Defining k in $G(k)$

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We show how the field of definition k, of a k-isotropic absolutely almost simple k-group G "lives" in the group $G(k)$ of k-rational points. The construction which is inspired by the fundamental work of Borel-Tits [BT3] is as follows: We choose an element inside the center of the unipotent radical of a minimal parabolic k-subgroup P; the orbit under the action of the center Z of a Levi k-subgroup of P generates a one-dimensional vector space which then carries the additive field structure in a natural way. The multiplicative structure is induced by the action of Z. If G is k-simple, our construction yields a finite extension l of k.

As an immediate consequence we obtain an answer to a question of Borovik-Nesin [BN] p. 367 under the additional assumption that G is k-isotropic:

Theorem *If G is a k-simple k-isotropic group such that $G(k)$ has finite Morley rank, then k is either algebraically closed or real closed. If G is absolutely simple k-isotropic, then k is algebraically closed.*

1 Preliminaries

The model theoretic notion of defining a field in a group means giving a first order formula in the language of groups (just using the group operation and possibly naming finitely many elements of the group as parameters) satisfied by a set of elements of the group such that the field operations on this set are again definable by first order formulas. In this way the field "lives" inside the group. Our construction yields just such a realization of the field in the group.

We assume that the reader is familiar with the basic theory of linear algebraic groups. Most of what is needed here can be found in [Bo] or [BT1]. Throughout this note, k is a field, $K \supseteq k$ its algebraic closure, and G is a reductive k-group. Even though from the model theoretic point of view the question of definability becomes

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somewhat trivial if k is finite, we still think it worthwhile having a uniform treatment for all fields.

A reductive group is always assumed to be connected. As usual let G_m and G_a denote the multiplicative and the additive group of K, respectively.

A group G is called k-almost simple if it has no proper normal connected subgroup which is defined over k. If G is almost simple, then it is of course k-almost simple; in this case, G is also called absolutely almost simple. The converse is false, as a k-simple group need not be absolutely simple; see the comments in Section 4 below.

Let T be a maximal torus of G defined over k. Then T decomposes into an almost direct product $T = T_a \cdot T_d$, where T_d is k-split and T_a is anisotropic.

A Borel subgroup of G is a maximal connected solvable subgroup of G; it contains some maximal torus of G. A closed subgroup of G containing some Borel subgroup is called parabolic.

The group G is called isotropic over k if some proper parabolic subgroup of G is defined over k; otherwise it is called anisotropic. If some Borel subgroup is defined over k, then G is called quasi-split; if every conjugacy class of parabolic subgroups contains a parabolic k-subgroup, then G is called split. A reductive k-group is isotropic if and only if its semisimple part (i.e. its commutator subgroup) contains a k-split torus.

Note that if k is finite or locally finite, then G is quasi-split and hence isotropic by [Bo], 16.6.

We recall some facts about the structure of parabolic k-subgroups, eg. see [BT1]. Let $P \subseteq G$ be a parabolic subgroup with unipotent radical $U = \mathcal{R}_u P$. A complement L of U in P is called a Levi subgroup of P. Levi subgroups in parabolic subgroups of reductive groups do always exist and are again reductive. Since U is normal in P, we have a semidirect product $P = L \ltimes U$. Moreover, if P is defined over k, then U is defined over k, and there exist Levi subgroups of P which are defined over k. If P is minimal among the parabolic k-subgroups, then the semisimple part L' of a Levi k-subgroup L is anisotropic; L' is also called the semisimple anisotropic kernel of G.

1.1 Roots and coroots Let $S \subseteq G$ be a k-torus. The character group $X^*(S) = \mathrm{Mor}(S, G_m)$ is free abelian and denoted additively. Dually, we consider the group $X_*(S) = \mathrm{Mor}(G_m, S)$ of all one-parameter subgroups in S. There is a pairing

$$\langle\ ,\ \rangle : X^*(S) \times X_*(S) \longrightarrow \mathbb{Z}$$

which is defined for $\alpha \in X^*(S), \beta \in X_*(S)$ by $\alpha \circ \beta(x) = x^{\langle \alpha, \beta \rangle}$ for all $x \in G_m$.

