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A CLASSIFICATION OF PARABOLIC SUBGROUPS OF CLASSICAL GROUPS WITH A FINITE NUMBER OF ORBITS ON THE UNIPOTENT RADICAL

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Abstract. Let G be a classical algebraic group defined over an algebraically closed field. We classify all instances when a parabolic subgroup P of G acts on its unipotent radical P_u, or on $\mathfrak{p}_u$, the Lie algebra of P_u, with only a finite number of orbits.

The proof proceeds in two parts. First we obtain a reduction to the case of general linear groups. In a second step, a solution for these is achieved by studying the representation theory of a particular quiver with certain relations.

Furthermore, for the general linear groups we obtain a combinatorial formula for the number of orbits in the finite cases.

1. Introduction

Let G be a reductive algebraic group defined over an algebraically closed field k. Let P be a parabolic subgroup of G with unipotent radical P_u. Then P acts on P_u via conjugation and on $\mathfrak{p}_u$, the Lie algebra of P_u, via the adjoint action. The modality of the action of P on $\mathfrak{p}_u$, denoted by $\operatorname{mod}(P : \mathfrak{p}_u)$, is the maximal number of parameters upon which a family of P-orbits on $\mathfrak{p}_u$ depends; likewise for $\operatorname{mod}(P : P_u)$, see Section 2 for a precise definition. The *modality* of P is defined as $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$. Note that $\operatorname{mod} P$ is zero precisely when P has only a finite number of orbits on $\mathfrak{p}_u$. By a fundamental result of R.W. Richardson P always admits a dense orbit on P_u [Ri]. Thus, if $\operatorname{mod} P$ is positive, an infinite family of orbits occurs in a subvariety of P_u of codimension at least 1. The general problem of classifying all parabolic subgroups of reductive algebraic groups G of modality zero was posed in [PR]. In this paper we present a classification for the classical groups. Let

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$\ell(P_u)$ denote the nilpotency class of P_u. Our chief result is

Theorem 1.1. *Let G be a simple classical algebraic group and P a parabolic subgroup of G. Suppose that* $\operatorname{char} k$ *is either zero or a good prime for G. The number of P-orbits on P_u, or $\mathfrak{p}_u$, is finite if and only if one of the following holds:*

(i) $\ell(P_u) \leq 4$;
(ii) *G is of type D_r, $\ell(P_u) = 5$, P is not invariant under the automorphism of G of order 2 stemming from the interchange of the simple roots α_{r-1} and α_r, and the derived subgroup of a Levi subgroup of P consists of two commuting simple parts.*

Under the assumption that $\operatorname{char} k = 0$ a partial classification of parabolic subgroups as in Theorem 1.1 was obtained in [HR1]. The cases for Borel subgroups, semisimple rank one, and semisimple rank two parabolic subgroups of all simple algebraic groups are classified in [Ka], [PR], and [JR], respectively. A detailed discussion of parabolic subgroups with an Abelian unipotent radical, including the structure of orbit closures, appears in [RRS]. Further results on parabolic subgroups with this finiteness property in exceptional groups can also be found in [JR].

Our proof of Theorem 1.1 proceeds in two steps. The first one consists in a reduction of the problem to general linear groups, and the second in a solution for this latter case. However, our result for the general linear groups is more general than the finiteness statement of Theorem 1.1, as in this instance we can relax the requirement on the underlying field k.

Theorem 1.2. *Let k be an infinite field and let V be a finite-dimensional k-vector space. Let P be a parabolic subgroup of $GL(V)$. Then the number of P-orbits on P_u or $\mathfrak{p}_u$ is finite if and only if $\ell(P_u) \leq 4$.*

Theorem 1.2 is proved in Section 4. This is accomplished by translating the problem into a question of the representation-type of a particular category of representations of a quiver with certain relations. In turn, this can be answered using a result from the theory of Δ-filtered modules over quasi-hereditary algebras from [DR]. Apart from the desired finiteness statement, this approach also leads to a formula for the number of orbits when $\ell(P_u) \leq 4$.

Theorem 1.2 is generalised in a forthcoming paper by T. Brüstle and the first author [BrHi], where all parabolic subgroups P of $GL(V)$ with a finite number of orbits on $\mathfrak{p}_u^{(l)}$ for $l \geq 1$ are classified, where $\mathfrak{p}_u^{(l)}$ denotes the l-th member of the descending central series of $\mathfrak{p}_u$.

In a related paper by the first author [Hi] the action of a maximal torus $T \subset B$ of $GL(V)$ on subalgebras of $\mathfrak{b}_u$ is investigated; a complete geometric description of the T-orbits is given in terms of the moduli spaces of the action of T.

The paper is structured as follows. After recalling some earlier results from [Ro1] and [Ro2] in Section 2, we proceed in the next part to prove one implication in Theorem 1.1, namely that $\operatorname{mod} P$ is positive for those P not satisfying the stated conditions. In Section 4 we prove Theorem 1.2 above. In the next section we complete the proof of Theorem 1.1 by reducing the finiteness statement to the case of the general linear groups $GL(V)$.

In Section 6 we present some applications of Theorem 1.1 to finite double coset decompositions in general linear groups and to tangent bundles on flag manifolds, and finally, we discuss some computational aspects related to the finite cases in $GL(V)$.

As a general reference for algebraic groups we cite Borel's book [Bo] and for information on root systems we refer the reader to Bourbaki [Bou]. The material needed here from the general theory of representations of finite-dimensional algebras can be found in the book by Gabriel and Roiter [GR] and Ringel's monograph [Rin]. Our standard reference on categories of Δ-filtered modules over quasi-hereditary algebras is [DR].

2. Notation and Recollection

Let R be an algebraic group acting morphically on an algebraic variety X. For x in X the R-orbit in X through x is denoted by $R \cdot x$. Recall that the *modality* of the action of R on X is defined as $\operatorname{mod}(R : X) = \max_Z \min_{z \in Z} \operatorname{codim}_Z R^0 \cdot z$, where Z runs through all irreducible R^0-invariant subvarieties of X, cf. [PR]. In case X is an irreducible variety let $k(X)^R$ denote the field of R-invariant rational functions on X. Since $\min_{x \in X} \operatorname{codim}_X R \cdot x = \operatorname{trdeg} k(X)^R$ (e.g. see [PV, 2.3]), $\operatorname{mod}(R : X)$ measures the maximal number of parameters upon which a family of R-orbits on X depends. The modality of the action of R on X is 0 precisely when R admits only a finite number of orbits on X. The notion of modality originates in the work of Arnold [Ar]; the definition here is due to Vinberg [Vi]; see also [PV, 5.2].

