ENUMERATING REPRESENTATIONS IN FINITE WREATH PRODUCTS

THOMAS MÜLLER†

Abstract. Let Γ be a group (finite or infinite), H a finite group, and let R_n denote the sequence $H \wr S_n$ of symmetric wreath products as well as certain variants of it (including in particular $H \wr A_n$ and W_n, the Weyl group of type D_n) . We compute the exponential generating function for the number $|\mathrm{Hom}(\Gamma, R_n)|$ of Γ-representations in R_n and for some refinements of this sequence under very mild finiteness assumptions on Γ (always met for instance if Γ is finitely generated). This generalizes in a uniform way the connection between the problem of counting finite index subgroups in a group Γ and the enumeration of Γ-actions on finite sets on the one hand, and the recent results of Chigira concerning solutions of the equation $x^m = 1$ in the groups $H \wr S_n$, $H \wr A_n$, and W_n on the other. We also study the asymptotics of the function $|\mathrm{Hom}(G, H \wr S_n)|$ for arbitrary finite groups G and H.

Introduction

For a group Γ denote by $s_\Gamma(n)$ the number of subgroups of index n in Γ. If Γ is finitely generated or of finite (subgroup) rank then $s_\Gamma(n)$ is finite for all n. There is a well-known connection between the problem of counting finite index subgroups in a group Γ and the enumeration of Γ-actions on finite sets (i.e. permutation representions of Γ), manifesting itself in a variety of results scattered throughout the literature. One of the earliest results in this direction is M. Hall's recursion formula

$$s_{F_r}(n) = n(n!)^{r-1} - \sum_{k=1}^{n-1} \big((n-k)!\big)^{r-1} s_{F_r}(k)$$

for the number of index n subgroups in the free group of (finite) rank r; cf. [8, Theorem 5.2]. Dey's corresponding formula [4, Theorem 6.10] for (non-trivial) free products is another relevant example. Denote by $\beta_m(B)$ the number of solutions of the equation $x^m = 1$ in a finite group B. Chowla, Herstein and Scott [3] obtain the exponential generating function of the sequence $\{\beta_m(S_n)\}_{n=0}^\infty$ as

$$\sum_{n=0}^{\infty} \frac{\beta_m(S_n)}{n!} z^n = \exp\left(\sum_{d|m} \frac{z^d}{d}\right).$$

The more general relation

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, S_n)|}{n!} z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d} z^d\right), \quad m = |G|$$

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for an arbitrary finite group G, which exhibits the exponential generating function of the sequence $\{|\mathrm{Hom}(G, S_n)|\}_0^\infty$ as a particular type of entire function, was a starting point in [14] for the asymptotic enumeration of finite group actions; cf. [14, Theorem 5]. The most general expression of this correspondence to be found in the literature appears to be [7, Prop. 1], which apart from incorporating the results mentioned, can also be used for example to count torsion-free subgroups of finite index. This latter aspect has served as a starting point for a detailed analysis of the growth behaviour and the asymptotics of the number of free subgroups of finite index in a finitely generated virtually free group; cf. [12] and [13].

The subject gained an interesting new perspective with the appearance of Chigira's paper [1]. In this paper Chigira studies the equation $x^m = 1$ in finite wreath products of the form $H \wr S_n$ and $H \wr A_n$, and in the Weyl groups W_n of type D_n, obtaining generating functions for the sequences $\{\beta_m(H \wr S_n)\}_{n=0}^\infty$, $\{\beta_m(H \wr A_n)\}_{n=0}^\infty$, and $\{\beta_m(W_n)\}_{n=0}^\infty$; cf. [1, Theorems 2, 4, and 5]. His result for the representation sequence $\{H \wr S_n\}$ for instance is that

$$\sum_{n=0}^{\infty} \frac{\beta_m(H \wr S_n)}{n!} z^n = \exp\left(\sum_{d|m} \frac{|H|^{d-1}\beta_{m/d}(H)}{d} z^d\right).$$

Furthermore, Chigira indicates how to modify Wilf's argument in [20] leading to an asymptotic formula for $\beta_m(S_n)$ with m fixed and $n \to \infty$ to obtain a similar asymptotic formula for $\beta_m(H \wr S_n)$; cf. [1, Theorem 3].
The purpose of the present paper is threefold: (i) to generalize Chigira's results [1, Theorems 2, 4, 5] from the consideration of cyclic groups to the enumeration of wreath product representations of (almost) arbitrary groups Γ, (ii) to replace Chigira's case-by-case study of the sequences $\{H \wr S_n\}$, $\{H \wr A_n\}$, and $\{W_n\}$ by an approach working uniformly for these and certain other representation sequences, and (iii) to study in detail the asymptotics of the function $|\mathrm{Hom}(G, H \wr S_n)|$ for arbitrary finite groups G and H.
Apart from their intrinsic interest, our results also provide a key ingredient for a substantial improvement over the results obtained in [15] concerning the subgroup growth of virtually free groups. This development ultimately leads to a solution of Poincaré's problem concerning the asymptotic distribution of the isomorphism types of finite index subgroups in the modular group (and other virtually free groups). These aspects will be explained in forthcoming work; cf. [16] and [17].