Let H be an S-invariant subgroup of G. Then S acts on the Lie algebra $\mathfrak{h}$ of H, and the weight spaces of $\mathfrak{h}$ with respect to S are $\mathfrak{h}_\alpha = \{X \in \mathfrak{h} |\ s \cdot X = \alpha(s)X$ for all $s \in S\}$ for $\alpha \in X^*(S)$. If $\alpha \neq 0$ and $\mathfrak{h}_\alpha \neq \{0\}$, then α is called a root; the set of all roots is denoted by $\Phi(S, H)$. There are two important root systems in G, the absolute root system $\Phi = \Phi(T, G)$, where T is a maximal k-torus, and the relative root system ${}_k\Phi = \Phi(T_d, G)$; the restriction map $X^*(T) \longrightarrow X^*(T_d)$ induces a surjection $\Phi \longrightarrow \{0\} \cup {}_k\Phi$. Note that in general $\alpha \in \Phi(T, G)$ is defined only over the separable closure k_s of k.

Dually to these roots there are the coroots, i.e., certain one-parameter subgroups $\alpha^\vee$ defined by $\langle\beta, \alpha^\vee\rangle\langle\alpha, \alpha\rangle = 2\langle\beta, \alpha\rangle$, where $(\alpha, \beta) \longmapsto \langle\alpha, \beta\rangle$ is a suitable inner product, cp. [Ti] 1.1.1. and [BT3] 8.2. Coroots exist for both the absolute root system Φ and the relative root system ${}_k\Phi$.

2 The setup

Assume that G is absolutely almost simple and isotropic, and let $T = T_a \cdot T_d$ be a maximal k-torus. If the root $\alpha \in \Phi(T, G)$ is defined over k (rather than k_s), then also the root group U_α and the dual root $\alpha^\vee$ are defined over k. There is some parabolic k-subgroup $P \subseteq G$ whose unipotent radical contains U_α. Note that the center of the Levi k-subgroup L of P is contained in T and hence acts as scalars on U_α which carries the structure of a one-dimensional vector space defined over k. Using the additive group structure of U_α and the multiplicative action of the torus one can then easily define a field structure isomorphic to k on $U_\alpha(k)$.

This approach works in particular if G is split or quasi-split over k, since in that case the highest root α with respect to some Borel k-subgroup B is defined over k: Every absolute root in $\Phi(T, G)$ is defined over the separable closure k_s of k, and the Galois group $\Gamma = Gal(k_s/k)$ acts on $\Phi(T, G)$. Since B is defined over k, Γ permutes the set of positive roots $\Phi(T, B) \subseteq \Phi(T, G)$, [Bo], 20.3. But the highest root α is unique in $\Phi(T, B)$, hence it is Γ-stable and thus defined over k, [Bo], 8.11.

In general there might not be an absolute root defined over k, and we have to use the following replacement in order to get a scalar action on some one-dimensional vector space: Let $P \subseteq G$ be a minimal parabolic k-subgroup, with Levi k-subgroup $L \subseteq P$ and unipotent radical U, and let again $T = T_a \cdot T_d \subseteq L$ be a maximal k-torus. The center $Z(U)$ of U carries the structure of a k-vector space (see [BT1], Thm. 3.17.) on which L acts linearly (see [ABS], Lemma 2). Note that the connected component Z^0 of Z is a subtorus of T which has nontrivial intersection with T_d since the derived Levi group L' is anisotropic. Suppose that U has the following property:

($*$) There exists a k-subgroup $V \subseteq Z(U)$ which is an irreducible L-module.

Assuming ($*$), the center Z of L acts on V as K-scalars, by Schur's Lemma. We require that $Z(k)$ acts on V via k-scalars: Let $f : L \to \mathbf{GL}(V)$ be the representation afforded by the action of L on V, and put $A = f(Z)$. By Schur's Lemma, A consists of multiples of the identity, $A \subseteq G_m$. On the other hand Z^0 acts nontrivially on V, whence $A = G_m$ is a one-dimensional torus. To see that also $Z(k)$ acts as k-scalars, albeit not necessarily as all scalars from k, we need that this torus is k-split: Let

$S = (Z^0)_d$ be the split part of the connected center of L. Then S acts by some nontrivial character on $Z(U)$. In particular, $f(S) = A$ and thus A is k-split.