Let G be a connected reductive algebraic group, T a maximal torus in G and Ψ the set of roots of G with respect to T. Fix a set of simple roots Π of Ψ and let B be the Borel subgroup of G corresponding to Π. A parabolic subgroup of G containing B is called standard. By $\Psi(P)$ we denote the set of roots of P. As every parabolic subgroup of G is conjugate to a standard one, we may if we wish assume that the parabolic subgroups under consideration are standard. For a parabolic subgroup P of G, we call $\operatorname{mod} P := \operatorname{mod}(P : \mathfrak{p}_u)$ the *modality* of P.

Let Θ be a semisimple automorphism of G and let P be a Θ-stable parabolic subgroup of G. Let G^Θ and P^Θ be the Θ-fixed point subgroups of G and P. Our next results are Theorem 1.1, Corollary 3.10, and Theorem 1.2 from [Ro2]; see also [PR].

Proposition 2.1. *Let Θ be a semisimple automorphism of G. Assume that P is a Θ-stable parabolic subgroup of G. Then $\operatorname{mod} P^\Theta \leq \operatorname{mod} P$.*

In our setting this result is applied to get finiteness results for fixed point subgroups of parabolic subgroups of $SL_n(k)$ under a graph automorphism.

Proposition 2.2. *Let G be a simple algebraic group and P a parabolic subgroup of G containing a maximal torus T of G. Let H be a Levi subgroup of G normalised by T, or the derived subgroup thereof. Set $Q := P \cap H$. Then* $\operatorname{mod} Q \leq \operatorname{mod} P$.

This fact is used in inductive arguments to yield infinite families of parabolic subgroups of positive modality, see Lemma 3.2 below.

Proposition 2.3. *Let G be a simple algebraic group, P a parabolic subgroup of G, and suppose that* $\operatorname{char} k$ *is zero or a good prime for G. Then* $\operatorname{mod}(P : P_u) = \operatorname{mod}(P : \mathfrak{p}_u)$.

So, provided $\operatorname{char} k$ is zero or a good prime for G, it suffices to work with the adjoint action of P on $\mathfrak{p}_u$ and one obtains the same modality result also for the action of P on P_u; this in particular applies to Theorem 1.1.

Remark 2.4. Suppose that $\operatorname{char} k$ does not divide the structure constants of the Chevalley commutator relations of G (for classical G this is precisely the case when $\operatorname{char} k$ is zero or a good prime for G). Then for P a standard parabolic subgroup of G the class of nilpotency of P_u can be determined as follows [BT, Prop. 4.7(iii)]. Let $\beta \in \Psi^+$. Write $\beta = \sum_{\alpha\in\Pi} c_\alpha(\beta)\alpha$, with $c_\alpha(\beta) \in \mathbb{Z}_0^+$ for each $\alpha \in \Pi$, and let ρ be the highest root of Ψ. Then $\ell(P_u) = \sum_{\alpha\in\Pi\setminus\Psi(P)} c_\alpha(\rho)$.

3. Parabolic subgroups in classical groups of positive modality

Throughout this section, G denotes a classical algebraic group. Our aim is to prove one implication in the statement of Theorem 1.1:

Proposition 3.1. *Let G be a simple classical algebraic group and P a parabolic subgroup of G. Then* $\operatorname{mod} P$ *is positive provided P does not satisfy the conditions of Theorem 1.1.*

Proposition 3.1 follows from our next result and [Ro1, Theorem 3.1]; the latter also applies in positive characterisitc by [Ro2, Remark 3.14]. Observe that there are no characteristic restrictions in Proposition 3.1 which provides a slightly stronger statement than what is required for Theorem 1.1.

Lemma 3.2. *Let G be a simple algebraic group of type D_r and P a parabolic subgroup of G satisfying the following conditions:*

(i) $\ell(P_u) = 5$;
(ii) *P is not invariant under the graph automorphism of G of order* 2 *stemming from the interchange of the simple roots α_{r-1} and α_r;*
(iii) *the semisimple part of a Levi subgroup of P consists of three nontrivial, pairwise commuting simple components.*

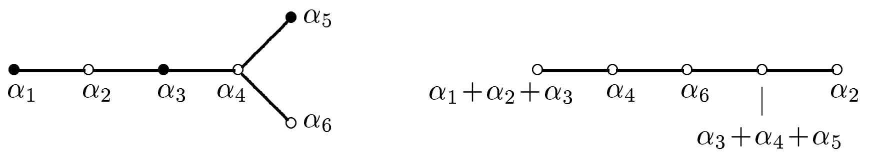
Then $\operatorname{mod} P$ *is positive.*

Proof. All these conditions combined imply that r is at least 6. For $r = 6$ there is just one case up to conjugacy by the graph automorphism: namely, for G of type D_6 the standard parabolic subgroup P of G with $\Psi(P) = \{-\alpha_1, -\alpha_3, -\alpha_5\} \cup \Psi^+$ satisfies the conditions of the lemma.

In this case let H be the (regular) simple subgroup of G relative to the subsystem of Ψ which is spanned by $\Pi(H) := \{\alpha_1 + \alpha_2 + \alpha_3, \alpha_4, \alpha_6, \alpha_3 + \alpha_4 + \alpha_5, \alpha_2\}$. Then H is the derived subgroup of a Levi subgroup of G of type A_5 and $Q = H \cap P$ is the standard Borel subgroup of H relative to $\Pi(H)$. It follows from [BH] that $\operatorname{mod} Q$ is positive and thus by Proposition 2.2 $\operatorname{mod} P$ is positive as well.

The general case where G is of type D_r for r greater than 6 and P is as in the statement above reduces inductively to this particular D_6 configuration just discussed by applying Proposition 2.2; e.g., see the proof of Proposition 4.3 in [Ro1]. This completes the proof of the lemma.

Below in Figure 1 we show the crucial D_6 case from Lemma 3.2. We indicate the standard Levi components of P by colouring the nodes corresponding to their simple roots in the Dynkin-diagram of G.



