Sensitivity of the solutions of some matrix equations

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Abstract

We develop a general method for obtaining bounds for the sensitivity of the solution of the stable Lyapunov equation. Our arguments not only work simultaneously for both the continuous and the discrete time Lyapunov equations, they work equally well for more general equations that have been studied. Further, we obtain bounds valid for all unitarily invariant norms.

§ 1 Introduction

In this note we consider the question of the sensitivity of the solution of some linear matrix equations. The equations we consider first are the continuous–time Lyapunov equation and the discrete–time Lyapunov equation. These are

$$A^*X + XA = W, \tag{1.1}$$

and

$$X - F^*XF = Q. \tag{1.2}$$

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We will assume that the matrix A is **positively stable**; i.e., all the eigenvalues of A lie in the open right half plane, and that the matrix F is **stable**; i.e., all the eigenvalues of F lie in the open unit disk. The equation (1.2) is also known as the Stein equation. Both the equations have been studied extensively because of their importance in differential equations and control theory. See, e.g. [11],[18]. More general equations have also been of interest. One that we will consider is

$$A^*X + XA + tA^{*1/2}XA^{1/2} = W, \tag{1.3}$$

where A is again positively stable, $A^{1/2}$ is its positively stable square root, and t is a real number (subject to some restrictions mentioned later). See [17] and the references therein for work on such equations.

Another equation of interest is the Sylvester equation

$$AX - XB = Y, \tag{1.4}$$

where A and B are any two matrices that have no eigenvalue in common. This equation includes the Lyapunov equation (1.1) as a special case, and has also been studied in great detail; see [6] for a recent survey.

In each of the equations (1.1)–(1.4), the matrix X is unknown and the other matrices are assumed to be given. The conditions that we have imposed ensure in each case the existence of a unique solution X. However, when the solution is computed numerically, the given matrices are subjected to perturbations. It is then of interest to obtain error bounds for the solution.

This problem too has been studied in detail. The present note supplements the existing literature, especially the work in [9], [14], [10], [8]. We make several observations that lead to a unification of some of the existing proofs, clarify some of the issues and bring in new methods that have not been employed in this context until very recently [2].

§ 2 The general framework

Consider the linear equation

$$Lx = w, \tag{2.1}$$

where x and w are vectors in a Banach space X, and L is an invertible bounded linear operator on X. Consider also the perturbed version of this equation

$$(L + \Delta L)\ (x + \Delta x) = w + \Delta w, \tag{2.2}$$

where ΔL and Δw represent small changes in L and w respectively. From these two equations we obtain

$$\Delta x = -L^{-1}\Delta L(x + \Delta x) + L^{-1}\Delta w.$$

Hence,

$$||\Delta x|| \leq ||L^{-1}||\ ||\Delta L||\ ||x + \Delta x||\ + ||L^{-1}||\ ||\Delta w||,$$

and therefore,

$$\frac{||\Delta x||}{||x+\Delta x||} \le ||L^{-1}|| \left(||\Delta L|| + \frac{||\Delta w||}{||x+\Delta x||} \right). \tag{2.3}$$

To achieve some symmetry we use the inequality

$$\frac{1}{||x+\Delta x||} \le \frac{||L+\Delta L||}{||w+\Delta w||}$$

that can be obtained from (2.2), and get from (2.3)

$$\frac{||\Delta x||}{||x+\Delta x||} \le ||L^{-1}|| \left(||\Delta L|| + \frac{||L+\Delta L|| \, ||\Delta w||}{||w+\Delta w||} \right) \tag{2.4}$$

or equivalently,

$$\frac{||\Delta x||}{||x+\Delta x||} \le ||L^{-1}|| \, ||L+\Delta L|| \left(\frac{||\Delta L||}{||L+\Delta L||} + \frac{||\Delta w||}{||w+\Delta w||} \right). \tag{2.5}$$

Interchanging the roles of (2.1) and (2.2), we could also write

$$\frac{||\Delta x||}{||x||} \le ||(L+\Delta L)^{-1}|| \, ||L|| \left(\frac{||\Delta L||}{||L||} + \frac{||\Delta w||}{||w||} \right). \tag{2.6}$$

These inequalities used routinely in the numerical analysis of linear systems give bounds for the relative perturbation Δx of the solution x .