In the setting above where G is quasi-split over k, $B \subseteq G$ is a Borel k-subgroup, and α is the highest root with respect to B, the root group U_α is contained in the center of the unipotent radical of B, and the Levi subgroup $L = T$ of B clearly acts irreducibly on $V = U_\alpha$.

In general, we apply the following result due to Azad-Barry-Seitz [ABS], §2 (here, the field of definition k does not enter).

2.1 Proposition [ABS] *Let $P \subseteq G$ be a parabolic subgroup of an almost simple group G, let $V = Z(U)$ be the center of the unipotent radical of P, and let $L \subseteq P$ be a Levi subgroup. Then V is an irreducible L-module, except possibly if* $\operatorname{char} k = 2$ *and G is of type B_n, C_n, F_4 or G_2, or* $\operatorname{char} k = 3$ *and G is of type G_2.*

2.2 Proposition *Let G be a k-isotropic absolutely almost simple k-group. Then every parabolic k-subgroup P of G has property $(*)$.*

Proof. If G is split, all absolute roots and their corresponding root groups and all parabolic subgroups are defined over k. So, for any parabolic subgroup $P \subseteq G$ with $U = \mathcal{R}_u P$ any root group $U_\alpha \subseteq Z(U)$ satisfies property $(*)$. If G is isotropic and of type G_2, then G is automatically split, see [Ti] p. 61. Hence, unless $\operatorname{char} k = 2$ and G is of type B_n, C_n or F_4, the claim follows from Proposition 2.1 and the discussion above.

We now investigate these remaining cases in characteristic 2. Here $Z(U)$ need no longer be irreducible for the action of a Levi subgroup of P. Instead we take an irreducible submodule of $Z(U)$ for our k-subgroup V in $(*)$.

We may suppose that G is a k-isotropic absolutely almost simple k-group which is not split nor quasi-split. We consult [Ti] pp. 55 - 60 for the type of a parabolic k-subgroup P of G which can occur in this instance. Let L be a Levi subgroup of P.

Consider the case when G is of type B_n and $\operatorname{char} k = 2$. Here the derived Levi subgroup L' is of type B_{n-r} where r is the index of the quadratic form q associated with G. For $r > 1$ the center of U is the one-dimensional trivial module for B_{n-r}. If $r = 1$, $Z(U) = U$ and this space affords the natural module for B_{n-1}. Since $\operatorname{char} k = 2$, this module is no longer irreducible. There is a one-dimensional trivial submodule afforded by the radical of the form on $Z(U)$. We may suppose that L' is defined over k. Thus the quadratic form associated to $L' \cong B_{n-1}$ is defined over k and so is its radical. Note that the center of L acts on the one-dimensional submodule via scalars in both instances. Thus the trivial module for $L'(k)$ is realized in $U(k)$ for any $r \geq 1$.

Next let G be of type C_n and $\operatorname{char} k = 2$. According to [Ti] p. 56 the type of P depends on an integer d. If $d = 1$, then L' is of type A_{n-1} and $U = Z(U)$ is isomorphic to the symmetric square of the natural module N of A_{n-1} (or its dual).

Since $\operatorname{char} k = 2$, this module is no longer irreducible. It admits $\Lambda^2 N$, the alternating square of N, and $N^{(2)}$, a Frobenius twist of N, as composition factors with socle $N^{(2)}$. Note that the center of L acts on $N^{(2)}$ as scalars. Clearly, N is defined over k, and so are $N^{(2)}$. If $d > 1$, then L' is of type $A_{d-1}^r \times C_{n-rd}$. It follows from [ABS] that $Z(U)$ is isomorphic to the symmetric square of the natural module (or its dual) of the first A_{d-1} factor of L' with the remaining components of L' acting trivially. The same argument applies in this instance.

Finally, let G be of type F_4 and $\operatorname{char} k = 2$. Since G is not split nor quasi-split, L' is of type B_3. The center of U is the natural module for L' of dimension 7. This module is no longer irreducible in characteristic 2. We argue as in the case $r = 1$ for type B_n. $\square$

3 Definability of k in $G(k)$

As above G is a k-isotropic absolutely almost simple k-group.