$G = D_6$, P, $\ell(P_u) = 5$ $\qquad$ $H \cong A_5$, $Q = H \cap P$, $\ell(Q_u) = 5$

Figure 1.

4. Parabolic subgroups in general linear groups

The finite cases

Let k be a field and V a finite-dimensional k-vector space. Let P be a parabolic subgroup of $GL(V)$. A Levi subgroup of P is (isomorphic to) a product of general linear groups, say $GL_{d_i}(k)$, for $1 \leq i \leq t$, with $\dim V = \sum d_i$. So, the ordered tuple $d = (d_1, \dots, d_t)$ determines the conjugate class of P in $GL(V)$. To indicate this, we often write $P = P(d)$. Our principal result of this section is

Theorem 4.1. *Let k be an infinite field and let V be a finite-dimensional k-vector space. Let $P = P(d)$ be a parabolic subgroup of $GL(V)$ with $d = (d_1, \dots, d_t)$. Then the number of P-orbits on $\mathfrak{p}_u$ is finite if and only if $t \leq 5$.*

Note that if $P = P(d)$ is as in Theorem 4.1, then $\ell(P_u) = t - 1$, whence Theorem 1.2 follows, cf. Remark 4.11.

We prove Theorem 4.1 by translating the question on the number of orbits into one concerning the number of indecomposable objects in a certain category.

For a fixed $t \in \mathbb{N}$ we define a category $\mathcal{F}(t)$ as follows. The objects are pairs (F, f), where F is a flag $\{0\} = V_0 \subseteq V_1 \subseteq \ldots \subseteq V_t = V$ of length t of subspaces of some finite-dimensional k-vector space V, and f is an endomorphism of V with $f(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq t$. In particular, f is nilpotent. Let (F, f) and (F', f') be in $\mathcal{F}(t)$. A morphism $\varphi : (F, f) \to (F', f')$ is a linear map $\varphi : V \to V'$ such that $\varphi(V_i) \subseteq V_i'$ for $1 \leq i \leq t$ and $\varphi f = f' \varphi$. For (F, f) in $\mathcal{F}(t)$ we set $d_i := \dim V_i - \dim V_{i-1}$ for $1 \leq i \leq t$ and call $d = (d_1, \ldots, d_t)$ the *dimension vector* of (F, f). Then, after choosing a suitable basis of V, $P = P(d)$ is the stabiliser of F in $GL(V)$ and f lies in $\mathfrak{p}_u$. We often also refer to d as the dimension vector of $P(d)$.

If $\varphi : (F, f) \xrightarrow{\sim} (F', f')$ is an isomorphism in $\mathcal{F}(t)$, then $d = d'$ and so $\dim V = \dim V'$. After identifying F with F' we see that $\varphi \in P = P(d)$ and $f' = \varphi f \varphi^{-1}$, i.e., f and f' are endomorphisms in $\mathfrak{p}_u$ conjugate under the action of P.

Lemma 4.2. *For any $t \in \mathbb{N}$ the isomorphism classes of objects in $\mathcal{F}(t)$ of fixed dimension vector d correspond bijectively to the P-orbits on $\mathfrak{p}_u$ for $P = P(d)$. This correspondence is induced by the map $(F, f) \mapsto P \cdot f$, where P is the stabiliser of the flag F and f lies in $\mathfrak{p}_u$.*

Proof. It follows from the remarks above that there is a well-defined map from the set of isomorphism classes of objects in $\mathcal{F}(t)$ of fixed dimension vector d to the set of P-orbits on $\mathfrak{p}_u$ for $P = P(d)$. Clearly, it is onto, as for any f in $\mathfrak{p}_u$, the pair (F, f) lies in $\mathcal{F}(t)$ and its class maps to the P-orbit through f. Moreover, since for any $(F, f), (F, f')$ in $\mathcal{F}(t)$ and $\varphi \in P = P(d)$ satisfying $f' = \varphi f \varphi^{-1}$ the linear map φ defines an isomorphism between (F, f) and (F, f'), this correspondence is injective.

In order to prove Theorem 4.1, it suffices to show that there are only finitely many indecomposable objects in $\mathcal{F}(t)$ for $t \leq 5$, because of this correspondence. In Corollary 4.4 below we shall see that $\mathcal{F}(t)$ always satisfies the Krull-Schmidt property. However, this fact is not required for this reduction.

Instead of working directly with $\mathcal{F}(t)$, we pass to an equivalent category. Let $\mathcal{M}(t)$ be the full subcategory of representations M of the quiver $\mathcal{Q}$

$$\circ \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} \circ \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \circ \quad \cdots \quad \circ \underset{\beta_{t-1}}{\overset{\alpha_{t-1}}{\rightleftarrows}} \circ$$

subject to the relations $\beta_1 \alpha_1 = 0$, $\beta_i \alpha_i = \alpha_{i-1} \beta_{i-1}$ for $1 < i < t$, and the condition that the linear maps $M(\alpha_i)$ are injective for $1 \leq i < t$.

Lemma 4.3. *The categories $\mathcal{F}(t)$ and $\mathcal{M}(t)$ are equivalent for any $t \in \mathbb{N}$.*

Proof. Let (F, f) be in $\mathcal{F}(t)$ and define a representation $M(F, f)$ of the quiver $\mathcal{Q}$ above by

$$V_1 \rightleftarrows V_2 \rightleftarrows V_3 \quad \cdots \quad V_{t-1} \rightleftarrows V_t \ ,$$

where the maps from left to right are simply the inclusions $V_{i-1} \hookrightarrow V_i$ and the maps in the opposite directions are the restrictions of f to V_i. One easily checks that $M(F, f)$ satisfies the conditions above, i.e., $M(F, f)$ is in $\mathcal{M}(t)$, and that the map $(F, f) \mapsto M(F, f)$ defines an equivalence of the two categories. Indeed, given such a representation of $\mathcal{Q}$ as above, the injectivity of the maps $M(\alpha_i)$ provides the desired flag structure on $V = V_t$ and furthermore, the relations show that for $f = M(\alpha_{t-1})M(\beta_{t-1}) \in \operatorname{End} V$ we get $f(V_i) \subseteq V_{i-1}$ for $1 \leq i \leq t$, as required.

