\mathcal{M}(E)$ can be identified with $\mathbb{R}^d_+$. Hence $\mathbb{M}$ is a continuous state branching process with state space $\mathbb{R}^d_+$ and (3.10) becomes

$$E_x[\exp(-\langle\lambda,X_t\rangle)] = \exp(-\langle\psi(t,\lambda),x\rangle)\,, t\geq 0, x\in\mathbb{R}^d_+, \lambda\in\mathbb{R}^d_+\,.$$

The corresponding (Q,Φ)*–superprocess with immigration rate* $\langle\cdot,\mathbb{1}\rangle$ is an $\mathbb{R}^d_+$–valued Markov process $\mathbb{M}^*=(\Omega^*,\mathcal{F}^*,(X^*_t)_{t\geq 0},(P^*_x)_{x\in\mathbb{R}^d_+})$ given by the Laplace functional

$$E^*_x[\exp(-\langle\lambda,X^*_t\rangle)] = \exp(-\int_0^t\langle\psi(s,\lambda),\mathbb{1}\rangle\,ds)\exp(-\langle\psi(t,\lambda),x\rangle)\,, \tag{3.11}$$

where $t\geq 0, x\in\mathbb{R}^d_+, \lambda\in\mathbb{R}^d_+$ (cf. [Daw, Subsection 6.2.4]). Let $(p^*_t)_{t\geq 0}$ denote the corresponding transition semigroup. Clearly $(p^*_t)_{t\geq 0}$ induces a C_0–semigroup of contractions $(T^*_t)_{t\geq 0}$ on $C_\infty(\mathbb{R}^d_+)$ such that $T^*_t(D)\subset D$, $t\geq 0$, where $D:=\operatorname{span}\{f_\lambda|\lambda\in\overset{\circ}{\mathbb{R}^d_+}\}$, $f_\lambda:=\exp(-\langle\lambda,\cdot\rangle)$. If $(A,D(A))$ denotes the generator it follows that $D\subset D(A)$ and

$$Af_\lambda = \sum_{i=1}^d x_i\frac{\partial^2 f_\lambda}{\partial x_i^2} + \sum_{i=1}^d\frac{\partial f_\lambda}{\partial x_i} + \sum_{i=1}^d (Q^Tx)_i\frac{\partial f_\lambda}{\partial x_i}\,, \lambda\in\overset{\circ}{\mathbb{R}^d_+}\,.$$

Moreover, $D\subset D(A)$ dense w.r.t. the graph norm by [ReSi2, Theorem X.49].

The singular diffusion equation corresponding to the operator A has been studied in the 1–dimensional case both from the analytic and the probabilistic point of view by W. Feller (cf. [Fe1] and [Fe2]), J. Lamperti and P. Ney (cf. [LaNe]) and K. Kawazu and S. Watanabe (cf. [KaWa]).

Note that the vector field Q^Tx satisfies (3.2)–(3.4). Hence there exists a generalized Dirichlet form $\mathcal{E}$ associated with $(\mathcal{A},\mathcal{V})$ and the closure of $\langle Q^Tx,\nabla u\rangle - d(L/2)u$, $u\in C_0^\infty(\mathbb{R}^d_+)$, in $\mathcal{H}$, where $L:=\|Q\|_{L(\mathbb{R}^d)}$. Let $(G_\alpha)_{\alpha>0}$ be the associated resolvent and $(G^*_\alpha)_{\alpha>0}$ be the resolvent of $(A-d(L/2),D(A))$. We will show that $G^*_\alpha f$ is an dx–version of $G_\alpha f$ for all $f\in C_\infty(\mathbb{R}^d_+)\cap\mathcal{H}$, $\alpha>0$. Indeed, if $(L,D(L))$ denotes the generator associated with $\mathcal{E}$ it is easy to see that $D\subset D(L)$ and $Lf_\lambda = Af_\lambda - d(L/2)f_\lambda$, $\lambda\in\overset{\circ}{\mathbb{R}^d_+}$. Hence $G^*_\alpha f$ is an dx–version of $G_\alpha f$ for all $f\in(\alpha-A)(D)$, $\alpha>0$. For arbitrary $f\in C_\infty(\mathbb{R}^d_+)\cap\mathcal{H}$ let $(f_n)_{n\geq 1}\subset(\alpha-A)(D)$ such that $\lim_{n\to\infty}f_n=f$ in $C_\infty(\mathbb{R}^d_+)$. Clearly $\lim_{n\to\infty}G^*_\alpha f_n = G^*_\alpha f$ in $C_\infty(\mathbb{R}^d_+)$ but also $\lim_{n\to\infty}G_\alpha f_n = G_\alpha f$ in $L^\infty(\mathbb{R}^d_+,dx)$ since αG_α is sub–Markovian. It is now easy to check that p^*_tf is an dx–version of $\exp(d(L/2)t)T_tf$ for all $f\in\mathcal{B}_b(\mathbb{R}^d_+)\cap\mathcal{H}$, $t\geq 0$. Hence we have proved the following

Proposition 3.5. *Let $\mathbb{M}^*$ be a (Q,Φ)–superprocess with immigration rate $\langle\cdot,\mathbb{1}\rangle$ and $(p^*_t)_{t\geq 0}$ be the transition semigroup of $\mathbb{M}^*$. Let $\mathcal{E}$ be the generalized Dirichlet form associated with the closure of $(\mathcal{A}, C_0^\infty(\overset{\circ}{\mathbb{R}^d_+}))$ (cf. (3.5)) and the closure of $\langle B,\nabla u\rangle - d(L/2)u$, $u\in C_0^\infty(\mathbb{R}^d_+)$, in $\mathcal{H}$. If $(T_t)_{t\geq 0}$ denotes the associated semigroup it follows that $p^*_t f$ is an dx–version of $\exp(d(L/2)t)T_t f$ for all $f\in\mathcal{B}^+_b(\mathbb{R}^d_+)\cap\mathcal{H}$, $t\geq 0$.*

(b) Singular diffusion operators with non–smooth coefficients

It is now easy to construct examples for generalized Dirichlet forms associated with singular diffusion operators with non–smooth coefficients. In particular, consider the bilinear form

$$
\begin{aligned}
\mathcal{E}(u,v) = {} & \sum_{i,j=1}^d \int_{\mathbb{R}^d_+} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\,dx + \sum_{i=1}^d \int_{\mathbb{R}^d_+} b_i^1 \frac{\partial u}{\partial x_i} v\,dx \\
& + \sum_{i=1}^d \int_{\mathbb{R}^d_+} d_i \frac{\partial v}{\partial x_i} u\,dx + \int_{\mathbb{R}^d_+} uvc\,dx \\
& - \sum_{i=1}^d \int_{\mathbb{R}^d_+} b_i \frac{\partial u}{\partial x_i} v\,dx + d(L/2)\int_{\mathbb{R}^d_+} uv\,dx\,;\, u,v\in C_0^\infty(\mathbb{R}^d_+)\,,
\end{aligned}
\tag{3.12}
$$

where $a_{ij}, b_i^1, d_i \in L^1_{\mathrm{loc}}(\mathbb{R}^d_+, dx)$, $a_{ij}=a_{ji}$, $1\leq i,j\leq d$, $c\in L^\infty(\mathbb{R}^d_+,dx)$, satisfy the following assumptions: There exist positive constants ν and C_0 such that

$$
\nu^{-1}\sum_{i=1}^d x_i\xi_i^2 \leq \sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \leq \nu\sum_{i=1}^d x_i\xi_i^2 \quad \text{for all } \xi=(\xi_1,\dots,\xi_d)\in\mathbb{R}^d\,, x\in\mathbb{R}^d_+\,,
\tag{3.13}
$$

$$
|b_i^1(x)|, |d_i(x)| \leq C_0\sqrt{x_i}\,, 1\leq i\leq d\,, x\in\mathbb{R}^d_+\,,
\tag{3.14}
$$

(3.15)

$$
0 \leq \int_{\mathbb{R}^d_+} u^2 c\,dx + \sum_{i,j=1}^d \int_{\mathbb{R}^d_+} (b_i^1 + d_i)\frac{\partial u}{\partial x_i} u\,dx \text{ and}
$$

$$
0 \leq \int_{\mathbb{R}^d_+} u^2 c\,dx + 2\sum_{i=1}^d \int_{\mathbb{R}^d_+} d_i\frac{\partial u}{\partial x_i} u\,dx \text{ if } u\in C_0^\infty(\mathbb{R}^d_+)\,,
$$

and $B=(b_1,\dots,b_d)$ is a smooth vector field satisfying (3.2)–(3.4).

Let

(3.16)

$$
\begin{aligned}
\mathcal{A}'(u,v) := {} & \sum_{i,j=1}^d \int_{\mathbb{R}^d_+} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} a_{ij}\,dx + \sum_{i=1}^d \int_{\mathbb{R}^d_+} b_i^1 \frac{\partial u}{\partial x_i} v\,dx \\
& + \sum_{i=1}^d \int_{\mathbb{R}^d_+} d_i \frac{\partial v}{\partial x_i} u\,dx + \int_{\mathbb{R}^d_+} uvc\,dx\;;\, u,v\in C_0^\infty(\overset{\circ}{\mathbb{R}^d_+})\,.
\end{aligned}
$$

Then

$$\nu^{-1}\mathcal{A}(u,u) \le \mathcal{A}'(u,u) \le \nu\mathcal{A}(u,u) + dC_0\mathcal{A}_1(u,u) + \|c\|_\infty \|u\|^2_{\mathcal{H}} \,, \tag{3.17}$$

and

$$\begin{aligned} |\check{\mathcal{A}}'(u,v)| &\le (1/2)\sum_{i=1}^d \left| \int_{\mathbb{R}^d_+} (b_i^1 - d_i)\Big(\frac{\partial u}{\partial x_i} v - \frac{\partial v}{\partial x_i} u\Big)\, dx \right| \\ &\le (C_0^2 + d)\mathcal{A}_1(u,u)^{1/2}\mathcal{A}_1(v,v)^{1/2} \end{aligned}$$

if $u, v \in C_0^\infty(\overset{\circ}{\mathbb{R}^d_+})$. Therefore, $(\mathcal{A}', C_0^\infty(\overset{\circ}{\mathbb{R}^d_+}))$ satisfies the weak sector condition and can be extended to a coercive closed form $\mathcal{A}'$ on $\mathcal{V}$ such that the $\mathcal{A}'_1$–norm is equivalent to the $\mathcal{A}_1$–norm. By (3.15) $(\mathcal{A}', \mathcal{V})$ is a Dirichlet form (cf. [MR1, Subsection II.2.d)]. Thus we have

Proposition 3.6. *Suppose that (3.13)–(3.15) holds and let $B = (b_1, \dots, b_d)$ be a smooth vector field satisfying (3.2)–(3.4). Then there exists a generalized Dirichlet form $\mathcal{E}'$ extending the bilinear form $\mathcal{E}_{d(L/2)}$ (cf. (3.12)). $\mathcal{E}'$ is given as the bilinear form associated with the closure of $(\mathcal{A}', C_0^\infty(\overset{\circ}{\mathbb{R}^d_+}))$ in $\mathcal{H}$ (cf. (3.16)) and the closure of $\langle Q^T x, \nabla u\rangle - d(L/2)u$, $u \in C_0^\infty(\mathbb{R}^d_+)$, in $\mathcal{H}$.*

II.4 Generalized Dirichlet forms in infinite dimensions

For the terminology in this section we refer to [MR1, Section II.3]. Throughout the whole section let E be a separable real Banach space and μ some finite positive measure on $\mathcal{B}(E)$ with supp $(\mu) \equiv E$.

(a) Let

$$\mathcal{F}C_b^\infty := \{f(\ell_1, \dots, \ell_m) \mid m \in \mathbb{N}, f \in C_b^\infty(\mathbb{R}^m), \ell_1, \dots, \ell_m \in E'\}$$

be the set of finitely based smooth functions, where $C_b^\infty(\mathbb{R}^m)$ denotes the space of all smooth functions f on $\mathbb{R}^m$ with f and all partial derivatives bounded.

For $u \in \mathcal{F}C_b^\infty$ and $k \in E$ let

$$\frac{\partial u}{\partial k}(z) := \frac{d}{ds} u(z + sk)|_{s=0} \,, z \in E \,,$$

be the Gâteaux derivative of u in direction k.

An element $k \in E$ is called *well–μ–admissible* if there exists $\beta_k \in L^2(E,\mu)$ such that for all $u, v \in \mathcal{F}C_b^\infty$

$$\int_E \frac{\partial u}{\partial k} v \, d\mu = -\int_E u \frac{\partial v}{\partial k}\, d\mu - \int_E uv\beta_k \, d\mu \,.$$

β_k is called the *logarithmic derivative* in direction $k \in E$.

Assume that $K_0 \subset E$ is a countable subset consisting of well–μ–admissible elements such that

$$\sum_{k \in K_0} {}_{E'}\langle \ell, k \rangle_E^2 < \infty \text{ for all } \ell \in E' . \tag{4.1}$$

Then we can define the bilinear form

$$D(u,v) := \sum_{k \in K_0} \int_E \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} \, d\mu; \;\; u, v \in \mathcal{F}C_b^\infty .$$

It is well–known that $(D, \mathcal{F}C_b^\infty)$ is closable on $L^2(E,\mu)$ and its closure $(D, \widetilde{V})$ a (symmetric) Dirichlet form.

Let $\varphi : \mathbb{R} \times E \to \mathbb{R}$ be some $\mathcal{B}(\mathbb{R} \times E)$ –measurable function such that

$$\begin{aligned} &\varphi(t_0, \cdot) \in \widetilde{V} \text{ with } \varphi(t_0,\cdot) > 0 \text{ for some } t_0 \in \mathbb{R} \text{ and} \\ &\varphi(t_0,\cdot)\beta_k \in L^2(E,\mu) \text{ for all } k \in K_0 , \end{aligned} \tag{4.2}$$

and

$$c^{-1}\varphi(t,\cdot) \le \varphi(t_0,\cdot) \le c\,\varphi(t,\cdot) \text{ for all } t \in \mathbb{R} , \tag{4.3}$$

for some constant $c > 0$.

It follows that the form

$$\mathcal{E}^{(t)}(u,v) := \sum_{k \in K_0} \int_E \frac{\partial u}{\partial k} \frac{\partial v}{\partial k} \varphi^2(t,\cdot) d\mu; \;\; u, v \in \mathcal{F}C_b^\infty ,$$

is closable on $H = L^2(E, \varphi^2(t_0,\cdot)\mu)$ for $t = t_0$ (and hence all t) and its closure $(\mathcal{E}^{(t)}, V)$ a (symmetric) Dirichlet form (cf. [MR1, Subsection II.3.d)]). According to Example I.4.9 (iii) the corresponding time dependent Dirichlet form is a generalized Dirichlet form.

(b) Assume next, that we are given a separable real Hilbert space H densely and continuously embedded into E. Identifying H with its dual we obtain $E' \subset H \subset E$ densely and continuously. It follows that for $u = f(\ell_1, \dots, \ell_m) \in \mathcal{F}C_b^\infty$ and $k \in H$ we have that

$$\frac{\partial u}{\partial k}(z) = \sum_{i=1}^m \frac{\partial f}{\partial x_i}(\ell_1(z), \dots, \ell_m(z)) \langle \ell_i, k \rangle_H \, , z \in E .$$

Consequently, $k \mapsto \frac{\partial u}{\partial k}(z)$ is continuous on H and we can define $\nabla u(z) \in H$ by

$$\langle \nabla u(z), k \rangle_H = \frac{\partial u}{\partial k}(z) .$$

We call ∇u the *H–gradient* of u.

Assume that μ is a Gaussian measure on $(E, \mathcal{B}(E))$ with covariance $\langle \cdot, \cdot \rangle_H$, i.e., (E, H, μ) is an abstract Wiener space. Since every $h \in H$ is well–μ–admissible (cf. [MR1, Theorem II.3.1]) it follows that

$$D(u, v) := \int_E \langle \nabla u, \nabla v \rangle_H \, d\mu; \; u, v \in \mathcal{F}C_b^\infty \ ,$$

is closable and its closure $(D, H^{1,2}(\mu))$ a (symmetric) Dirichlet form on $L^2(E, \mu)$.

Lemma 4.1. *Let $V : E \to \mathbb{R}$ be measurable such that $e^{|V|} \in L^1(E, \mu)$. If $u \in \mathcal{F}C_b^\infty$ then*

$$|\int_E u^2 V \, d\mu| \le (2 + \|e^{|V|}\|_1) D_1(u, u) \ .$$

Proof. Let $u \in \mathcal{F}C_b^\infty$ such that $\|u\|_2 = 1$. By Young's inequality, $xy \le x \log x - x + e^y$ for all $x, y \ge 0$, we obtain that

$$\int_E u^2 |V| \, d\mu \le \int_E u^2 \log u^2 \, d\mu - \int_E u^2 d\mu + \int_E e^{|V|} \, d\mu \ .$$

By the logarithmic Sobolev inequality (cf. [Gr]), i.e.,

$$\int_E f^2 \log |f| \, d\mu \le D(f, f) + \|f\|_2^2 \log \|f\|_2 \text{ for all } f \in \mathcal{F}C_b^\infty \ ,$$

we obtain that

$$\begin{aligned} \int_E u^2 |V| \, d\mu &\le 2D(u, u) + 2\|u\|_2^2 \log \|u\|_2 - \|u\|_2^2 + \|e^{|V|}\|_1 \\ &\le (2 + \|e^{|V|}\|_1) D_1(u, u) \ . \end{aligned}$$

For arbitrary $u \in \mathcal{F}C_b^\infty$, $u \ne 0$, consider now $u/\|u\|_2$. $\square$

Let $b, d : \mathbb{R} \times E \to H$ be measurable, $c : \mathbb{R} \times E \to \mathbb{R}$ measurable such that for some $\varepsilon > 0$ $e^{\varepsilon \|b(t,\cdot)\|_H^2}$, $e^{\varepsilon \|d(t,\cdot)\|_H^2}$, $e^{\varepsilon |c(t,\cdot)|^2} \in L^1(E, \mu)$ and, moreover, for some constant L_0

$$\|e^{\varepsilon \|b(t,\cdot)\|_H^2}\|_1, \|e^{\varepsilon \|d(t,\cdot)\|_H^2}\|_1, \|e^{\varepsilon |c(t,\cdot)|^2}\|_1 \le L_0 \text{ for all } t \in \mathbb{R} \ . \tag{4.4}$$

Assume that

$$\int_E u^2 c(t, \cdot) \, d\mu + \int_E \langle b(t, \cdot) + d(t, \cdot), \nabla u \rangle_H u \, d\mu \ge 0 \tag{4.5}$$

$$\text{and } \int_E u^2 c(t, \cdot) \, d\mu + 2 \int_E \langle d(t, \cdot), \nabla u \rangle_H u \, d\mu \ge 0 \text{ for all } u \in \mathcal{F}C_b^\infty \, , t \in \mathbb{R} \ .$$

Let $A : \mathbb{R} \times E \to L(H)$ be measurable such that for some positive constants ν, M

$$\nu^{-1}\|h\|_H^2 \le \langle A(t,z)h, h\rangle_H \le \nu\|h\|_H^2 \text{ for all } t \in \mathbb{R}, z \in E\,, \text{ and} \tag{4.6}$$

$$\|\check{A}(t,z)\|_{L(H)} \le M \text{ for all } t \in \mathbb{R}, z \in E, \text{ where } \check{A} := 1/2(A - \widehat{A})\ . \tag{4.7}$$

Let

$$\begin{aligned}\mathcal{E}^{(t)}(u,v) := &\int_E \langle A(t,\cdot)\nabla u, \nabla v\rangle_H\, d\mu + \int_E \langle b(t,\cdot), \nabla u\rangle_H v\, d\mu \\ &+ \int_E u\langle d(t,\cdot), \nabla v\rangle_H\, d\mu + \int_E uvc(t,\cdot)\, d\mu\ ; \\ &u, v \in \mathcal{F}C_b^\infty\ .\end{aligned} \tag{4.8}$$

By (4.5) and (4.6) we obtain that

$$\nu^{-1}D(u,u) \le \mathcal{E}^{(t)}(u,u)\,, u \in \mathcal{F}C_b^\infty\ , \tag{4.9}$$

and by the last lemma and (4.4)

(4.10)

$$\begin{aligned}|\int_E \langle b(t,\cdot), \nabla u\rangle_H v\, d\mu| &\le \varepsilon^{-1/2}(2 + \|e^{\varepsilon\|b(t,\cdot)\|_H^2}\|_1)^{1/2} D(u,u)^{1/2} D_1(v,v)^{1/2} \\ &\le \varepsilon^{-1/2}(2+L_0)^{1/2} D_1(u,u)^{1/2} D_1(v,v)^{1/2}\ ,\end{aligned}$$

similarly

$$|\int_E u\langle d(t,\cdot), \nabla v\rangle_H\, d\mu| \le \varepsilon^{-1/2}(2+L_0)^{1/2} D_1(u,u)^{1/2} D_1(v,v)^{1/2} \tag{4.11}$$

and

$$|\int_E uvc(t,\cdot)\, d\mu| \le (1/\varepsilon)(2+L_0) D_1(u,u)^{1/2} D_1(v,v)^{1/2} \tag{4.12}$$

for all $u, v \in \mathcal{F}C_b^\infty$ and $t \in \mathbb{R}$. By (4.6), (4.10), (4.11) and (4.12) there exists a constant C_0 with

$$\mathcal{E}^{(t)}(u,u) \le C_0 D_1(u,u) \text{ for all } u \in \mathcal{F}C_b^\infty\,, t \in \mathbb{R}, \tag{4.13}$$

and by (4.7), (4.10) and (4.11) there exists a constant K with

$$|\check{\mathcal{E}}^{(t)}(u,v)| \le K D_1(u,u)^{1/2} D_1(v,v)^{1/2} \text{ for all } u, v \in \mathcal{F}C_b^\infty\,, t \in \mathbb{R}. \tag{4.14}$$

It is now easy to see that $(\mathcal{E}^{(t)}, \mathcal{F}C_b^\infty)$ satisfies the weak sector condition (with some sector constant independent of t) and hence can be extended to a coercive closed form $\mathcal{E}^{(t)}$ on $H^{1,2}(\mu)$ by (4.9) and (4.13). By (4.5) we obtain that $(\mathcal{E}^{(t)}, H^{1,2}(\mu))$ is a Dirichlet form in the sense of Definition I.4.3 by the results in [MR1, Subsection II.2.d)]. According to Example I.4.9 (iii) the corresponding time dependent Dirichlet form is a generalized Dirichlet form.

For more examples of time dependent Dirichlet forms in infinite dimensions we refer to [St3].

II.5 Transformations of generalized Dirichlet forms

Throughout the whole section let $(E, \mathcal{B}, m)$ be a σ–finite measure space such that $\mathcal{H} := L^2(E, m)$ is separable, $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on $\mathcal{H}$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying D1. Let $(U_t)_{t\geq 0}$ denote the C_0–semigroup of contractions corresponding to $(\Lambda, D(\Lambda, \mathcal{H}))$.

(a) Transformations of generalized Dirichlet forms via Subordination

Throughout this subsection we assume that $\|U_t\|_{L(\mathcal{V})}$ is bounded.

Let $(\mu_t)_{t>0}$ be a vaguely continuous convolution semigroup of sub–probability measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ supported by $[0, \infty)$, i.e.,

(5.1) $\operatorname{supp}(\mu_t) \subset [0, \infty)$ and $\mu_t([0, \infty)) \leq 1$ for all $t > 0$,

(5.2) $\mu_t * \mu_s = \mu_{t+s}$ for all $s, t > 0$,

(5.3) $\lim_{t\downarrow 0} \mu_t = \delta_0$ vaguely,

where $*$ denotes convolution of probability measures on $\mathbb{R}$ and δ_0 the Dirac measure in zero. In the following let $\mu_0 := \delta_0$.

Note that the map $[0, \infty) \to \mathcal{H}$, $s \mapsto U_s h$, is integrable in the sense of Bochner against any finite measure on $([0, \infty), \mathcal{B}([0, \infty)))$ for any $h \in \mathcal{H}$. In particular, $U_t^\mu h := \int U_s h \, \mu_t(ds)$, is well–defined and it is easy to see that $(U_t^\mu)_{t\geq 0}$ is again a C_0–semigroup of contractions on $\mathcal{H}$ which can be restricted to a C_0–semigroup on $\mathcal{V}$. It follows that the corresponding generator $(\Lambda^\mu, D(\Lambda^\mu, \mathcal{H}))$ satisfies D1.

Assume now that $(\mathcal{A}, \mathcal{V})$ is a Dirichlet form and $(\Lambda, D(\Lambda, \mathcal{H}))$ a Dirichlet operator. We obtain that $(\Lambda^\mu, D(\Lambda^\mu, \mathcal{H}))$ is a Dirichlet operator again, since $(U_t)_{t\geq 0}$ is sub–Markovian and thus $(U_t^\mu)_{t\geq 0}$ is sub–Markovian too (cf. [MR1, Proposition I.4.3]). By Proposition I.4.7 the bilinear form $\mathcal{E}^\mu$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda^\mu, D(\Lambda^\mu, \mathcal{H}))$ is a generalized Dirichlet form.