3.1 Lemma *Let k be infinite and let $L \subseteq P$ be a Levi k-subgroup of a parabolic k-subgroup P of G. Then there exists an element $x \in G(k)$ such that the group $L(k)$ of k-rational points of L is the centralizer of x in $G(k)$; in particular, $L(k)$ is definable in $G(k)$. Let Z denote the center of L. Then $Z(k)$ is the center of $L(k)$, and thus both $L(k)$ and $Z(k)$ are definable subgroups of $G(k)$.*

Proof. Let $S = Z^0$ be the connected center of L. By [Bo] Prop. 20.6 L is the G-centralizer of S, and S is a k-torus. By *loc. cit.* Prop. 8.18 there exists an element $x \in S(k)$ such that L is the G-centralizer of x (note that k is infinite). Therefore, $L(k) = L \cap G(k)$ is the $G(k)$-centralizer of x. Clearly, this is a definable subgroup of $G(k)$. Since L is reductive, the center Z of L is defined over k, and since $L(k)$ is dense in L, the center of $L(k)$ is precisely $Z(k)$. $\square$

3.2 Theorem *If G is an isotropic absolutely almost simple k-group, then k is definable in $G(k)$.*

Proof. Let P be a minimal parabolic k-subgroup of G, and let V be the irreducible L-module from $(*)$ whose existence is guaranteed by Proposition 2.2. Let Z denote the center of a Levi k-subgroup of P.

Let $v \in V(k)$ be a nontrivial element. The orbit $Z(k) \cdot v$ is contained in a one-dimensional k-subspace of $V(k)$. We now show that it contains all elements of the form $t^2 v$, where $t \in k^*$. For, let $S \subseteq Z^0$ be a one-dimensional isotropic torus which acts by some nontrivial k-root $\alpha \in {}_k\Phi$ on V. Let $\alpha^\vee : G_m \longrightarrow T$ be the corresponding k-coroot. Since P is minimal among the parabolic k-subgroups, the semisimple part L' of L is anisotropic. Therefore, $\alpha^\vee(G_m)$ is contained in the split torus T_d, see *loc.cit.*

8.15. But L' is anisotropic, hence $T_d \cap L'$ is finite and thus $\alpha^\vee(G_m) \subseteq T_d \subseteq Z^0$. Now $\langle \alpha, \alpha^\vee \rangle = 2$. Clearly, $\alpha^\vee(k^*) \subseteq Z(k)$, and for $t \in k^*$ we have

$$\alpha^\vee(t) \cdot v = \alpha(\alpha^\vee(t))v = t^{\langle \alpha, \alpha^\vee \rangle} v = t^2 v.$$

Now we define a field k' isomorphic to k as follows: consider the subset $k' \subseteq V(k)$ consisting of all elements of the form $x \cdot v - y \cdot v$ with $x, y \in Z(k)$ (resp. $x, y \in S(k)$, if k is finite). Clearly, k' is definable by 3.1 (resp. the fact that $S(k)$ is a finite cyclic group). If $\operatorname{char} k \neq 2$, then every element of k is a difference of two squares ($4z = (1+z)^2 - (1-z)^2$), hence $k' = kv$. If $\operatorname{char} k = 2$, then $k' = k^2 v$ so in any case $(k', +)$ is an isomorphic copy of $(k, +)$. A multiplication in k' is defined by $(x \cdot v)(y \cdot v) = (xy) \cdot v$, for $x, y \in Z(k)$, and by extending this rule linearly to k'. Thus we defined an isomorphic copy of k in the group $G(k)$. □

3.3 Corollary *Let G be a k-isotropic absolutely simple group. If $G(k)$ has finite Morley rank, then k is algebraically closed.*

Proof. This follows immediately from Theorem 3.2 and the fact that the only fields of finite Morley rank are algebraically closed ones. □

4 The k-simple case

A simple complex algebraic group G can also be viewed as a real algebraic group. However, considered as a real group it is only **R**-simple, but not absolutely simple and **R** is clearly not definable in G. In that sense **R** is the "wrong" field of definition of G.

We return to the general situation. Let l/k be a separable field extension of finite degree. There is a functor $R_{l/k}$ which assigns, by restriction of scalars, to each affine l-group H an affine k-group $G = R_{l/k}H$ such that $H(l) \cong G(k)$.