Let $\{\Pi_n\}$ denote either the sequence $\{A_n\}$ of alternating groups or the sequence $\{S_n\}$ of symmetric groups. Moreover, given $\{\Pi_n\}$, a finite group H, and a normal subgroup $N \trianglelefteq H$ with H/N abelian put

$$(H \wr \Pi_n)_N := \Big\{(f, \pi) \in H \wr \Pi_n : \prod_i f(i) \in N\Big\}, \quad n \geq 0.$$

Our main counting result (Theorem 1) computes the exponential generating function of the sequence $\{|\mathrm{Hom}(\Gamma, (H \wr \Pi_n)_N)|\}_{n=0}^\infty$ and of certain refinements of it whenever $(H : N) \leq 2$ and Γ satisfies some very mild finiteness assumptions (always met for instance if Γ is finitely generated).

The proof of Theorem 1 occupies the first two sections of this paper. In Section 1, given (among other things) a group Γ, a finite group H, and a finite set Ω, we study a certain complex $\mathcal{R}(\Omega)$ of representations $\Gamma \to H \wr S(\Omega)$ inducing a transitive Γ-action on Ω and satisfying some further (more technical) conditions. These extra conditions enable us in particular to controll the process of glueing together such transitive representations to obtain a representation of Γ in $(H \wr \Pi_n)_N$. Our principal tool is Proposition 1 which calculates the cardinalities of the sets $\mathcal{R}(\Omega)$ whenever they turn out to be finite. Building on this result the proof of Theorem 1 is then completed in Section 2. In Section 3 we consider the special cases of Theorem 1 related to symmetric and alternating wreath products, and the Weyl groups of type D_n, illustrating each case with some specific examples. In particular, we recover the enumerative results mentioned earlier in the introduction. Finally, in Section 4 we combine one of our present results (formula (23)) with some machinery developed in [14] to obtain a complete asymptotic expansion of the function $|\mathrm{Hom}(G, H \wr S_n)|$ for arbitrary finite groups G and H; cf. Theorem 4. We also formulate there a conjecture, relating the asymptotic behaviour of the functions $|\mathrm{Hom}(G, W_n)|$ and $|\mathrm{Hom}(G, H \wr A_n)|$ to that of the functions $|\mathrm{Hom}(G, C_2 \wr S_n)|$ respectively $|\mathrm{Hom}(G, H \wr S_n)|$. Our final result (Proposition 3) verifies this conjecture in some special cases.

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1. Counting transitive representations

Denote by $S(\Omega)$ the symmetric group on the set Ω and put $S([n]) = S_n$ for the standard set $[n] := \{1, \ldots, n\}$. Throughout this paper we choose algebraic multiplication in $S(\Omega)$, i.e., the product $\pi_1 \cdot \pi_2$ of permutations $\pi_1, \pi_2 \in S(\Omega)$ is defined via

$$(\pi_1 \cdot \pi_2)(\omega) := \pi_2(\pi_1(\omega)), \quad \omega \in \Omega.$$

Consequently, group actions on sets will always be right actions, and, for a finite group H and a permutation group $\Pi(\Omega)$ on the finite set Ω, multiplication in the wreath product

$$H \wr \Pi(\Omega) = \big\{(f, \pi) : \ f : \Omega \to H,\ \pi \in \Pi(\Omega)\big\}$$

is given by the formulae

$$\begin{gathered}(f_1, \pi_1) \cdot (f_2, \pi_2) := (f, \pi_1 \cdot \pi_2)\\ f(\omega) := f_1(\omega)\, f_2(\pi_1(\omega)), \quad \omega \in \Omega.\end{gathered}$$

The canonical projection of $H \wr \Pi(\Omega)$ onto $\Pi(\Omega)$ will be denoted by ϵ.

1.1. **The sets $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$.** Our set-up for defining and investigating the sets $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$ consists of the following:

- a group Γ
- a finite group H
- a normal subset Σ of $\Gamma - \{1\}$
- a normal subgroup N of H with H/N abelian
- a normal functor $\Pi(-) : \mathrm{Ens} \to \mathrm{Groups}$.