Example 5.1. (fractional diffusion operators) Consider the situation of Example I.4.9 (iii) with $X = \mathbb{R}^d$ and $\mu = dx$, i.e., $(\mathcal{E}^{(t)}, V)_{t\in\mathbb{R}}$ is a family of Dirichlet forms on $H := L^2(\mathbb{R}^d, dx)$ with sector constant K independent of t satisfying I.4.9 (iii)(a) and (b) and $\bar{\mathcal{A}}(u, v) := \int \mathcal{E}^{(t)}(u(t), v(t))\, dt$, $u, v \in \overline{\mathcal{V}} := L^2(\mathbb{R}, V, dt)$. By the unique isomorphism $L^2(\mathbb{R}, H, dt) \cong L^2(\mathbb{R} \times \mathbb{R}^d, dt \otimes dx) =: \mathcal{H}$ $(\bar{\mathcal{A}}, \overline{\mathcal{V}})$ can be identified with a Dirichlet form $(\mathcal{A}, \mathcal{V})$ in $\mathcal{H}$. Let $(U_t)_{t\geq 0}$ be the semigroup corresponding to $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t}, \mathcal{H}))$ and let $(\eta_t^\alpha)_{t>0}$ denote the one–sided stable semigroup of order

$\alpha \in (0,1)$. If $(U_t^\alpha)_{t\geq 0}$ is the semigroup subordinated to $(U_t)_{t\geq 0}$ via $(\eta_t^\alpha)_{t>0}$ then the corresponding generator $\left(\left(\frac{\partial}{\partial t}\right)^\alpha, D\left(\left(\frac{\partial}{\partial t}\right)^\alpha, \mathcal{H}\right)\right)$, which is just the time derivative of power α, satisfies D1. The bilinear form $\mathcal{E}^\alpha$ associated with $(\mathcal{A},\mathcal{V})$ and $\left(\left(\frac{\partial}{\partial t}\right)^\alpha, D\left(\left(\frac{\partial}{\partial t}\right)^\alpha, \mathcal{H}\right)\right)$ is a generalized Dirichlet form.

In the special case where $V = H^{1,2}\left(\mathbb{R}^d\right)$ and $\mathcal{E}^{(t)}(u,v) = \int \langle \nabla u, \nabla v\rangle\, dx$ for all $t \in \mathbb{R}$, the generator $(L^\alpha, D\left(L^\alpha\right))$ associated with $\mathcal{E}^\alpha$ is an extension of the fractional diffusion operator $(\frac{\partial}{\partial t})^\alpha u + \Delta u$, $u \in C_0^\infty\left(\mathbb{R}^{d+1}\right)$ (cf. [SWy]).

(b) Transformation of generalized positivity preserving forms via excessive functions

In this subsection we suppose that the bilinear form associated with $(\mathcal{A},\mathcal{V})$ and $(\Lambda, D(\Lambda,\mathcal{H}))$ is a generalized positivity preserving form. Let $(G_\beta)_{\beta>0}$ denote the associated resolvent. Let $h \in \mathcal{H}$, $h > 0$, be an α–excessive element for some $\alpha \geq 0$, i.e., $\beta G_{\beta+\alpha} h \leq h$ for all $\beta > 0$ (cf. Definition III.1.1).

Then we can define the h–transform $\mathcal{E}^h$ of $\mathcal{E}$ on $\mathcal{H}^h := L^2(E, h^2\mu)$ as follows:

Let $(\mathcal{A}^h,\mathcal{V}^h)$ be the coercive closed form given by

$$\mathcal{A}^h(u,v) := \mathcal{A}_\alpha(uh, vh) \text{ if } u,v \in D(\mathcal{V}^h) := \{u \in \mathcal{H}^h \mid uh \in \mathcal{V}\},$$

and $(\Lambda^h, D(\Lambda^h,\mathcal{H}^h))$ be the linear operator given by

$$\Lambda^h u := \frac{1}{h}\Lambda(uh) \text{ if } u \in D(\Lambda^h,\mathcal{H}^h) := \{u \in \mathcal{H}^h \mid uh \in D(\Lambda,\mathcal{H})\}\ .$$

It is easy to see that $(\Lambda^h, D(\Lambda^h,\mathcal{H}^h))$ generates a C_0–semigroup of contractions and that $\mathcal{V}^h$ is Λ^h–admissible. If $\mathcal{E}^h$ denotes the bilinear form associated with $(\mathcal{A}^h,\mathcal{V}^h)$ and $(\Lambda^h, D(\Lambda^h,\mathcal{H}^h))$ and $(G_\beta^h)_{\beta>0}$ the associated resolvent it follows that $G_\beta^h u = (1/h)G_{\beta+\alpha}(uh)$, $u \in \mathcal{H}^h$, $\beta > 0$, and consequently, $(G_\beta^h)_{\beta>0}$ is sub–Markovian which implies that $\mathcal{E}^h$ is a generalized Dirichlet form.

Example 5.2. (Brownian Bridge) Let $d \geq 1$, $(D, H^{1,2}(\mathbb{R}^d))$ be the usual gradient form on $H^{1,2}(\mathbb{R}^d)$, i.e., $D(u,v) = (1/2)\int_{\mathbb{R}^d}\langle\nabla u, \nabla v\rangle\, dx\,; u,v \in H^{1,2}(\mathbb{R}^d)$, and $T > 0$. Let $\mathcal{E}$ be the time dependent Dirichlet form associated with $(D, H^{1,2}(\mathbb{R}^d))_{t\in[0,T)}$, $(G_\alpha)_{\alpha>0}$ be the associated resolvent and $(T_t)_{t\geq 0}$ the associated semigroup. Let $t > 0$ and $\overline{p}_t f(s,x) := \int_{\mathbb{R}^d} f(s+t,y)p_t(x,y)\, dy$ if $s+t < T$ and $\overline{p}_t f(s,x) := 0$ if $s+t \geq T, x \in \mathbb{R}^d$, $f \in \mathcal{B}_b([0,T)\times\mathbb{R}^d)$, where $p_t(x,y) = (2\pi t)^{-d/2}\exp(-\frac{|x-y|^2}{2t})$, $t > 0, x,y \in \mathbb{R}^d$. It follows that $\overline{p}_t f$ is an $dt\otimes dx$–version of $T_t f$ if $f \in \mathcal{B}_b([0,T)\times\mathbb{R}^d)\cap L^2([0,T)\times\mathbb{R}^d, dt\otimes dx)$, $t \geq 0$.

Let $h(s,x) := p_{T-s}(0,x), 0 \leq s < T, x \in \mathbb{R}^d$, and note that $\overline{p}_t h \leq h$, $t \geq 0$. Clearly $h \in L^2([0,T)\times\mathbb{R}^d, dt\otimes dx)$ only if $d = 1$. However, since $(T_t)_{t\geq 0}$ (hence $(G_\alpha)_{\alpha>0}$ too) operates on $L^1([0,T)\times\mathbb{R}^d, dt\otimes dx)$ as well, $h \in L^1([0,T)\times\mathbb{R}^d, dt\otimes dx)$

and $\alpha G_\alpha h \leq h$ for all $\alpha > 0$ we can define the h–transform $\mathcal{E}^h$ of $\mathcal{E}$ in $L^2([0,T) \times \mathbb{R}^d, h^2 dt \otimes dx)$, which is again a generalized Dirichlet form, in the same way as above. If $(T_t^h)_{t\geq 0}$ denotes the associated semigroup it is easy to see that $\overline{p}_t^h f := \frac{1}{h}\overline{p}_t(fh)$ is a $dt\otimes dx$–version of $T_t^h f$ for all $f \in \mathcal{B}_b([0,T)\times\mathbb{R}^d) \cap L^2([0,T)\times\mathbb{R}^d, h^2 dt\otimes dx)$, $t \geq 0$. Note that $(\overline{p}_t^h)_{t\geq 0}$ is the transition semigroup of the forward space–time process corresponding to the Brownian Bridge to 0 on $[0,T]$ (cf. [Sh, Exercise 62.25] and [KShr, Subsection 5.6]).

III. Analytic Potential Theory of Generalized Dirichlet Forms

We develop the analytic potential theory of generalized positivity preserving forms and generalized Dirichlet forms. We introduce the notion of excessive and reduced functions as well as the concept of $\mathcal{E}$–nests and $\mathcal{E}$–exceptional sets. Moreover, we will construct Choquet capacities whose zero sets are exactly the $\mathcal{E}$–exceptional sets. These capacities will be an essential tool in the construction and analysis of associated processes.

Throughout the whole chapter let $(E, \mathcal{B}, m)$ be a σ–finite measure space such that the corresponding L^2–space $\mathcal{H} := L^2(E, m)$ is separable. Let $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on $\mathcal{H}$ with sector constant $K \geq 1$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator on $\mathcal{H}$ satisfying D1. Let $\mathcal{E}$ denote the bilinear form associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$. In the first two subsections we only assume that $\mathcal{E}$ is a generalized positivity preserving form. Let $(G_\alpha)_{\alpha>0}$ (resp. $(\widehat{G}_\alpha)_{\alpha>0}$) be the associated resolvent (resp. coresolvent) and $(T_t)_{t\geq 0}$ (resp. $(\widehat{T}_t)_{t\geq 0}$) the associated semigroup (resp. cosemigroup). $(T_t)_{t\geq 0}$ is a positivity preserving semigroup since $(G_\alpha)_{\alpha>0}$ is positivity preserving (cf. [MR2, Theorem 1.7]). Note that by Remark I.4.2 $(\widehat{G}_\alpha)_{\alpha>0}$ and $(\widehat{T}_t)_{t\geq 0}$ are positivity preserving too.

III.1 Excessive, coexcessive, 1–reduced and 1–coreduced functions

Definition 1.1. Let $\alpha > 0$. An element $u \in \mathcal{H}$ is called *α–excessive* (resp. *α–coexcessive*) if $\beta G_{\beta+\alpha} u \leq u$ (resp. $\beta \widehat{G}_{\beta+\alpha} u \leq u$) for all $\beta \geq 0$.

Let $\mathcal{P}_\alpha$ (resp. $\widehat{\mathcal{P}}_\alpha$), $\alpha > 0$, denote the set of all α–excessive (resp. α–coexcessive) elements in $\mathcal{V}$.

Lemma 1.2. *For $u \in \mathcal{H}$ and $\alpha > 0$ the following statements are equivalent:*

(i) u is α–excessive.

(ii) $e^{-\alpha t} T_t u \leq u$ for all $t \geq 0$.

The corresponding statements for α–coexcessive elements are also equivalent.

Proof. (i) $\Rightarrow$ (ii) Since $(\widehat{T}_t)_{t\geq 0}$ is positivity preserving it follows for $v \in \mathcal{H}\,, v \geq 0$,

$$\widehat{G}_\alpha v - e^{-\alpha t} \widehat{T}_t \widehat{G}_\alpha v = \int_0^t e^{-\alpha s} \widehat{T}_s v \, ds \geq 0\,,$$

and therefore,

$$\begin{aligned}
\left(\beta G_{\beta+\alpha} u - e^{-\alpha t} T_t \left(\beta G_{\beta+\alpha} u\right), v\right)_{\mathcal{H}} &= \left(\beta G_{\beta+\alpha} u, v - e^{-\alpha t} \widehat{T}_t v\right)_{\mathcal{H}} \\
&= \mathcal{E}_\alpha \left(\beta G_{\beta+\alpha} u, \widehat{G}_\alpha v - e^{-\alpha t} \widehat{T}_t \widehat{G}_\alpha v\right) \\
&= \beta \left(u - \beta G_{\beta+\alpha} u, \widehat{G}_\alpha v - e^{-\alpha t} \widehat{T}_t \widehat{G}_\alpha v\right)_{\mathcal{H}} \geq 0\ .
\end{aligned}$$

Hence $e^{-\alpha t}T_t\,(\beta G_{\beta+\alpha}u) \le \beta G_{\beta+\alpha}u \le u$ for $\beta \ge 0$. Since $\lim\limits_{\beta\to\infty} \beta G_{\beta+\alpha}u = u$ the last inequality implies (ii).
(ii) $\Rightarrow$ (i) Clearly $\beta G_{\beta+\alpha}u = \beta\int_0^\infty e^{-(\beta+\alpha)t}T_t u\,dt \le \beta\int_0^\infty e^{-\beta t}u\,dt = u.$ $\square$

In the theory of Dirichlet forms there is a complete characterization of α–excessive functions in terms of the associated Dirichlet form. This fact can be carried over to the case of generalized positivity preserving forms.

Proposition 1.3. *For $u \in \mathcal{V}$ and $\alpha > 0$ the following statements are equivalent:*

(i) u is α–excessive.

(ii) $\mathcal{E}_\alpha(u, v) \ge 0$ for all $v \in \widehat{\mathcal{F}}\,, v \ge 0$.

Proof. (i) $\Rightarrow$ (ii) Let $v \in \widehat{\mathcal{F}}\,, v \ge 0$. Since $\lim_{\beta\to\infty} \beta G_{\beta+\alpha}u = u$ in $\mathcal{V}$ we obtain that

$$\begin{aligned}\mathcal{E}_\alpha(u, v) &= \lim_{\beta\to\infty} \mathcal{E}_\alpha(\beta G_{\beta+\alpha}u, v)\\ &= \lim_{\beta\to\infty} \beta(u - \beta G_{\beta+\alpha}u, v)_{\mathcal{H}} \ge 0\,.\end{aligned}$$

(ii) $\Rightarrow$ (i) Let $v \in \mathcal{H}\,, v \ge 0$. Then

$$\beta(u - \beta G_{\beta+\alpha}u, v)_{\mathcal{H}} = \beta(v - \beta\widehat{G}_{\beta+\alpha}v, u)_{\mathcal{H}} = \mathcal{E}_\alpha(u, \beta\widehat{G}_{\beta+\alpha}v) \ge 0$$

since $\beta\widehat{G}_{\beta+\alpha}v \ge 0$. $\square$

Proposition 1.4. *For $u \in \widehat{\mathcal{F}}$ and $\alpha > 0$ the following statements are equivalent:*

(i) u is α–coexcessive.

(ii) $\mathcal{E}_\alpha(v, u) \ge 0$ for all $v \in \mathcal{V}\,, v \ge 0$.

Proof. (i)$\Rightarrow$ (ii) Let $v \in \mathcal{V}$, $v \ge 0$. Since $\lim_{\beta\to\infty} \beta G_{\beta+\alpha}v = v$ in $\mathcal{V}$ it follows that

$$\begin{aligned}\mathcal{E}_\alpha(v, u) &= \lim_{\beta\to\infty} \mathcal{E}_\alpha(\beta G_{\beta+\alpha}v, u) = \lim_{\beta\to\infty} \beta(v - \beta G_{\beta+\alpha}v, u)_{\mathcal{H}}\\ &= \lim_{\beta\to\infty} \beta(u - \beta\widehat{G}_{\beta+\alpha}u, v)_{\mathcal{H}} \ge 0\,.\end{aligned}$$

(ii) $\Rightarrow$ (i) This inclusion can be proved in the same way as (ii) $\Rightarrow$ (i) in Proposition 1.3. $\square$

1–reduced and 1–coreduced functions

Notation. For an element $h \in \mathcal{H}$ let $\mathcal{L}_h := \{v | v \in \mathcal{H}, v \ge h\}$ and $\beta_h : \mathcal{H} \to \mathcal{H}$ be defined by $\beta_h(v) := (v - h)^-$.

Lemma 1.5. *Let $h \in \mathcal{H}$. Then*

(i) β_h is continuous and bounded.

(ii) $-\beta_h$ is monotone, i.e., $-(\beta_h(u) - \beta_h(v), u - v)_{\mathcal{H}} \geq 0$ for all $u, v \in \mathcal{H}$.

(iii) $-(\beta_h(u) - \beta_h(v), (u-v)^+)_{\mathcal{H}} \geq 0$ for all $u, v \in \mathcal{H}$.

(iv) $\beta_h(v) \geq 0$ and $\beta_h(v) = 0 \ \Leftrightarrow\ v \in \mathcal{L}_h$.

Proof. (i) β_h is bounded since

$$\begin{aligned} \|\beta_h(v)\|_{\mathcal{H}}^2 &= \int ((v-h)^-)^2\, dm \leq \|v-h\|_{\mathcal{H}}^2 \\ &\leq (\|v\|_{\mathcal{H}} + \|h\|_{\mathcal{H}})^2\,, \end{aligned} \tag{1.1}$$

and a contraction, and hence in particular continuous, since

$$\begin{aligned} \|\beta_h(u) - \beta_h(v)\|_{\mathcal{H}}^2 &= \int ((u-h)^- - (v-h)^-)^2\, dm \\ &\leq \int (u-v)^2\, dm = \|u-v\|_{\mathcal{H}}^2\,. \end{aligned} \tag{1.2}$$

(ii) For arbitrary real numbers a, b and c we have that

$$((a-c)^- - (b-c)^-)(a-b) \leq 0 \tag{1.3}$$

since $(a-c)^- \geq (b-c)^-$ for $a \leq b$ and $(a-c)^- \leq (b-c)^-$ for $a > b$. Consequently,

$$(\beta_h(u) - \beta_h(v), u-v)_{\mathcal{H}} = \int ((u-h)^- - (v-h)^-)(u-v)\, dm \leq 0\,.$$

(iii) For arbitrary real numbers a, b and c inequality (1.3) implies

$$((a-c)^- - (b-c)^-)(a-b)^+ \leq 0$$

and therefore,

$$(\beta_h(u) - \beta_h(v), (u-v)^+)_{\mathcal{H}} \leq 0\,.$$

(iv) Obvious. □

Proposition 1.6. *For $h \in \mathcal{H}$ and $\alpha > 0$ there exist $h_\alpha \in \mathcal{F}$ and $\widehat{h}_\alpha \in \widehat{\mathcal{F}}$ such that*

$$\begin{aligned} &\mathcal{E}_1(h_\alpha, v) = \alpha(\beta_h(h_\alpha), v)_{\mathcal{H}} \text{ for all } v \in \mathcal{V}\,, \text{ and} \\ &\mathcal{E}_1(v, \widehat{h}_\alpha) = \alpha(\beta_h(\widehat{h}_\alpha), v)_{\mathcal{H}} \text{ for all } v \in \mathcal{V}\,. \end{aligned}$$

h_α and $\hat{h}_\alpha$ are uniquely determined.

Proof. Restricting β_h to $\mathcal{V}$ one obtains a continuous and bounded map $\beta_h : \mathcal{V} \to \mathcal{V}'$ since (1.1) and (1.2) imply

$$\|\beta_h(u) - \beta_h(v)\|_{\mathcal{V}'} \leq \|\beta_h(u) - \beta_h(v)\|_{\mathcal{H}} \leq \|u-v\|_{\mathcal{H}} \leq \|u-v\|_{\mathcal{V}}\,,$$

and

$$\|\beta_h(u)\|_{\mathcal{V}'} \leq \|\beta_h(u)\|_{\mathcal{H}} \leq \|u-h\|_{\mathcal{H}} \leq \|u\|_{\mathcal{V}} + \|h\|_{\mathcal{H}}\,.$$

Similar to Section I.3 define $M_1, \widehat{M}_1 : \mathcal{V} \to \mathcal{V}'$ by $\langle M_1 u, v\rangle = \mathcal{A}_1(u,v) = \langle \widehat{M}_1 v, u\rangle$. M_1 and $\widehat{M}_1$ are obviously continuous and linear and the map $M_1 - \alpha\beta_h$ (resp. $\widehat{M}_1 - \alpha\beta_h$) satisfies (M.1) - (M.3) since (M.1) and (M.2) are obvious by Lemma 1.5 (ii) and (iii) and (M.3) is satisfied because

$$\begin{aligned}
-(\beta_h(u), u)_{\mathcal{H}} &= -(\beta_h(u), u-h)_{\mathcal{H}} - (\beta_h(u), h)_{\mathcal{H}} \\
&= \|(u-h)^-\|^2_{\mathcal{H}} - (\beta_h(u), h)_{\mathcal{H}} \\
&\geq -(\|u\|_{\mathcal{H}} + \|h\|_{\mathcal{H}})\|h\|_{\mathcal{H}} \\
&\geq -\|u\|_{\mathcal{V}}\|h\|_{\mathcal{H}} - \|h\|^2_{\mathcal{H}}
\end{aligned}$$

and therefore,

$$\frac{\langle M_1 u - \alpha\beta_h(u), u\rangle}{\|u\|_{\mathcal{V}}} \geq \frac{\langle M_1 u, u\rangle}{\|u\|_{\mathcal{V}}} - \alpha\|h\|_{\mathcal{H}} - \alpha\frac{\|h\|^2_{\mathcal{H}}}{\|u\|_{\mathcal{V}}} \to \infty$$

$$\left(\text{ resp. } \frac{\langle \widehat{M}_1 u - \alpha\beta_h(u), u\rangle}{\|u\|_{\mathcal{V}}} \geq \frac{\langle \widehat{M}_1 u, u\rangle}{\|u\|_{\mathcal{V}}} - \alpha\|h\|_{\mathcal{H}} - \alpha\frac{\|h\|^2_{\mathcal{H}}}{\|u\|_{\mathcal{V}}} \to \infty\right)$$

if $\|u\|_{\mathcal{V}} \to \infty$. By Proposition I.3.2 (resp. Proposition I.3.3) there is a uniquely determined $h_\alpha \in \mathcal{F}$ (resp. $\widehat{h}_\alpha \in \widehat{\mathcal{F}}$) such that $M_1 h_\alpha - \Lambda h_\alpha - \alpha\beta_h(h_\alpha) = 0$ (resp. $\widehat{M}_1\widehat{h}_\alpha - \widehat{\Lambda}\widehat{h}_\alpha - \alpha\beta_h(\widehat{h}_\alpha) = 0$) and consequently, $\mathcal{E}_1(h_\alpha, v) = \alpha(\beta_h(h_\alpha), v)_{\mathcal{H}}$ (resp. $\mathcal{E}_1(v, \widehat{h}_\alpha) = \alpha(\beta_h(\widehat{h}_\alpha), v)_{\mathcal{H}}$) for all $v \in \mathcal{V}$. $\square$

Proposition 1.7. *Let $h \in \mathcal{H}$. Suppose $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ (resp. $\mathcal{L}_h \cap \widehat{\mathcal{F}} \neq \emptyset$). Then there exists $e_h \in \mathcal{L}_h \cap \mathcal{P}_1$ (resp. $\hat{e}_h \in \mathcal{L}_h \cap \widehat{\mathcal{P}}_1$) such that:*

(i) $\lim_{\alpha\to\infty} h_\alpha = e_h$ *(resp.* $\lim_{\alpha\to\infty}\widehat{h}_\alpha = \widehat{e}_h$*) strongly in $\mathcal{H}$ and weakly in $\mathcal{V}$ (where $h_\alpha \in \mathcal{F}$ (resp. $\widehat{h}_\alpha \in \widehat{\mathcal{F}}$) is as in Proposition 1.6).*

(ii) $e_h \leq u$ for all $u \in \mathcal{L}_h$, u 1–excessive (resp. $\hat{e}_h \leq u$ for all $u \in \mathcal{L}_h$, u 1–coexcessive).

(iii) $\mathcal{E}_1(e_h, v) \geq \mathcal{A}_1(e_h, e_h)$ for all $v \in \mathcal{L}_h \cap \widehat{\mathcal{F}}$
(resp. $\mathcal{E}_1(v, \hat{e}_h) \geq \mathcal{A}_1(\hat{e}_h, \hat{e}_h)$ for all $v \in \mathcal{L}_h \cap \mathcal{F}$).

Proof. 1. Step: $h_{\alpha_1} \leq h_{\alpha_2}$ (resp. $\widehat{h}_{\alpha_1} \leq \widehat{h}_{\alpha_2}$) if $\alpha_1 \leq \alpha_2$.
Since

$$\begin{aligned}
0 &= \mathcal{E}_1\left(h_{\alpha_1} - h_{\alpha_2}, (h_{\alpha_1} - h_{\alpha_2})^+\right) - \alpha_1\left(\beta_h\left(h_{\alpha_1}\right), (h_{\alpha_1} - h_{\alpha_2})^+\right)_{\mathcal{H}} \\
&\quad + \alpha_2\left(\beta_h\left(h_{\alpha_2}\right), (h_{\alpha_1} - h_{\alpha_2})^+\right)_{\mathcal{H}} \\
&= \mathcal{E}_1\left(h_{\alpha_1} - h_{\alpha_2}, (h_{\alpha_1} - h_{\alpha_2})^+\right) - \alpha_1\left(\beta_h\left(h_{\alpha_1}\right) - \beta_h\left(h_{\alpha_2}\right), (h_{\alpha_1} - h_{\alpha_2})^+\right)_{\mathcal{H}} \\
&\quad + (\alpha_2 - \alpha_1)\left(\beta_h\left(h_{\alpha_2}\right), (h_{\alpha_1} - h_{\alpha_2})^+\right)_{\mathcal{H}} \\
&\geq \|\left(h_{\alpha_1} - h_{\alpha_2}\right)^+\|^2_{\mathcal{H}}\,,
\end{aligned}$$

by Lemma 1.5 and Proposition I.4.4, it follows $(h_{\alpha_1} - h_{\alpha_2})^+ = 0$, i.e., $h_{\alpha_1} \le h_{\alpha_2}$. The corresponding statement for $\widehat{h}_{\alpha_1}$ and $\widehat{h}_{\alpha_2}$ can be proved similarly.

2. Step: Let $\alpha > 0$. Then $h_\alpha \le u$ for $u \in \mathcal{L}_h$, u 1–excessive (resp. $\widehat{h}_\alpha \le u$ for $u \in \mathcal{L}_h$, u 1–coexcessive).
Since u is 1–excessive (resp. u 1–coexcessive) we have that

$$\mathcal{E}_1(\beta G_{\beta+1}u, (h_\alpha - \beta G_{\beta+1}u)^+) = \beta(u - \beta G_{\beta+1}u, (h_\alpha - \beta G_{\beta+1}u)^+)_{\mathcal{H}} \ge 0$$

$$(\text{ resp. } \mathcal{E}_1((\widehat{h}_\alpha - \beta \widehat{G}_{\beta+1}u)^+, \beta \widehat{G}_{\beta+1}u) \ge 0\,).$$

Consequently,

$$\begin{aligned} \|\,(h_\alpha - u)^+\,\|_{\mathcal{H}}^2 &= \lim_{\beta\to\infty} \left(h_\alpha - \beta G_{\beta+1}u, (h_\alpha - \beta G_{\beta+1}u)^+\right)_{\mathcal{H}} \\ &\le \limsup_{\beta\to\infty} \mathcal{E}_1\left(h_\alpha - \beta G_{\beta+1}u, (h_\alpha - \beta G_{\beta+1}u)^+\right) \\ &\le \limsup_{\beta\to\infty} \mathcal{E}_1\left(h_\alpha, (h_\alpha - \beta G_{\beta+1}u)^+\right) \\ &= \limsup_{\beta\to\infty} \alpha\left(\beta_h\left(h_\alpha\right), (h_\alpha - \beta G_{\beta+1}u)^+\right)_{\mathcal{H}} \\ &= \alpha\left((h_\alpha - h)^-, (h_\alpha - u)^+\right)_{\mathcal{H}} = 0\ . \end{aligned}$$

Hence $(h_\alpha - u)^+ = 0$, i.e., $h_\alpha \le u$. The statement $\widehat{h}_\alpha \le u$ can be proved similarly.