The functor preserves parabolic subgroups and central isogenies, see [BT1]. In the isogeny class of a semisimple k-group G there are two special groups, the *simply connected group* $\widetilde{G}$ and the *adjoint group* $\overline{G}$, along with corresponding central isogenies $\widetilde{G} \longrightarrow G \longrightarrow \overline{G}$. We need the following result:

4.1 Proposition [BT1],[Ti] *Let G be an almost simple k-group. If G is either adjoint or simply connected, then there exists a separable field extension l/k of finite degree and an absolutely almost simple l-group H such that*

$$R_{l/k}H \cong G.$$

Defining the field in the absolutely simple case relied on the fact that we have an irreducible action on (some subgroup of) the center of the unipotent radical. This is not true anymore if the group is only k-simple rather than absolutely simple. Nevertheless, our construction yields a field also in this more general context - - and in some sense this is the "correct" one.

4.2 Theorem *If G is an almost k-simple k-isotropic group, then there is a finite separable extension field l of k such that l is definable in $G(k)$.*

Proof. If G is either adjoint or simply connected, then by Proposition 4.1 there is a finite separable extension l of k and an absolutely simple l-group H with $H(l) \cong G(k)$ as groups. By Theorem 3.2, l is definable in $H(l)$ using only the group structure, which gives a definition for l in $G(k)$ as desired.

If G is neither adjoint nor simply connected, there might not be any absolutely simple group corresponding to G. Let $\overline{G}$ and $\widetilde{G}$ be the adjoint and simply connected groups isogenous with G under central isogenies ϕ and ϕ', respectively. Let l be the finite separable extension and $\widetilde{H}$ the absolutely simple simply connected l-group as in Proposition 4.1. The adjoint group $\overline{H}$ isogenous with $\widetilde{H}$ under some central isogeny ψ corresponds under the functor $R_{l/k}$ to the adjoint group $\overline{G}$ and, by the uniqueness of isogenies (up to k-isomorphism), the following diagram commutes:

$$\begin{tikzcd}
\widetilde{H} \arrow[r, "R_{l/k}"] \arrow[dd, "\psi"] & \widetilde{G} \arrow[dd, "R_{l/k}\psi"] \arrow[dr, "\phi"] & \\
& & G \arrow[dl, "\phi'"] \\
\overline{H} \arrow[r, "R_{l/k}"] & \overline{G} &
\end{tikzcd}$$

Let P be a minimal parabolic k-subgroup of G with Levi k-subgroup L, $U = \mathcal{R}_u P$, and $Z = Z(L)$. Note that $Z(k)$ is definable in $G(k)$ as in the absolutely simple case.

Since central isogenies preserve parabolic subgroups and induce isomorphisms on the unipotent radicals, we may identify U with its image in $\overline{G}$ and its preimage in $\widetilde{G}$. Clearly, the Levi complements of these parabolic subgroups are preserved as well and by $\widetilde{Z}$ and $\overline{Z}$ we denote their respective centers. As before, we can define l in $\widetilde{G}(k)$ and in $\overline{G}(k)$ using the action of $\widetilde{Z}(k)$ and $\overline{Z}(k)$, respectively, on the same element $v \in U$. Let M, $\widetilde{M}$ and $\overline{M}$ denote the orbits of v under $Z(k)$, $\widetilde{Z}(k)$ and $\overline{Z}(k)$, respectively. Define $\Delta M = \{a - b \,|\, a, b \in M\}$, and likewise for $\widetilde{M}$ and $\overline{M}$. Then the set $\Delta\widetilde{M} = \Delta\overline{M} \subseteq U$ carries the desired field structure. However, we have $\widetilde{M} \subseteq M \subseteq \overline{M}$, since $\phi(\widetilde{Z}(k)) \subseteq Z(k)$ and $\phi'(Z(k)) \subseteq \overline{Z}(k)$. Hence the set $\Delta M \subseteq U$ together with addition and multiplication as defined in Theorem 3.2 defines the field l in $G(k)$, as desired. $\square$

4.3 Corollary *Let G be k-simple and k-isotropic. If $G(k)$ has finite Morley rank, then k is algebraically closed or real closed.*

Proof. By the Artin-Schreier Theorem, the only fields which admit nontrivial algebraically closed extensions of finite degree are the real closed ones. The rest follows from Theorem 4.2. $\square$

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