By definition, a normal functor $\Pi(-) : \mathrm{Ens} \to \mathrm{Groups}$ is a covariant functor from the category Ens consisting of finite sets and bijections to the category Groups of groups and homomorphisms satisfying

$$\Pi(\Omega) \trianglelefteq S(\Omega)$$
$$\Pi(\alpha)(\pi) = \iota_\alpha(\pi)$$

for all finite sets Ω, Ω_1, Ω_2, every bijection $\alpha : \Omega_1 \to \Omega_2$, and each element $\pi \in \Pi(\Omega_1)$. Here $\iota_\alpha : S(\Omega_1) \to S(\Omega_2)$ is the isomorphism given by $\iota_\alpha(\pi) = \alpha\pi\alpha^{-1}$. Such functors correspond in a one-to-one fashion to sequences $\{\Pi_n\}_0^\infty$, where Π_n is a normal subgroup of S_n for each $n \geq 0$.

The set $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$ and the group $U(\Omega, \omega_0)$

For a finite set Ω denote by $(H \wr \Pi(\Omega))_N$ the kernel of the homomorphism $H \wr S(\Omega) \to (H/N) \times (S(\Omega)/\Pi(\Omega))$ mapping (f, π) to $(\prod_\omega f(\omega) N, \pi\Pi(\Omega))$. Given a non-empty finite set Ω and two normal subgroups $\Gamma_\Pi, \Gamma_N \trianglelefteq \Gamma$ we define $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$ to be the set of all representations $\tau : \Gamma \to H \wr S(\Omega)$ such that

(i) $\epsilon\tau$ is transitive,
(ii) $\epsilon\tau$ is Σ-free (i.e., $\epsilon\tau(\sigma)$ is a fixed-point-free permutation for each $\sigma \in \Sigma$),
(iii) $(\epsilon\tau)^{-1}(\Pi(\Omega)) = \Gamma_\Pi$,
(iv) $\tau^{-1}((H \wr S(\Omega))_N) = \Gamma_N$.

Also, given a base point $\omega_0 \in \Omega$, let $U(\Omega, \omega_0)$ denote the subgroup of $H \wr S(\Omega)$ consisting of those elements (f, π) such that $f(\omega_0) = 1$ and $\pi(\omega_0) = \omega_0$. Clearly, $U(\Omega, \omega_0) \cong H \wr S(\Omega - \{\omega_0\})$ and hence

$$|U(\Omega, \omega_0)| = |H|^{|\Omega|-1}(|\Omega| - 1)!.$$

The action of $H \wr S(\Omega)$ by conjugation on the set $\mathrm{Hom}(\Gamma, H \wr S(\Omega))$, i.e.,

$$(\tau \cdot x)(\gamma) := x^{-1}\tau(\gamma)x$$

for $\tau \in \mathrm{Hom}(\Gamma, H \wr S(\Omega))$, $x \in H \wr S(\Omega)$, and $\gamma \in \Gamma$, restricts to an action of $H \wr S(\Omega)$ (and hence of $U(\Omega, \omega_0)$) on the complex $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$. We claim that, as a consequence of transitivity, this action of $U(\Omega, \omega_0)$ on $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$ is in fact *free*. Indeed, suppose that $\tau \cdot x = \tau$ for some $x = (f_x, \pi_x) \in U(\Omega, \omega_0)$ and some $\tau \in \mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$. By definition this means that

$$[x, \tau(\gamma)] = 1, \quad \gamma \in \Gamma. \tag{1}$$

Letting $\tau(\gamma) = (f_\gamma, \pi_\gamma)$ we infer in particular from (1) that

$$f_\gamma(\omega) f_x(\pi_\gamma(\omega)) = f_x(\omega) f_\gamma(\pi_x(\omega)) \quad (\gamma \in \Gamma,\ \ \omega \in \Omega). \tag{2}$$

Since $\epsilon\tau$ is transitive we can find elements $\gamma_\omega \in \Gamma$ for $\omega \in \Omega$ such that

$$\epsilon\tau\,(\gamma_\omega)\,(\omega_0) = \omega, \quad \omega \in \Omega.$$

Specializing (2) to $\omega = \omega_0$ and $\gamma = \gamma_{\omega'}$ for some $\omega' \in \Omega$ gives $f_x(\omega') = 1$, since $f_x(\omega_0) = 1$, $\pi_x(\omega_0) = \omega_0$, and $\pi_{\gamma_{\omega'}} = \omega'$. Moreover, using (1) again, we have for $\omega \in \Omega$

$$\begin{aligned}
\pi_x(\omega) &= \epsilon(x)(\epsilon\tau(\gamma_\omega)(\omega_0))\\
&= \epsilon\tau(\gamma_\omega)(\epsilon(x)(\omega_0))\\
&= \epsilon\tau(\gamma_\omega)(\omega_0)\\
&= \omega,
\end{aligned}$$

i.e., $\pi_x = \mathrm{id}$, and hence $x = 1$ as claimed.

