PARTIAL NORMS AND THE CONVERGENCE OF GENERAL PRODUCTS OF MATRICES

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29 October 1997

Abstract

Motivated by the theory of inhomogeneous Markov chains, we determine a sufficient condition for the convergence to 0 of a general product formed from a sequence of real or complex matrices. When the matrices have a common invariant subspace H, we give a sufficient condition for the convergence to 0 on H of a general product. Our result is applied to obtain a condition for the weak ergodicity of an inhomogeneous Markov chain. We compare various types of contractions which may be defined for a single matrix, such as paracontraction, l–contraction, and H–contraction, where H is an invariant subspace of the matrix.

*Work supported by NSF Grant DMS-9306357.

†Research supported in part by NSF Grant DMS-9424346. This research was partly performed during a visit by this author to the University of Bielefeld, Germany, supported by the Sonderforschungsbereich “Diskrete Strukturen in der Mathematik”, Universität Bielefeld.

1 Introduction

Recently there has been much interest in conditions for the convergence of infinite products of real or complex matrices. Several investigations have concentrated on products taken in one direction – left or right, see for example the recent papers by Beyn and Elsner [2] and Hartfiel and Rothblum [4]. However, in this paper, we are concerned with *general products* formed from a given infinite sequence of matrices. These are defined further on in the paper and they have previously been considered for nonnegative and for stochastic matrices by Seneta in [16, Chapters 3.1 and 4.6].

Our principal result is a sufficient condition for the convergence to 0 of infinite general products of matrices. We pay particular attention to the case where there is a common invariant subspace for all the matrices in the product. As a special case, we obtain a result on the weak ergodicity of an inhomogeneous Markov chain.

The investigations described above are preceded by a study of the interrelations of several types of contractions which may be defined for a single matrix.

The motivation for our study comes from the theory of inhomogeneous Markov chains, see [15] and [16, Chapter 4] for much further background material. This theory naturally leads to the study of the convergence, ergodicity and weak ergodicity of products of stochastic matrices.

We now describe our paper in more detail. First, in Section 2, we sharpen several observations concerning paracontraction. This we shall achieve by introducing a partial norm on the matrix restricted to an invariant subspace H and the consequent notion of *H–contraction.* We relate this concept to the notion of paracontraction as introduced by Nelson and Neumann in [13] and to the notion of l–contraction recently introduced by Beyn and Elsner in [2].

Second, in Section 3, which is essentially independent of Section 2, we turn to our main results. We formulate our sufficient conditions depending on norms for the convergence to 0 of infinite general products of matrices which are formed from a given infinite sequence of matrices. We then apply the results to examine the convergence to 0 of infinite general products on a common invariant subspace of the matrices.

In Section 4 we specialize our results and we investigate products of stochastic matrices to obtain a suffcient condition for weak ergodicity. The ℓ_1 coefficient of ergodicity due to Bauer, Deutsch, Stoer [1] plays a special role here.

In Section 5 we give bounds on the algebraic and geometric multiplicities of the eigenvalues of the restriction of a matrix to an invariant subspace in terms of these quantities for the entire matrix. These are related to familiar results on stochastic matrices.

2 Partial norms and Paracontractions

In this paper $\mathbb{F}$ will stand for the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. We begin by recalling the following definition:

Definition 2.1 (Nelson and Neumann [13]) Let $A \in \mathbb{F}^{n,n}$ and let ν be a vector norm on $\mathbb{F}^n$. Then A is *paracontracting with respect to* ν if

$$\nu(Ax) \ < \nu(x) \ \text{ whenever } Ax \neq x. \tag{1}$$

In particular we see that one immediate consequence of the definition is that $\nu^0(A) \leq 1$, where $\nu^0(A)$ denotes the operator norm corresponding to the norm ν. For a given matrix norm let $\mathcal{N}_\nu$ denote the set of all paracontracting matrices with respect to ν in $\mathbb{F}^n$. Nelson and Neumann show in [13] that if $A \in \mathcal{N}_\nu$, then $\lim_{i\to\infty} A^i$ exists and that $AB \in \mathcal{N}_\nu$ if $A, B \in \mathcal{N}_\nu$.