Now let X be the space of $n \times n$ complex matrices $\mathbb{M}$. We will consider various norms on $\mathbb{M}$. The symbol $|||\cdot|||$ will denote any **unitarily invariant norm**, viz. one that satisfies the properties

$$|||UAV||| = |||A||| \text{ for all } A \in \mathbb{M} \text{ and unitary } U, V,$$

and

$$|||E||| = 1 \text{ for any rank one orthogonal projection } E.$$

Special examples of such norms are the Schatten p –norms defined as

$$||A||_p = \left(\sum_{j=1}^{n} s_j^p(A) \right)^{1/p}, \quad 1 \le p \le \infty,$$

where $s_1(A) \ge \cdots \ge s_n(A)$ are the singular values of A . In this notation

$$||A||_\infty = s_1(A) = \sup_{\substack{x \in \mathbb{C}^n \\ ||x||=1}} ||Ax||,$$

is the operator norm (or the spectral norm) of A . We shall denote this simply by $||A||$.

We will also consider another more general class of norms called **weakly unitarily invariant norms**. A member $||\cdot||_{wui}$ of this family is characterized by the property

$$||UAU^*||_{wui} \;=\; ||A||_{wui} \;\text{ for all }\; A \in \mathbb{M} \;\text{ and unitary }\; U.$$

This family includes all the unitarily invariant norms and several others. For a detailed exposition of these norms see [3].

Now return to the matrix equations in Section 1. Each of them has the form

$$\mathcal{L}(X) = W, \tag{2.7}$$

where $\mathcal{L}$ is a linear operator on the space $\mathbb{M}$. Thus, for example, if we put

$$\mathcal{L}_A(X) = A^*X + XA, \tag{2.8}$$

then the equation (1.1) can be written as $\mathcal{L}_A(X) = W$.

To use a perturbation bound like (2.3)–(2.6) we have to evaluate the norms of $\mathcal{L}$ and $\mathcal{L}^{-1}$. The norm of $\mathcal{L}$ is defined as its norm as a linear operator on the space $\mathbb{M}$ which in turn is equipped with one of the unitarily (or weakly unitarily invariant) norms. Thus, we introduce the notation

$$|||\mathcal{L}||| = \sup_{|||X|||=1} |||\mathcal{L}(X)|||, \tag{2.9}$$

for each unitarily invariant norm $|||\cdot|||$.

The norm of $\mathcal{L}$ is easy to estimate. Thus, for example

$$|||\mathcal{L}_A||| = \sup_{|||X|||=1} |||A^*X + XA||| \leq \; 2||A||, \tag{2.10}$$

because of the triangle inequality and the property $|||ST||| \leq ||S||\,|||T|||$ of every unitarily invariant norm ([3], p. 94).

It is not so easy to find good bounds for the norm of $\mathcal{L}^{-1}$. The rest of the paper is devoted to this issue.

§ 3 Positive Maps

For brevity, we will use the term **positive matrix** to mean a positive semidefinite matrix.

A linear map $\Phi : \mathbb{M} \to \mathbb{M}$ is said to be **positive** if the matrix $\Phi(A)$ is positive whenever A is positive. Positive maps on C^*–algebras have received great attention in recent decades. The space $\mathbb{M}$ equipped with the norm $||\cdot||$ is a C^*–algebra, and we will make use of a basic elementary fact about positive maps known for general C^*–algebras. This says that every positive linear map attains its norm at the identity matrix:

$$||\Phi|| = ||\Phi(I)||. \tag{3.1}$$

See ([15], Exercise 10.5.10).

It is instructive to see a special instance of this general theorem in a setting more familiar in matrix theory. Let Z be a positive matrix. Let $\Phi(X) = Z \circ X$, the **Schur product** or the **Hadamard product** of Z and X. By a well–known theorem of Schur this is a positive linear map on $\mathbb{M}$, and we have

$$||Z \circ X|| \leq \max_i \ z_{ii} \ ||X|| \tag{3.2}$$

for all X. See ([12], p. 95, [13], p. 343). It is easy to see that in this case the assertion (3.1) follows from this.

Another important notion is that of a **completely positive map**. Let $[A_{ij}]_{N\times N}$ denote an $N \times N$ block matrix whose entries A_{ij} come from $\mathbb{M}$. We say that a linear map Φ on $\mathbb{M}$ is **completely positive** if for every N the matrix $[\Phi(A_{ij})]_{N\times N}$ is positive whenever $[A_{ij}]_{N\times N}$ is positive. Such maps have turned out to be of crucial importance in the theory of C^*–algebras. A fundamental structure theorem for completely positive maps on $\mathbb{M}$ was proved by Choi [7].