3. Step: $\sup_{\alpha>0} \|h_\alpha\|_{\mathcal{V}} < +\infty$ (resp. $\sup_{\alpha>0} \|\widehat{h}_\alpha\|_{\mathcal{V}} < +\infty$).
For $v \in \mathcal{L}_h \cap \mathcal{F}$ (resp. $v \in \mathcal{L}_h \cap \widehat{\mathcal{F}}$) we have that

$$\begin{aligned} \|v - h_\alpha\|_{\mathcal{V}}^2 &\le \mathcal{E}_1\left(v - h_\alpha, v - h_\alpha\right) = \mathcal{E}_1\left(v, v - h_\alpha\right) - \alpha\left(\beta_h\left(h_\alpha\right), v - h_\alpha\right)_{\mathcal{H}} \\ &\le \mathcal{E}_1\left(v, v - h_\alpha\right) \le 2K\|v\|_{\mathcal{F}}\|v - h_\alpha\|_{\mathcal{V}} \end{aligned}$$

$$(\text{ resp. } \|v - \widehat{h}_\alpha\|_{\mathcal{V}}^2 \le \mathcal{E}_1\left(v - \widehat{h}_\alpha, v - \widehat{h}_\alpha\right) \le 2K\|v\|_{\widehat{\mathcal{F}}}\|v - \widehat{h}_\alpha\|_{\mathcal{V}}\,)$$

and therefore, $\sup_{\alpha>0} \|h_\alpha\|_{\mathcal{V}} \le \sup_{\alpha>0} \|h_\alpha - v\|_{\mathcal{V}} + \|v\|_{\mathcal{V}} < +\infty$ (resp. $\sup_{\alpha>0} \|\widehat{h}_\alpha\|_{\mathcal{V}} \le \sup_{\alpha>0} \|\widehat{h}_\alpha - v\|_{\mathcal{V}} + \|v\|_{\mathcal{V}} < +\infty$).

By the monotone convergence theorem there exists an element $e_h \in \mathcal{H}$ (resp. $\hat{e}_h \in \mathcal{H}$) such that $\lim_{\alpha\to\infty} h_\alpha = e_h$ (resp. $\lim_{\alpha\to\infty} \widehat{h}_\alpha = \widehat{e}_h$) in $\mathcal{H}$. Consequently, $e_h \in \mathcal{V}$ and $\lim_{\alpha\to\infty} h_\alpha = e_h$ weakly in $\mathcal{V}$ (resp. $\widehat{e}_h \in \mathcal{V}$ and $\lim_{\alpha\to\infty} \widehat{h}_\alpha = \widehat{e}_h$ weakly in $\mathcal{V}$) by [MR1, Lemma I.2.12]. Clearly $e_h \in \mathcal{P}_1$ (resp. $\widehat{e}_h \in \widehat{\mathcal{P}}_1$) since $h_\alpha \in \mathcal{P}_1$ (resp. $\widehat{h}_\alpha \in \widehat{\mathcal{P}}_1$), $\alpha > 0$, and $e_h \le u$ for all $u \in \mathcal{L}_h$, u 1–excessive (resp. $\widehat{e}_h \le u$ for all $u \in \mathcal{L}_h$, u 1–coexcessive) by the 2. Step. Since

$$\begin{aligned} \left(\beta_h\left(e_h\right), \widehat{G}_1 v\right)_{\mathcal{H}} &= \lim_{\alpha\to\infty} \left(\beta_h\left(h_\alpha\right), \widehat{G}_1 v\right)_{\mathcal{H}} \\ &= \lim_{\alpha\to\infty} (1/\alpha)\mathcal{E}_1\left(h_\alpha, \widehat{G}_1 v\right) \\ &= \lim_{\alpha\to\infty} (1/\alpha)(h_\alpha, v)_{\mathcal{H}} = 0\ , \end{aligned}$$

for all $v \in \mathcal{H}$, and $\widehat{G}_1(\mathcal{H})$ dense in $\mathcal{H}$ it follows $\beta_h(e_h) = 0$, i.e., $e_h \in \mathcal{L}_h$. The statement $\beta_h(\widehat{e}_h) = 0$ is proved similarly.

For $v \in \mathcal{L}_h \cap \widehat{\mathcal{F}}$ we obtain that

$$\begin{aligned}\mathcal{A}_1(e_h, e_h) &\leq \liminf_{\alpha\to\infty} \mathcal{A}_1(h_\alpha, h_\alpha) \leq \liminf_{\alpha\to\infty} \mathcal{E}_1(h_\alpha, h_\alpha) \\ &= \liminf_{\alpha\to\infty} \alpha(\beta_h(h_\alpha), h_\alpha)_{\mathcal{H}} \leq \liminf_{\alpha\to\infty} \alpha(\beta_h(h_\alpha), v)_{\mathcal{H}} \\ &= \liminf_{\alpha\to\infty} \mathcal{E}_1(h_\alpha, v) = \mathcal{E}_1(e_h, v) \ .\end{aligned}$$

The statement $\mathcal{A}_1(\widehat{e}_h, \widehat{e}_h) \leq \mathcal{E}_1(v, \widehat{e}_h)$ for all $v \in \mathcal{L}_h \cap \mathcal{F}$ can be proved similarly. □

Definition 1.8. e_h (resp. $\widehat{e}_h$) is called *1–reduced* (resp. *1–coreduced)* function of h.

Remark 1.9. Let $h \in \mathcal{H}$. The following example shows that an element $w \in \mathcal{L}_h \cap \mathcal{V}$ with $\mathcal{E}_1(w, v) \geq \mathcal{A}_1(w, w)$ for all $v \in \mathcal{L}_h \cap \widehat{\mathcal{F}}$ is not uniquely determined in general. Let $\mathcal{E}$ be the generalized Dirichlet form corresponding to the uniform motion on the real line, i.e., $\mathcal{A} = 0$ on $\mathcal{H} = L^2(\mathbb{R}, dx)$ and $(\Lambda, D(\Lambda, \mathcal{H})) = (\frac{d}{dt}, H^{1,2}(\mathbb{R}))$. Let $h = \mathbb{1}_{[0,1]}$ and $w_\beta(t) := e^{\beta t} \wedge \mathbb{1}_{(-\infty,1]}(t)\,, \beta \in [1/2, 1]$. It is easy to see that $w_\beta \in \mathcal{L}_h \cap \mathcal{V} = \mathcal{L}_h$ and $\mathcal{E}_1(w_\beta, v) \geq \mathcal{A}_1(w_\beta, w_\beta)$ for all $v \in \mathcal{L}_h \cap \widehat{\mathcal{F}}$.

III.2 Capacities of generalized positivity preserving forms

For the rest of this chapter we assume that E is a Hausdorff topological space, and $\mathcal{B}$ its Borel σ–algebra. For an open subset $U \subset E$ and an element $f \in \mathcal{H}$ with $\mathcal{L}_{f\cdot\mathbb{1}_U} \cap \mathcal{F} \neq \emptyset$ (resp. $\mathcal{L}_{f\cdot\mathbb{1}_U} \cap \widehat{\mathcal{F}} \neq \emptyset$) let $f_U := e_{(f\cdot\mathbb{1}_U)}$ be the 1–reduced function (resp. $\widehat{f}_U := \widehat{e}_{(f\cdot\mathbb{1}_U)}$ the 1–coreduced function) of $f \cdot \mathbb{1}_U$. Let f_U^α (resp. $\widehat{f}_U^\alpha$), $\alpha > 0$, denote the element $(f \cdot \mathbb{1}_U)_\alpha$ (resp. $(\widehat{f \cdot \mathbb{1}_U})_\alpha$) (cf. Proposition 1.6).

Lemma 2.1. *Let $u \in \mathcal{H}$.*

(i) If u is 1–excessive (resp. 1–coexcessive) and $\mathcal{L}_u \cap \mathcal{F} \neq \emptyset$ (resp. $\mathcal{L}_u \cap \widehat{\mathcal{F}} \neq \emptyset$) then $u \in \mathcal{V}$.

(ii) If $U \subset E$ open and is 1–excessive (resp. 1–coexcessive) such that $\mathcal{L}_u \cap \mathcal{F} \neq \emptyset$ (resp. $\mathcal{L}_u \cap \widehat{\mathcal{F}} \neq \emptyset$) then $u = u_U$ (resp. $u = \widehat{u}_U$) on U.

Proof. (i) Let $f \in \mathcal{L}_u \cap \mathcal{F}$. Then

$$\begin{aligned}\mathcal{A}_1(\beta G_{\beta+1}u - f, \beta G_{\beta+1}u - f) &\leq \mathcal{E}_1(\beta G_{\beta+1}u - f, \beta G_{\beta+1}u - f) \\ &= \beta(u - \beta G_{\beta+1}u, \beta G_{\beta+1}u - f)_{\mathcal{H}} - \mathcal{E}_1(f, \beta G_{\beta+1}u - f) \\ &\leq 2K\|f\|_{\mathcal{F}}\|\beta G_{\beta+1}u - f\|_{\mathcal{V}}\,.\end{aligned}$$

Consequently, $\sup_{\beta>0}\|\beta G_{\beta+1}u\|_{\mathcal{V}} < \infty$ which implies $u \in \mathcal{V}$. The dual statement can be shown in a similar way.
(ii) $u_U \wedge u$ is 1–excessive since u and u_U are 1–excessive and consequently, $u \cdot \mathbb{1}_U \leq u_U \leq u_U \wedge u$ by Proposition 1.7 which implies $u = u_U$ on U. The dual statement for $\widehat{u}_U$ can be shown similarly. □

Lemma 2.2.

(i) Let $U \subset E$, U open. Then

$$\|f_U\|_{\mathcal{V}} \leq 3K\|f\|_{\mathcal{F}} \text{ for all } f \in \mathcal{F}$$
$$\left(\text{ resp. } \|\widehat{f}_U\|_{\mathcal{V}} \leq 3K\|\widehat{f}\|_{\widehat{\mathcal{F}}} \text{ for all } \widehat{f} \in \widehat{\mathcal{F}}\right) .$$

(ii) Let $f \in \mathcal{F}$. Then $\lim_{\alpha\to\infty} e_{(\alpha G_{\alpha+1}f-f)} = 0$ in $\mathcal{H}$.

Proof. (i) For $f \in \mathcal{F}$

$$\begin{aligned}\mathcal{A}_1(f - f_U, f - f_U) &\leq \liminf_{\alpha\to\infty} \mathcal{A}_1(f - f_U^\alpha, f - f_U^\alpha) \\ &\leq \liminf_{\alpha\to\infty} \mathcal{E}_1(f - f_U^\alpha, f - f_U^\alpha) \\ &\leq \liminf_{\alpha\to\infty} \mathcal{E}_1(f, f - f_U^\alpha) \\ &\leq 2K\|f\|_{\mathcal{F}}\|f - f_U\|_{\mathcal{V}} .\end{aligned}$$

Hence $\|f_U\|_{\mathcal{V}} \leq 3K\|f\|_{\mathcal{F}}$. The dual statement can be proved similarly.
(ii) First assume that $f = G_1 h$ for some $h \in \mathcal{H}$. Then $\lim_{\alpha\to\infty} \alpha G_{\alpha+1} f = f$ in $\mathcal{V}$ by Proposition I.3.7 and moreover,

$$\begin{aligned}|\langle\Lambda(\alpha G_{\alpha+1}f - f), v\rangle| &= |\mathcal{E}_1(\alpha G_{\alpha+1}f - f, v) - \mathcal{A}_1(\alpha G_{\alpha+1}f - f, v)| \\ &\leq (\|\alpha G_{\alpha+1}h - h\|_{\mathcal{H}} + K\|\alpha G_{\alpha+1}f - f\|_{\mathcal{V}})\|v\|_{\mathcal{V}}\end{aligned}$$

for all $v \in \mathcal{V}$ which implies that $\lim_{\alpha\to\infty} \|\Lambda(\alpha G_{\alpha+1}f - f)\|_{\mathcal{V}'} = 0$ and consequently, $\lim_{\alpha\to\infty} \|\alpha G_{\alpha+1}f - f\|_{\mathcal{F}} = 0$. Since $f_E = e_f$ we obtain by (i) that $\lim_{\alpha\to\infty} \|e_{(\alpha G_{\alpha+1}f-f)}\|_{\mathcal{V}} = 0$.
For arbitrary $f \in \mathcal{F}$ and $\varepsilon > 0$ let $g \in G_1(\mathcal{H})$ with $\|f - g\|_{\mathcal{F}} \leq \varepsilon/(3K)$. Let $\alpha_0 > 0$ be such that $\|e_{\alpha G_{\alpha+1}g-g}\|_{\mathcal{V}} \leq \varepsilon/3$ for all $\alpha \geq \alpha_0$. Since $e_{f-g} \in \mathcal{P}_1$ and $e_{f-g} \geq \alpha G_{\alpha+1}e_{f-g} \geq \alpha G_{\alpha+1}(f - g)$ it follows from Proposition 1.7 that $e_{\alpha G_{\alpha+1}(f-g)} \leq e_{f-g}$, $\alpha > 0$. Thus $\alpha G_{\alpha+1}f - f \leq e_{f-g} + e_{g-f} + e_{\alpha G_{\alpha+1}g-g}$ and Proposition 1.7 now implies that $e_{\alpha G_{\alpha+1}f-f} \leq e_{f-g} + e_{g-f} + e_{\alpha G_{\alpha+1}g-g}$, $\alpha > 0$. By (i)

$$\|e_{\alpha G_{\alpha+1}f-f}\|_{\mathcal{H}} \leq \|e_{f-g}\|_{\mathcal{H}} + \|e_{g-f}\|_{\mathcal{H}} + \|e_{\alpha G_{\alpha+1}g-g}\|_{\mathcal{H}} \leq \varepsilon$$

for all $\alpha \geq \alpha_0$. □

Definition 2.3.

(i) An increasing sequence of closed subsets $(F_k)_{k\geq1}$ is called an $\mathcal{E}$–*nest* if for every element $u \in \mathcal{F} \cap \mathcal{P}_1$ it follows that $\lim_{k\to\infty} u_{E\setminus F_k} = 0$ in $\mathcal{H}$.

(ii) An $\mathcal{E}$–nest $(F_k)_{k\geq0}$ is called a *regular $\mathcal{E}$–nest* if for all $k \geq 1$, $U \subset E$, U open, $m(U \cap F_k) = 0$ implies that $U \subset E \setminus F_k$.

(iii) A subset $N \subset E$ is called $\mathcal{E}$–*exceptional* if there is an $\mathcal{E}$–nest $(F_k)_{k\geq1}$ such that $N \subset \bigcap_{k\geq1} E \setminus F_k$.

(iv) We say that a property of points in E holds $\mathcal{E}$–*quasi everywhere ($\mathcal{E}$–q.e.)* if the property holds for all points outside some $\mathcal{E}$–exceptional set.

Lemma 2.4. *Let $(F_k)_{k\geq 1}$ be an $\mathcal{E}$–nest, such that the relative topology on F_k is strongly Lindeloef for all k, and define $\tilde{F}_k := supp(\mathbb{1}_{F_k} \cdot m)$. Then $(\tilde{F}_k)_{k\geq 1}$ is a regular $\mathcal{E}$–nest with $\tilde{F}_k \subset F_k$ for all k.*

Proof. Note that $\mathrm{supp}(\mathbb{1}_{F_k} \cdot m)$ exists, and that $m(F_k \setminus \tilde{F}_k) = 0$, since the relative topology on F_k is strongly Lindeloef. Consequently, $\mathcal{L}_{f\cdot \mathbb{1}_{F_k^c}} = \mathcal{L}_{f\cdot \mathbb{1}_{\tilde{F}_k^c}}$ which implies $f_{F_k^c} = f_{\tilde{F}_k^c}$ and thus $\lim_{k\to\infty} f_{\tilde{F}_k^c} = 0$ in $\mathcal{H}$ for all $f \in \mathcal{F} \cap \mathcal{P}_1$. Therefore, $(\tilde{F}_k)_{k\geq 1}$ is an $\mathcal{E}$–nest and for arbitrary $U \subset E$, U open, with $m(U \cap \tilde{F}_k) = 0$ we obtain that $U \subset \tilde{F}_k^c$. $\square$

Construction of a Choquet capacity

Throughout this subsection let $\varphi \in \mathcal{H}$, $0 < \varphi$, $h := G_1\varphi$ and $\widehat{g} := \widehat{G}_1\varphi$. Since $\mathcal{L}_h \cap \mathcal{F} \neq \emptyset$ and $\mathcal{L}_{\widehat{g}} \cap \widehat{\mathcal{F}} \neq \emptyset$ the 1–reduced function h_U and the 1–coreduced function $\widehat{g}_U$ for all open subsets $U \subset E$ are well–defined.

Definition 2.5. For $U \subset E$, U open, let $\mathrm{Cap}_\varphi(U) := (h_U, \varphi)_{\mathcal{H}}$ and for arbitrary $A \subset E$ let $\mathrm{Cap}_\varphi(A) := \inf\limits_{\substack{A\subset U\\ U\text{open}}} \mathrm{Cap}_\varphi(U)$.

Remark 2.6. (i) $\mathrm{Cap}_\varphi(A) = 0$ for $A \in \mathcal{B}(E)$ implies $m(A) = 0$ since

$$0 = \inf_{\substack{A\subset U\\ U\text{open}}} (h_U, \varphi) \geq (\mathbb{1}_A \cdot h, \varphi)_{\mathcal{H}} = \left(\widehat{G}_1\left(\mathbb{1}_A\varphi\right), \varphi\right)_{\mathcal{H}}$$

which implies $\widehat{G}_1\left(\mathbb{1}_A\varphi\right) = 0$, thus $\mathbb{1}_A\varphi = 0$ and consequently, $m(A) = 0$.
(ii) In the situation of Example I.4.9 (i), i.e., $(\mathcal{A}, \mathcal{V})$ is a Dirichlet form on $\mathcal{H}$ and $\Lambda = 0$, hence $\mathcal{F} = \mathcal{V} = \widehat{\mathcal{F}}$ and $\mathcal{E} = \mathcal{A}$ let $u \in \mathcal{F}$. It follows from Proposition 1.7 (iii), that for any open set U, the 1–reudced function u_U coincides with the 1–reduced function as defined in [MR1, Proposition III.1.5]. It follows that in this special situation an increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is an $\mathcal{A}$–nest in the sense of [MOvR, Definition 2.9] if and only if $(F_k)_{k\geq 1}$ is an $\mathcal{A}$–nest in the sense of Definition 2.3 (cf. [MOvR, Lemma 2.10 and Theorem 2.14]). Moreover, Cap_φ as defined in Definition 2.5 coincides with the φ–capacity in [MOvR]. In particular, if $(\mathcal{A}, \mathcal{V})$ is a Dirichlet form in the sense of [MR1], i.e., both $(\mathcal{A}, \mathcal{V})$ and the coform $(\widehat{\mathcal{A}}, \mathcal{V})$ defined by $\widehat{\mathcal{A}}(u, v) := \mathcal{A}(v, u)\,; u, v \in \mathcal{V}$, are Dirichlet forms in the sense of Definition I.4.3, Cap_φ coincides with $\mathrm{Cap}_{h,g}$ as defined in [MR1, Section III.2].

Denote by $(L, D(L))$ the Dirichlet operator corresponding to $(\mathcal{A}, \mathcal{V})$. As mentioned in Remark I.4.10 we can define another generalized Dirichlet form $\mathcal{E}'$ given by $\mathcal{E}'(u, v) := (-Lu, v)_{\mathcal{H}}$ if $u \in D(L), v \in \mathcal{H}$, and $\mathcal{E}'(u, v) := (u, -\widehat{L}v)_{\mathcal{H}}$ if $u \in \mathcal{H}, v \in D(\widehat{L})$. We claim that the 1–reduced function of h on U, which we denote in this case by h'_U, $U \subset E$, U open, coincides with the 1–reduced function in the first case. Indeed, since the associated resolvents in both cases coincide, the set of 1–excessive functions w.r.t. $\mathcal{A}$ and $\mathcal{E}'$ coincide. It is easy to see that Proposition

1.7 (ii) now implies $h'_U = h_U$. As a consequence we obtain that the φ–capacity, in this case denoted by Cap'_φ, coincides with Cap_φ.

Lemma 2.7. *(i) If $(U_n)_{n\geq 1}$ is an increasing sequence of open subsets then*

$$\mathrm{Cap}_\varphi(\bigcup_{n\geq 1} U_n) = \sup_{n\geq 1} \mathrm{Cap}_\varphi(U_n).$$

(ii) For $U, V \subset E$ open we have

$$\mathrm{Cap}_\varphi(U\cap V) + \mathrm{Cap}_\varphi(U\cup V) \leq \mathrm{Cap}_\varphi(U) + \mathrm{Cap}_\varphi(V)\ .$$

("strong subadditivity").

Proof. (i) Let $U := \bigcup_{n\geq 1} U_n$. Obviously $h_{U_n} \leq h_{U_{n+1}} \leq h_U$, $n \geq 1$, and therefore, $\lim_{n\to\infty} h_{U_n} = h_\infty$ in $\mathcal{H}$ for some $h_\infty \in \mathcal{H}$ with $h_\infty \leq h_U$ by the monotone convergence theorem. Since h_∞ is 1–excessive and $h_\infty \in \mathcal{L}_{h\cdot \mathbb{1}_U}$ it follows from Proposition 1.7 that $h_U \leq h_\infty$. Hence $h_U = h_\infty$ and

$$\mathrm{Cap}_\varphi(\bigcup_{n\geq 1} U_n) = (h_U, \varphi)_{\mathcal{H}} = \sup_{n\geq 1}(h_{U_n}, \varphi)_{\mathcal{H}} = \sup_{n\geq 1} \mathrm{Cap}_\varphi(U_n)\ .$$

(ii) Clearly $h_{U\cap V} \leq h_U \wedge h_V$ and $h_{U\cup V} = h_U$ on U, $h_{U\cup V} = h_V$ on V, and $\widehat{g} = \widehat{g}_{U\cup V}$ on $U\cup V$. Hence it follows that

$$\begin{aligned}
\mathrm{Cap}_\varphi(U\cap V) + \mathrm{Cap}_\varphi(U\cup V) &= \mathcal{E}_1(h_{U\cap V} + h_{U\cup V}, \widehat{g}) \\
&= \lim_{\alpha\to\infty} \mathcal{E}_1(h^\alpha_{U\cap V} + h^\alpha_{U\cup V}, \widehat{g}) \\
&= \lim_{\alpha\to\infty} \mathcal{E}_1(h^\alpha_{U\cap V} + h^\alpha_{U\cup V}, \widehat{g}_{U\cup V}) \\
&= \lim_{\alpha\to\infty} \lim_{\beta\to\infty} \mathcal{E}_1(h^\alpha_{U\cap V} + h^\alpha_{U\cup V}, \widehat{g}^\beta_{U\cup V}) \\
&\leq \limsup_{\alpha\to\infty} \limsup_{\beta\to\infty} \mathcal{E}_1(h_U + h_V, \widehat{g}^\beta_{U\cup V}) \\
&\leq \limsup_{\alpha\to\infty} \limsup_{\beta\to\infty} \mathcal{E}_1(h_U + h_V, \widehat{g}) \\
&= \mathrm{Cap}_\varphi(U) + \mathrm{Cap}_\varphi(V)\ . \quad \square
\end{aligned}$$

By the last lemma the following proposition is an easy consequence (cf. [MR1, Theorem III.2.8]).

Proposition 2.8.

(i) Cap_φ is a Choquet capacity, i.e., it has the following properties:

(a) If $(A_n)_{n\geq 1}$ is an increasing sequence of arbitrary subsets of E then

$$\mathrm{Cap}_\varphi(\bigcup_{n\geq 1} A_n) = \sup_{n\geq 1} \mathrm{Cap}_\varphi(A_n).$$

(b) If $(K_n)_{n\geq 1}$ is a decreasing sequence of compact subsets of E then

$$\mathrm{Cap}_\varphi(\bigcap_{n\geq 1} K_n) = \inf_{n\geq 1} \mathrm{Cap}_\varphi(K_n).$$

(ii) For any sequence $(A_n)_{n\geq 1}$ of arbitrary subsets A_n of E we have that

$$\mathrm{Cap}_\varphi(\bigcup_{n\geq 1} A_n) \leq \sum_{n\geq 1} \mathrm{Cap}_\varphi(A_n)\ .$$

Lemma 2.9. *Let $f \in \mathcal{F} \cap \mathcal{P}_1$, $\widehat{f} \in \widehat{\mathcal{F}} \cap \widehat{\mathcal{P}}_1$ and $U \subset E$ open. Then $\mathcal{E}_1(f_U, \widehat{f}) = \mathcal{E}_1(f, \widehat{f}_U)$.*

Proof.

$$\begin{aligned}\mathcal{E}_1(f_U, \widehat{f}) &= \lim_{\alpha\to\infty} \mathcal{E}_1(f_U^\alpha, \widehat{f}) = \lim_{\alpha\to\infty} \mathcal{E}_1(f_U^\alpha, \widehat{f}_U)\\ &= \lim_{\alpha\to\infty} \lim_{\beta\to\infty} \mathcal{E}_1(f_U^\alpha, \widehat{f}_U^\beta)\\ &\leq \limsup_{\alpha\to\infty} \limsup_{\beta\to\infty} \mathcal{E}_1(f, \widehat{f}_U^\beta) = \mathcal{E}_1(f, \widehat{f}_U)\end{aligned}$$

by Proposition 1.4 and conversely,

$$\begin{aligned}\mathcal{E}_1(f, \widehat{f}_U) &= \lim_{\alpha\to\infty} \mathcal{E}_1(f, \widehat{f}_U^\alpha) = \lim_{\alpha\to\infty} \mathcal{E}_1(f_U, \widehat{f}_U^\alpha)\\ &\leq \mathcal{E}_1(f_U, \widehat{f})\end{aligned}$$

by Proposition 1.3. □

Proposition 2.10. *Let $(F_k)_{k\geq 1}$ be an increasing sequence of closed subsets. Then $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest if and only if $\lim\limits_{k\to\infty} \mathrm{Cap}_\varphi(F_k^c) = 0$.*

Proof. Suppose $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest. Then $\lim_{k\to\infty} h_{F_k^c} = 0$ in $\mathcal{H}$ and therefore, $\lim_{k\to\infty} \mathrm{Cap}_\varphi(F_k^c) = 0$.