We now introduce the concept of a partial norm with respect to a subspace and the concept of H–contraction:

Definition 2.2 Let H be a subspace of $\mathbb{F}^n$ and let $A \in \mathbb{F}^{nn}$ be a matrix which leaves H invariant. Let ν be a norm on $\mathbb{F}^n$ and define ν^0 to be the operator norm induced by ν. Then

$$\nu^0_H(A) \ = \ \sup_{0\neq x\in H} \frac{\nu(Ax)}{\nu(x)}. \tag{2}$$

We call $\nu_H^0(A)$ a *partial norm of A with respect to the invariant subspace H* of A.

Definition 2.3 The matrix $A \in \mathbb{F}^{n,n}$ is *nonexpansive* with respect to the vector norm ν on $\mathbb{F}^n$ if $\nu^0(A) \leq 1$. The matrix A is an *H–contractor* if it is nonexpansive, H is an invariant subspace of A, and $\nu_H^0(A) < 1$.

We shall denote the range of $A \in \mathbb{F}^{nn}$ by $\mathcal{R}(A)$ and the nullspace of A by $\mathcal{N}(A)$.

Remark 2.4 If $A \in \mathbb{F}^{n,n}$ is nonexpansive with respect to ν then the spectral radius $\rho(A)$ of A satisfies $\rho(A) \leq \nu^0(A) \leq 1$. If 1 is an eigenvalue of A, then we must have that $\nu^0(A) = \rho(A)$ and it follows that all Jordan blocks of A corresponding to 1 are 1×1, (viz. $\mathrm{index}_0(A) = 1$), e.g., Mott and Schneider [12]. Hence $K = \mathcal{N}(I - A)$ and $H = \mathcal{R}(I - A)$ are complementary invariant subspaces.

We next show the connection between paracontraction and H–contraction.

Lemma 2.5 *Let $A \in \mathbb{F}^{n,n}$ be paracontracting with respect to ν and let K be the subspace of all its fixed points. Then $H = \mathcal{R}(I - A)$ is invariant under A, complementary to K, and such that $\nu_H^0(A) < 1$ so that A is an H–contractor.*

Proof: It follows by Remark 2.4 that H is invariant under A and complementary to K. Since A is paracontracting with respect to ν, we have that $\nu^0(A) \leq 1$. In view of (1), it now follows that $\nu(Ax) < \nu(x)$, for all $x \in H$, and so since the unit ball $\mathcal{U}$ of ν_H^0 in H is compact,

$$\max_{x \in \mathcal{U}} \nu(Ax) \ < \ 1.$$

Hence $\nu_H^0(A) < 1$. □

Recently Beyn and Elsner [2] have introduced the notion of l–paracontraction:

Definition 2.6 ([2]) A matrix $A \in \mathbb{F}^{n,n}$ is l–paracontracting with respect to the vector norm ν on $\mathbb{F}^n$ if there exists a $\gamma > 0$ such that

$$\nu(Ax) \ \leq \ \nu(x) - \gamma\nu(Ax - x), \quad \forall x \in \mathbb{F}^n. \tag{3}$$

In their paper Beyn and Elsner establish several conditions which are equivalent to l–paracontraction. Clearly l–paracontraction implies paracontraction. We next show that H–contraction implies l–paracontraction for a suitable chosen norm and a suitably chosen subspace H.

Theorem 2.7 *Let $A \in \mathbb{F}^{n,n}$ and suppose that A is nonexpansive with respect to the norm ν on $\mathbb{F}^n$. Let $H = \mathcal{R}(I-A)$ and let $K = \mathcal{N}(I-A)$. If the norm ν satisfies*

$$\nu(y+z) = \nu(y) + \nu(z), \text{ for } y \in H,\ z \in K, \tag{4}$$

then the following are equivalent:

(i) *A is l–paracontracting with respect to ν.*

(ii) *A is paracontracting with respect to ν.*

(iii) *A is an H–contraction with respect to ν.*

Proof: (i) $\Longrightarrow$ (ii): Obvious.

(ii) $\Longrightarrow$ (iii): By Lemma 2.5.