The Schur product map $\Phi(X) = Z \circ X$ where Z is positive, is an example of a completely positive map. This is so because for each N the $N \times N$ block matrix with all entries Z is again positive.

A linear map Φ on $\mathbb{M}$ is said to be **unital** if $\Phi(I) = I$, and **trace preserving** if $\operatorname{tr} \Phi(A) = \operatorname{tr} A$, for all A. A positive, unital, trace preserving map is said to be a **doubly stochastic map** on $\mathbb{M}$. See ([1], Section 7) for an admirably concise introduction to the properties of such maps.

If Z is a positive matrix with unit diagonal; i.e., $z_{ii} = 1$ for all i, then the Schur product map $\Phi(X) = Z \circ X$ is a doubly stochastic map on $\mathbb{M}$.

The inequality (3.2) has a generalization: if Z is positive then for every unitarily invariant norm

$$|||Z \circ X||| \leq \max \ z_{ii} \ |||X||| \tag{3.3}$$

for all X . See ([13], p. 343).

More generally, if Φ is any doubly stochastic map on $\mathbb{M}$, then for every unitarily invariant norm

$$|||\Phi(X)||| \leq |||X||| \quad \text{for all } X. \tag{3.4}$$

See ([1], Corollary 7.8). More can be said when X is Hermitian. Then we have

$$||\Phi(X)||_{wui} \leq ||X||_{wui}, \tag{3.5}$$

for every weakly unitarily invariant norm. This follows from Theorem 7.1 in [1].

§ 4 The Lyapunov Equations

Consider now the equations (1.1) and (1.2). We have already introduced the notation $\mathcal{L}_A$ in (2.8). Let us write

$$\mathcal{S}_F(X) = X - F^*XF. \tag{4.1}$$

Then the equation (1.2) can be written as $\mathcal{S}_F(X) = Q$.

One of the fundamental facts about these equations is that when A is positively stable and F stable, the operators $\mathcal{L}_A^{-1}$ and $\mathcal{S}_F^{-1}$ are positive maps on $\mathbb{M}$. This can be seen in different ways. One way is to note this from the explicit solutions

$$\mathcal{L}_A^{-1}(W) = \int_0^\infty e^{-A^*t}\, We^{-At}\, \mathrm{dt}, \tag{4.2}$$

$$\mathcal{S}_F^{-1}(Q) = \sum_{k=0}^\infty F^{*k}QF^k. \tag{4.3}$$

See ([18] , p. 443, [8]). Another way is to see this first when A and F are diagonal. If $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ then the solution X of (1.1) can be written as

$$x_{ij} = \frac{w_{ij}}{\bar{\lambda}_i + \lambda_j},$$

or as,

$$X = Z \circ W,$$

where Z is the matrix with entries $z_{ij} = (\bar{\lambda}_i + \lambda_j)^{-1}$. This matrix is positive, because it can be expressed as a Gram matrix: if $f_i(t) = e^{-t\lambda_i},\ 0 \leq t < \infty$, then f_i are in the space $L_2([0,\infty))$, and in this space we have the inner product

$$< f_i, f_j > = \int_0^\infty e^{-t(\bar{\lambda}_i+\lambda_j)}\, \mathrm{dt} = \frac{1}{\bar{\lambda}_i + \lambda_j} = z_{ij}.$$

See ([12], p. 135) and ([13], p. 347) from where we have taken this argument. So, $\mathcal{L}_A^{-1}$ is a positive map when A is diagonal. One can then prove that this is also the case when A is similar to a diagonal matrix; see ([13], p. 301). By a continuity argument the same is true for all A.

A similar argument can be made for the equation (1.2). If $F = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ then the solution of (1.2) can be written as

$$x_{ij} = \frac{q_{ij}}{1 - \bar{\lambda}_i \lambda_j}.$$

If λ_i is any point in the open unit disk then there is a unique point μ_i in the open right half plane such that

$$\lambda_i = \frac{\mu_i - 1}{\mu_i + 1}. \tag{4.4}$$

Using this correspondence, we can write

$$\frac{1}{1 - \bar{\lambda}_i \lambda_j} = \frac{(\bar{\mu}_{i+1})(\mu_{j+1})}{2(\bar{\mu}_i + \mu_j)}. \tag{4.5}$$

Let $D =$ $\mathrm{diag}(\mu_1 + 1, \dots, \mu_n + 1)$. Then the equation (4.5) says that the matrix with entries $\frac{1}{1-\bar{\lambda}_i\lambda_j}$ can be written as $\frac{1}{2}$ D^*ZD, where Z is the matrix with entries $\frac{1}{\bar{\mu}_i + \mu_j}$. Since Z is positive, so is D^*ZD. ¿From this one can see that $\mathcal{S}_F^{-1}$ is a positive map in the same way as one did for $\mathcal{L}_A^{-1}$.