On the other hand suppose $\lim\limits_{k\to\infty} \mathrm{Cap}_\varphi(F_k^c) = 0$. Since $(\widehat{g}_{F_k^c})_{k\geq 1}$ is decreasing, there exists some $\widehat{g}_\infty \in \mathcal{H}$ such that $\lim_{k\to\infty} \widehat{g}_{F_k^c} = \widehat{g}_\infty$ in $\mathcal{H}$. By the last lemma we obtain that

$$\begin{aligned}(\varphi, \hat{g}_\infty)_\mathcal{H} &= \lim_{k\to\infty} (\varphi, \widehat{g}_{F_k^c})_\mathcal{H} = \lim_{k\to\infty} \mathcal{E}_1(h, \widehat{g}_{F_k^c})\\ &= \lim_{k\to\infty} \mathcal{E}_1(h_{F_k^c}, \widehat{g}) = \lim_{k\to\infty} \mathrm{Cap}_\varphi(F_k^c) = 0\end{aligned}$$

and therefore, $\widehat{g}_\infty = 0$. Since $\sup_{k\geq 1} \|\widehat{g}_{F_k^c}\|_{\mathcal{V}} < \infty$ we conclude that $\lim_{k\to\infty} \widehat{g}_{F_k^c} = 0$ weakly in $\mathcal{V}$.

Let $f \in \mathcal{F} \cap \mathcal{P}_1$ be arbitrary. In the same manner as above we can show the existence of a function $f_\infty \in \mathcal{H}$ with $\lim_{k\to\infty} f_{F_k^c} = f_\infty$ in $\mathcal{H}$. Since

$$
\begin{aligned}
(f_\infty, \varphi)_{\mathcal{H}} &= \lim_{k\to\infty} (f_{F_k^c}, \varphi)_{\mathcal{H}} = \lim_{k\to\infty} \mathcal{E}_1(f_{F_k^c}, \widehat{g}) \\
&= \lim_{k\to\infty} \mathcal{E}_1(f, \widehat{g}_{F_k^c}) = 0
\end{aligned}
$$

it follows that $f_\infty = 0$ which implies the assertion. □

For $A \subset E$ closed and a linear subspace $W \subset \mathcal{H}$ let

$$
W_A := \{u \mid u \in W, u = 0 \quad m - \text{a.e. on } E \setminus A\}. \tag{2.1}
$$

Remark 2.11. Let $(F_k)_{k\geq 1}$ be an increasing sequence of closed subsets. If $\bigcup_{k\geq 1} \mathcal{F}_{F_k}$ is dense in $\mathcal{F}$ it follows that $(F_k)_{k\geq 1}$ is an $\mathcal{E}$–nest.

Proof. The existence of an element $\widehat{g}_\infty \in \mathcal{V}$ such that $\lim_{k\to\infty} \widehat{g}_{F_k^c} = \widehat{g}_\infty$ in $\mathcal{H}$ and weakly in $\mathcal{V}$ follows in the same way as in the proof of Proposition 2.10. Clearly for $f \in \mathcal{F}_{F_k}$ and $\ell \geq k$ we have

$$
\mathcal{E}_1(f, \widehat{g}_{F_\ell^c}) = \lim_{\alpha\to\infty} \mathcal{E}_1(f, \widehat{g}^\alpha_{F_\ell^c}) = 0
$$

and therefore, $\mathcal{E}_1(f, \widehat{g}_\infty) = \lim_{\ell\to\infty} \mathcal{E}_1(f, \widehat{g}_{F_\ell^c}) = 0$ for all $f \in \bigcup_{k\geq 1} \mathcal{F}_{F_k}$. Therefore, $\widehat{g}_\infty = 0$ since $\bigcup_{k\geq 1} \mathcal{F}_{F_k}$ is dense in $\mathcal{F}$. Thus

$$
\lim_{k\to\infty} \operatorname{Cap}_\varphi(F_k^c) = \lim_{k\to\infty} \mathcal{E}_1(h_{F_k^c}, \widehat{g}) = \lim_{k\to\infty} \mathcal{E}_1(h, \widehat{g}_{F_k^c}) = 0\ . \quad \square
$$

Lemma 2.12. *Let $(U_n)_{n\geq 1}$ be an increasing sequence of open subsets in E. Let $U := \bigcup_{n\geq 1} U_n$ and $u \in \mathcal{P}_1$ such that $\mathcal{L}_u \cap \mathcal{F} \neq \emptyset$. Then $\lim_{n\to\infty} u_{U_n} = u_U$ in $\mathcal{H}$.*

Proof. Clearly $u_{U_n} \leq u_{U_{n+1}} \leq u_U$, $n \geq 1$, and therefore, $\lim_{n\to\infty} u_{U_n} = u_\infty$ in $\mathcal{H}$ for some $u_\infty \in \mathcal{H}$ with $u_\infty \leq u_U$ by the monotone convergence theorem. Since u_∞ is 1–excessive and $u_\infty \in \mathcal{L}_{u 1\!\!1_U}$ it follows from Proposition 1.7 that $u_U \leq u_\infty$. Consequently, $u_U = u_\infty$ which implies the assertion. □

III.3 $\mathcal{E}$–quasi continuity

Throughout this section we suppose that $\mathcal{E}$ is a generalized Dirichlet form. In particular the associated resolvent $(G_\alpha)_{\alpha>0}$ is sub–Markovian. Let $\varphi \in \mathcal{H}$, $0 < \varphi \leq 1$, $h := G_1\varphi$ and $\widehat{g} := \widehat{G}_1\varphi$. Since $(G_\alpha)_{\alpha>0}$ is sub–Markovian we obtain that $0 \leq h \leq 1$.

Definition 3.1.

(i) A function f defined up to some $\mathcal{E}$–exceptional set $N \subset E$ is called *$\mathcal{E}$–quasi–continuous ($\mathcal{E}$–q.c.)* (resp. *$\mathcal{E}$–quasi–lower–semicontinuous ($\mathcal{E}$–q.l.s.c.)*) if there is some $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ such that $\bigcup\limits_{k\geq 1} F_k \subset E \setminus N$ and $f_{|F_k}$ is continuous (resp. lower semicontinuous) for all k.

(ii) A sequence of $\mathcal{E}$–q.e. defined functions $(f_k)_{k\geq 1}$ converges *$\mathcal{E}$–quasi–uniformly* to some $\mathcal{E}$–q.e. defined function f if there is some $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ such that f_k, $k \geq 1$, and f are defined on $\bigcup_{k\geq 1} F_k$ and $\lim_{k\to\infty} f_k = f$ uniformly on F_k for all k.

For an $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ let

$$\begin{aligned} C(\{F_k\}) =& \{f : A \to \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E, f_{|F_k} \text{ is continuous } \forall k\} \\ C_l(\{F_k\}) =& \{f : A \to \mathbb{R} \mid \bigcup_{k\geq 1} F_k \subset A \subset E, f_{|F_k} \text{ is lower semicontinuous } \forall k\} \ . \end{aligned}$$

Proposition 3.2. *Let $(F_k)_{k\geq 1}$ be a regular $\mathcal{E}$–nest and $f \in C_l(\{F_k\})$. If $U \subset E$ open and $f \leq 0$ on U m–a.e. then $f(x) \leq 0$ for all $x \in \bigcup_{k\geq 1}(F_k \cap U)$.*

Proof. The set $V := (\{f > 0\} \cup F_k^c) \cap U$ is open in E. Now the assertion follows as in [MR1, Proposition III.3.9]. □

Corollary 3.3. *Let $(F_k)_{k\geq 1}$ be a regular $\mathcal{E}$–nest and $f \in C_l(\{F_k\})$. If $f \leq 0$ m–a.e. then $f \leq 0$ $\mathcal{E}$–q.e.*

Proof. The assertion follows immediately by Proposition 3.2. □

Corollary 3.4. *Let $(F_k)_{k\geq 1}$ be a regular $\mathcal{E}$–nest and $f, g \in C(\{F_k\})$. If $f = g$ m–a.e. then $f = g$ $\mathcal{E}$–q.e.*

Proof. Since $f - g$ and $g - f$ are both $\mathcal{E}$–q.l.s.c. the assertion follows by the last corollary. □

Lemma 3.5. *Let S be a countable family of $\mathcal{E}$–q.c. (resp. $\mathcal{E}$–q.l.s.c.) functions. Then there exists an $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ such that $S \subset C(\{F_k\})$ (resp. $S \subset C_l(\{F_k\})$).*

Proof. By Proposition 2.10 the assertion can be proved in the same way as in [MR1, Proposition III.3.3]. □

We can prove now the following Chebychev–type inequality:

Proposition 3.6. *For $u \in \mathcal{P}_1$ with $\mathcal{E}$–q.l.s.c. m–version $\tilde{u}$ and $\lambda > 0$*

$$Cap_\varphi(\{\tilde{u} > \lambda\}) \leq (1/\lambda)\ \|\varphi\|_{\mathcal{H}} \|u\|_{\mathcal{H}} \ . \tag{3.1}$$

Proof. Suppose $\tilde{u} \in C_l(\{F_k\})$. Then $\{\tilde{u} > \lambda\} \cup F_k^c$ is open and

$$h \leq \frac{u}{\lambda} + h_{F_k^c} \text{ on } \{\tilde{u} > \lambda\} \cup F_k^c \text{ for all } k.$$

Hence

$$\begin{aligned}\mathrm{Cap}_\varphi(\{\tilde{u} > \lambda\}) \leq & (u/\lambda + h_{F_k^c}, \varphi)_\mathcal{H} \\ \leq & (1/\lambda)\|u\|_\mathcal{H}\|\varphi\|_\mathcal{H} + \mathrm{Cap}_\varphi(F_k^c) \text{ for all } k. \quad \square\end{aligned}$$

Proposition 3.7. *Let $u_n \in \mathcal{H}$ with $\mathcal{E}$–q.c. m–version $\tilde{u}_n$, $n \geq 1$, such that $\lim_{n\to\infty} e_{u_n - u} + e_{u-u_n} = 0$ in $\mathcal{H}$ for some $u \in \mathcal{H}$. Then there exist a subsequence $(\tilde{u}_{n_k})_{k\geq 1}$ and an $\mathcal{E}$–q.c. m–version $\tilde{u}$ of u such that $\lim_{k\to\infty} \tilde{u}_{n_k} = \tilde{u}$ $\mathcal{E}$–quasi–uniformly.*

Proof. Let $v_n := e_{u_n - u} + e_{u-u_n}$, $n \geq 1$. Let $(F_k)_{k\geq 1}$ be an $\mathcal{E}$–nest such that $\{\tilde{u}_n | n \geq 1\} \subset C(\{F_k\})$. Then $\{|\tilde{u}_n - \tilde{u}_m| > \lambda\} \cup F_k^c$ is open for all $\lambda > 0$ and

$$h \leq (1/\lambda)(v_n + v_m) + h_{F_k^c} \text{ on } \{|\tilde{u}_n - \tilde{u}_m| > \lambda\} \cup F_k^c$$

for all k. Hence

$$\begin{aligned}\mathrm{Cap}_\varphi(\{|\tilde{u}_n - \tilde{u}_m| > \lambda\}) \leq & (1/\lambda)(v_n + v_m, \varphi)_\mathcal{H} + (h_{F_k^c}, \varphi)_\mathcal{H} \\ \leq & (1/\lambda)(\|v_n\|_\mathcal{H} + \|v_m\|_\mathcal{H})\|\varphi\|_\mathcal{H} + \mathrm{Cap}_\varphi(F_k^c)\end{aligned}$$

for all k which implies that

$$\mathrm{Cap}_\varphi(\{|\tilde{u}_n - \tilde{u}_m| > \lambda\}) \leq (1/\lambda)(\|v_n\|_\mathcal{H} + \|v_m\|_\mathcal{H})\|\varphi\|_\mathcal{H} \, .$$

Let $(n_k)_{k\geq 1}$ be a subsequence with $\sum_{k=1}^\infty 2^k \|v_{n_k}\|_\mathcal{H} < \infty$ and let

$$A_k := \bigcup_{i\geq k} \{|\tilde{u}_{n_{i+1}} - \tilde{u}_{n_i}| > 2^{-i}\} \, , k \geq 1 \, ,$$

then $F_k' := F_k \cap A_k^c$ is an increasing sequence of closed subsets with

$$\begin{aligned}\mathrm{Cap}_\varphi(E \setminus F_k') \leq & \mathrm{Cap}_\varphi(F_k^c) + \mathrm{Cap}_\varphi(A_k) \\ \leq & \mathrm{Cap}_\varphi(F_k^c) + \sum_{i\geq k} 2^i (\|v_{n_{i+1}}\|_\mathcal{H} + \|v_{n_i}\|_\mathcal{H})\|\varphi\|_\mathcal{H} \\ \leq & \mathrm{Cap}_\varphi(F_k^c) + (3/2)\|\varphi\|_\mathcal{H} \sum_{i\geq k} 2^i \|v_{n_i}\|_\mathcal{H} \to 0 \, , k \to \infty \, ,\end{aligned}$$

hence $(F_k')_{k\geq 1}$ is an $\mathcal{E}$–nest. Since $(\tilde{u}_{n_k})_{k\geq 1}$ converges uniformly on F_k' we obtain that

$$\tilde{u}(x) := \begin{cases} \lim_{k\to\infty} \tilde{u}_{n_k}(x) & \text{if } x \in \bigcup_{k\geq 1} F_k' \\ 0 & \text{else} \end{cases}$$

is $\mathcal{E}$–q.c. Since $|u_n - u| \leq v_n$, hence $\lim_{n\to\infty} u_n = u$ in $\mathcal{H}$, and $m(\bigcap_{k\geq 1} E \setminus F_k') = 0$ it follows that $\tilde{u}$ is an $\mathcal{E}$–q.c. m–version of u. $\square$

Corollary 3.8. *Let $u_n \in \mathcal{F}$ with $\mathcal{E}$–q.c. m–version $\tilde{u}_n$, $n \geq 1$, such that $\lim_{n\to\infty} u_n = u$ in $\mathcal{F}$. Then there exist a subsequence $(\tilde{u}_{n_k})_{k\geq 1}$ and an $\mathcal{E}$–q.c. m–version $\tilde{u}$ of u such that $\lim_{k\to\infty} \tilde{u}_{n_k} = \tilde{u}$ $\mathcal{E}$–quasi–uniformly.*

Proof. By Lemma 2.2 $\lim_{n\to\infty} e_{u_n-u}+e_{u-u_n} = 0$ in $\mathcal{V}$. Hence the assertion follows from Proposition 3.7. □

Lemma 3.9. *Let $u_n \in \mathcal{P}_1$ with $\mathcal{E}$–q.l.s.c. m–versions $\tilde{u}_n$, $n \geq 1$, such that $\lim_{n\to\infty} u_n = 0$ in $\mathcal{H}$. Then there exists a subsequence $(\tilde{u}_{n_k})_{k\geq 1}$ such that $\limsup_{k\to\infty} \tilde{u}_{n_k} \leq 0$ $\mathcal{E}$–q.e.*

Proof. By (3.1) we can find a subsequence $(u_{n_k})_{k\geq 1}$ with $\mathrm{Cap}_\varphi(\{\tilde{u}_{n_k} > 2^{-k}\}) \leq 2^{-k}$, $k \geq 1$. Let $u_\infty := \limsup_{k\to\infty} \tilde{u}_{n_k}$. Since for $n \geq \ell$

$$\mathrm{Cap}_\varphi(\bigcup_{m\geq n} \{\tilde{u}_{k_m} > 2^{-\ell}\}) \leq \sum_{m\geq n} \mathrm{Cap}_\varphi\left(\{\tilde{u}_{k_m} > 2^{-m}\}\right) \leq 2^{-(n-1)}$$

it follows that

$$\begin{aligned}\mathrm{Cap}_\varphi\left(\{u_\infty > 2^{-\ell}\}\right) &\leq \mathrm{Cap}_\varphi(\bigcap_{n\geq 1}\bigcup_{m\geq n} \{\tilde{u}_{k_m} > 2^{-\ell}\}) \\ &\leq \inf_{n\geq 1} \mathrm{Cap}_\varphi(\bigcup_{m\geq n} \{\tilde{u}_{k_m} > 2^{-\ell}\}) \\ &\leq \inf_{n\geq \ell} \mathrm{Cap}_\varphi(\bigcup_{m\geq n} \{\tilde{u}_{k_m} > 2^{-\ell}\}) = 0 \ .\end{aligned}$$

Consequently, $\mathrm{Cap}_\varphi(\{u_\infty > 0\}) = \sup_{\ell\geq 1} \mathrm{Cap}_\varphi(\{u_\infty > 2^{-\ell}\}) = 0$ which implies the assertion by Proposition 2.10. □

Lemma 3.10. *Let $(F_k)_{k\geq 1}$ be an $\mathcal{E}$–nest and $\tilde{h} \in C(\{F_k\})$ be an $\mathcal{E}$–q.c. m–version of h. Then there exists an $\mathcal{E}$–nest $(\tilde{F}_k)_{k\geq 1}$ such that $\tilde{F}_k \subset F_k$, $k \geq 1$, and $\tilde{h} \geq 1/k$ on $\tilde{F}_k$ for all k.*

Proof. Let $\tilde{F}_k := \{\tilde{h} \geq 1/k\} \cap F_k$. Then $(\tilde{F}_k)_{k\geq 1}$ is an increasing sequence of closed subsets and $h \leq h \wedge (1/k) + h_{F_k^c}$ on $\tilde{F}_k^c$. Hence

$$\mathrm{Cap}_\varphi(\tilde{F}_k^c) = (h_{\tilde{F}_k^c}, \varphi)_\mathcal{H} \leq (h \wedge (1/k), \varphi)_\mathcal{H} + \mathrm{Cap}_\varphi(F_k^c) \ .$$

Since the right hand side converges to zero as $k \to \infty$ we obtain the assertion by Proposition 2.10. □

IV. Generalized Dirichlet Forms and their Probabilistic Counterpart

Throughout this chapter let E be a Hausdorff topological space and assume that its Borel σ–algebra $\mathcal{B}(E)$ is generated by the set $C(E)$ of all continuous functions on E. Let m be a σ–finite measure on $(E, \mathcal{B}(E))$ such that $\mathcal{H} := L^2(E, m)$ is a separable (real) Hilbert space.

In this chapter we study the question which generalized Dirichlet forms are associated with "nice" Markov processes. To this end we introduce a class of Markov processes, so called *m–tight special standard processes* (cf. Definition 1.6 below) which is the right class for our purposes. It is well–known by now, that in the theory of Dirichlet forms there is a complete analytic characterization of all Dirichlet forms that are associated with m–tight special standard processes (cf. [AMR], [MR1]). In this chapter we will extend this characterization to the framework of generalized Dirichlet forms. To this end we introduce the notion of *quasi–regularity* into the framework of generalized Dirichlet forms (cf. Section 1). Under some additional assumption on the domain $\mathcal{F}$ of a quasi–regular generalized Dirichlet form $\mathcal{E}$ which is always fulfilled if $\mathcal{E}$ is a Dirichlet form or a time dependent Dirichlet form, but also in the case of the examples in Sections II.1–II.3, we construct in Section 2 an associated m–tight special standard process. In Section 3 we show that quasi–regularity of a generalized Dirichlet form $\mathcal{E}$ is necessary for the existence of an m–tight m–special standard process. In Section 4 we discuss the quasi–regularity property in the case of the examples given in I.4.9 and Chapter II.

In the first three sections let $(\mathcal{A}, \mathcal{V})$ be a coercive closed form on $\mathcal{H}$ with sector constant $K \geq 1$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ be a linear operator satisfying D1, and suppose that the associated bilinear form $\mathcal{E}$ is a generalized Dirichlet form. Denote by $(G_\alpha)_{\alpha>0}$ the associated resolvent and by $(T_t)_{t\geq 0}$ the associated semigroup.

IV.1 Association of right processes and quasi–regular generalized Dirichlet forms

In order to state our results precisely we recall some notions from the general theory of Markov processes. Adjoin Δ as an isolated point to E and let $E_\Delta := E \cup \{\Delta\}$. Denote by $\mathcal{P}(E_\Delta)$ the set of all probability measures on $(E_\Delta, \mathcal{B}(E_\Delta))$. As usual we extend every function f defined on a subset $A \subset E$ to $A \cup \{\Delta\}$ by setting $f(\Delta) := 0$. Denote by $\mathcal{B}(E_\Delta)^*$ the σ–algebra of universally measurable sets.

There are different versions of the notions of a Markov process and a right process in the literature. Hence we give a precise definition of the notions of a Markov process and a right process which are used in this work, and which coincide with the ones in [MR1].

Definition 1.1. $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ is called a *(time–homogeneous) Markov process* with state space E, life time ζ, and corresponding filtration $(\mathcal{M}_t)_{t\geq 0}$ if

(M.1) $X_t : \Omega \to E_\Delta$ is $\mathcal{M}_t/\mathcal{B}(E_\Delta)$ –measurable for all $t \geq 0$, and $X_t(\omega) = \Delta \quad \Leftrightarrow$ $t \geq \zeta(\omega)$ for all $\omega \in \Omega$.

(M.2) For all $t \geq 0$ there exists a map $\theta_t : \Omega \to \Omega$ such that $X_s \circ \theta_t = X_{s+t}$ for all $s \geq 0$.

(M.3) $(P_x)_{x\in E_\Delta}$ is a family of probability measures on $(\Omega, \mathcal{M})$, such that $x \mapsto P_x[B]$ is $\mathcal{B}(E_\Delta)^*$ –measurable for all $B \in \mathcal{M}$ and $\mathcal{B}(E_\Delta)$–measurable for all $B \in \sigma(X_t|t \geq 0)$ and $P_\Delta[X_0 = \Delta] = 1$.

(M.4) For all $A \in \mathcal{B}(E_\Delta)$, $s, t \geq 0$, and $x \in E_\Delta$

$$P_x[X_{t+s} \in A|\mathcal{M}_t] = P_{X_t}[X_s \in A] \quad P_x - a.s.$$

If $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ is a Markov process and μ a positive measure on $(E_\Delta, \mathcal{B}(E_\Delta))$ let $P_\mu := \int P_x \, \mu(dx)$. For a sub–$\sigma$–algebra $\mathcal{A} \subset \mathcal{M}$ let $\mathcal{A}^{P_\mu}$ be its P_μ–completion in $\mathcal{M}$.

Definition 1.2. A Markov process $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with state space E, life time ζ, and corresponding filtration $(\mathcal{M}_t)_{t\geq 0}$ is called a *right process* if

(M.5) $P_x[X_0 = x] = 1$ for all $x \in E_\Delta$.

(M.6) $t \mapsto X_t(\omega)$ is right continuous on $[0, \infty)$ for all $\omega \in \Omega$.

(M.7) $(\mathcal{M}_t)_{t\geq 0}$ is right continuous and for any $(\mathcal{M}_t)_{t\geq 0}$–stopping time τ and $\mu \in \mathcal{P}(E_\Delta)$

$$P_\mu[X_{\tau+s} \in A|\mathcal{M}_\tau] = P_{X_\tau}[X_s \in A] \quad P_\mu\text{–a.s.}$$

for all $A \in \mathcal{B}(E_\Delta)$ and $s \geq 0$.

Given a right process $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with state space E the family $(p_t)_{t\geq 0}$ (resp. $(R_\alpha)_{\alpha>0}$) of kernels on $(E, \mathcal{B}(E))$ defined by $p_t f(x) := E_x[f(X_t)]$ (resp. $R_\alpha f(x) := \int_0^\infty e^{-\alpha t} p_t f(x)\, dt$), $x \in E$, $f \in \mathcal{B}_b(E)$, is called the *transition semigroup* (resp. *resolvent*) *of* $\mathbb{M}$. Note that $R_\alpha f$ is well–defined and $\mathcal{B}(E)$–measurable for all $f \in \mathcal{B}_b(E)$ because of our assumption $\mathcal{B}(E) = \sigma(C(E))$.

For a subset $A \in \mathcal{B}(E)$ let $\sigma_A := \inf\{t > 0 \mid X_t \in A\}$ be the *first hitting time* (w.r.t. $\mathbb{M}$).

Remark 1.3.

(i) Given a right process $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with state space E and life time ζ, the filtration $(\mathcal{F}_t)_{t\geq 0}$ defined by $\mathcal{F}_t := \bigcap_{\mu\in\mathcal{P}(E_\Delta)} (\mathcal{F}_t^0)^{P_\mu}$, where $\mathcal{F}_t^0 := \sigma(X_s|s \in [0, t])$, is called the *natural filtration*. A right process w.r.t. some filtration $(\mathcal{M}_t)_{t\geq 0}$ is always a right process w.r.t. the natural filtration.

(ii) If E is a Radon topological space in the sense of [Sh] (i.e. homeomorphic to a universally measurable subset of a compact metric space, e.g. a locally compact separable metric space) then any right process $\mathbb{M}$ with state space E w.r.t. some filtration $(\mathcal{M}_t)_{t\geq 0}$ (satisfying $\theta_{t+s} = \theta_t \circ \theta_s$ for all $s, t \geq 0$) is a right process w.r.t. the natural filtration in the sense of [Sh]. Conversely, any right process $\mathbb{M}$ w.r.t. some filtration $(\mathcal{M}_t)_{t\geq 0}$ in the sense of [Sh] is a right process w.r.t. $(\mathcal{M}_t)_{t\geq 0}$ in the sense of Definition 1.2 if the corresponding transition semigroup $(p_t)_{t\geq 0}$ satisfies $p_t(\mathcal{B}_b(E_\Delta)) \subset \mathcal{B}_b(E_\Delta)$, $t \geq 0$.