(iii) $\Longrightarrow$ (i): Since A is nonexpansive, we have by Remark 2.3 that $K \oplus H = \mathbb{F}^n$. For $x \in \mathbb{F}^n$, write

$$x = y + z, \qquad y \in H,\ z \in K.$$

Let $k = \nu_H^0(A)$ and note that $k < 1$ by (iii). Since $x - Ax = y - Ay \in H$ it follows that

$$\nu(x - Ax) = \nu((I-A)y) \ \leq\ \nu_H^0(I-A)\nu(y) \ \leq\ (1+k)\nu(y). \tag{5}$$

Moreover, using (4), we obtain

$$\nu(x) - \nu(Ax) \ =\ \nu(y+z) - \nu(Ay+z) \ =\ \nu(y) - \nu(Ay) \ \geq\ (1-k)\nu(y). \tag{6}$$

We combine (5) and (6) and we deduce that

$$\nu(x - Ax) \ \leq\ (1+k)\nu(y) \ \leq\ \frac{1+k}{1-k}(\nu(x) - \nu(Ax)).$$

and this is equivalent to (3) with $\gamma = (1-k)/(1+k)$. $\square$

Remark 2.8 If ν' is a norm on $\mathbb{F}^n$, then we may define a norm ν that satisfies (4) and agrees with ν' on H and K by setting $\nu(y+z) = \nu'(y) + \nu'(z)$ for $y \in H$ and $z \in K$.

Remark 2.9 The implication (iii) $\Longrightarrow$ (ii) of Theorem 2.7 holds under weaker assumptions on the norm ν. Specifically, if A is an H–contraction with respect to ν and the condition

$$\nu(y) < \nu(y') \Rightarrow \nu(y+z) < \nu(y'+z), \ y \in H, \ z \in K$$

is satisfied, then A can be shown to be paracontracting with respect to ν.

3 Convergence of Infinite Products

In this section we develop our main results concerning the convergence of products of complex matrices taken in an arbitrary order from an infinite sequence of matrices. Such products were considered (in a slightly less general form) in Seneta [16, Section 4.6] in the case of stochastic matrices, see also [8] and [14].

Let $A_1, A_2, \ldots$ be a sequence of complex matrices. We shall consider products of matrices obtained from the sequence in the following manner: First choose some permutation of the given infinite sequence to obtain a sequence $B_1, B_2, \ldots$. Then form the products $C_{p,r}$ of the matrices $B_{p+1}, \ldots, B_{p+r}$ in some order. We shall call $C_{p,r}$ a *general product* from the sequence $A_1, A_2, \ldots$ and we shall consider the existence of $\lim_{r\to\infty} C_{p,r}$. If this limit is 0, for all permutations of $A_1, A_2, \ldots$ and all $p, \ p \geq 0$, then we shall say that all general products from the the sequence $A_1, A_2, \ldots$ converge to 0.

As an example of a sequence of general product suppose the chosen order is $A_9, A_7, A_5, \ A_{14}, A_2, \ldots$. Then the sequence of $(C_{2,1}, C_{2,2}, \ldots)$ may begin thus:

$$\begin{aligned} C_{2,1} &= A_7, \\ C_{2,2} &= A_7 A_5, \\ C_{2,3} &= A_5 A_2 A_7, \\ C_{2,4} &= A_2 A_7 A_{14} A_5. \end{aligned}$$

Note that, for a given sequence $C_{p,1}, C_{p,2}, \ldots$ of general products each factor of $C_{p,r}$ occurs in $C_{p,r+1}$, but the order in which the factors occur in $C_{p,r}$ is arbitrary.