Corollary 4.4. *Let $t \in \mathbb{N}$. Then $\mathcal{F}(t)$ is a Krull-Schmidt category, i.e., every object in $\mathcal{F}(t)$ has a unique decomposition into a direct sum of indecomposable ones (up to the order of the summands).*

Proof. Being a full subcategory of a module category, $\mathcal{M}(t)$ admits the Krull-Schmidt property for each $t \in \mathbb{N}$, whence so does $\mathcal{F}(t)$ by Lemma 4.3.

We observe that $\mathcal{M}(t)$ is precisely the category $\mathcal{F}(\Delta)$ of Δ-filtered modules over the Auslander algebra of the representation-finite algebra $k[x]/(x^t)$, cf. [DR, §6 - 7]. The key result we need here is Proposition 7.2 from [DR]:

Theorem 4.5. *Let $t \in \mathbb{N}$. Then the representation type of $\mathcal{M}(t)$ is finite precisely when $t \leq 5$; it is tame for $t = 6$, and wild whenever $t \geq 7$.*

The first part of the statement follows from the finiteness of the Auslander-Reiten quivers of $\mathcal{M}(t)$ for $t \leq 5$ as exhibited in [DR]. Because of the equivalence above, the same result holds for $\mathcal{F}(t)$, and so Theorem 4.1 follows from the first part of Theorem 4.5 and Lemmas 4.2 and 4.3.

We discuss further consequences of the results above for our setting. Let $I(t)$ be a complete set of representatives of isomorphism classes of indecomposable objects in $\mathcal{F}(t)$. According to Lemma 4.3 and Theorem 4.5, $I(t)$ is a finite set for $t \leq 5$, and here we identify $I(t)$ with the set $\{1, 2, \ldots, |I(t)|\}$. Let $\underline{d}^i$ be the dimension vector of $i \in I(t)$. The dimension vectors $\underline{d}^i$ are listed in the appendix to [HR2] for all $i \in I(t)$ and $2 \leq t \leq 5$.

Now we present a formula for the number of orbits when $t \leq 5$. The proof follows at once from the first part of Theorem 4.5 and the previous Lemmas. In particular, this requires the fact that $\mathcal{F}(t)$ is a Krull-Schmidt category.

Corollary 4.6. *Let V be a finite-dimensional k-vector space and let $P = P(d)$ be a parabolic subgroup of $GL(V)$ with $d = (d_1, \ldots, d_t)$ and $t \leq 5$. Then the number of P-orbits on $\mathfrak{p}_u$ equals the number of tuples $(a_1, a_2, \ldots) \in \mathbb{N}_0{}^{|I(t)|}$ such that $d = \sum_{i \in I(t)} a_i \underline{d}^i$ with the $\underline{d}^i$'s and $I(t)$ as above.*

Our next result shows that the number of orbits only depends on the conjugacy class of a Levi subgroup of $P = P(d)$ in $GL(V)$ rather than on P itself in case $t \leq 5$. Let $n(d)$ denote the number of P-orbits on $\mathfrak{p}_u$.

Corollary 4.7. *Let V be a finite-dimensional k-vector space, $P = P(d)$, and $P' = P(d')$ parabolic subgroups of $GL(V)$, where both d and d' are of length $t \leq 5$. If P and P' have conjugate Levi subgroups, then $n(d) = n(d')$.*

Proof. The fact that P and P' have conjugate Levi subgroups implies that there is a permutation σ of $\{1, \dots, t\}$ such that $d' = \sigma d = (d_{\sigma 1}, \dots, d_{\sigma t})$. Perusing the dimension vectors $\underline{d}^i$ for i in $I(t)$ and $t \leq 5$ in the appendix to [HR2] one observes that the multiplicity of each t-tuple $\underline{d}^i$ equals that of $\sigma \underline{d}^i$ for any permutation σ of $\{1, \dots, t\}$ for $t \leq 5$. The claim now follows from the formula in Corollary 4.6.

The tame cases

For our purpose we need to further analyse the tame situation of Theorem 4.5. More specifically, we need some insight regarding the structure of families of indecomposable objects in $\mathcal{F}(6)$. By $\mathcal{M}(t)(d)$ we denote the subcategory of $\mathcal{M}(t)$ of all objects of a fixed dimension vector d; likewise for $\mathcal{F}(t)$.

By Theorem 4.5 the module category $\mathcal{M}(6)$ is of tame representation type and so is $\mathcal{F}(6)$ by Lemma 4.3. So, a family of non-isomorphic indecomposable objects in $\mathcal{F}(6)$ depends only on a single parameter. According to [DR, Prop. 7.2], $\mathcal{M}(6)$ is of *tubular* type. This implies that there exist two dimension vectors in $\mathcal{M}(6)$, say ∂^1 and ∂^2, with the property that whenever $\mathcal{M}(6)(d)$ admits a one-parameter family of indecomposable non-isomorphic objects, d is of the form $d = a_1\partial^1 + a_2\partial^2$ for some non-negative integers a_1 and a_2, cf. [Rin, §5]. In this case we simply say that d admits a one-parameter family of indecomposable non-isomorphic objects in $\mathcal{M}(6)$. It turns out that ∂^1 and ∂^2 are linearly dependent in this case. Thus the same holds in $\mathcal{F}(6)$ by Lemma 4.3. The following result is crucial for our further study.

Proposition 4.8. *A dimension vector in $\mathcal{F}(6)$ which admits a nontrivial one-parameter family of indecomposable objects is an integer multiple of $(1, 1, 1, 1, 1, 1)$.*