It is interesting to note that the maps $\mathcal{L}_A^{-1}$ and $\mathcal{S}_F^{-1}$ are not just positive, they are completely positive. Again one can see this in different ways. The very form of the solutions (4.2) and (4.3) and Choi's characterisation [7] immediately give this fact. Another illuminating way of looking at this is the following. Given any matrix A and a positive integer N let $\widetilde{A}$ be the direct sum of N copies of A; i.e., $\widetilde{A}$ is a block diagonal matrix with N copies of A down its diagonal. If A is positively stable then so is $\widetilde{A}$. The N^2 equations

$$A^* X_{ij} + X_{ij} A = W_{ij} \quad i, j = 1, 2, \dots, N$$

can be written as a single matrix equation

$$\widetilde{A}^*[X_{ij}] + [X_{ij}]\widetilde{A} = [W_{ij}],$$

where, as before $[X_{ij}]$ is the $N \times N$ block matrix with entries X_{ij}. The positivity of $(\mathcal{L}_{\widetilde{A}})^{-1}$ for all N is precisely the complete positivity of $\mathcal{L}_A^{-1}$. The same argument shows that $\mathcal{S}_F^{-1}$ is completely positive.

Now let H, K, H_*, K_* be the matrices defined as

$$\begin{aligned} H &= \mathcal{L}_A^{-1}(I), \\ K &= \mathcal{S}_F^{-1}(I), \\ H_* &= \mathcal{L}_{A^*}^{-1}(I), \\ K_* &= \mathcal{S}_{F^*}^{-1}(I). \end{aligned} \tag{4.6}$$

In other words, H, K are the solutions of (1.1) and (1.2) respectively, when $W = I$ and $Q = I$. The matrices H_*, K_* are the solutions of the equations obtained from these when A and A^* and F and F^* are interchanged. The following theorem describes some known results.

Theorem 4.1 *With the notations introduced above, we have*

$$||\mathcal{L}_A^{-1}|| = ||H||, \tag{4.7}$$

$$||\mathcal{S}_F^{-1}|| = ||K||, \tag{4.8}$$

$$||\mathcal{L}_A^{-1}||_1 = ||H_*||, \tag{4.9}$$

$$||\mathcal{S}_F^{-1}||_1 = ||K_*||. \tag{4.10}$$

The relation (4.7) was proved in [10] and (4.8) in [8]. We remark that both are immediate consequences of the fact that $\mathcal{L}_A^{-1}$ and $\mathcal{S}_F^{-1}$ are positive maps, and the relation (3.1). This was noted in [2] for (4.7). We thus have a unified view of (4.7) and (4.8). Further, this proof works equally well in C^* –algebras. The norm $||A||_1 = \sum_{j=1}^{n} s_j(A)$ is the dual of the operator norm $||A||$. The relations (4.9) and (4.10) then follow from (4.7) and (4.8) by an elementary functional analysis argument; see [2]. The relation (4.9) was first proved in [16].

Our next theorem says that when A (or F) is normal then the above theorem can be generalised to all unitarily invariant norms. Note that when A is normal we have $H = H_*$, and when F is normal we have $K = K_*$. This follows from the equations (4.2) and (4.3).

Theorem 4.2 *Let A be normal. Then for every unitarily invariant norm*

$$|||\mathcal{L}_A^{-1}||| = ||H||. \tag{4.11}$$

Let F be normal. Then for every unitarily invariant norm

$$|||\mathcal{S}_F^{-1}||| = ||K||. \tag{4.12}$$

Proof Because of unitary invariance of the norms involved, we may assume without loss of generality that $A = \text{diag}(\lambda_1, \dots, \lambda_n)$. We have observed earlier that the solution X of (1.1) can then be expressed as $X = Z \circ W$, where Z is the positive matrix $z_{ij} = (\bar{\lambda}_i + \lambda_j)^{-1}$. Because of the inequality (3.3) we have, therefore,

$$|||X||| \;\leq\; \frac{1}{2 \text{ min Re } \lambda_i} \, |||W||| \;=\; ||H|| \; |||W|||.$$

In other words, $|||\mathcal{L}_A^{-1}||| \;\leq\; ||H||$. Now let j be the index for which $\min_i \text{Re } \lambda_i = \text{Re } \lambda_j$. Choose W to be a diagonal matrix with $w_{jj} = 1$ and all other entries 0 . Then

$|||W||| = 1$ for every unitarily invariant norm and $|||X||| = \frac{1}{2 \text{ Re } \lambda_j}$. Hence $|||\mathcal{L}_A^{-1}||| = ||H||$. This proves (4.11).