Let μ be a matrix norm (viz. a submultiplicative norm on $\mathbb{F}^{nn}$) and denote

$$\mu^+(P) \;=\; \max(\mu(P), 1) \qquad \text{and} \qquad \mu^-(P) \;=\; \min(\mu(P), 1).$$

Now let $A_1, A_2, \ldots$ be a sequence of matrices in $\mathbb{F}^{nn}$ and let μ be a matrix norm. We now define two conditions:

$\boxed{\textbf{Condition (C)}}$: We say that the sequence $A_1, A_2, \ldots,$ satisfies Condition **(C)** for the norm μ if

$$\sum_{i=1}^{\infty} (\mu^+(A_i) - 1) \text{ converges.} \tag{7}$$

$\boxed{\textbf{Condition (D)}}$: We say that the sequence $A_1, A_2, \ldots$ satisfies Condition **(D)** for the norm μ if

$$\sum_{j=1}^{\infty} (1 - \mu^-(A_i)) \text{ diverges.} \tag{8}$$

We are now ready to prove the following result:

Proposition 3.1 *Let $A_1, A_2, \ldots$ be a sequence of matrices in $\mathbb{F}^{nn}$. Let μ be a matrix norm on $\mathbb{F}^{nn}$. Suppose that the sequence $A_1, A_2, \ldots$ satisfies Condition* **(C)** *for the norm μ. Then all general products from $A_1, A_2, \ldots$ are bounded.*

Proof: Let $B_1, B_2, \ldots$ be a permutation of $A_1, A_2, \ldots$ and let $C_{p,r}$ be a product of $B_{p+1}, \ldots, B_{p+r}$ in some order. By Condition **(C)**and [6, Theorem 14], $\sum_{i=1}^{\infty} (\mu^+(B_i) - 1)$ converges and hence $\sum_{i=1}^{\infty} (\mu^+(B_{p+i}) - 1)$ also converges.

Thus, by [6, Theorem 51], the product $\prod_{i=1}^{\infty} \mu^{+}(B_{p+i})$ converges and so there exists a positive constant M such that $\prod_{i=1}^{r} \mu^{+}(B_{p+i}) \leq M$, for each $r \in \{1, 2, \ldots\}$. It follows that

$$\begin{aligned}
\mu\left(C_{p,r}\right) &\leq \mu\left(B_{p+1}\right) \cdots \mu\left(B_{p+r}\right) \\
&= \left[\prod_{i=1}^{r} \mu^{-}\left(B_{p+i}\right)\right]\left[\prod_{i=1}^{r} \mu^{+}\left(B_{p+i}\right)\right] \\
&\leq M\left[\prod_{i=1}^{r} \mu^{-}\left(B_{p+i}\right)\right] \\
&\leq M.
\end{aligned} \tag{9}$$

□

The above proposition allows us to prove a stronger result under an additional conditions. Note that in the theory of infinite products of nonnegative numbers it is customary to speak of *divergence* to 0, see e.g. [6, p. 93].

Theorem 3.2 *Let $A_1, A_2, \ldots$ be a sequence of matrices in $\mathbb{F}^{nn}$. Let μ be a matrix norm on $\mathbb{F}^{nn}$. Suppose that the sequence $A_1, A_2, \ldots$ satisfies Conditions* **(C)** *and* **(D)** *for the norm μ. Then all general products from $A_1, A_2, \ldots$ converge to* 0.

<u>Proof</u>: Let $B_1, B_2, \ldots$ be a permutation of $A_1, A_2, \ldots$ and let $C_{p,r}$ be a product of $B_{p+1}, \ldots, B_{p+r}$ in some order. As in the proof of Proposition(3.1), we have that

$$\mu\left(C_{p,r}\right) \leq M\left[\prod_{i=1}^{r} \mu^{-}\left(B_{p+i}\right)\right]. \tag{10}$$

By Condition **(D)**, the sum $\sum_{i=1}^{\infty}(1-\mu^{-}(A_i))$ diverges and so, by [6, Theorem 14], $\sum_{i=1}^{\infty}(1-\mu^{-}(B_i))$ diverges. Thus $\sum_{i=1}^{\infty}(1-\mu^{-}(B_{p+i}))$ also diverges.