Proof. We use the machinery and terminology from [DR]. The category $\mathcal{M}(t)$ of Δ-filtered modules of the Auslander algebra $\mathcal{A}(t)$ of $k[x]/(x^t)$ is equivalent to the category of finite-dimensional modules of the preprojective algebra $\mathcal{P}(t-1)$ of a path algebra of a quiver of Dynkin type A_{t-1}, modulo finitely many objects in $\mathcal{M}(t)$, see [DR, Theorem 3.4]. This allows us to work with $\mathcal{P}(t-1)$-mod instead of $\mathcal{M}(t)$. (For a definition of a preprojective algebra see [GP] or [DR, §6].) This category is of tubular type for $t = 6$ and the minimal dimension vectors of families of indecomposable objects in $\mathcal{P}(5)$ are $\partial^1_{\mathcal{P}(5)} = (1, 2, 2, 2, 1)$ and $\partial^2_{\mathcal{P}(5)} = (2, 3, 4, 3, 2)$. These vectors can be obtained as follows. We consider the $\mathbb{Z}$-cover $\widetilde{\mathcal{P}(5)}$ of $\mathcal{P}(5)$. Using covering theory [BG] and the associated push down functor $\widetilde{\mathcal{P}(5)}$-mod $\to \mathcal{P}(5)$-mod, it suffices to determine the minimal tame subalgebras of $\widetilde{\mathcal{P}(5)}$ along with their minimal dimension vectors of families of indecomposable modules. For our case, we can use the list of minimal tame algebras, presented in [HV], the so called *tame concealed* ones. According to [Rin] the category $\widetilde{\mathcal{P}(5)}$-mod is of tubular type and there are exactly two such subalgebras up to $\mathbb{Z}$-shifts. Moreover, a minimal tame subalgebra on the list of Happel and Vossieck [HV] has exactly one minimal dimension vector which corresponds to a nontrivial one-parameter family. The ones which occur in $\widetilde{\mathcal{P}(5)}$ are precisely those presented in Figure 2. From these one readily obtains the minimal dimension vectors $\partial^1_{\mathcal{P}(5)}$ and $\partial^2_{\mathcal{P}(5)}$ above for $\mathcal{P}(5)$ by means of covering theory [BG].

$$\begin{tikzcd}[column sep=small]
 & \overset{1}{\circ} \arrow[dl] \arrow[dr] & & \overset{1}{\circ} \arrow[dl] \arrow[dr] & \\
\overset{1}{\circ} \arrow[dr] & & \overset{2}{\circ} \arrow[dl] \arrow[dr] & & \overset{1}{\circ} \arrow[dl] \\
 & \overset{1}{\circ} & & \overset{1}{\circ} &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
\overset{1}{\circ} \arrow[dr] & & \overset{2}{\circ} \arrow[dl] \arrow[dr] & & \overset{1}{\circ} \arrow[dl] \\
 & \overset{3}{\circ} \arrow[dl] \arrow[dr] & & \overset{3}{\circ} \arrow[dl] \arrow[dr] & \\
\overset{1}{\circ} & & \overset{2}{\circ} & & \overset{1}{\circ}
\end{tikzcd}$$

Figure 2.

The tame concealed subalgebras of $\widetilde{\mathcal{P}(5)}$ with their respective minimal dimension vectors of families of indecomposable modules.

By $\underline{\dim}\, M$ we denote the dimension vector of a module M in $\mathcal{M}(t)$, $\mathcal{A}(t)$-mod, or $\mathcal{P}(t-1)$-mod. From our discussion above we know the minimal dimension vectors of families of indecomposable modules in $\mathcal{P}(5)$. In order to get the corresponding ones in $\mathcal{M}(6)$ we need to study the functor $\mathcal{M}(t) \to \mathcal{P}(t-1)$-mod which is given by $M \mapsto M/\eta_T M$, where $T = \bigoplus_{i=1}^t T(i)$ is the *characteristic module* of $\mathcal{A}(t)$ (cf. [DR, §3]) and $\eta_T M := \operatorname{Im}\big(\bigoplus_{i=1}^t (\operatorname{Hom}(T(i), M) \otimes T(i)) \to M\big)$ is the maximal submodule of M generated by all the images of the summands $T(i)$ of T, called the *trace* of T in M (cf. [DR, §1]).

Our next lemma provides a formula to compute $\underline{\dim}\, \eta_T M$. Thus, this yields $\underline{\dim}\, M$ provided $\underline{\dim}(M/\eta_T M)$ is known, since $\underline{\dim}\, M = \underline{\dim}\, \eta_T M + \underline{\dim}(M/\eta_T M)$. Note that an $\mathcal{A}(t)$-module N is a t-tuple of k-vector spaces; analogously for $\mathcal{P}(t-1)$; by N_i we denote the i-th component of N. We write $\operatorname{soc}(N)$ for the socle of N and $\operatorname{soc}(N)_i$ for the i-th component of $\operatorname{soc}(N)$.

Lemma 4.9. *Let M be an indecomposable module in $\mathcal{M}(t)$ and $N = M/\eta_T M$ its image in $\mathcal{P}(t-1)$. Then either $N = 0$ and $M \cong T(i)$ for some i, or else $\eta_T M = \bigoplus_{i=1}^{t-1} \operatorname{soc}(N)_i \otimes T(i+1)$. In particular, $\underline{\dim}\, \eta_T M = \sum_{i=1}^{t-1} \dim \operatorname{soc}(N)_i \cdot \underline{\dim}\, T(i+1)$.*

Proof. Because there are surjective homomorphisms $T(i) \to T(j)$ for $i \leq j$, we have injective maps $\mathrm{Hom}(T(j), M) \to \mathrm{Hom}(T(i), M)$ for $i \leq j$. Consequently, we may define $M/\eta_T M$ inductively via

$$M^t := M/\eta_{T(t)} M, \quad M^i := M^{i+1}/\eta_{T(i)} M^{i+1}, \quad \text{thus} \quad M^1 = M/\eta_T M.$$

Further, $\dim \mathrm{Hom}(T(i), M^{i+1}) = \dim \mathrm{soc}(M^i)_{i-1} = \dim \mathrm{soc}(M/\eta_T M)_{i-1}$, whenever M^{i+1} does not contain any $T(i)$ as a direct summand. Thus the map which sends a $\mathcal{P}(t-1)$-module N to M is obtained by an extension of the form

$$0 \to \bigoplus_{i=1}^{t-1} \mathrm{soc}(N)_i \otimes T(i+1) \to M \to N \to 0.$$

This completes the proof of Lemma 4.9.

We return again to the calculation of the minimal dimension vectors of families of indecomposable modules in $\mathcal{M}(6)$. Above we determined these for $\mathcal{P}(5)$, and now we compute the corresponding dimension vectors in $\mathcal{A}(6)$-mod using the equivalence between $\mathcal{A}(t)$-mod and $\mathcal{P}(t-1)$-mod from the proof of [DR, Prop. 7.2]: Modules over $\mathcal{P}(t-1)$ can be considered as modules over $\mathcal{A}(t)$ by dropping the injectivity condition and by requiring $V_t = \{0\}$. The dimension vectors $\mathrm{soc}(\partial^1_{\mathcal{P}(5)})$ and $\mathrm{soc}(\partial^2_{\mathcal{P}(5)})$ of the socles of the indecomposable modules, with dimension vectors as above, are $(0,1,0,1,0)$ and $(1,0,2,0,1)$, respectively. This follows readily from the dimension vectors of the families of the subalgebras in $\widetilde{\mathcal{P}(5)}$ in Figure 2 above.