Definition 1.4. A right process $\mathbb{M}$ with state space E and resolvent $(R_\alpha)_{\alpha>0}$ is called *associated with* $\mathcal{E}$ if $R_\alpha f$ is an m–version of $G_\alpha f$ for all $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$. $\mathbb{M}$ is called *properly associated in the resolvent sense with* $\mathcal{E}$ if in addition $R_\alpha f$ is $\mathcal{E}$–q.c. for $\alpha > 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$.

If $\mathbb{M}$ is a right process with state space E, and $(p_t)_{t\geq 0}$ denotes the transition semigroup of $\mathbb{M}$, it is easy to see that $\mathbb{M}$ is associated with $\mathcal{E}$ if and only if $p_t f$ is an m–version of $T_t f$ for all $t \geq 0$ and $f \in \mathcal{B}_b(E) \cap \mathcal{H}$.

Example 1.5. Consider the generalized Dirichlet form of Example III.1.8, i.e., $\mathcal{E}$ is the generalized Dirichlet form associated with the uniform motion $\mathbb{M}$ on the real line. Let $(p_t)_{t\geq 0}$ be the transition semigroup of $\mathbb{M}$, i.e. $p_t f(x) = f(x+t)$, $f \in \mathcal{B}_b(\mathbb{R})$, $t \geq 0$. Clearly $\mathbb{M}$ is properly associated in the resolvent sense with $\mathcal{E}$, but in general, $p_t f$ is not $\mathcal{E}$–q.c. if $t > 0$ and $f \in \mathcal{B}_b(\mathbb{R}) \cap \mathcal{H}$.

Definition 1.6. Let $\mathbb{M} = (\Omega, \mathcal{F}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta}$ be a right process with state space E and life time ζ. Let μ be a σ–finite positive measure on $(E_\Delta, \mathcal{B}(E_\Delta))$.

(i) $\mathbb{M}$ is called *μ–tight*, if there exists an increasing sequence $(K_n)_{n\geq 1}$ of compact metrizable sets in E such that

$$P_\mu\left[\lim_{n\to\infty} \sigma_{E\setminus K_n} < \zeta\right] = 0.$$

(ii) $\mathbb{M}$ is called *μ–special standard*, if

(M.8) $X_{t-} := \lim_{\substack{s\uparrow t\\ s<t}} X_s$ exists in E for all $t \in (0, \zeta)$ P_μ–a.s.

(M.9) $\lim_{n\to\infty} X_{\tau_n} = X_\tau$ P_μ–a.s. on $\{\tau < \zeta\}$ and X_τ is $\bigvee_{n\geq 1} \mathcal{F}_{\tau_n}^{P_\mu}$–measurable for every increasing sequence $(\tau_n)_{n\geq 1}$ of $(\mathcal{F}_t^{P_\mu})_{t\geq 0}$–stopping times with limit τ.

(iii) $\mathbb{M}$ is called *special standard*, if $\mathbb{M}$ is μ–special standard for all $\mu \in \mathcal{P}(E_\Delta)$.

(iv) $\mathbb{M}$ is called a *Hunt process*, if (M.8) and (M.9) hold with ζ replaced by ∞ and E by E_Δ for all $\mu \in \mathcal{P}(E_\Delta)$.

In the next step we introduce the notion of quasi–regularity for generalized Dirichlet forms and discuss some simple consequences.

Definition 1.7. The generalized Dirichlet form $\mathcal{E}$ associated with $(\mathcal{A}, \mathcal{V})$ and $(\Lambda, D(\Lambda, \mathcal{H}))$ is called *quasi–regular* if:

(i) There exists an $\mathcal{E}$–nest $(E_k)_{k\geq 1}$ consisting of compact sets.

(ii) There exists a dense subset of $\mathcal{F}$ whose elements have $\mathcal{E}$–q.c. m–versions.

(iii) There exist $u_n \in \mathcal{F}$, $n \in \mathbb{N}$, having $\mathcal{E}$–q.c. m–versions $\tilde{u}_n$, $n \in \mathbb{N}$, and an $\mathcal{E}$–exceptional set $N \subset E$ such that $\{\tilde{u}_n \mid n \in \mathbb{N}\}$ separates the points of $E \setminus N$.

Proposition 1.8. *Let $\mathcal{E}$ be quasi–regular. Then every $u \in \mathcal{F}$ has an $\mathcal{E}$–q.c. m–version $\tilde{u}$.*

Proof. Immediate consequence of 1.7 (ii) and Corollary III.3.8. $\square$

Proposition 1.9. *Let $\mathcal{E}$ be quasi–regular. Then*

(i) For every dense subset $D \subset \mathcal{F}$ there exist an $\mathcal{E}$–exceptional set $N \subset E$ and $\mathcal{E}$–q.c. m–versions $\tilde{u}$ of $u \in D$, such that $\{\tilde{u} | u \in D\}$ separates the points of $E \setminus N$.

(ii) There exist a countable subset of bounded, 1–excessive elements, such that $D - D$ is a dense subset of $\mathcal{F}$, and an $\mathcal{E}$–exceptional set $N \subset E$ and $\mathcal{E}$–q.c. m–versions $\tilde{u}$ for $u \in D$ such that $\{\tilde{u} | u \in D\}$ separates the points of $E \setminus N$.

Proof. (i) By Corollary III.3.8 the proof is similar to [MR1, Proposition IV.3.4 (i)].
(ii) Let $(u_n)_{n\geq 1} \subset \mathcal{H} \cap L^\infty(E, m)$ be a dense subset of $\mathcal{H}$ and $D := \{G_1 u_n^+, G_1 u_n^- | n \geq 1\}$. Clearly D is a subset of bounded, 1–excessive elements such that $D - D$ is dense in $\mathcal{F}$. By (i) there exist an $\mathcal{E}$–exceptional set $N \subset E$ and $\mathcal{E}$–q.c. m–versions $\tilde{u}$ of $u \in D$ such that $\{\tilde{u} - \tilde{v} | u, v \in D\}$ separates the points of $E \setminus N$. But then $\{\tilde{u} | u \in D\}$ too, separates the points of $E \setminus N$. $\square$

Lemma 1.10. *Let $\mathcal{E}$ be quasi–regular and $(\tilde{E}_k)_{k\geq 1}$ be an $\mathcal{E}$–nest of compact sets. Then there exists an $\mathcal{E}$–nest $(E_k)_{k\geq 1}$ of compact metrizable sets E_k with $E_k \subset \tilde{E}_k$, $k \geq 1$.*

Proof. Let $(\tilde{u}_n)_{n\geq 1}$ be a countable subset of $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{F}$ and $N \subset E$ be an $\mathcal{E}$–exceptional set such that $(\tilde{u}_n)_{n\geq 1}$ separates the points of $E \setminus N$. Let $(E_k)_{k\geq 1}$ be an $\mathcal{E}$–nest such that $N \subset \bigcap_{k\geq 1} E \setminus E_k$, $(\tilde{u}_n)_{n\geq 1} \subset C(\{E_k\})$, and $E_k \subset \tilde{E}_k$, $k \geq 1$. Then

$$\rho(x, y) := \sum_{n=1}^{\infty} 2^{-n} \frac{|\tilde{u}_n(x) - \tilde{u}_n(y)|}{1 + |\tilde{u}_n(x) - \tilde{u}_n(y)|}$$

defines a metric on E_k. The corresponding topology coincides with the original trace topology on E_k induced by E since E_k is compact. $\square$

IV.2 Construction of associated processes

Consider now the following assumption on $\mathcal{E}$:

D3 There exists a linear subspace $\mathcal{Y} \subset \mathcal{H} \cap L^\infty(E,m)$ such that $\mathcal{Y} \cap \mathcal{F}$ is dense in $\mathcal{F}$, $\lim_{\alpha\to\infty} e_{\alpha G_\alpha u - u} = 0$ in $\mathcal{H}$ for all $u \in \mathcal{Y}$ and for the closure $\overline{\mathcal{Y}}$ of $\mathcal{Y}$ in $L^\infty(E,m)$ it follows that $u \wedge \alpha \in \overline{\mathcal{Y}}$ for $u \in \mathcal{Y}$ and $\alpha \geq 0$.

Remark 2.0. Let $\mathcal{Y}$ be as in D3 and $\overline{\mathcal{Y}}$ be the closure of $\mathcal{Y}$ in $L^\infty(E,m)$. Then $u \wedge \alpha \in \overline{\mathcal{Y}}$ for all $u \in \overline{\mathcal{Y}}$ and $\alpha \geq 0$. Indeed, let $(u_n)_{n\geq 1} \subset \mathcal{Y}$ be such that $\|u_n - u\|_\infty \to 0$, $n \to \infty$, then $u_n \wedge \alpha \in \overline{\mathcal{Y}}$ and $\|u_n \wedge \alpha - u \wedge \alpha\|_\infty \to 0$, $n \to \infty$, which implies the assertion.

A sufficient criterion for D3 to hold, and which is easy to check in applications is obtained in the following

Proposition 2.1. *Let $\mathcal{Y} \subset \mathcal{F} \cap L^\infty(E,m)$ be a linear subspace which is dense in $\mathcal{F}$ and closed under multiplication. Then $\mathcal{Y}$ satisfies D3.*

Proof. Clearly, $\lim_{\alpha\to\infty} e_{\alpha G_\alpha u - u} = 0$ in $\mathcal{H}$ for all $u \in \mathcal{Y}$ by Lemma III.2.2. Let $u \in \mathcal{Y}$, $\alpha \geq 0$, and $M := \|u\|_\infty$. By the Weierstrass Polynomial Approximation Theorem there exists a polynomial P_ε on $\mathbb{R}$ for $\varepsilon > 0$ with $|x \wedge \alpha - P_\varepsilon(x)| \leq \varepsilon$ for $x \in [-M,M]$. We may assume that $P_\varepsilon(0) = 0$, hence $P_\varepsilon(u) \in \mathcal{Y}$ again and $\|u \wedge \alpha - P_\varepsilon(u)\|_\infty \leq \varepsilon$. Hence $u \wedge \alpha \in \overline{\mathcal{Y}}$ (= the closure of $\mathcal{Y}$ in $L^\infty(E,m)$). $\square$

We can now state the following

Theorem 2.2. *Let $\mathcal{E}$ be a quasi–regular generalized Dirichlet form satisfying D3. Then there exists an m–tight special standard process $\mathbb{M}$ which is properly associated in the resolvent sense with $\mathcal{E}$.*

The rest of this section is devoted to the proof of Theorem 2.2. We start with an $\mathcal{E}$–nest $(E_k)_{k\geq 1}$ of compact metrizable sets. By [Sch, p. 102, Corollary 2], $Y := \bigcup_{k\geq 1} E_k$ is a Lusin space.

Proposition 2.3. *For $\alpha > 0$ there exists a kernel $\widetilde{R}_\alpha$ from $(E, \mathcal{B}(E))$ to $(Y, \mathcal{B}(Y))$ with the following properties:*

(i) $\widetilde{R}_\alpha f$ is an $\mathcal{E}$–q.c. m–version of $G_\alpha f$ for all $f \in \mathcal{H}$.

(ii) $\alpha \widetilde{R}_\alpha(z, Y) \leq 1$ for all $z \in E$.

The kernel $\widetilde{R}_\alpha$ is unique in the sense that, if K is another kernel from $(E, \mathcal{B}(E))$ to $(Y, \mathcal{B}(Y))$ satisfying (i) and (ii), it follows that $K(z,\cdot) = \widetilde{R}_\alpha(z,\cdot)$ $\mathcal{E}$–q.e.

Proof. Let $\alpha > 0$ and let $qC(E)$ denote the set of $\mathcal{E}$–q.c. functions defined $\mathcal{E}$–q.e. on E. The map $T : \mathcal{H} \to qC(E)$, $f \mapsto \tilde{G}_\alpha f$, where $\tilde{G}_\alpha f$ denotes an $\mathcal{E}$–q.c. m–version of $G_\alpha f$, is quasi–linear in the sense of [AM, Definition 1.2] by Corollary

III.3.3, Corollary III.3.4, and Corollary III.3.8. By [AM, Theorem 4.2] there exists a kernel K from $(E, \mathcal{B}(E))$ to $(Y, \mathcal{B}(Y))$ such that $Kf = Tf$, $\mathcal{E}$–q.e., for all $f \in \mathcal{H}$. Let $(A_k)_{k\geq 1}$ be an increasing sequence of $\mathcal{B}(Y)$–measurable subsets of Y such that $m(A_k) < \infty$. Then $\alpha K \mathbb{1}_{A_k} \leq \mathbb{1}$ $\mathcal{E}$–q.e. for all $\alpha > 0$ which implies the existence of an $\mathcal{E}$–exceptional set $N \in \mathcal{B}(E)$ such that $\alpha K(x, Y) \leq 1$ for all $x \in E \setminus N$. Then $\tilde{R}_\alpha(x, \cdot) := \mathbb{1}_{E\setminus N} K(x, \cdot)$ satisfies (i) and (ii). The uniqueness of $\tilde{R}_\alpha$ follows from [AM, Theorem 4.2]. $\square$

If u is an m–version of an element in $\mathcal{P}_1$ we can define some $\mathcal{E}$–q.l.s.c. regularization by

$$\tilde{u} := \sup_{n\geq 1} n\widetilde{R}_{n+1} u \ .$$

Assume that u is $\mathcal{E}$–q.c. Then $\tilde{u} \leq u$ $\mathcal{E}$–q.e. by Corollary III.3.3 since $n\widetilde{R}_{n+1}u \leq \tilde{u}$ m–a.e. In order to construct an associated process without branching points it is now necessary to find "enough" elements in $\mathcal{P}_1$ with $\mathcal{E}$–q.c. m–versions u such that $\tilde{u} = u$ $\mathcal{E}$–q.e. We will prove in the next proposition that this is possible because of assumption D3. To this end let $\mathcal{Y} \subset \mathcal{H} \cap L^\infty(E, m)$ be a linear subspace satisfying D3. Note that by Proposition III.3.7 and quasi–regularity any $u \in \mathcal{Y}$ admits an $\mathcal{E}$–q.c. m–version.

Proposition 2.4. *Let u be an $\mathcal{E}$-q.c. m–version of some element in $\mathcal{P}_1$ with $\mathcal{L}_u \cap \mathcal{F} \cap \mathcal{P}_1 \neq \emptyset$. Suppose there is a sequence $(u_n)_{n\geq 1}$ of $\mathcal{E}$-q.c. m–versions of elements in $\mathcal{Y}$ converging to u $\mathcal{E}$-quasi uniformly. Then $u = \sup_{n\geq 1} n\widetilde{R}_{n+1} u$ $\mathcal{E}$-q.e.*

For the proof of Proposition 2.4 we need two lemmas:

Lemma 2.5. *Let u, v be $\mathcal{E}$-q.c. m–versions of elements in $\mathcal{Y}$ and $(\alpha_n)_{n\geq 1} \subset \mathbb{N}$ be a sequence with $\lim\limits_{n\to\infty} \alpha_n = +\infty$. Then there is a subsequence $(\alpha_{n_\ell})_{\ell\geq 1}$ with*

$$\lim_{\ell\to\infty} \alpha_{n_\ell} \widetilde{R}_{\alpha_{n_\ell}+1} u \wedge v^+ = u \wedge v^+ \ \mathcal{E}\text{-q.e.}$$

Proof. Since $u \wedge v^+ \in \overline{\mathcal{Y}}$ by Remark 2.0 we can find for $\varepsilon > 0$ an $\mathcal{E}$–q.c. m–version f_ε of some element in $\mathcal{Y}$ such that $|f_\varepsilon - u \wedge v^+| \leq \varepsilon$ m–a.e. and hence $\mathcal{E}$–q.e. by Corollary III.3.3. Consequently, there exists an $\mathcal{E}$–exceptional set N_1 with $\lim_{k\to\infty} f_{1/k}(x) = u \wedge v^+(x)$ for all $x \in E \setminus N_1$. By Proposition III.3.7 and the usual diagonal argument we can find a subsequence $(\alpha_{n_\ell})_{\ell\geq 1}$ and an $\mathcal{E}$–exceptional set $N_2 \supset N_1$ such that

$$\lim_{\ell\to\infty} \alpha_{n_\ell} \tilde{R}_{\alpha_{n_\ell}+1} f_{1/k}(x) = f_{1/k}(x) \text{ for all } k \geq 1,\ x \in E \setminus N_2 \ .$$

Hence $\lim\limits_{\ell\to\infty} \alpha_{n_\ell} \widetilde{R}_{\alpha_{n_\ell}+1} u \wedge v^+(x) = u \wedge v^+(x)$ for all $x \in E \setminus N_2$. $\square$

Lemma 2.6. *Let u be an $\mathcal{E}$–q.c. m–version of some element in $\mathcal{F}$, v be an $\mathcal{E}$–q.c. m–version of some element in $\mathcal{Y}$, and $(\alpha_n)_{n\geq 1} \subset \mathbb{N}$ be a sequence with $\lim_{n\to\infty} \alpha_n = +\infty$. Then there is a subsequence $(\alpha_{n_\ell})_{\ell\geq 1}$ with*

$$\lim_{\ell\to\infty} \alpha_{n_\ell} \widetilde{R}_{\alpha_{n_\ell}+1} u \wedge v^+ = u \wedge v^+ \ \mathcal{E}\text{–}q.e.$$

Proof. By Proposition 1.8, Corollary III.3.8, and Corollary III.3.4 there exist $\mathcal{E}$–q.c. m–versions u_n of elements in $\mathcal{Y} \cap \mathcal{F}$ such that $u_n \to u$ in $\mathcal{F}$ and $\mathcal{E}$–quasi uniformly. Hence $v_n := e_{u-u_n} + e_{u_n-u} \to 0$ in $\mathcal{V}$ by Lemma III.2.2. Denote by $\tilde{v}_n$ some $\mathcal{E}$–q.l.s.c. regularization of v_n. By Lemma III.3.9 there is a subsequence $(n_k)_{k\geq 1} \subset \mathbb{N}$ with

$$\limsup_{k\to\infty} \tilde{v}_{n_k} \leq 0 \ \mathcal{E}\text{–q.e.}$$

Since $\tilde{v}_{n_k} \geq 0$ $\mathcal{E}$–q.e. there exists some $\mathcal{E}$–exceptional set N_1 such that

$$\lim_{k\to\infty} \tilde{v}_{n_k}(x) = 0 \text{ for all } x \in E \setminus N_1 .$$

Note that $|\alpha \widetilde{R}_{\alpha+1} u \wedge v^+ - \alpha \widetilde{R}_{\alpha+1} u_n \wedge v^+| \leq \alpha \widetilde{R}_{\alpha+1}|u - u_n| \leq \tilde{v}_n$ $\mathcal{E}$–q.e. Hence we can find an $\mathcal{E}$–exceptional set $N_2 \supset N_1$ with

$$|m\widetilde{R}_{m+1} u \wedge v^+(x) - m\widetilde{R}_{m+1} u_n \wedge v^+(x)| \leq \tilde{v}_n(x) \text{ for all } n, m \geq 1,\ x \in E \setminus N_2 \ .$$

By Lemma 2.5 and the usual diagonal argument we obtain a subsequence $(\alpha_{n_\ell})_{\ell\geq 1}$ and some $\mathcal{E}$–exceptional set $N_3 \supset N_2$ such that

$$\lim_{\ell\to\infty} \alpha_{n_\ell} \widetilde{R}_{\alpha_{n_\ell}+1} u_m \wedge v^+(x) = u_m \wedge v^+(x) \text{ for all } m \geq 1,\ x \in E \setminus N_3 \ .$$

Clearly $\lim_{\ell\to\infty} \alpha_{n_\ell} \widetilde{R}_{\alpha_{n_\ell}+1} u \wedge v^+(x) = u \wedge v^+(x)$ for all $x \in E \setminus N_3$ which implies the assertion. $\square$

Proof *(of Proposition 2.4).* We only have to show $\sup_{n\geq 1} n\widetilde{R}_{n+1} u \geq u$ $\mathcal{E}$–q.e. Let $\varphi \in \mathcal{B}_b(E) \cap \mathcal{H}, 0 < \varphi \leq 1$, and $h := \widetilde{R}_1 \varphi$. By quasi–regularity of $\mathcal{E}$ we can find an $\mathcal{E}$–nest $(F''_k)_{k\geq 1}$ of compact metrizable sets with $\{h\} \cup \{u_n | n \in \mathbb{N}\} \subset C(\{F''_k\})$ and $u_n \to u$ uniformly on F''_k, $k \geq 1$. By Lemma III.3.10 we can find an $\mathcal{E}$–nest $(F'_k)_{k\geq 1}$ with $F'_k \subset F''_k$, $k \geq 1$, and $h \geq 1/k$ on F'_k, $k \geq 1$, and subsequently a regular $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ with $F_k \subset F'_k$ by Lemma III.2.4. For every k there is some $n_k \in \mathbb{N}$ such that $|u_{n_k} - u| \leq (1/k)h$ on F_k, hence we may assume $|u_k - u| \leq (1/k)h$ on F_k. Let f be an $\mathcal{E}$–q.c. m-version of some element in $\mathcal{L}_u \cap \mathcal{F} \cap \mathcal{P}_1$. Clearly, there is some $\mathcal{E}$–exceptional set N_1 with $\lim_{n\to\infty} f \wedge u_n^+(x) = u(x)$ for all $x \in E \setminus N_1$. Note that $v_k := (1/k)h + f_{F_k^c} \to 0$ in $\mathcal{H}$. Let $\tilde{v}_k$ denote some $\mathcal{E}$–q.l.s.c. regularization of v_k, $k \geq 1$. By Lemma III.3.9 there is a subsequence $(n_k)_{k\geq 1} \subset \mathbb{N}$ and an $\mathcal{E}$–exceptional set $N_2 \supset N_1$ with

$$\lim_{k\to\infty} \tilde{v}_{n_k}(x) = 0 \text{ for all } x \in E \setminus N_2$$

since $\tilde{v}_k \geq 0$ $\mathcal{E}$–q.e. Note that $|\alpha\widetilde{R}_{\alpha+1}f \wedge u_n^+ - \alpha\widetilde{R}_{\alpha+1}u| \leq \alpha\widetilde{R}_{\alpha+1}|f \wedge u_n^+ - u| \leq \tilde{v}_n$ $\mathcal{E}$–q.e. Hence there is some $\mathcal{E}$–exceptional set $N_3 \supset N_2$ such that

$$|m\widetilde{R}_{m+1}f \wedge u_n^+(x) - m\widetilde{R}_{m+1}u(x)| \leq \tilde{v}_n(x) \text{ for all } n, m \geq 1,\ x \in E \setminus N_3.$$

By Lemma 2.6 and the usual diagonal argument we can find a sequence $(n_\ell)_{\ell\geq 1} \subset \mathbb{N}$ with $\lim\limits_{\ell\to\infty} n_\ell = +\infty$ and an $\mathcal{E}$–exceptional set $N_4 \supset N_3$ with

$$\lim_{\ell\to\infty} n_\ell\widetilde{R}_{n_\ell+1}f \wedge u_k^+(x) = f \wedge u_k^+(x) \text{ for all } k \geq 1, x \in E \setminus N_4.$$

Hence $\lim\limits_{\ell\to\infty} n_\ell\widetilde{R}_{n_\ell+1}u(x) = u(x)$ for all $x \in E \setminus N_4$ and thus

$$\sup_{n\geq 1} n\widetilde{R}_{n+1}u(x) \geq \lim_{\ell\to\infty} n_\ell\widetilde{R}_{n_\ell+1}u(x) = u(x) \text{ for all } x \in E \setminus N_4$$

which implies the assertion. $\square$

Construction of a Ray resolvent on some compactification of a subset of E

Let $(E_k)_{k\geq 1}$ be an $\mathcal{E}$–nest of compact metrizable sets, and $\mathcal{U}_0$ be a countable family of open sets in E containing $E \setminus E_k$, $k \geq 1$, such that $\{U \cap E_k \mid U \in \mathcal{U}_0\}$ forms a base for the trace topology on E_k for all k. Let $\mathcal{U}_1 = \{U = \bigcup\limits_{i=1}^{n} U_i \mid U_i \in \mathcal{U}_0\} \cup \{E\}$, and $\widetilde{D}$ be a countable subset of $\mathcal{E}$–q.c. m–versions of bounded 1–excessive elements as in Proposition 1.9 (ii). Let $\varphi \in \mathcal{B}_b(E) \cap \mathcal{H}$, $0 < \varphi \leq 1$, $h := \widetilde{R}_1\varphi$, and $\mathbb{Q}^+ := \mathbb{Q} \cap (0, \infty)$.

Define J_0 to be the smallest family of $\mathcal{E}$–q.c. functions such that

(2.1) $\widetilde{D} \subset J_0, h \in J_0$.

(2.2) $\widetilde{R}_\alpha u_U \in J_0, \tilde{R}_\alpha u \in J_0$ for $u \in J_0, U \in \mathcal{U}_1, \alpha \in \mathbb{Q}^+$.

(2.3) $u \wedge v, u \wedge 1, (u+1) \wedge v \in J_0$ for $u, v \in J_0$.

(2.4) J_0 is a $\mathbb{Q}^+$–cone, i.e., $c_1u + c_2v \in J_0$ for $u, v \in J_0, c_1, c_2 \in \mathbb{Q}^+$.