We again apply [6, Theorem 51], and we obtain that $\prod_{i=1}^{\infty} \mu^{-}(B_{p+i})$ diverges. But since $\mu^{-}(B_i) \leq 1$, the last product must diverge to 0 and the proof is done. □

If H is a subspace of $\mathbb{F}^{n}$ which is invariant under $A \in \mathbb{F}^{nn}$, we denote the restriction of A to H by $A_{|_H}$. As an immediate application of Theorem 3.2 we obtain the following result:

Corollary 3.3 *Let $A_1, A_2, \ldots$ be a sequence of matrices and let $C_{p,1}, C_{p,2}, \ldots$ be sequence of general products from the $A_1, A_2, \ldots$. Let H be a subspace of $\mathbb{F}^n$ which is invariant under each A_i, $i = 1, \ldots$. Let ν be a norm on $\mathbb{F}^n$. Suppose that the sequence $((A_i)_{|H})$ satisfies conditions* **(C)** *and* **(D)** *for the norm ν_H^0. If $x \in H$, then*

$$\lim_{r\to\infty} C_{p,r}x \;=\; 0.$$

Proof: Immediate by Theorem 3.2. $\square$

Since $\nu_H^0(A) \le \nu^0(A)$ when H is an invariant subspace of A, it follows easily that under the conditions of Corollary(3.3), Condition **(C)**(resp. Condition **(D)**) on the sequence of (A_i) implies Condition **(C)**(resp. Condition **(D)**) on the sequence of $((A_i)_{|H})$.

4 Applications to Stochastic Matrices

In this section we apply the foregoing results to stochastic matrices. In order to be consistent with our previous section we consider *column* stochastic matrices.Thus "stochastic matrix" will mean "column stochastic matrix".

Let $e = (1, \ldots, 1)^T \in \mathbb{R}^n$. Hence in this section we shall assume that

$$H \;=\; \{x \in \mathbb{R}^n \;:\; e^T x = 0\}. \tag{11}$$

If A is a stochastic matrix in $\mathbb{R}^{n,n}$, then H is invariant under A, but note that H need not be a complement to the space of fixed points of A. If ν is a norm on $\mathbb{R}^n$ and H is given by (11), then the corresponding partial norm ν_H^0 on $\mathbb{R}^{nn}$ will be called a *coefficient of ergodicity* as is usual in the literature on Markov chains.

The ℓ_1 norm on $\mathbb{R}^n$ plays a special role in this theory and we shall denote it henceforth by ω. The corresponding coefficient of ergodicity was apparently first computed in Bauer, Deutsch, and Stoer [1], see also [17], and equals

$$\omega_H^0(A) \;=\; (1/2) \max_{i,k\in\{1,\ldots,n\}} \sum_{j=1}^{n} |a_{i,j} - a_{k,j}|.$$

It is known that $\omega_H^0(A) \le 1$ for all stochastic matrices A and ω_H^0 is the only coefficient of ergodicity that satisfies this inequality, [7] and [9], but see also

[14].

Remark 4.1 We comment that the only stochastic matrix A which is para-contracting with respect to the ℓ_1 norm ω is the identity matrix. This follows easily from the facts that for any nonnegative vector u with $\omega(u) = 1$, we also have that $\omega(Au) = 1$. On the other hand, every stochastic matrix that has a row with two or more positive elements is H–contracting for the subspace H given in (11) and the norm ω.

Definition 4.2 Let $P_1, P_2, \ldots$ be a sequence of $n \times n$ stochastic matrices. We shall say that all general products formed from this sequence are *weakly ergodic* if for all general products $B_{p,1}, B_{p,2}, \ldots$, we have that

$$lim_{r\to\infty} B_{p,r} x \;=\; 0, \quad \text{for all} \;\; x \in H. \tag{12}$$

.

Since every $x \in H$ can be written $x = c(u - v)$, where $u, v \in \mathbb{R}^n$ are nonnegative and $e^T u = e^T v = 1$ and $c \in \mathbb{R}$, it is easily seen that, for each product considered, our definition is equivalent to that in [3], [11], or [16, Defn. 3.3].