Observe that the indecomposable summands $T(i)$ of T have dimension vector $\underline{\dim}\, T(i) = (0, \ldots, 1, 2, \ldots, t-i+1)$ for $1 \leq i \leq t$. Finally, knowing $\underline{\dim}\, T(i)$, $\mathrm{soc}\, \partial^1_{\mathcal{P}(5)}$ and $\mathrm{soc}\, \partial^2_{\mathcal{P}(5)}$, we can compute the desired dimension vectors in $\mathcal{M}(6)$ using the dimension formula from Lemma 4.9:
$\partial^1 = (1,2,2,2,1,0) + (0,0,1,2,3,4) + (0,0,0,0,1,2) = (1,2,3,4,5,6)$ and
$\partial^2 = (2,3,4,3,2,0) + (0,1,2,3,4,5) + 2 \cdot (0,0,0,1,2,3) + (0,0,0,0,0,1) = 2 \cdot \partial^1$.

Consequently, the dimension vectors of the corresponding flags in $\mathcal{F}(6)$ are $(1,1,1,1,1,1)$ and $(2,2,2,2,2,2)$. This completes the proof of the proposition.

For the remainder of this section we discuss the modality of those parabolic subgroups $P = P(d)$ of $GL(V)$ which correspond to the tame category $\mathcal{F}(6)$. While it follows from Theorem 4.1 that $\mathrm{mod}\, P$ is positive whenever t is greater than 5, in this instance we are able to determine $\mathrm{mod}\, P$ precisely.

Proposition 4.10. *Let k be an infinite field and V a finite-dimensional k-vector space. Let $P = P(d)$ be a parabolic subgroup of $GL(V)$ with $d = (d_1, \ldots, d_6)$. Then $\mathrm{mod}\, P$ equals the minimum of the d_i's for $1 \leq i \leq 6$.*

Proposition 4.10 is a special case of a more general result. Namely, let $\mathcal{C}$ be a Krull-Schmidt category and assume that each subcategory $\mathcal{C}(d)$ of objects of a fixed dimension vector d admits the structure of an algebraic variety on which an algebraic group $G(d)$ acts morphically, such that the orbits of this action are exactly the isomorphism classes of objects of dimension vector d in $\mathcal{C}$. Of course, the situation we have in mind is precisely like the one we consider above. If $\mathcal{C}$ is of finite representation type, then the modality of the action of $G(d)$ on $\mathcal{C}(d)$ is zero for any d. Often the converse also holds, e.g., when $\mathcal{C}$ is a category of modules.

The situation is more complicated if $\mathcal{C}$ is tame. There is a natural partial ordering on the set of dimension vectors of fixed cardinality. Namely, we write $d \leq d'$ provided $d_j \leq d'_j$ for all j. Tameness forces that all *indecomposable* objects of a given dimension vector can be parameterised by a finite number of one-parameter families. The modality is the maximal number of parameters which arise from families of *all* objects in our category. Consequently, we have

$$\operatorname{mod}(G(d) : \mathcal{C}(d)) = \max\{\sum a_i \mid \sum a_i \underline{d}^i \leq d\},$$

where the sum runs over all dimension vectors $\underline{d}^i$ admitting a one-parameter family of non-isomorphic objects. This formula remains valid if we only sum over those dimension vectors $\underline{d}^i$ which are minimal with respect to the partial ordering above. These remarks together with this formula and Proposition 4.8 imply Proposition 4.10.

Some complements

Remark 4.11. Note that for any parabolic subgroup P of $GL(V)$ the map $x \mapsto 1 + x$ is a P-equivariant morphism from $\mathfrak{p}_u$ to P_u. Thus we obtain the same results as above for the action of P on P_u instead of $\mathfrak{p}_u$, as claimed.

Remark 4.12. Let n be in $\mathbb{N}$. Suppose that the equation $x^n = \mu$ can be solved in k for every μ in k. Then $GL_n(k) = SL_n(k) \cdot D$, where $D = \{\operatorname{diag}(\lambda, \ldots, \lambda) \mid \lambda \in k^* = k \setminus \{0\}\}$ is a one-dimensional central torus of $GL_n(k)$ and $SL_n(k) \cap D$ is finite. Since D is central in $GL_n(k)$, it acts trivially on both $\mathfrak{p}_u$ and P_u. Whence in this case the statements of the results of this section also hold for $SL(V)$ in place of $GL(V)$. In general, if P is a parabolic subgroup of $GL(V)$ such that P acts on $\mathfrak{p}_u$ with an infinite number of orbits, then so does $SL_n(k) \cap P$. However, the converse need not hold. For instance, let B be a Borel subgroup of $GL_2(\mathbb{Q})$. Then B has two orbits on $\operatorname{Lie} B_u$, while $B \cap SL_2(\mathbb{Q})$ admits an infinite number of orbits on the Lie algebra of its unipotent radical, as $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ is infinite.

Remark 4.13. The categories $\mathcal{F}(t)$ and $\mathcal{M}(t)$ can be defined for an arbitrary field k and Theorem 4.5 is still valid. Thus all the results above can be applied and interpreted for finite fields as well.

Let k be algebraically closed of characteristic p and σ a Frobenius endomorphism of $G = GL(V)$. Let $P = P(d)$ be a parabolic subgroup of G and suppose that $t \leq 5$. Then, by Theorem 4.5 and Corollary 4.6, the number of orbits of P^σ on $\mathfrak{p}_u^\sigma$ equals the number of orbits of P on $\mathfrak{p}_u$. It follows from [SS, 3.4] that the centralizer in P of an element in $\mathfrak{p}_u$ is connected; likewise for the action of P on P_u. This connectedness property is well known for the centralizers of unipotent elements in G, cf. [Sp].