Clearly J_0 is a countable set of $\mathcal{E}$–q.c. m–versions of bounded 1–excessive elements and we have

Proposition 2.7. *For every $u \in J_0$, $\mathcal{L}_u \cap \mathcal{F} \cap \mathcal{P}_1 \neq \emptyset$, and there is a sequence $(f_n)_{n\geq 1}$ of $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{Y}$ such that $f_n \to u$ $\mathcal{E}$–quasi uniformly.*

Proof. The assertion is obvious for any $u \in J_0$ which is the m–version of some element in $\mathcal{F}$. Therefore, it remains to show that if $u, v \in J_0$, $(f_n)_{n\geq 1}$ and $(g_n)_{n\geq 1}$ are $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{Y}$ with $f_n \to u$ and $g_n \to v$ $\mathcal{E}$–quasi uniformly, there is a sequence $(h_n)_{n\geq 1}$ of $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{Y}$ such that $h_n \to u \wedge v$ (resp. $u \wedge 1$, resp. $(u+1) \wedge v$) $\mathcal{E}$–quasi uniformly. Since $u \wedge v \in \overline{\mathcal{Y}}$ (resp. $u \wedge 1 \in \overline{\mathcal{Y}}$, resp. $(u+1) \wedge v = (v-u) \wedge 1 + u \in \overline{\mathcal{Y}}$) there exist $\mathcal{E}$–q.c. m–versions h_n of elements in $\mathcal{Y}$ such that $|h_n - g_n \wedge f_n| \leq 1/n$ (resp. $|h_n - f_n \wedge 1| \leq 1/n$, resp. $|h_n - (f_n+1) \wedge g_n| \leq 1/n$) $\mathcal{E}$–q.e. Since $f_n \wedge g_n \to u \wedge v$ (resp. $f_n \wedge 1 \to u \wedge 1$, resp. $(f_n+1) \wedge g_n \to (u+1) \wedge v$) $\mathcal{E}$–quasi uniformly, the assertion follows. □

Proposition 2.8. *There is a regular $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ of compact metrizable sets $F_k \subset E_k$ such that*

(i) $J_0 \subset C(\{F_k\})$.

(ii) $\inf\limits_{x \in F_k} h(x) > 0$ *for all* k.

(iii) $\lim\limits_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}^+}} \alpha \tilde{R}_{\alpha+1} u(x) = u(x)$ *for all* $u \in J_0, x \in Y_1 := \bigcup\limits_{k\geq 1} F_k$.

(iv) J_0 *separates the points of* Y_1.

Proof. J_0 separates the points up to some $\mathcal{E}$–exceptional set N_1. Since J_0 is countable there is some $\mathcal{E}$–exceptional set $N_2 \supset N_1$ such that $\lim\limits_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}^+}} \alpha \tilde{R}_{\alpha+1} u(x) = u(x)$ for all $x \in E \setminus N_2$ by Proposition 2.4 and Proposition 2.7. By Lemma III.3.5 there exists an $\mathcal{E}$–nest $(F_k'')_{k\geq 1}$ of compact sets with $J_0 \subset C(\{F_k''\})$, $F_k'' \subset E_k$, $k \geq 1$, and $\bigcup\limits_{k\geq 1} F_k'' \subset E \setminus N_2$. Using Lemma III.3.10 we can find an $\mathcal{E}$–nest $(F_k')_{k\geq 1}$ with $F_k' \subset F_k''$ and $h \geq 1/k$ on F_k', $k \geq 1$, and subsequently a regular $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ with $F_k \subset F_k'$, $k \geq 1$, (cf. Lemma III.2.4). □

In the same way as in [MR1, Lemma IV.3.11] we can find some set $Y_2 \in \mathcal{B}(E), Y_2 \subset Y_1$ such that $E \setminus Y_2$ is $\mathcal{E}$–exceptional and $\widetilde{R}_\alpha(x, Y \setminus Y_2) = 0$ for all $x \in Y_2, \alpha \in \mathbb{Q}^+$.

Adjoin Δ as an isolated point to Y_2 and let $Y_\Delta := Y_2 \cup \{\Delta\}$. Each function f on Y_2 will be regarded as a function on Y_Δ by setting $f(\Delta) := 0$.

Define for $\alpha \in \mathbb{Q}^+$ and $A \in \mathcal{B}(Y_\Delta)$

$$R_\alpha(x, A) := \begin{cases} \widetilde{R}_\alpha(x, A \cap Y_2) + ((1/\alpha) - \widetilde{R}_\alpha(x, Y_2))\delta_\Delta(A) & \text{if } x \in Y_2 \\ (1/\alpha)\delta_\Delta(A) & \text{if } x = \Delta\ . \end{cases}$$

Then $(R_\alpha)_{\alpha\in\mathbb{Q}^+}$ is a Markovian resolvent of kernels on $(Y_\Delta, \mathcal{B}(Y_\Delta))$. Let $J := \{u + c \cdot 1\!\!1_{Y_\Delta} \mid u \in J_0, c \in \mathbb{Q}^+\}$. Then J is a $\mathbb{Q}^+$–cone, inf–stable, i.e., $u \wedge v \in J$, if $u, v \in J$, and $R_\alpha u \in J$ if $u \in J$, $\alpha \in \mathbb{Q}^+$. J separates the points of Y_Δ, each function $u \in J$ is 1–supermedian w.r.t. $(R_\alpha)_{\alpha\in\mathbb{Q}^+}$, i.e., $\alpha R_{\alpha+1} u \leq u$ if $u \in J$, $\alpha \in \mathbb{Q}^+$, and $\lim\limits_{\substack{\alpha\to\infty \\ \alpha\in\mathbb{Q}^+}} \alpha R_{\alpha+1} u(x) = u(x)$ for all $u \in J, x \in Y_\Delta$.

Let $J = \{u_n \mid n \in \mathbb{N}\}$ and define the metric

$$\rho(x,y) := \sum_{n=1}^{\infty} 2^{-n} \frac{|u_n(x) - u_n(y)|}{1 + |u_n(x) - u_n(y)|} \text{ on } Y_\Delta \ .$$

Denote with the prefix "$\rho-$" concepts which are meant relative to the topology induced by the metric ρ and with $\overline{Y}$ the completion of Y_Δ w.r.t. ρ. $\overline{Y}$ is a compact metric space and for every $u \in J$ there is a unique ρ–continuous extension $\overline{u}$. Let $\overline{J}$ be the set of all ρ-continuous extensions $\overline{u}$ of functions $u \in J$. Note that $\overline{J} - \overline{J}$ is dense in $C(\overline{Y})$ by the Stone–Weierstrass–Theorem because $\overline{J}$ separates the points of $\overline{Y}$ and $\overline{J} - \overline{J}$ is inf–stable since $(\overline{u} - \overline{u_1}) \wedge (\overline{v} - \overline{v_1}) = (\overline{u} + \overline{v_1}) \wedge (\overline{v} + \overline{u_1}) - (\overline{u_1} + \overline{v_1}) \in \overline{J}$, if $\overline{u}, \overline{u_1}, \overline{v}, \overline{v_1} \in \overline{J}$.

For $\alpha \in \mathbb{Q}^+$ and $\overline{u} \in \overline{J}$ define $\overline{R}_\alpha \overline{u} := \overline{R_\alpha u}$ and extend $\overline{R}_\alpha$ to $\overline{J} - \overline{J}$ by linearity. Since $\|\overline{R}_\alpha \overline{u}\|_\infty = \|R_\alpha u\|_\infty \leq 1/\alpha \|u\|_\infty = 1/\alpha \|\overline{u}\|_\infty$, we can extend $\overline{R}_\alpha$ to a linear bounded positive operator on $C(\overline{Y})$. The family of operators $(\overline{R}_\alpha)_{\alpha \in \mathbb{Q}^+}$ satisfies the resolvent equation, so $\alpha \mapsto \overline{R}_\alpha f$ is locally uniformly continuous on $\mathbb{Q}^+$, and consequently, we can define $\overline{R}_\alpha f := \lim\limits_{\substack{\beta \to \alpha \\ \beta \in \mathbb{Q}^+}} \overline{R}_\beta f$ for arbitrary $\alpha > 0$. Denote by $(\overline{R}_\alpha)_{\alpha > 0}$ the corresponding Markovian resolvent of kernels. Clearly $(\overline{R}_\alpha)_{\alpha > 0}$ is a Ray resolvent (cf. [MR1, Definition IV.1.19]).

The corresponding right process $\overline{\mathbb{M}}$

Let $D := \{x \in \overline{Y} \mid \lim\limits_{\alpha \to \infty} \alpha \overline{R}_\alpha f(x) = f(x), \ \forall f \in C(\overline{Y})\}$ be the set of non–branching points. Since $\lim\limits_{\alpha \to \infty} \alpha \overline{R}_\alpha f(x) = f(x)$ for all $f \in \overline{J} - \overline{J}$ and $x \in Y_\Delta$ we obtain that $Y_\Delta \subset D$.

By [G, Theorem 3.6] (cf. also [MR1, Theorem IV.1.20]) there exists a unique semigroup $(\overline{p}_t)_{t \geq 0}$ of Markovian kernels on $(\overline{Y}, \mathcal{B}(\overline{Y}))$ such that

$$\overline{R}_\alpha f = \int_0^\infty e^{-\alpha t} \overline{p}_t f \, dt \text{ for all } \alpha > 0, \ f \in C(\overline{Y}) \ . \tag{2.5}$$

By [Sh, Theorem 9.13] (cf. also [MR1, Theorem IV.1.22]) there exists a right process $\overline{\mathbb{M}} = (\overline{\Omega}, \overline{\mathcal{F}}, (\overline{X}_t)_{t \geq 0}, (\overline{P}_x)_{x \in D})$ with state space D and transition semigroup $(\overline{p}_t)_{t \geq 0}$ such that for all $\omega \in \overline{\Omega}, t \mapsto \overline{X}_t(\omega)$ has left limits in $\overline{Y}$ for $t \in (0, \infty)$. Let $(\overline{\mathcal{F}}_t)_{t \geq 0}$ denote the natural filtration corresponding to the right process $\overline{\mathbb{M}}$ and $(\overline{\theta}_t)_{t \geq 0}$ be the corresponding family of shift operators.

Proposition 2.9. *Let $F \subset Y_2$ be compact, $F \subset F_k$ for some $k, A := \overline{Y} \setminus F$ and $u \in J_0$. Then there exist an $\mathcal{E}$–exceptional set $N \subset E$ and an $\mathcal{E}$–q.l.s.c. regularization $\tilde{u}_{E \setminus F}$ of $u_{E \setminus F}$ such that*

$$\overline{E}_x \left[e^{-\sigma_A} \overline{u}(\overline{X}_{\sigma_A}) \right] \leq \tilde{u}_{E \setminus F}(x) \ \textit{for all } x \in Y_\Delta \setminus N \ .$$

Proof. Since $F \subset F_k$ for some k there is an increasing sequence $(U_n)_{n\geq 1}$ of open sets in $\mathcal{U}_1$ such that $\bigcup_{n\geq 1} U_n = E \setminus F =: U$. Hence $(\alpha\overline{R_{\alpha+1}u_{U_n}}(x))_{\alpha\in\mathbb{Q}^+, n\geq 1}$ is increasing in α and n for all $x \in \overline{Y}$. Let $f = \sup_{\alpha\in\mathbb{Q}^+}\sup_{n\geq 1} \alpha\overline{R_{\alpha+1}u_{U_n}}(\leq \overline{u})$. Since $\tilde{R}_{\alpha+1}u_{U_n}(x) - \tilde{R}_{\beta+1}u_{U_n}(x) = (\beta-\alpha)\tilde{R}_{\alpha+1}\tilde{R}_{\beta+1}u_{U_n}(x)$ if $\alpha, \beta \in \mathbb{Q}^+, x \in Y_2, n \geq 1$, it follows that $R_{\alpha+1}u_{U_n}(x) - R_{\beta+1}u_{U_n}(x) = (\beta-\alpha)R_{\alpha+1}R_{\beta+1}u_{U_n}(x)$ if $\alpha, \beta \in \mathbb{Q}^+, x \in Y_\Delta, n \geq 1$, and therefore,

$$\begin{aligned}
\lim_{\substack{\alpha\to\infty\\ \alpha\in\mathbb{Q}^+}} \alpha\overline{R}_{\alpha+1}f(x) &= \lim_{\substack{\alpha\to\infty\\ \alpha\in\mathbb{Q}^+}}\lim_{n\to\infty}\lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{Q}^+}} \beta\alpha\overline{R}_{\alpha+1}\overline{R_{\beta+1}u_{U_n}}(x) \\
&= \lim_{\substack{\alpha\to\infty\\ \alpha\in\mathbb{Q}^+}}\lim_{n\to\infty}\lim_{\substack{\beta\to\infty\\ \beta\in\mathbb{Q}^+}} \frac{\beta\alpha}{\beta-\alpha}(\overline{R_{\alpha+1}u_{U_n}}(x) - \overline{R_{\beta+1}u_{U_n}}(x)) \\
&= \lim_{\substack{\alpha\to\infty\\ \alpha\in\mathbb{Q}^+}}\lim_{n\to\infty} \alpha\overline{R_{\alpha+1}u_{U_n}}(x) = f(x)
\end{aligned}$$

for all $x \in \overline{Y}$. Note that

$$\overline{E}_x\left[\int_0^\infty e^{-\alpha t}g(\overline{X}_t)\,dt\right] = R_\alpha(g_{|Y_\Delta})(x) \text{ for all } g \in \overline{J}, x \in Y_\Delta. \tag{2.6}$$

Since $\sigma(\overline{J}) = \mathcal{B}(\overline{Y})$ a monotone class argument implies that (2.6) holds for all bounded $\mathcal{B}(\overline{Y})$–measurable functions g. Consequently, for $g := (\overline{u} - f)\mathbb{1}_A \in \mathcal{B}_b^+(\overline{Y})$ we obtain that $\overline{R}_\alpha g(x) = R_\alpha(g_{|Y_\Delta})(x) = \widetilde{R}_\alpha(g_{|Y_2})(x)$ for $x \in Y_2$, since $g(\Delta) = 0$, and therefore, $\overline{R}_\alpha g_{|Y_2} = 0$ $\mathcal{E}$–q.e. since $g_{|Y_2} = 0$ m–a.e. by Lemma III.2.12. Clearly $\overline{R}_\alpha g(\Delta) = R_\alpha(g_{|Y_\Delta})(\Delta) = 0$. Hence there is an $\mathcal{E}$–exceptional set N with $\overline{R}_\alpha g(x) = 0$ for all $\alpha \in \mathbb{Q}^+, x \in Y_\Delta \setminus N$.

Let $\mathbb{Q}^+ = \{t_n \mid n \in \mathbb{N}\}$ and $\sigma_n := \min\{t_i \mid i \leq n, \overline{X}_{t_i} \in A\}$ (with the convention $\min\emptyset = +\infty$). Then

$$\overline{E}_x\left[\int_0^\infty e^{-\alpha t}\overline{E}_{\overline{X}_t}\left[e^{-\sigma_n}g(\overline{X}_{\sigma_n})\right]dt\right] = 0 \text{ for all } x \in Y_\Delta \setminus N\ . \tag{2.7}$$

Since $\left(e^{-t}\alpha\overline{R_{\alpha+1}u_{U_n}}(\overline{X}_t), \overline{\mathcal{F}}_t, \overline{P}_x\right)_{t\in[0,\infty]}$ (where we set $e^{-\infty}\alpha\overline{R_{\alpha+1}u_{U_n}}(\overline{X}_\infty) := 0$) is a right continuous supermartingale for $x \in D$ we obtain by the optional sampling theorem

$$\overline{E}_x\left[e^{-\sigma_n}\alpha\overline{R_{\alpha+1}u_{U_m}}(\overline{X}_{\sigma_n})\right] \leq \alpha\overline{R_{\alpha+1}u_{U_m}}(x) \ \text{ for all } x \in D\ .$$

It follows from the monotone convergence theorem

$$\overline{E}_x\left[e^{-\sigma_n}f(\overline{X}_{\sigma_n})\right] \leq f(x) \ \text{ for all } x \in D\ . \tag{2.8}$$

Combining (2.7) and (2.8) we get

$$\overline{E}_x\left[\int_0^\infty e^{-\alpha t}\overline{E}_{\overline{X}_t}\left[e^{-\sigma_n}\overline{u}(\overline{X}_{\sigma_n})\right]dt\right] \leq \overline{R}_\alpha f(x) \text{ for all } x \in Y_\Delta \setminus N\ .$$

Since $F \subset F_k$ for some k and F_k is a compact subset of $\overline{Y}$ we obtain that $A = \overline{Y} \setminus F$ is an open subset of $\overline{Y}$ and hence $\sigma_n \to \sigma_A$ for $n \to \infty$ which implies

$$\overline{E}_x \left[\int_0^\infty e^{-\alpha t} \overline{E}_{\overline{X}_t} \left[e^{-\sigma_A} \overline{u}(\overline{X}_{\sigma_A}) \right] dt \right] \le \overline{R}_\alpha f(x) \text{ for all } x \in Y_\Delta \setminus N \ .$$

Since $t \mapsto \overline{X}_t$ is right continuous it follows $\overline{X}_{t+\sigma_A \circ \overline{\theta}_t} \to \overline{X}_{\sigma_A}$ for $t \downarrow 0$ and thus $\overline{E}_x \left[e^{-\sigma_A \circ \overline{\theta}_t} \overline{u}(\overline{X}_{t+\sigma_A \circ \overline{\theta}_t}) \right] \to \overline{E}_x \left[e^{-\sigma_A} \overline{u}(\overline{X}_{\sigma_A}) \right]$ by Lebesgue's theorem. Hence for $x \in Y_\Delta \setminus N$

$$\begin{aligned} \overline{E}_x \left[e^{-\sigma_A} \overline{u}(\overline{X}_{\sigma_A}) \right] &= \lim_{\substack{\alpha \to \infty \\ \alpha \in \mathbb{Q}^+}} \int_0^\infty \alpha e^{-(\alpha+1)t} \overline{E}_x \left[e^{-\sigma_A \circ \overline{\theta}_t} \overline{u}(\overline{X}_{t+\sigma_A \circ \overline{\theta}_t}) \right] dt \\ &= \lim_{\substack{\alpha \to \infty \\ \alpha \in \mathbb{Q}^+}} \alpha \overline{E}_x \left[\int_0^\infty e^{-(\alpha+1)t} \overline{E}_{\overline{X}_t} \left[e^{-\sigma_A} \overline{u}(\overline{X}_{\sigma_A}) \right] dt \right] \\ &\le \lim_{\substack{\alpha \to \infty \\ \alpha \in \mathbb{Q}^+}} \alpha \overline{R}_{\alpha+1} f(x) = f(x) \ . \end{aligned}$$

Since $f_{|Y_2}$ is an $\mathcal{E}$–q.l.s.c. regularization of u_U the assertion follows. $\square$

Let $\zeta := \sigma_{\overline{Y} \setminus Y_2}$. ζ is an $(\overline{\mathcal{F}}_t)_{t \ge 0}$-stopping time, since $Y_2 \in \mathcal{B}(\overline{Y})$ (cf. [MR1, Lemma IV.3.13]). Since $(e^{-t}\overline{u}(\overline{X}_t), \overline{\mathcal{F}}_t, \overline{P}_x)_{t \in [0,\infty]}$ is a right continuous nonnegative supermartingale for $x \in D$ (where we set again $e^{-\infty}\overline{u}(\overline{X}_\infty) := 0$) if $\overline{u} \in \overline{J}$ the set $\overline{\Omega}_t := \{\inf_{s \le t} e^{-s}\overline{u}(\overline{X}_s) = 0\} \cap \{e^{-t}\overline{u}(\overline{X}_t) > 0\}$ is $\overline{\mathcal{F}}_t$–measurable and $\overline{P}_x[\overline{\Omega}_t] = 1$, $t \ge 0$, $x \in D$ (cf. [MR1, Lemma IV.3.19]). This implies that the set

$$\Omega_0 := \{\omega \in \overline{\Omega} \mid \inf_{s \le t} e^{-s}\overline{u}(\overline{X}_s)(\omega) > 0 \text{ or } e^{-t}\overline{u}(\overline{X}_t)(\omega) = 0 \, \forall t \in \mathbb{Q}^+, \overline{u} \in \overline{J}\}$$

is $\overline{\mathcal{F}}$–measurable, shift–invariant, i.e., $\overline{\theta}_t(\Omega_0) \subset \Omega_0$, $t \ge 0$, and satisfies $\overline{P}_x[\Omega_0] = 1$ for all $x \in D$.

Let $(K_n)_{n \ge 1}$ be an $\mathcal{E}$–nest of compact metrizable sets such that $K_n \subset F_n \cap Y_2$ and let $\sigma := \sup_{n \ge 1} \sigma_{\overline{Y} \setminus K_n}$. By Proposition 2.9 we can find an $\mathcal{E}$–exceptional set $N' \subset E$ and $\mathcal{E}$–q.l.s.c. regularizations $\tilde{u}_{E \setminus K_n}$ for $u_{E \setminus K_n}$ if $u \in J_0$, $n \ge 1$, such that

$$\overline{E}_x \left[e^{-\sigma_{\overline{Y} \setminus K_n}} \overline{u}(\overline{X}_{\sigma_{\overline{Y} \setminus K_n}}) \right] \le \tilde{u}_{E \setminus K_n}(x) \text{ for all } u \in J_0, n \ge 1, x \in Y_\Delta \setminus N'.$$

Fix $u \in J_0$ and let $f \in \mathcal{L}_u \cap \mathcal{F} \cap \mathcal{P}_1$ (cf. Proposition 2.7). Since $f_{E \setminus K_n} \to 0$, $n \to \infty$, in $\mathcal{H}$ and $u_{E \setminus K_n} \le f_{E \setminus K_n}$, we obtain that $u_{E \setminus K_n} \to 0$, $n \to \infty$, in $\mathcal{H}$. By Lemma III.3.9 we conclude that $\limsup_{n \to \infty} \tilde{u}_{E \setminus K_n} \le 0$ $\mathcal{E}$–q.e. Since this is true for any $u \in J_0$ there is some $\mathcal{E}$–exceptional set $N \supset N'$, $N \in \mathcal{B}(Y_2)$, such that

$$\lim_{n \to \infty} \overline{E}_x \left[e^{-\sigma_{\overline{Y} \setminus K_n}} \overline{u}(\overline{X}_{\sigma_{\overline{Y} \setminus K_n}}) \right] \le \inf_{n \ge 1} \tilde{u}_{E \setminus K_n}(x) \le 0 \text{ for all } u \in J_0, x \in Y_\Delta \setminus N.$$

Therefore the set

$$\Omega_1 := \{\omega \in \Omega_0 \mid e^{-\sigma(\omega)}\overline{u}(\rho - \lim_{n \to \infty} \overline{X}_{\sigma_{\overline{Y} \setminus K_n}}(\omega)) = 0 \ \forall u \in J_0\}$$

is $\overline{\mathcal{F}}$–measurable, shift–invariant, and satisfies $\overline{P}_x[\Omega_1] = 1$ for all $x \in Y_\Delta \setminus N$.

Clearly $e^{-t}\overline{u}(\overline{X}_t)(\omega) = 0$ for all $t \geq \sigma(\omega)$ and $\omega \in \Omega_1$. In particular, $e^{-t}\overline{h}(\overline{X}_t)(\omega) = 0$ for all $t \geq \sigma(\omega)$ and $\omega \in \Omega_1$ which implies that $\overline{X}_t(\omega) \in \overline{Y} \setminus Y_2$, hence $\sigma(\omega) \geq \zeta(\omega)$. On the other hand clearly $\sigma_{\overline{Y}\setminus K_n}(\omega) \leq \zeta(\omega)$ and therefore, $\zeta = \sigma$ on Ω_1. This implies

$$(2.9)\quad R_\alpha u(x) = \overline{R}_\alpha \overline{u}(x) = \overline{E}_x\left[\int_0^\zeta e^{-\alpha t}\overline{u}(\overline{X}_t)\,dt\right] \;\forall x \in Y_\Delta \setminus N,\; \alpha \in \mathbb{Q}^+,\; u \in J_0\,.$$

Since K_n is compact in E the trace topologies on K_n induced by $\overline{Y}$ and E coincide. Hence $\overline{X}_t(\omega)$ is right continuous on $[0, \zeta(\omega))$ and has left limits on $(0, \zeta(\omega))$ in $\bigcup_{n\geq 1} K_n$ w.r.t. the original topology for $\omega \in \Omega_1$.

Let $Z_1 := Y_2 \setminus N$. Repeating the argument above we find decreasing sequences $(Z_n)_{n\geq 1} \subset \mathcal{B}(Y_2)$ and $(\Omega_n)_{n\geq 1} \subset \overline{\mathcal{F}}$ such that $E \setminus Z_n$ is $\mathcal{E}$–exceptional, $\overline{P}_x[\Omega_n] = 1$ for all $x \in Z_n \cup \{\Delta\}$, Ω_n is shift–invariant, $\overline{X}_t(\omega) \in Z_{n-1}$ for $t \in [0, \zeta(\omega))$ and $\omega \in \Omega_n$, and $\overline{X}_{t-}(\omega) \in Z_{n-1}$ for $t \in (0, \zeta(\omega))$ and $\omega \in \Omega_n$. Hence for $S := \bigcap_{n\geq 1} Z_n \in \mathcal{B}(Y_2)$, $S_\Delta := S \cup \{\Delta\}$, and $\Omega := \bigcap_{n\geq 1} \Omega_n \in \overline{\mathcal{F}}$ it follows that

(2.10) $E \setminus S$ is $\mathcal{E}$–exceptional.

(2.11) $\overline{P}_x[\Omega] = 1$ for all $x \in S_\Delta$.

(2.12) Ω is shift–invariant.

(2.13) $\overline{X}_t(\omega)$ is right continuous and has left limits in S for $t \in (0, \zeta(\omega))$ w.r.t. the original topology on E for all $\omega \in \Omega$.

(2.14) $R_\alpha u(x) = \overline{E}_x\left[\int_0^\zeta e^{-\alpha t}\overline{u}(\overline{X}_t)dt\right]$ for all $\alpha \in \mathbb{Q}^+, u \in J_0, x \in S_\Delta$.

(2.15) $\lim_{n\to\infty} \sigma_{\overline{Y}\setminus K_n}(\omega) = \zeta(\omega)$ for all $\omega \in \Omega$.

(2.16) $\zeta(\omega) > 0$ for $\omega \in \{\overline{X}_0 = x\} \cap \Omega$, $x \in S$ and $\zeta(\omega) = 0$ for $\omega \in \{\overline{X}_0 = \Delta\} \cap \Omega$.