By Theorem 3.2 we now immediately obtain:

Theorem 4.3 *Let ν be a norm on $\mathbb{R}^n$ and let ν_H^0 be the corresponding coefficient of ergodicity. Let $P_1, P_2, \ldots$ be a sequence of $n \times n$ stochastic matrices. Then all general products formed from this sequence are weakly ergodic if*

$$\sum_{i=1}^{\infty} ({\nu_H^0}^+(P_i) - 1) \text{ converges} \tag{13}$$

and

$$\sum_{i=1}^{\infty} (1 - {\nu_H^0}^-(P_i)) \text{ diverges.} \tag{14}$$

Note that (13) is automatically satisfied if $\nu = \omega$, the ℓ_1-norm. This special case of Theorem 4.3 is observed in [16, Exercise 4.36]. Results related to this case (and which therefore involve only (14) explicitly) are to be found in [15, Theorem 1] and in [11]. The theorem in the latter paper is there illustrated by an example of a sequence of stochastic matrices that satisfies (14) for the norm ω, see [11, p. 333].

The following corollary is due to Rhodius [14, Thm. 3, Part I] in the case of $\nu = \omega$, see Leizarowitz [8, Thm A (i)] for a related result.

Corollary 4.4 *Let $P_1, P_2, \ldots$ be a sequence of stochastic matrices. If $\nu_H^0(P_i) \leq 1$ for all i, $i = 1, 2, \ldots$, and there exists a point of accumulation c of the sequence $\nu_H^0(P_1), \nu_H^0(P_2), \ldots$ such that $c < 1$, then all general products of the sequence are weakly ergodic.*

Proof: Clearly condition (13) holds and there is an infinite subsequence $j_1, j_2, \ldots$ such that $\omega_H^0(P_{i_j}) < (1+c)/2 < 1$, $j = 1, 2, \ldots$. Then condition (14) holds for this subsequence. The result follows from Theorem (4.3). □

Another corollary of Theorem 4.3 is [8, Thm A (ii)]:

Corollary 4.5 *Let $P_1, P_2, \ldots$ be a sequence of stochastic matrices and let ν be a norm on $\mathbb{R}^n$. If all points of accumulation c of the sequence $\nu_H^0(P_1)$, $\nu_H^0(P_2), \ldots$ satisfy $c < 1$, then all general products of the sequence are weakly ergodic.*

Proof: Since the set of accumulation points of a bounded sequence is compact, there exists $d < 1$ such that only a finite number of terms of the sequence $\nu_H^0(P_1), \nu_H^0(P_2), \ldots$ exceed d. Hence (13) and (14) hold for the sequence of ergodicity coefficients and the corollary follows from Theorem 4.3. □

5 Bounds for eigenvalues

Theorem 5.1 *Let H be an invariant subspace for $A \in \mathbb{C}^{n,n}$. Let $\alpha_\lambda(A)$ and $\gamma_\lambda(A)$ be the algebraic and geometric multiplicities of $\lambda \in \mathbb{C}^{nn}$ as an eigenvalue of A, respectively. Then*

$$\alpha_\lambda(A) \;\leq\; \alpha_\lambda(A_{|_H}) + n - \dim(H)$$

and

$$\gamma_\lambda(A) \;\leq\; \gamma_\lambda(A_{|_H}) + n - \dim(H).$$

Further, if μ is a matrix norm and $|\lambda| > \mu_H^0(A)$, then $\alpha_\lambda(A) \leq n - \dim(H)$ and $\gamma_\lambda(A) \leq n - \dim(H)$.

Proof: By a proper choice of basis for $\mathbb{C}^n$ we may put A in the form

$$A \;=\; \begin{pmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{pmatrix} \tag{15}$$

and the first inequality follows immediately.

The second inequality is an immediate consequence of [10, Theorem 2.1], see also [5, Cor. 5.8].

Finally, since all eigenvalues λ of A_{22} satisfy $|\lambda| \leq \mu_H^0(A)$, the last part of the theorem follows. $\square$

Remark 5.2 Theorem (5.1) implies several known results. We now give an example below which could also be derived from [4, Theorem 2]. Let A be a stochastic matrix, let H be defined by (11), and suppose that a coefficient of ergodicity satisfies $\nu_H^0(A) < 1$. Since, in this case, the matrix $A_{1,1}$ in (15) is 1×1, it follows that 1 is an algebraically simple eigenvalue of A and all other eigenvalues λ of A satisfy $\nu_H^0(A) \geq |\lambda|$.

Acknowledgment

We would like to thank Wenchao Huang for some helpful remarks. We thank Olga Holtz for her careful reading of the manuscript.

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