The same argument works to give (4.12). ■

We have noted in the course of the proof that for normal A with eigenvalues $\lambda_1, \dots, \lambda_n$,

$$||H|| = \frac{1}{2 \min \text{Re } \lambda_i}. \tag{4.13}$$

We should also note that for normal F with eigenvalues $\lambda_1, \dots, \lambda_n$

$$||K|| = \frac{1}{1 - \max \ |\lambda_i|^2} = \frac{1}{1 - (\rho(F))^2}, \tag{4.14}$$

where $\rho(F)$ is the spectral radius of F .

For the case of the Frobenius norm, the result of Theorem 4.2 is known. See, e.g. ([8], p. 1211). In [2] a different proof of the relation $|||\mathcal{L}_A^{-1}||| \le ||H||$ was given; the equality was not noticed.

We have already estimated the norm $|||\mathcal{L}_A|||$ in (2.10). The same reasoning shows

$$|||\mathcal{L}_{A+\Delta A} - \mathcal{L}_A||| \le 2|||\Delta A|||. \tag{4.15}$$

Note also that

$$|||\mathcal{S}_F||| = \sup_{|||X|||=1} |||X - F^* X F||| \le 1 + ||F||^2, \tag{4.16}$$

$$\begin{aligned} |||\mathcal{S}_{F+\Delta F} - \mathcal{S}_F||| &= \sup_{|||X|||=1} |||F^* X \Delta F + (\Delta F)^* X F + (\Delta F)^* X \Delta F||| \\ &\le 2 \, ||F|| \, ||\Delta F|| + ||\Delta F||^2 = ||\Delta F|| \, (2||F|| + \|\Delta F\|). \end{aligned} \tag{4.17}$$

We now have all the information required to write down perturbation bounds for (1.1) and (1.2). We first write a master theorem and then its corollaries.

Theorem 4.3 *Let A be a positively stable matrix. Then for every unitarily invariant norm, we have the following bound for the sensitivity of the solution X of the equation (1.1)*

$$\frac{|||\Delta X|||}{|||X + \Delta X|||} \le 2|||\mathcal{L}_A^{-1}||| \, ||A + \Delta A|| \left\{ \frac{||\Delta A||}{||A + \Delta A||} + \frac{|||\Delta W|||}{|||W + \Delta W|||} \right\}. \tag{4.18}$$

Let F be a stable matrix. Then for the sensitivity of the solution X of the equation (1.2) we have the bound

$$\frac{|||\Delta X|||}{|||X + \Delta X|||} \le |||\mathcal{S}_F^{-1}||| \left\{ \alpha \, \frac{||\Delta F||}{||F + \Delta F||} + \beta \, \frac{|||\Delta Q|||}{|||Q + \Delta Q|||} \right\}, \tag{4.19}$$

where

$$\alpha = 2(||F|| + ||\Delta F||)^2, \quad \beta = 1 + ||F + \Delta F||^2. \tag{4.20}$$

Proof Proceed as in the derivation of the inequality (2.4). Use the inequalities (4.15)—(4.17) at the appropriate stages. We leave the details as they are routine. ■

Now we can write down the following special cases of this:

1. If the norm $|||\cdot|||$ in (4.18) and (4.19) is restricted to be the spectral norm $||\cdot||$, we can substitute for $||\mathcal{L}_A^{-1}||$ and $||\mathcal{S}_F^{-1}||$ the values $||H||$ and $||K||$, respectively, given by Theorem 4.1. This is exactly the main result (2.2) of [10] and (2.11) of [8].

2. In the same way if $|||\cdot|||$ is the trace norm $||\cdot||_1$, we can substitute for $||\mathcal{L}_A^{-1}||_1$ and $||\mathcal{S}_F^{-1}||_1$ in (4.18) and (4.19) the values $||H_*||$ and $||K_*||$, given by Thereom 4.1.

3. If A or F is normal, we can use Theorem (4.2) and obtain from (4.18) and (4.19) a result valid for all unitarily invariant norms.