Remark 4.14. In [KP, §3] representations of the quiver $\mathcal{Q}$ with the same relations but without the injectivity condition demanded for $\mathcal{M}(t)$ are used to describe closures of nilpotent conjugacy classes for general linear groups.

5. Parabolic subgroups of classical groups of modality zero

Throughout this section suppose that G is a simple classical algebraic group and $\operatorname{char} k$ is zero or a good prime for G and V is a finite-dimensional k-vector space. The aim of this section is to complete the proof of Theorem 1.1 by showing the finiteness statement for those parabolic subgroups satisfying the given conditions.

Remark 5.1. Since k is algebraically closed, $GL(V)$ and $SL(V)$ only differ by central elements, and thus by Remark 4.12 the statements of the previous section also hold for $SL(V)$ in place of $GL(V)$. If $P = P(d)$ is a parabolic subgroup of $SL(V)$ with $d = (d_1, \dots, d_t)$, then $\ell(P_u) = t - 1$. Hence, Theorem 4.1 implies the desired finiteness statement of Theorem 1.1 for $SL(V)$.

Lemma 5.2. *Let $G = SL(V)$. Then each of the classical groups $SO(V)$ and $Sp(V)$ can be realized as a fixed point subgroup of G for a suitable semisimple automorphism Θ of G, i.e., G^Θ equals $SO(V)$ or $Sp(V)$. Moreover, each parabolic subgroup of G^Θ can be realized as a fixed point subgroup P^Θ for some Θ-invariant parabolic subgroup P of G. In case G^Θ is $SO(V)$ and $\dim V$ is even, P^Θ is only determined up to equivalence under the graph automorphism of G^Θ.*

For a proof of the assertion on G^Θ see Steinberg [St, §11 p. 169]. The remaining statements follow easily from the explicit description of Θ in *loc. cit.* Note that $\ell(P_u^\Theta) \leq \ell(P_u)$. In general, this inequality may be strict.

We formulate a first consequence for the other simple classical groups from the finiteness result for $SL(V)$.

Corollary 5.3. *Let H be a simple algebraic group of type B_r, C_r, or D_r and let Q be a parabolic subgroup of H. Suppose $\operatorname{char} k \neq 2$. Then $\operatorname{mod} Q = 0$ provided*

(i) *H is of type B_r or C_r and $\ell(Q_u) \leq 4$;*
(ii) *H is of type D_r and $\ell(Q_u) \leq 3$;*
(iii) *H is of type D_r, $\ell(Q_u) = 4$, and Q is stable under the automorphism of H of order 2 stemming from the interchange of the simple roots α_{r-1} and α_r.*

Proof. Let $G = SL(V)$ and let Θ be the automorphism of G as in Lemma 5.2 such that G^Θ equals H. Since Θ has order two, the assumption on the characteristic of k ensures that Θ is semisimple. Thus Proposition 2.1 can be applied in this instance. Let Q be a parabolic subgroup of H as in Corollary 5.3. By Lemma 5.2. we can obtain Q as a fixed point subgroup P^Θ for some Θ-invariant parabolic subgroup P of G. In each of the cases to be considered, this can be done for a suitable P with $\ell(P_u) \leq 4$. The desired result for H then follows from Proposition 2.1 and the finiteness result for G. For some examples see [HR1, Ex. 3.5].

Corollary 5.3 covers all cases of Theorem 1.1 for groups of type B_r and C_r, but only partially those of type D_r. Next we address the remaining D_r cases.

Proposition 5.4. *Let H be a simple classical algebraic group of type D_r, $r \geq 5$, and let Q be a parabolic subgroup of H such that $\ell(Q_u) = 4$ or 5 and Q is not invariant under the automorphism of H of order 2 stemming from the interchange of the simple roots α_{r-1} and α_r. In addition, if $\ell(Q_u) = 5$, assume that the derived subgroup of a Levi subgroup of Q consists of precisely two commuting simple components. Then $\operatorname{mod} Q = 0$.*

Proof. We argue as in the proof of Corollary 5.3. Let $G = SL(V)$ and let Θ be the semisimple automorphism of G as above such that H is equal to G^Θ. As asserted by Lemma 5.2, we can obtain Q as a fixed point subgroup P^Θ for some Θ-invariant parabolic subgroup P of G. However, in all of these cases $\ell(P_u) = 5$ and the number of P-orbits on $\mathfrak{p}_u$ is infinite by Proposition 3.1 and thus Proposition 2.1 does not yield the desired finiteness statement for $P^\Theta = Q$.

The parabolic subgroups $P = P(d)$ of G that occur in this way have the feature that d is of the form $(1, a, b, b, a, 1)$ or $(a, 1, b, b, 1, a)$, with $a, b \in \mathbb{N}$, depending on $\ell(Q_u)$ being 4 or 5, respectively. Thus $\operatorname{mod} P = 1$, by Proposition 4.10, and the only minimal dimension vector admitting a one-parameter family of orbits which is a summand of such a particular d is $(1, 1, 1, 1, 1, 1)$, by Proposition 4.8. This dimension vector corresponds to the standard Borel subgroup B in $SL_6(k)$. In this instance there is a unique one-parameter family of B-orbits on $\mathfrak{b}_u$ with representatives of the form

$$\begin{pmatrix} . & 1 & . & . & . & . \\ . & . & . & 1 & \lambda & . \\ . & . & . & 1 & 1 & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 1 \\ . & . & . & . & . & . \end{pmatrix}$$

where the parameter λ is taken from the field k and the dots represent zero entries. This one-parameter family already appears in [BH]. Now one easily checks that no B-conjugate of any member of this family is invariant under

$d\Theta_e$, where $d\Theta_e$ denotes the differential of Θ: a necessary condition for $d\Theta_e$-invariance is that the entries of the second main diagonal are all equal to zero. But the 1 in position $(3, 4)$ can not be removed using elements from B. Embedding this family into $\mathfrak{p}_u$ by taking direct sums, gives rise to a family of P-orbits on $\mathfrak{p}_u$ with the same property, that no P-conjugate of any of its members is $d\Theta_e$-invariant. Consequently, the intersection of the single nontrivial one-parameter family of P-orbits on $\mathfrak{p}_u$ with $\mathfrak{q}_u$ is empty. Because a P-orbit on $\mathfrak{p}_u$ intersects $\mathfrak{q}_u$ either trivially or in a finite union of Q-orbits (this follows from the proof of Proposition 2.1, cf. [Ro2, Theorem 1.1]), Q in turn only has a finite number of orbits on $\mathfrak{q}_u$. This completes the proof of the proposition.