The next lemma can be proved as Lemma IV.3.21 in [MR1].

Lemma 2.10. *Let $(\tau_n)_{n\geq 1}$ be an increasing sequence of $(\overline{\mathcal{F}}_t)_{t\geq 0}$–stopping times and $\tau := \sup_{n\geq 1} \tau_n$. Then*

$$\overline{P}_x\left[\rho - \lim_{n\to\infty} \overline{X}_{\tau_n} \neq \overline{X}_\tau,\; \rho - \lim_{n\to\infty} \overline{X}_{\tau_n} \in Y_2,\; \tau < \infty\right] = 0 \textit{ for all } x \in S_\Delta.$$

Let $\mathbb{M}_S = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (P_x)_{x\in S_\Delta})$ be defined by $\mathcal{F} := \overline{\mathcal{F}} \cap \Omega, \mathcal{F}_t := \overline{\mathcal{F}}_t \cap \Omega, P_x := \overline{P}_{x|\mathcal{F}}$ for $x \in S_\Delta$ and

$$X_t(\omega) := \begin{cases} \overline{X}_t(\omega) & \text{if } t \in [0, \zeta(\omega)) \\ \Delta & \text{if } t \geq \zeta(\omega)\ . \end{cases}$$

Lemma 2.11.

(i) $\mathbb{M}_S$ *is a special standard process with state space* S *and life time* ζ.

(ii) $E_x\left[\int_0^\infty e^{-\alpha t} f(X_t)dt\right] = \widetilde{R}_\alpha f(x)$ *for all* $f \in \mathcal{B}_b(Y_2), x \in S, \alpha > 0$.

Proof. (i) It is easy to see that $\mathbb{M}_S$ is a right process w.r.t. the right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ with state space S and life time ζ. Moreover, $X_t(\omega)$ has left limits in S for $t \in (0, \zeta(\omega))$ for all $\omega \in \Omega$.

Let $\mu \in \mathcal{P}(S_\Delta)$ and τ be an $(\mathcal{F}_t^{P_\mu})_{t\geq 0}$–stopping time. Then $\overline{\tau}(\omega) = \tau(\omega)$ if $\omega \in \Omega$ and $\overline{\tau}(\omega) = 0$ if $\omega \in \overline{\Omega} \setminus \Omega$ is an $(\overline{\mathcal{F}}_t^{\overline{P}_\mu})_{t\geq 0}$–stopping time. By [FOT, Lemma A.2.3] there exists an $(\overline{\mathcal{F}}_t)_{t\geq 0}$–stopping time $\tilde{\tau}$ such that $\overline{P}_\mu[\tilde{\tau} \neq \overline{\tau}] = 0$ and consequently, $P_\mu[\tilde{\tau} \neq \tau] = 0$.

Hence, if $(\tau_n)_{n\geq 1}$ is an increasing sequence of $(\mathcal{F}_t^{P_\mu})_{t\geq 0}$–stopping times with limit τ, there exists an increasing sequence of $(\overline{\mathcal{F}}_t^{\overline{P}_\mu})_{t\geq 0}$–stopping times $(\tilde{\tau}_n)_{n\geq 1}$ with $P_\mu[\tilde{\tau}_n \neq \tau_n] = 0$, $n \geq 1$. Hence $X_{\tilde{\tau}} = X_\tau$ P_μ–a.s., where $\tilde{\tau} = \sup_{n\geq 1} \tilde{\tau}_n$, and by (2.13) and Lemma 2.10

$$\begin{aligned}
P_\mu[\lim_{n\to\infty} X_{\tau_n} \neq X_\tau, \tau < \zeta] &= \overline{P}_\mu[\lim_{n\to\infty} \overline{X}_{\tilde{\tau}_n} \neq \overline{X}_{\tilde{\tau}}, \tilde{\tau} < \zeta] \\
&= \overline{P}_\mu[\rho - \lim_{n\to\infty} \overline{X}_{\tilde{\tau}_n} \neq \overline{X}_{\tilde{\tau}}, \rho - \lim_{n\to\infty} \overline{X}_{\tilde{\tau}_n} \in S, \tilde{\tau} < \zeta] \\
&= 0\ .
\end{aligned}$$

We obtain that $\mathbb{M}_S$ is quasi–left continuous up to ζ. Let

$$V(\omega) := \begin{cases} \rho - \lim_{n\to\infty} X_{\tau_n}(\omega) & \text{if } \omega \in \{\rho - \lim_{n\to\infty} X_{\tau_n} \in S,\ \tau < \infty\} \\ \Delta & \text{if } \omega \in \{\rho - \lim_{n\to\infty} X_{\tau_n} \notin S\} \cup \{\tau = \infty\}. \end{cases}$$

Since $\mathbb{M}_S$ is quasi–left continuous up to ζ and $X_{\tau_n}(\omega) = \Delta = X_\tau(\omega)$ if $\tau_n(\omega) \geq \zeta(\omega)$ we conclude

$$P_\mu\left[V \neq X_\tau\right] = P_\mu\left[V \neq X_\tau,\ \tau = \zeta < \infty\right]\ .$$

By Lemma 2.10 and the fact that $\overline{X}_t(\omega) \notin S$ if $t \geq \zeta(\omega)$ and $\omega \in \Omega$ it follows that

$$\begin{aligned}
&P_\mu[\ V \neq X_\tau,\ \tau = \zeta < \infty] \\
&\quad = P_\mu\left[\rho - \lim_{n\to\infty} X_{\tau_n} \in S,\ \tau = \zeta < \infty\right] \\
&\quad = \overline{P}_\mu\left[\rho - \lim_{n\to\infty} \overline{X}_{\tilde{\tau}_n} \in S, \tilde{\tau} = \zeta < \infty\right] \\
&\quad \leq \overline{P}_\mu\left[\rho - \lim_{n\to\infty} \overline{X}_{\tilde{\tau}_n} \in S,\ \overline{X}_{\tilde{\tau}} \notin S,\ \tilde{\tau} < \infty\right] \\
&\quad = 0\ .
\end{aligned}$$

Since V is $\bigvee_{n\geq 1} \mathcal{F}_{\tau_n}^{P_\mu}$–measurable we obtain that X_τ is $\bigvee_{n\geq 1} \mathcal{F}_{\tau_n}^{P_\mu}$–measurable. Since μ was arbitrary we conclude that $\mathbb{M}_S$ is special standard.

(ii) Note that for $\alpha \in \mathbb{Q}^+$, $f \in J_0$, $x \in S_\Delta$ we have by (2.14) that

$$E_x\left[\int_0^\infty e^{-\alpha t} f(X_t)\,dt\right] = R_\alpha f(x) = \tilde{R}_\alpha f(x). \tag{2.17}$$

This equality holds in particular for $f = \sup_{n\geq 1}(nh)\wedge \mathbb{1} = \mathbb{1}_{Y_2}$. Since Y_2 is the Borel subset of a Lusin space and J_0 separates the points of Y_2 it follows $\sigma(J_0) = \mathcal{B}(Y_2)$ by [Sch, p.108, Lemma 18]. Hence monotone class arguments imply that (2.17) is true for all $f \in \mathcal{B}_b(Y_2)$ and $\alpha \in \mathbb{Q}^+$. Finally, (2.17) also holds for all $\alpha > 0$ and $f \in \mathcal{B}_b(Y_2)$ by Lebesgue's theorem. □

Denote by $\mathbb{M} = (\Omega', \mathcal{F}', (X'_t)_{t\geq 0}, (P'_x)_{x\in E_\Delta})$ the *trivial extension* of $\mathbb{M}_S$ to E, i.e.,

$$\begin{aligned}
\Omega' &:= \Omega \cup N\,, \text{ where } N := E\setminus S\,,\\
\mathcal{F}' &:= \{A_1\cup A_2 |\, A_1 \in \mathcal{F},\ A_2 \in \mathcal{B}(N)\}\,,\\
X'_t(\omega) &:= \begin{cases} X_t(\omega) & \text{if } \omega \in \Omega\\ x & \text{if } \omega = x \in N\,,\end{cases}\\
P'_x[A] &:= \begin{cases} P_x[A\cap\Omega] & \text{if } x \in S_\Delta\\ \delta_x(A) & \text{if } x\in N\,,\end{cases}\,,
\end{aligned} \tag{2.18}$$

then $\mathbb{M}$ is special standard again and m–tight by (2.15). This completes the proof of Theorem 2.2.

IV.3 Necessity of quasi–regularity

In this section we will prove the following converse result to Theorem 2.2:

Theorem 3.1. *Assume that there exists an m–tight m–special standard process $\mathbb{M}$ with state space E that is associated with $\mathcal{E}$. Then $\mathcal{E}$ is quasi–regular, and, moreover, $\mathbb{M}$ is properly associated in the resolvent sense with $\mathcal{E}$.*

Together with Theorem 2.2 we obtain the following

Corollary 3.2. *Let $\mathcal{E}$ satisfy D3. Then $\mathcal{E}$ is quasi–regular if and only if there exists an m–tight special standard process $\mathbb{M}$ associated with $\mathcal{E}$.*

For the proof of Theorem 3.1 let $\mathbb{M} = (\Omega, \mathcal{M}, (X_t)_{t\geq 0}, (P_x)_{x\in E_\Delta})$ with life time ζ be an m–tight m–special standard process associated with $\mathcal{E}$. Let $(R_\alpha)_{\alpha>0}$ (resp. $(p_t)_{t\geq 0}$) denote the resolvent (resp. transition semigroup) of $\mathbb{M}$.

(a) Proof of 1.7 (i)

Let $\tilde{C}$ denote the set of all $f \in \mathcal{B}_b(E)$ such that $t \mapsto f(X_t)$ is right continuous on $[0,\infty)$ and $t\mapsto f(X_{t-})$ is left continuous on $(0,\zeta\,)P_m$–a.s.

Proposition 3.3. *$R_\alpha f \in \tilde{C}$ for all $f \in \mathcal{B}_b(E)$, $\alpha > 0$.*

For the proof of Proposition 3.3 we can carry over the proof of [MR1, Proposition IV.5.14] which extends immediately to our more general situation.

A function $f \in \mathcal{B}_b(E)$ is called *1–supermedian for $(R_\alpha)_{\alpha>0}$* if $\alpha R_{\alpha+1} f \leq f$, $\alpha \geq 0$. Note that 1–supermedian functions $f \in \mathcal{B}_b(E) \cap \mathcal{H}$ are m–versions of 1–excessive elements in $\mathcal{H}$. $f \in \mathcal{B}_b(E)$ is called *1–excessive for $(R_\alpha)_{\alpha>0}$* if it is 1–supermedian for $(R_\alpha)_{\alpha>0}$ and $\lim_{\alpha\to\infty} \alpha R_{\alpha+1} f = f$.

For $g \in \mathcal{B}_b(E)$ and $U \subset E$ open let $H^U g(x) := E_x[e^{-\sigma_U} g(X_{\sigma_U})]$.

Proposition 3.4. *Let $f \in \mathcal{B}_b^+(E) \cap \mathcal{H}$ and $U \subset E$ open. Then $H^U R_1 f$ is an m–version of $(G_1 f)_U$.*

Proof. 1. Step: $H^U R_1 f \geq (G_1 f)_U$ m–a.e.

The strong Markov property of $\mathbb{M}$ implies

$$
\begin{aligned}
H^U R_1 f(x) &= E_x\left[e^{-\sigma_U} E_{X_{\sigma_U}}\left[\int_0^\infty e^{-t} f(X_t)\,dt\right]\right] \\
&= E_x\left[\int_{\sigma_U}^\infty e^{-t} f(X_t)\,dt\right] .
\end{aligned} \tag{3.1}
$$

Since $\sigma_U \leq \sigma_U \circ \theta_t + t$ we obtain

$$
\begin{aligned}
e^{-t} p_t H^U R_1 f(x) &= E_x\left[e^{-t} E_{X_t}\left[\int_{\sigma_U}^\infty e^{-s} f(X_s)\,ds\right]\right] \\
&= E_x\left[\int_{\sigma_U\circ\theta_t + t}^\infty e^{-s} f(X_s)\,ds\right] \\
&\leq E_x\left[\int_{\sigma_U}^\infty e^{-s} f(X_s)\,ds\right] = H^U R_1 f(x) \text{ for all } x .
\end{aligned}
$$

It follows that $H^U R_1 f$ is 1–supermedian for $(R_\alpha)_{\alpha>0}$ because

$$
\alpha R_{\alpha+1} H^U R_1 f(x) = \int_0^\infty \alpha e^{-(\alpha+1)t} p_t H^U R_1 f(x)\,dt \leq H^U R_1 f(x) . \tag{3.2}
$$

Since $H^U R_1 f \leq R_1 f$ (cf. (3.1)) we obtain that $H^U R_1 f$ is the m–version of some 1–excessive element in $\mathcal{H}$.

Clearly $\sigma_U = 0$ P_x–a.s. for $x \in U$ and therefore, $H^U R_1 f(x) = R_1 f(x)$. Hence $H^U R_1 f \geq (G_1 f)_U$ m–a.e. by Proposition III.1.7.

2. Step: $H^U R_1 f \leq (G_1 f)_U$ m–a.e.

Let $\{t_i | i \geq 1\} \subset (0,\infty)$ dense and define the stopping times

$$\sigma_n := \min\{t_i \mid 1 \leq i \leq n, X_{t_i} \in U\}, n \geq 1 ,$$

(with the convention $\min\emptyset = +\infty$). Set $e^{-\infty}g(X_\infty) := 0$ for all $g \in \mathcal{B}_b(E)$. Note that

$$E_x\left[e^{-\sigma_n}g(X_{\sigma_n})\right] = 0 \quad m\text{–a.e.} \tag{3.3}$$

for $g \in \mathcal{B}_b(E)\cap\mathcal{H}$ with $g = 0$ m–a.e. on U since

$$\begin{aligned}\left|E_x\left[e^{-\sigma_n}g(X_{\sigma_n})\right]\right| &\le \sum_{i=1}^n \left|E_x\left[e^{-t_i}g(X_{t_i})\mathbb{1}_{\{\sigma_n=t_i\}}\right]\right| \\ &\le \sum_{i=1}^n E_x\left[e^{-t_i}|g\mathbb{1}_U|(X_{t_i})\right] \\ &= \sum_{i=1}^n e^{-t_i}p_{t_i}|g\mathbb{1}_U|(x) = 0 \quad m\text{–a.e.}\end{aligned}$$

because $p_t|g\mathbb{1}_U|$ is an m–version of $T_t|g\mathbb{1}_U|$.

Observe that $(e^{-t}R_1g(X_t), \mathcal{F}_t, P_x)_{t\in[0,\infty]}$ is a nonnegative supermartingale for all x and $g \in \mathcal{B}_b^+(E)$ (where $(\mathcal{F}_t)_{t\ge0}$ denotes the natural filtration corresponding to $\mathbb{M}$) since $e^{-t}p_tR_1g \le R_1g$ and therefore,

$$\begin{aligned}E_x\left[e^{-(t+s)}R_1g(X_{t+s})\middle|\mathcal{F}_s\right] &= E_{X_s}\left[e^{-(t+s)}R_1g(X_t)\right] \\ &= e^{-(t+s)}p_tR_1g(X_s) \le e^{-s}R_1g(X_s)\ .\end{aligned}$$

Now, let $g \in \mathcal{B}_b^+(E)$ be an m–version of $(G_1f)_U$. Since $(G_1f)_U$ is 1–excessive, we obtain that $f_n := n(g - nR_{n+1}g) \ge 0$ m–a.e., so that $\lim_{n\to\infty} R_1(f_n^+) = g$ m–a.e. Hence there is a set $A \in \mathcal{B}(E)$ with $m(E\setminus A) = 0$ and $\lim_{n\to\infty} R_1(f_n^+)(x) = g(x)$ for all $x \in A$. Therefore, $\overline{g} := g\mathbb{1}_A$ is an m–version of $(G_1f)_U$ too, and consequently, $\overline{g} = R_1f$ m–a.e. on U. By Proposition 3.3, (3.3) and the discrete version of the optional sampling theorem (cf. [Ba, Satz 17.4]) for m–a.e. $x \in A$

$$\begin{aligned}H^UR_1f(x) &= E_x\left[e^{-\sigma_U}R_1f(X_{\sigma_U})\right] \\ &= \lim_{n\to\infty} E_x\left[e^{-\sigma_n}R_1f(X_{\sigma_n})\right] \\ &= \lim_{n\to\infty} E_x\left[e^{-\sigma_n}\overline{g}(X_{\sigma_n})\right] \\ &= \lim_{n\to\infty}\lim_{m\to\infty} E_x\left[e^{-\sigma_n}R_1(f_m^+)\mathbb{1}_A(X_{\sigma_n})\right] \\ &\le \lim_{n\to\infty}\limsup_{m\to\infty} E_x\left[e^{-\sigma_n}R_1(f_m^+)(X_{\sigma_n})\right] \\ &\le \lim_{n\to\infty}\limsup_{m\to\infty} R_1(f_m^+)(x) = g(x)\ .\end{aligned}$$

Hence $H^UR_1f \le (G_1f)_U$. $\square$

Since $\mathbb{M}$ is m–tight there exists an increasing sequence $(K_n)_{n\ge1}$ of compact metrizable sets with

$$P_m\left[\lim_{n\to\infty}\sigma_{E\setminus K_n} < \zeta\right] = 0\ .$$

Proposition 3.5. *$(K_n)_{n\geq 1}$ is an $\mathcal{E}$–nest.*

Proof. Let $\varphi \in \mathcal{B}_b^+(E) \cap \mathcal{H}$, $0 < \varphi$, $h := R_1\varphi$, and $U_n := E \setminus K_n$. By Proposition III.2.10 it is enough to show that $\lim\limits_{n\to\infty} \operatorname{Cap}_\varphi(U_n) = 0$. But Proposition 3.4 implies

$$\begin{aligned}
\operatorname{Cap}_\varphi(U_n) =&(h_{U_n}, \varphi)_{\mathcal{H}} \\
=& \int_E E_x\left[e^{-\sigma_{U_n}} R_1\varphi\left(X_{\sigma_{U_n}}\right)\right] \varphi(x) m(dx) \\
=& \int_E E_x\left[\int_{\sigma_{U_n}}^{\zeta} e^{-s}\varphi(X_s)\, ds\right] \varphi(x) m(dx)\ .
\end{aligned}$$

Hence it suffices to show that

$$E_x\left[\int_{\sigma_{U_n}}^{\zeta} e^{-s}\varphi(X_s)\, ds\right] \to 0 \text{ in } \mathcal{H}\ .$$

Clearly $P_x\left[\lim\limits_{n\to\infty} \sigma_{U_n} < \zeta\right] = 0 \quad m$–a.e., implies

$$\int_{\sigma_{U_n}}^{\zeta} e^{-s}\varphi(X_s)\, ds \to 0 \quad P_x\text{–a.s.} \quad m\text{–a.e.}$$

Therefore, $E_x\left[\int_{\sigma_{U_n}}^{\zeta} e^{-s}\varphi(X_s)\, ds\right] \to 0 \quad m$–a.e. by Lebesgue's theorem and consequently,

$$E_x\left[\int_{\sigma_{U_n}}^{\zeta} e^{-s}\varphi(X_s)\, ds\right] \to 0 \text{ in } \mathcal{H}$$

by Lebesgue's theorem again. $\square$

Clearly Proposition 3.5 implies 1.7 (i).

Remark 3.6. The proof of Proposition 3.5 shows that an increasing sequence of closed sets $(E_n)_{n\geq 1}$ is an $\mathcal{E}$–nest if and only if $P_m[\lim_{n\to\infty} \sigma_{E\setminus E_n} < \zeta] = 0$.

(b) Proof of 1.7 (ii)

For arbitrary $A \subset E$ let $\tau_A := \inf\{0 \leq t < \zeta \mid \overline{X_0^t} \cap A \neq \emptyset\} \wedge \zeta$ denote the *first touching time of* A, where $\overline{X_0^t}(\omega)$ is the closure of $\{X_s(\omega) \mid 0 \leq s \leq t\}$ in E. Since $\mathbb{M}$ is m–special it follows that $\tau_A = \inf\{0 \leq t < \zeta \mid X_t \in A \text{ or } X_{t-} \in A\} \wedge \zeta$. Hence τ_A is an $(\mathcal{F}_t^{P_m})$–stopping time if $A \in \mathcal{B}(E)$ (cf. [DMe, IV.50]). A subset $N \subset E$ is called $\mathbb{M}$–*exceptional* if there exists $\tilde{N} \in \mathcal{B}(E)$ such that $N \subset \tilde{N}$ and $P_m[\tau_{\tilde{N}} < \zeta] = 0$. Note that this implies $m(\tilde{N}) = 0$, so that $\mathbb{M}$–exceptional sets $N \in \mathcal{B}(E)$ are zero sets w.r.t. m.

Proposition 3.7. *Let $f \in \tilde{C}$, then there exists an $\mathcal{E}$–q.c. function $\tilde{f}$ on E such that $f = \tilde{f}$ outside some $\mathbb{M}$–exceptional set.*

The proof of Proposition 3.7 is the same as the proof of Proposition IV.5.24 in [MR1] if one replaces $\mathrm{Cap}_{h,g}$ in [MR1] by Cap_φ, where $\varphi \in \mathcal{B}_b^+(E) \cap \mathcal{H}$, $0 < \varphi$, and $\int \varphi \, dm = 1$.

Since $\{G_\alpha f | f \in \mathcal{H}\} \subset \mathcal{F}$ dense by Remark I.3.5, Propositions 3.3 and 3.7 imply 1.7 (ii).

(c) Proof of 1.7 (iii)

Theorem 3.8. *(i) A subset N of E is $\mathbb{M}$–exceptional if and only if it is $\mathcal{E}$–exceptional.*

(ii) If $f \in \tilde{C}$ then f is $\mathcal{E}$–q.c. In particular, $R_\alpha f$ is $\mathcal{E}$–q.c. for all $f \in \mathcal{B}_b(E), \alpha > 0$, and $\mathbb{M}$ is properly associated in the resolvent sense with $\mathcal{E}$.

The proof of Theorem 3.8 is the same as the proof of Theorem IV.5.29 in [MR1] if one replaces $\mathrm{Cap}_{h,g}$ in [MR1] by Cap_φ, where $\varphi \in \mathcal{B}_b^+(E) \cap \mathcal{H}$, $0 < \varphi$, and $\int \varphi \, dm = 1$. Note that Cap_φ is a (finite) Choquet capacity by Proposition III.2.8.

Lemma 3.9. *Let $f \in \mathcal{B}_b^+(E) \cap \mathcal{H}$ and $U \subset E$ open. Then $H^U R_1 f$ is an $\mathcal{E}$–q.l.s.c. m–version of $(G_1 f)_U$.*

Proof. We already showed that $H^U R_1 f$ is 1–supermedian for $(R_\alpha)_{\alpha > 0}$ (cf. (3.2)). Since $\lim\limits_{t \downarrow 0} \sigma_U \circ \theta_t(\omega) + t = \sigma_U(\omega)$ for all ω we obtain by Lebesgue's theorem

$$\begin{aligned} \lim_{t \downarrow 0} e^{-t} p_t H^U R_1 f(x) &= \lim_{t \downarrow 0} E_x \left[\int_{\sigma_U \circ \theta_t + t}^{\infty} e^{-s} f(X_s) \, ds \right] \\ &= E_x \left[\int_{\sigma_U}^{\infty} e^{-s} f(X_s) \, ds \right] = H^U R_1 f(x) \text{ for all } x \ . \end{aligned}$$

It is easy to see that this implies

$$\lim_{\alpha \to \infty} \alpha R_{\alpha+1} H^U R_1 f(x) = H^U R_1 f(x) \text{ for all } x \ , \tag{3.4}$$

and therefore, $\sup\limits_{n \geq 1} n R_{n+1} H^U R_1 f = H^U R_1 f$. Since $\mathbb{M}$ is properly associated in the resolvent sense with $\mathcal{E}$ we obtain that $H^U R_1 f$ is $\mathcal{E}$–q.l.s.c. $\square$

Lemma 3.10. *Let $(E_n)_{n \geq 1}$ be an $\mathcal{E}$–nest. Then $P_x \left[\lim\limits_{n \to \infty} \sigma_{E \setminus E_n} < \zeta \right] = 0$ $\mathcal{E}$–q.e.*

Proof. Let φ be as in the proof of Proposition 3.5, $h = R_1 \varphi$, and $U_n := E \setminus E_n$. Clearly $H^{U_n} h \to 0$ in $\mathcal{H}$ (cf. the proof of Proposition 3.5 and Remark 3.6). Since $H^{U_n} h$ is $\mathcal{E}$–q.l.s.c. there is a subsequence $\left(H^{U_{n_k}} h\right)_{k \geq 1}$ such that

$$\limsup_{k \to \infty} H^{U_{n_k}} h \leq 0 \quad \mathcal{E}\text{–q.e.}$$

by Lemma III.3.9. But $H^{U_{n_k}}h \geq 0$ for all k implies $\lim_{k\to\infty} H^{U_{n_k}}h = 0 \quad \mathcal{E}$–q.e. Since $H^{U_{n+1}}h \leq H^{U_n}h$ we conclude that $\lim_{n\to\infty} H^{U_n}h = 0 \quad \mathcal{E}$–q.e. It follows $P_x[\lim_{n\to\infty} \sigma_{U_n} < \zeta] = 0 \quad \mathcal{E}$–q.e. because

$$H^{U_n}h(x) = E_x\left[\int_{\sigma_{U_n}}^{\zeta} e^{-t}\varphi(X_t)\,dt\right]$$

and $\varphi > 0$ on E. $\square$

Let $(K_n)_{n\geq 1}$ be as in Proposition 3.5 and $N \subset E$ be an $\mathcal{E}$–exceptional set such that

$$P_x\left[\lim_{n\to\infty} \sigma_{E\setminus K_n} < \zeta\right] = 0 \text{ for all } x \in E \setminus N\ .$$

Since K_n is a compact metrizable set (w.r.t. the trace topology induced by E) there exist open sets $U_m \subset E$ such that $(U_m \cap K_n)_{m\geq 1}$ separates the points of K_n for all n (i.e., for $x, y \in K_n$, $x \neq y$, there exist $U, V \in \mathcal{U} := \{U_n \mid n \geq 1\}$ such that $U \cap K_n \cap V = \emptyset$ and $x \in U$, $y \in V$).