The interpretation of various quantities that occur in these inequalities and their relation to other objects is given in great detail in [10] and [8].

§ 5 Remarks

5.1 The method we have used in this paper can be applied to variants of the Lyapunov equation. It is shown in [17] that for $t \in [0,2]$ the equation (1.3) has a positive solution X for each positive W. When A itself is positive this is true for all $t \in (-2,2]$. Other theorems of this type are also proved by Kwong. Our method for estimating $||\mathcal{L}_A^{-1}||$ depended only on the fact that X is positive when W is. So, it can be applied in all such cases.

5.2 With H and K as in (4.6), define the maps Γ_A and Σ_F on $\mathbb{M}$ as

$$\begin{aligned} \Gamma_A(Y) &= H^{-1/2}\mathcal{L}_A^{-1}(Y)H^{-1/2}, \\ \Sigma_F(Y) &= K^{-1/2}\mathcal{S}_F^{-1}(Y)K^{-1/2}. \end{aligned}$$

By the argument given in ([2], Theorem 2) Γ_A is doubly stochastic when A is normal and Σ_F is doubly stochastic when F is normal. (The condition is necessary and sufficient). For doubly stochastic maps we have the inequality (3.4) for all weakly unitarily invariant norms. Thus if W in (1.1) or Q in (1.2) were Hermitian we could obtain more information using this observation. This special case does arise in applications. See [14].

5.3 We take this opportunity to indicate how recent results on the Sylvester equation (1.4) can be used to obtain bounds like the one we have derived.

Let $\mathcal{J}_{A,B}(X) = AX - XB$. It is clear that

$$|||\mathcal{J}_{A,B}||| \leq ||A|| + ||B||.$$

Bounds for $||\mathcal{J}_{A,B}^{-1}|||$ when A and B are normal have been obtained in [4] and [5]. See also [6]. These can be used to write down bounds for the sensitivity of the solution X following our arguments.

References

[1] T. Ando, *Majorization, doubly stochastic matrices and comparison of eigenvalues*, Linear Algebra Appl. 118:163–248 (1989).

[2] R. Bhatia, *A note on the Lyapunov Equation*, Linear Algebra Appl. 259:71–76 (1997).

[3] R. Bhatia, *Matrix Analysis*, Springer, New York (1997).

[4] R. Bhatia, C. Davis and A. McIntosh, *Perturbation of spectral subspaces and solution of linear operator equations*, Linear Algebra Appl. 52/53:45–67 (1983).

[5] R. Bhatia, C. Davis and P. Koosis, *An extremal problem in Fourier analysis with applications in operator theory*, J. Funct. Anal. 82:138–150 (1989).

[6] R. Bhatia and P. Rosenthal, *How and why to solve the operator equation* $AX - XB = Y$, Bull. London Math. Soc. 29:1–21 (1997).

[7] M.D. Choi, *Completely positive linear maps on complex matrices*, Linear Algebra Appl. 10:285–290 (1975).

[8] P.M. Gahinet, A. Laub, C.S. Kenney and G. Hewer, *Sensitivity of the stable discrete–time Lyapunov equation*, IEEE Trans. Automat. Control 35:1209–1217 (1990).

[9] S.J. Hammarling, *Numerical solution of the stable, non–negative definite Lyapunov equation*, IMA J. Numer. Anal. 2:303–323 (1982).

[10] G. Hewer and C. Kenney, *The sensitivity of the stable Lyapunov equation*, SIAM J. Control Optim. 26:321–344 (1988).

[11] N.J. Higham, *Accuracy and stability of numerical algorithms*, Society for Industrial and Applied Mathematics (1996).

[12] R. Horn, *The Hadamard product*, in: Matrix Theory and Applications, Proceedings of Symposia in Applied Math., Vol. 20, American Mathematical Society (1990).

[13] R. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press (1991).

[14] E.A. Jonckheere, *New bound on the sensitivity of the solution of the Lyapunov equation*, Linear Algebra Appl. 60:57–64 (1984).

[15] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Vol. 4, Birkhäuser, Boston (1992).

[16] C. Kenney and G. Hewer, *Trace norm bounds for stable Lyapunov equations*, Linear Algebra Appl. 221:1–18 (1995).

[17] M.K. Kwong, *On the definiteness of the solutions of certain matrix equations*, Linear Algebra Appl. 108:177-197 (1988).

[18] P. Lancaster and M. Tismenetsky, *The Theory of Matrices*, 2nd ed., Academic Press, New York (1985).