Finally, we have all the ingredients to prove Theorem 1.1. It now follows from Proposition 3.1, Theorem 4.1 (and Remark 5.1), Corollary 5.3, and Proposition 5.4.

Examples 5.5. Assume that G, P, $H = G^{\Theta}$, and $Q = P^{\Theta}$ are as in Proposition 5.4. The coloured nodes indicate the simple roots in $\Psi(P)$; likewise for $\Psi(Q)$. In our first example the class of nilpotency of Q_u is 4, and 5 in the second, cf. Remark 2.4. For an explicit description of Θ in terms of matrices consult [St, §11]. Our examples are the minimal rank cases of Proposition 5.4. The corresponding dimension vectors d of the groups $P(d)$ in $SL_{10}(k)$ are $d = (1, 2, 2, 2, 2, 1)$ and $d = (2, 1, 2, 2, 1, 2)$, respectively. The class of nilpotency of P_u equals 5 in both incidences. The general cases are essentially the same, merely with larger simple components occurring in the Levi subgroups.

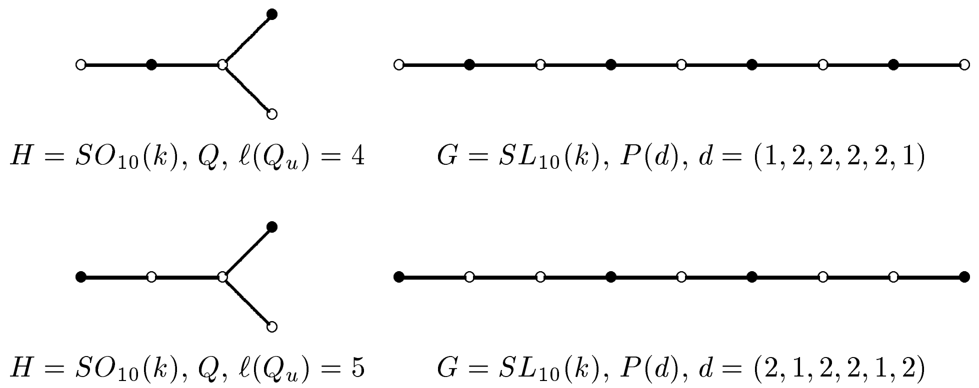


Figure 3.

6. Some consequences of Theorem 1.1

Let P be a parabolic subgroup of some reductive algebraic group G. Consider the representation $\phi : P \to GL(\mathfrak{p}_u)$ afforded by the adjoint action of P on $\mathfrak{p}_u$, and let R be the stabiliser in $GL(\mathfrak{p}_u)$ of a one-dimensional subspace of $\mathfrak{p}_u$ (R is a maximal parabolic subgroup of $GL(\mathfrak{p}_u)$). It can be shown that $\operatorname{mod}(P : \mathfrak{p}_u) = 0$ if and only if P acts on $\mathbb{P}(\mathfrak{p}_u)$, the set of one-dimensional subspaces of $\mathfrak{p}_u$, with a finite number of orbits. Since $GL(\mathfrak{p}_u)$ is transitive on $\mathbb{P}(\mathfrak{p}_u)$, we see that $\operatorname{mod} P = 0$ if and only if the set of double cosets $\phi(P) \backslash GL(\mathfrak{p}_u)/R$ is finite. Thus Theorem 1.1 provides a complete list of all cases when there are only finitely many such double cosets. Because of Richardson's dense orbit theorem [Ri], there is always an open dense P-orbit on $\mathbb{P}(\mathfrak{p}_u)$, and thus an open dense $\phi(P)$-R double coset in $GL(\mathfrak{p}_u)$.

Also, Theorem 1.1 provides a classification of all instances of parabolic subgroups P of classical algebraic groups G when G operates on the tangent bundle $\mathcal{T}$ or the cotangent bundle $\mathcal{T}^*$ of the flag manifold G/P with a finite number of orbits. By construction, the G-orbit of a point in $\mathcal{T}^* := G \times_P \mathfrak{p}_u$ meets the fibre over some point in G/P (which is isomorphic to $\mathfrak{p}_u$) in a single P-orbit on $\mathfrak{p}_u$, affording a canonical bijection between the G-orbits on $G \times_P \mathfrak{p}_u$ and the P-orbits on $\mathfrak{p}_u$; likewise for $\mathcal{T} := G \times_P \mathfrak{p}_u^*$.

7. Some complements

The number of isomorphism classes of indecomposable objects in $\mathcal{M}(t)$ for $t \leq 5$ and their dimension vectors can be determined from the Auslander-Reiten quivers listed in [DR]. There are 7, 16, and 45 such classes in $\mathcal{M}(t)$ for $t = 3, 4, 5$, respectively. For an explicit list of the corresponding dimension vectors in $\mathcal{F}(t)$ for $2 \leq t \leq 5$ consult the appendix in [HR2].

Matrix representatives of all the indecomposable objects in $\mathcal{F}(t)$ for $t \leq 5$ can be computed explicitly using the Auslander-Reiten quivers in [DR]. Then for a fixed dimension vector d one can get representatives for all orbits of $P = P(d)$ on $\mathfrak{p}_u$ by means of taking direct sums according to the formula in Corollary 4.6. This corresponds to taking direct sums of objects in $\mathcal{F}(t)$, cf. Lemma 4.2.

The fact that $\mathcal{F}(t)$ only has a finite number of indecomposable objects when $t \leq 5$ can be recovered using a computer program written by T. Brüstle and D. Guhe. The program explicitly determines all indecomposable objects of $\mathcal{F}(t)$ for $t \leq 5$, one from each equivalence class, along with their dimension vectors and explicit representatives in terms of matrices. Their algorithm is based on matrix reduction techniques, cf. [GR]. Another program which is due to R. Tiefenbrunner provides similar information utilising methods from the theory of bimodules.

Finally, we should like to mention that the formula of Corollary 4.6 can be used to explicitly compute $n(d)$, the number of orbits of $P(d)$ on the Lie algebra of its unipotent radical. A computer program which calculates $n(d)$ based on this formula was written by U. Jürgens.

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