Lemma 3.11. $(h - H^U h)_{U\in\mathcal{U}}$ *separates the points of* $\bigcup_{n\geq 1} K_n \cap (E \setminus N)$.

Proof. Let $x, y \in \bigcup_{n\geq 1} K_n \cap (E \setminus N)$, $\quad x \neq y$. Then we may assume $x, y \in K_m$ for some m with $P_y\left[\sigma_{E\setminus K_m} \wedge \zeta > 0\right] > 0$ and that there exist $U, V \in \mathcal{U}$ such that $x \in U$, $y \in V$ and $U \cap K_m \cap V = \emptyset$. Let $\overline{\Omega} := \{\omega \mid \sigma_{E\setminus K_m}(\omega) \wedge \zeta(\omega) > 0, X_0(\omega) = y\}$. For $\omega \in \overline{\Omega}$ the map $X_\cdot(\omega) : [0, \sigma_{E\setminus K_m}(\omega) \wedge \zeta(\omega)) \to K_m$ is right continuous w.r.t. the trace topology on K_m. Since $V \cap K_m$ is open in K_m and $U \cap K_m \cap V = \emptyset$ this implies $\sigma_U(\omega) > 0$. Hence $P_y[\sigma_U > 0] > 0$ and therefore,

$$H^U h(y) = E_y\left[\int_{\sigma_U}^{\infty} e^{-t}\varphi(X_t)\,dt\right] < E_y\left[\int_0^{\infty} e^{-t}\varphi(X_t)\,dt\right] = h(y)\ .$$

On the other hand clearly $H^U h(x) = h(x)$ which implies the assertion. $\square$

Since $\lim_{n\to\infty} nR_{n+1}(h - H^U h)(x) = (h - H^U h)(x)$ for all x (cf. (3.4)), we obtain from the last lemma that

$$\{nR_{n+1}(h - H^U h) \mid U \in \mathcal{U}, n \geq 1\}$$

separates the points of $\bigcup_{n\geq 1} K_n \cap (E \setminus N)$ too. Since $N \cup \left(\bigcap_{n\geq 1} (E \setminus K_n)\right)$ is $\mathcal{E}$–exceptional and $nR_{n+1}(h - H^U h)$ $\mathcal{E}$–q.c. (by Theorem 3.8) we see that 1.7 (iii) holds.

IV.4 Examples of quasi–regular generalized Dirichlet forms

(a) Quasi–regularity in the case of Example I.4.9

(i) If $(\mathcal{A}, \mathcal{V})$ is a quasi–regular Dirichlet form on $L^2(E, m)$ in the sense of [MOvR], and $\Lambda = 0$ it follows immediately that the corresponding generalized Dirichlet form is quasi–regular (cf. Remark III.2.6 (ii)).

Moreover, assumption D3 is satisfied for $\mathcal{Y} = \mathcal{V} \cap L^\infty(E, m)$ by [MR1, Theorem I.4.4].

(ii) In the case of Example I.4.9 (ii) we consider the following (typical) example. Let E be a locally compact separable metric space. Assume that $(\bar{T}_t)_{t \geq 0}$ is a C_0–semigroup of contractions on $C_\infty(E)$ with $0 \leq \bar{T}_t f \leq 1$ if $0 \leq f \leq 1$, hence a Feller semigroup, with generator $(\bar{\Lambda}, D(\bar{\Lambda}))$ and that m is a finite invariant measure for $(\bar{T}_t)_{t \geq 0}$, i.e., $\int \bar{T}_t f \, dm = \int f \, dm$ for all $f \in C_\infty(E)$. It follows that $(\bar{T}_t)_{t \geq 0}$ induces a sub–Markovian C_0–semigroup of contractions $(T_t)_{t \geq 0}$ on $L^2(E, m)$. Let $(\Lambda, D(\Lambda))$ be the corresponding generator and let $\mathcal{A} = 0$. To prove that the generalized Dirichlet form $\mathcal{E}$ associated with $\mathcal{A}$ and $(\Lambda, D(\Lambda))$ is quasi–regular let $\varphi \in C_\infty(E)$ such that $\varphi > 0$ everywhere on E and $h := G_1 \varphi$, where $(G_\alpha)_{\alpha > 0}$ denotes the associated resolvent. Since $h \in C_\infty(E)$ we have that $E_n := \{h \geq 1/n\}$ is an increasing sequence of compact sets with $\lim_{n \to \infty} \mathrm{Cap}_\varphi(E_n^c) = 0$ (cf. Lemma III.3.10). Hence $\mathcal{E}$ satisfies 1.7 (i). For the proof of 1.7 (ii) note that $D(\bar{\Lambda}) \subset D(\Lambda)$ and $\Lambda = \bar{\Lambda}$ on $D(\bar{\Lambda})$. Moreover, $D(\bar{\Lambda}) \subset D(\Lambda)$ dense w.r.t. the graph norm by [ReSi2, Theorem X.49] since $T_t(D(\bar{\Lambda})) \subset D(\bar{\Lambda})$, $t \geq 0$. This implies 1.7 (ii). 1.7 (iii) is obvious by the topological assumptions on E. Hence $\mathcal{E}$ is quasi–regular.

Moreover, assumption D3 is satisfied for $\mathcal{Y} = D(\bar{\Lambda})$ since $D(\bar{\Lambda}) \subset C_\infty(E)$ dense.

(iii) In the case of Example I.4.9 (iii) consider the case where $I = \mathbb{R}$.

Lemma 4.1. *Let $(\mathcal{E}^{(t)}, V)$ be quasi–regular for some $t \in \mathbb{R}$. Then the corresponding time dependent Dirichlet form $\mathcal{E}$ is quasi–regular again.*

Proof. Let

$$\mathcal{E}'(u,v) := \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right)_{\mathcal{H}} + \mathcal{A}(u,v); \; u, v \in D(\mathcal{E}') := D(\frac{\partial}{\partial t}, \mathcal{H}) \cap \mathcal{V} \, .$$

Clearly $(\mathcal{E}', D(\mathcal{E}'))$ is a Dirichlet form on $\mathcal{H}$. To prove quasi–regularity of $\mathcal{E}$ it is now enough to show that $(\mathcal{E}', D(\mathcal{E}'))$ is quasi–regular (cf. Remark III.2.6 (ii) and Remark III.2.11). To this end let $(E_n)_{n \geq 1}$ be an $\mathcal{E}^{(t)}$–nest and $\widetilde{D} :=$ span $\{fu | f \in C_0^1(\mathbb{R}), \; u \in \bigcup_{n \geq 1} V_{E_n}\}$. Then $\widetilde{D} \subset \mathcal{V}$ dense since $\bigcup_{n \geq 1} V_{E_n}$ is dense in V (cf. Remark III.2.6 (ii)) , $U_t(\widetilde{D}) \subset \widetilde{D}$ for all $t \geq 0$ and $\widetilde{D} \subset D(\frac{\partial}{\partial t}, \mathcal{V})$. Thus $\widetilde{D} \subset D(\frac{\partial}{\partial t}, \mathcal{V})$ dense by [ReSi2, Theorem X.49]. Since $\widetilde{D} \subset \bigcup_{n \geq 1} D(\mathcal{E}')_{K_n}$, where $K_n := [-n, n] \times E_n$, we have that $(K_n)_{n \geq 1}$ is an $\mathcal{E}'$–nest. Hence

(4.1) If $(E_n)_{n\geq 1}$ is an $\mathcal{E}^{(t)}$–nest of compact sets, $(K_n)_{n\geq 1}$ is an $\mathcal{E}'$–nest of compact sets.

(4.2) If $\tilde{u} \in C(\{E_n\})$ is the $\mathcal{E}^{(t)}$–q.c. μ–version of some element $u \in V$ and $f \in C_0^1(\mathbb{R})$ then $f\tilde{u} \in C(\{K_n\})$ is the $\mathcal{E}'$–q.c. $dt \otimes \mu$–version of $fu \in D(\mathcal{E}')$.

(4.3) If $(\tilde{u}_n)_{n\geq 1} \subset C(\{E_n\})$ separates the points of $\bigcup\limits_{n\geq 1} E_n$ such that for all $x \in \bigcup\limits_{n\geq 1} E_n$ there is some $\tilde{u}_n$ with $\tilde{u}_n(x) \neq 0$ and $(f_n)_{n\geq 1} \subset C_0^1(\mathbb{R})$ separates the points of $\mathbb{R}$, such that for all $s, t \in \mathbb{R}$ there is some f_n with $f_n(s) = f_n(t) \neq 0$, then $(f_n\tilde{u}_m)_{n,m\geq 1}$ separates the points of $\bigcup\limits_{n\geq 1} K_n$.

Clearly (4.1)–(4.3) imply that $(\mathcal{E}', D(\mathcal{E}'))$ is quasi–regular. $\square$

Moreover, assumption D3 is satisfied for $\mathcal{Y} = D(\frac{\partial}{\partial t}, \mathcal{H}) \cap \mathcal{V} \cap L^\infty(\mathbb{R}\times X, dt\otimes d\mu)$.

(b) Quasi–regularity in the case of Example II.1

Let $(U_t)_{t\geq 0}$ denote the semigroup corresponding to $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t}, \mathcal{H}))$. Since $U_t(C_0^\infty(\mathbb{R}^{d+1})) \subset C_0^\infty(\mathbb{R}^{d+1})$, $t \geq 0$, and $C_0^\infty(\mathbb{R}^{d+1}) \subset D(\frac{\partial}{\partial t}, \mathcal{V})$ it follows from [ReSi2, Theorem X.49] that $C_0^\infty(\mathbb{R}^{d+1}) \subset D(\frac{\partial}{\partial t}, \mathcal{V})$ dense w.r.t. the graph norm. In particular, $C_0^\infty(\mathbb{R}^{d+1}) \subset \mathcal{F}$ dense (cf. Lemma I.2.5). Hence if $(K_n)_{n\geq 1}$ is an increasing sequence of compact subsets with $\mathbb{R}^{d+1} = \bigcup_{n\geq 1} K_n$ and $K_n \subset \overset{\circ}{K}_{n+1}$, $n \geq 1$, it follows that $C_0^\infty(\mathbb{R}^{d+1}) \subset \bigcup_{n\geq 1} \mathcal{F}_{K_n}$ and consequently, $(K_n)_{n\geq 1}$ is an $\mathcal{E}$–nest by Remark III.2.11. Thus $\mathcal{E}$ satisfies 1.7 (i). But 1.7 (ii) and 1.7 (iii) are obvious. Hence $\mathcal{E}$ is a quasi–regular generalized Dirichlet form. Moreover, D3 is satisfied for $\mathcal{Y} = C_0^\infty(\mathbb{R}^{d+1})$.

(c) Quasi–regularity in the case of Example II.2 (b)

Consider the situation of Proposition II.2.6. To prove quasi–regularity in this case we need two lemmas.

Lemma 4.2. *Let $f \in D(\Lambda, \mathcal{H})$ and $u \in C_0^\infty(\mathbb{R}^d)$. Then $uf \in D(\Lambda, \mathcal{H})$ and $\Lambda(uf) = \langle B, \nabla u\rangle f + u\Lambda f$.*

Proof. Since $C_0^\infty(\mathbb{R}^d) \subset D(\Lambda, \mathcal{H})$ dense w.r.t. the graph norm there exist $f_n \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, such that $f_n \to f$ in $D(\Lambda, \mathcal{H})$. Clearly $\lim_{n\to\infty} uf_n = uf$ in $\mathcal{H}$, $uf_n \in D(\Lambda, \mathcal{H})$ and $\Lambda(uf_n) = \langle B, \nabla u\rangle f_n + u\Lambda f_n$. Since B is locally bounded we obtain that $\lim_{n\to\infty} \Lambda(uf_n) = \lim_{n\to\infty}\langle B, \nabla u\rangle f_n + u\Lambda f_n = \langle B, \nabla u\rangle f + u\Lambda f$ in $\mathcal{H}$. Since $(\Lambda, D(\Lambda, \mathcal{H}))$ is closed we conclude that $uf \in D(\Lambda, \mathcal{H})$ and $\Lambda(uf) = \langle B, \nabla u\rangle f + u\Lambda f$. $\square$

Lemma 4.3. *$C_0^\infty(\mathbb{R}^d) \subset \mathcal{F}$ dense.*

Proof. By the construction of $\mathcal{F}$ it suffices to show that $C_0^\infty(\mathbb{R}^d) \subset D(\Lambda, \mathcal{H}) \cap$

$H^{1,2}(\mathbb{R}^d)$ dense w.r.t. the $\mathcal{F}$–norm. Let $\chi_n \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, $1\!\!1_{B_n}(0) \leq \chi_n \leq 1\!\!1_{\mathbb{R}^d}$, $|\nabla\chi_n(x)| \leq 1/f(|x|)$ for all $x \in \mathbb{R}^d$. Since $|B(x)| \leq f(|x|)$, $x \in \mathbb{R}^d$, it follows for $f \in D(\Lambda,\mathcal{H}) \cap H^{1,2}(\mathbb{R}^d)$ that $\lim_{n\to\infty}\chi_n f = f$ in $H^{1,2}(\mathbb{R}^d)$, $\chi_n f \in D(\Lambda,\mathcal{H})$ and $\Lambda(\chi_n f) = \langle B,\nabla\chi_n\rangle f + \chi_n\Lambda f$ by Lemma 4.2. Hence $\lim_{n\to\infty}\Lambda(\chi_n f) = \lim_{n\to\infty}\langle B,\nabla\chi_n\rangle f + \chi_n\Lambda f = \Lambda f$ in $\mathcal{H}$ which implies that $\lim_{n\to\infty}\chi_n f = f$ in $\mathcal{F}$. Hence we may assume that supp $(f\,dx)$ is compact. Since $f \in H^{1,2}(\mathbb{R}^d)$ there exist $f_n \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, with supp $(f_n) \subset B_R(0)$ for some $R > 0$, and $\lim_{n\to\infty} f_n = f$ in $H^{1,2}(\mathbb{R}^d)$. Since B is locally bounded it follows that $\lim_{n\to\infty}\Lambda f_n = \lim_{n\to\infty}\langle B,\nabla f_n\rangle = \langle B,\nabla f\rangle$ in $\mathcal{H}$ which implies $\lim_{n\to\infty} f_n = f$ in $\mathcal{F}$. $\square$

Similar to the proof of the corresponding statement in Subsection (b) it follows that $\mathcal{E}$ is quasi–regular. Moreover, D3 is satisfied for $\mathcal{Y} = C_0^\infty(\mathbb{R}^d)$.

(d) Quasi–regularity in the case of Example II.2 (c)

Consider the situation of Proposition II.2.12 and assume in addition that for some constant M $|B(t,x)| \leq M(|(t,x)| + 1)$ if $t \in \mathbb{R}$, $x \in \mathbb{R}^d$. In particular we obtain that B is locally bounded.

Lemma 4.4.

(i) Let $f \in D(\Lambda,\mathcal{H})$ and $u \in C_0^\infty(\mathbb{R}^{d+1})$. Then $uf \in D(\Lambda,\mathcal{H})$ and $\Lambda(uf) = (\frac{\partial u}{\partial t} + \langle B,\nabla u\rangle)f + u\Lambda f$.

(ii) Let $f \in D(\Lambda,\mathcal{H})$ and $\varphi \in C^\infty(\mathbb{R})$, $|\varphi(t)| \leq |t|$ and $|\varphi'(t)| \leq 1$ for all $t \in \mathbb{R}$. Then $\varphi(f) \in D(\Lambda,\mathcal{H})$ and $\Lambda\varphi(f) = \varphi'(f)\Lambda f$.

Proof. The proof of (i) is similar to the proof of Lemma 4.2. Hence we omit it. (ii) Let $f_n \in C_0^\infty(\mathbb{R}^{d+1})$, $n \geq 1$, such that $\lim_{n\to\infty} f_n = f$ in $D(\Lambda,\mathcal{H})$. Clearly $\varphi(f_n) \in C_0^\infty(\mathbb{R}^{d+1})$ and $\Lambda\varphi(f_n) = \varphi'(f_n)\Lambda f_n$, $n \geq 1$. Since $\lim_{n\to\infty}\varphi(f_n) = \varphi(f)$ in $\mathcal{H}$ and $\lim_{n\to\infty}\Lambda\varphi(f_n) = \varphi'(f)\Lambda f$ in $\mathcal{H}$ at least along some subsequence, we conclude that $\varphi(f) \in D(\Lambda,\mathcal{H})$ and $\Lambda\varphi(f) = \varphi'(f)\Lambda f$. $\square$

Lemma 4.5. *$C_0^\infty(\mathbb{R}^{d+1}) \subset \mathcal{F}$ dense.*

Proof. By the construction of $\mathcal{F}$ it suffices to show that $C_0^\infty(\mathbb{R}^{d+1}) \subset D(\Lambda,\mathcal{H}) \cap \mathcal{V}$ dense w.r.t. the $\mathcal{F}$–norm. Let $f \in D(\Lambda,\mathcal{H}) \cap \mathcal{V}$. Similar to the proof of Lemma 4.3 we may suppose that supp $(f\,dy)$ is compact. By Lemma 4.4 (ii) we may assume in addition that f is bounded. Let $f_n \in C_0^\infty(\mathbb{R}^d)$, $n \geq 1$, such that $\lim_{n\to\infty} f_n = f$ in $D(\Lambda,\mathcal{H})$. By Lemma 4.4 we may suppose that $\sup_{n\geq 1}\|f_n\|_\infty < \infty$ and supp $(f_n) \subset B_R(0)$ for some $R > 0$. Let $v \in C_0^\infty(\mathbb{R}^{d+1})$. Since by Remark II.2.11

$$|\int_{\mathbb{R}^{d+1}}\langle B,\nabla(f_n v)\rangle\,dy| \leq (d^2\|L_\infty\|_\infty + 2c)\|f_n\|_{\mathcal{H}}\|v\|_{\mathcal{H}} \\ + d^2\|L_2\|_2\sup_{t\in\mathbb{R}}\|f_n(t,\cdot)\|_2^2\|v\|_{\mathcal{H}}$$

and B is locally bounded we conclude that for some constant m

$$
\begin{aligned}
&|(\frac{\partial f_n}{\partial t}, v)_{\mathcal{H}}| \leq |(\Lambda f_n, v)_{\mathcal{H}}| + |(\langle B, \nabla f_n\rangle, v)_{\mathcal{H}}| \\
&\leq |(\Lambda f_n, v)_{\mathcal{H}}| + |\int_{\mathbb{R}^{d+1}} \langle B, \nabla(f_n v)\rangle\, dy| + |(\langle B, \nabla v\rangle, f_n)_{\mathcal{H}}| \\
&\leq m\|v\|_{\mathcal{V}}\ .
\end{aligned}
$$

Therefore, $(\frac{\partial f_n}{\partial t})_{n\geq 1}$ is bounded in $\mathcal{V}'$.

Denote the time dependent Dirichlet form corresponding to $(D, H^{1,2}(\mathbb{R}^d))_{t\in\mathbb{R}}$ by $\mathcal{E}'$ and the semigroup of $(\frac{\partial}{\partial t}, D(\frac{\partial}{\partial t}, \mathcal{H}))$ by $(U_t)_{t\geq 0}$. Since both $(U_t)_{t\geq 0}$ and the adjoint semigroup can be extended to C_0–semigroups on $\mathcal{V}'$ (cf. Example I.4.9 (iii)) it follows from Remark I.2.10 that $\mathcal{F}' = \mathcal{V}\cap D(\frac{\partial}{\partial t}, \mathcal{V}')$, where $\mathcal{F}'$ is the domain of the closure of $\frac{\partial}{\partial t} : \mathcal{V}\cap D(\frac{\partial}{\partial t}, \mathcal{H}) \to \mathcal{V}'$. Note that $(\frac{\partial f_n}{\partial t})_{n\geq 1}$ bounded in $\mathcal{V}'$ and $\lim_{n\to\infty} f_n = f$ in $\mathcal{H}$, hence in particular in $\mathcal{V}'$, implies that $f \in D(\frac{\partial}{\partial t}, \mathcal{V}')$. Since $f\in\mathcal{V}$ we conclude that $f\in\mathcal{F}'$. By the results of Subsection (a) there exist $g_n \in C_0^\infty(\mathbb{R}^{d+1})$ with $\lim_{n\to\infty} g_n = f$ in $\mathcal{F}'$ (indeed, we may take as $\tilde{D}$ in Lemma 4.1 the linear subspace $\tilde{D} := \operatorname{span}\{hu | h\in C_0^\infty(\mathbb{R}), u\in C_0^\infty(\mathbb{R}^d)\}$). Let $\chi\in C_0^\infty(\mathbb{R}^{d+1})$ such that $\chi = 1$ on some open set containing $\operatorname{supp}(f dy)$ and $0\leq\chi\leq 1$. Note that $\chi v\in\mathcal{V}$ if $v\in\mathcal{V}$ and $\mathcal{A}(\chi v, \chi v)\leq C\mathcal{A}_1(v,v)$ for some positive constant C. It follows that $\lim_{n\to\infty}\chi g_n = f$ in $\mathcal{V}$ and $\lim_{n\to\infty}\langle B, \nabla(\chi g_n)\rangle = \langle B, \nabla f\rangle$ in $\mathcal{H}$ since B is locally bounded. Moreover, for all $v\in C_0^\infty(\mathbb{R}^{d+1})$

$$
\begin{aligned}
\langle\frac{\partial \chi g_n}{\partial t} - \frac{\partial f}{\partial t}, v\rangle &= -(\frac{\partial v}{\partial t}, \chi g_n - f)_{\mathcal{H}} \\
&= -(\frac{\partial v}{\partial t}, \chi(g_n - f))_{\mathcal{H}} \\
&= -(\frac{\partial \chi v}{\partial t}, g_n - f)_{\mathcal{H}} + (\frac{\partial\chi}{\partial t}v, g_n - f)_{\mathcal{H}} \\
&\leq \|\frac{\partial g_n}{\partial t} - \frac{\partial f}{\partial t}\|_{\mathcal{V}'}\|\chi v\|_{\mathcal{V}} + \|\frac{\partial\chi}{\partial t}v\|_{\mathcal{H}}\|g_n - f\|_{\mathcal{H}}\,,
\end{aligned}
$$

which implies that

$$
\|\frac{\partial\chi g_n}{\partial t} - \frac{\partial f}{\partial t}\|_{\mathcal{V}'} \leq ((C+1)^{\frac{1}{2}} + \|\frac{\partial\chi}{\partial t}\|_\infty)\|g_n - f\|_{\mathcal{F}'} \to 0\,, n\to\infty\,,
$$

and thus $\lim_{n\to\infty}\chi g_n = f$ in $\mathcal{F}$. $\square$

Similar to the proof of the corresponding statement in Subsection (b) it follows that $\mathcal{E}$ is quasi–regular. Moreover, D3 is satisfied for $\mathcal{Y} = C_0^\infty(\mathbb{R}^{d+1})$.

(e) Quasi–regularity in the case of Example II.3

Consider the situation of Proposition II.3.6. Let $(U_t)_{t\geq 0}$ denote the semigroup corresponding to $(\Lambda, D(\Lambda, \mathcal{H}))$. Since $U_t(C_0^\infty(\mathbb{R}_+^d)) \subset C_0^\infty(\mathbb{R}_+^d)$, $t \geq 0$, and $\Lambda u \in C_0^\infty(\mathbb{R}_+^d)$ if $u \in C_0^\infty(\mathbb{R}_+^d)$ we obtain that $C_0^\infty(\mathbb{R}_+^d) \subset \mathcal{F}$ dense similar to the proof of the corresponding statement in Subsection (b). Let $K_n := \{x | x = (x_1, \dots, x_d) \in \mathbb{R}_+^d\,, x_i \leq n\,, 1 \leq i \leq d\}$. Then $(K_n)_{n\geq 1}$ is an increasing sequence of compact sets with $C_0^\infty(\mathbb{R}_+^d) \subset \bigcup_{n\geq 1} \mathcal{F}_{K_n}$. Hence $(K_n)_{n\geq 1}$ is an $\mathcal{E}$–nest by Remark III.2.11 and $\mathcal{E}$ satisfies 1.7 (i). 1.7 (ii) and 1.7 (iii) are obvious. It follows that $\mathcal{E}$ is quasi–regular. Moreover, D3 is satisfied for $\mathcal{Y} = C_0^\infty(\mathbb{R}_+^d)$.

(f) Quasi–regularity in the case of Example II.4

Subsection II.4 (a)

Assume that K_0 satisfies instead of II (4.1) the stronger assumption

$$\exists\, c > 0 \text{ such that } \sum_{k\in K_0} {}_{E'}\langle \ell, k\rangle_E^2 \leq c^2 \|\ell\|_{E'}^2 \text{ for all } \ell \in E'\ . \tag{4.4}$$

Then the form $(\mathcal{E}^{(t)}, V)$ is quasi–regular for all $t \in \mathbb{R}$ (cf. [MR1, Subsection IV.4b)]) and by the results of Subsection (a) it follows that the corresponding time dependent Dirichlet form is quasi–regular. Moreover, assumption D3 is also satisfied.

Subsection II.4 (b)

Recall that in the situation of Example II.4 (b) any $h \in H$ is well–μ–admissible since (E, H, μ) is an abstract Wiener space. It follows that for any orthonormal basis K_0 of H we have that

$$\sum_{k\in K_0} {}_{E'}\langle \ell, k\rangle_E^2 = \|\ell\|_H^2 \text{ for all } \ell \in E'$$

and therefore, (4.4) is satisfied. Hence $(\mathcal{E}^{(t)}, V)$ is quasi–regular for all $t \in \mathbb{R}$ and it follows that the corresponding time dependent Dirichlet form is quasi–regular. Again, assumption D3 is also satisfied.

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