The map $\Psi_{\Gamma_\Pi}^{\Gamma_N}(\Omega, \omega_0)$

Let $\tau : \Gamma \to H \wr S(\Omega)$ be an element of $\mathcal{R}_{\Gamma_\Pi}^{\Gamma_N}(\Omega)$. Then the stabilizer $\Gamma' = \mathrm{stab}_{\epsilon\tau}(\omega_0)$ is a subgroup of index $|\Omega|$ in Γ which avoids Σ, $\Gamma' \cap \Sigma = \emptyset$, and satisfies $\tau_{\Gamma'}^{-1}(\Pi(\Gamma'\backslash\Gamma)) = \Gamma_\Pi$, where $\tau_{\Gamma'} : \Gamma \to S(\Gamma'\backslash\Gamma)$ describes the action of Γ by right multiplication on the set $\Gamma'\backslash\Gamma$ of right cosets of Γ modulo Γ'. For $\gamma \in \Gamma$ put $\tau(\gamma) = (f_\gamma, \pi_\gamma)$ and define a map $\chi_\tau : \Gamma' \to H$ via $\chi_\tau(\gamma') := f_{\gamma'}(\omega_0)$. One checks that χ_τ is in fact a homomorphism and that $\chi_{\tau_1} = \chi_{\tau_2}$ if τ_1 and τ_2 are equivalent under the action of $U(\Omega, \omega_0)$. Choose elements $\gamma_\omega \in \Gamma$ such that

$$\epsilon\tau(\gamma_\omega)(\omega_0) = \omega, \quad \omega \in \Omega,$$

and for $\gamma \in \Gamma$ and $\omega \in \Omega$ define $\gamma' \in \Gamma'$ and $\omega' \in \Omega$ by the equation $\gamma_\omega\gamma = \gamma'\gamma_{\omega'}$. We have, for $\gamma \in \Gamma$ and $\omega \in \Omega$,

$$\begin{aligned}
f_{\gamma_\omega}(\omega_0) f_\gamma(\omega) &= f_{\gamma_\omega\gamma}(\omega_0)\\
&= f_{\gamma'\gamma_{\omega'}}(\omega_0)\\
&= \chi_\tau(\gamma') f_{\gamma_{\omega'}}(\omega_0).
\end{aligned}$$

Note that if in the equation $\gamma_\omega\gamma = \gamma'\gamma_{\omega'}$ the element γ is fixed and ω runs through Ω then ω' will attain each value in Ω exactly once. It follows that modulo N for $\gamma \in \Gamma$

$$\begin{aligned}
\prod_\omega f_\gamma(\omega) &= \prod_\omega \left((f_{\gamma_\omega}(\omega_0))^{-1} \chi_\tau(\gamma') f_{\gamma_{\omega'}}(\omega_0) \right)\\
&\equiv \Big(\prod_\omega f_{\gamma_\omega}(\omega_0) \Big)^{-1} \prod_\omega f_{\gamma_{\omega'}}(\omega_0) \prod_\omega \chi_\tau(\gamma')\\
&\equiv \prod_\omega \chi_\tau(\gamma'),
\end{aligned}$$

i.e., we have

$$\prod_\omega f_\gamma(\omega) \equiv \prod_\omega \chi_\tau(\gamma_\omega\gamma\gamma_{\omega'}^{-1}) \mod N, \quad \gamma \in \Gamma. \tag{3}$$

Given a subgroup Γ' of finite index in Γ, a homomorphism $\chi : \Gamma' \to H$, and a right transversal $\{\gamma_1, \ldots, \gamma_{(\Gamma:\Gamma')}\}$ for Γ' in Γ we define

$$\Gamma_\chi := \Big\{\gamma \in \Gamma : \quad \prod_j \chi(\gamma_j \gamma \overline{\gamma_j \gamma}^{-1}) \in N\Big\},$$

where $^- : \Gamma \to \{\gamma_1, \ldots, \gamma_{(\Gamma:\Gamma')}\}$ associates with every element γ the representative $\overline{\gamma}$ of the coset $\Gamma'\gamma$. Consider the map $\psi : \Gamma \to H/N$ sending γ to $\prod_j \chi(\gamma_j\gamma\overline{\gamma_j\gamma}^{-1})N$. Since H/N is abelian the composition of χ with the canonical projection $H \to H/N$ factors through $\Gamma'/[\Gamma', \Gamma']$ inducing a homomorphism $\chi' : \Gamma'/[\Gamma', \Gamma'] \to H/N$, and we have $\psi = \chi' \circ V_{\Gamma\to\Gamma'}$ where $V_{\Gamma\to\Gamma'} : \Gamma \to \Gamma'/[\Gamma', \Gamma']$ is the transfer of Γ into Γ'. Since the transfer is a homomorphism and independent of the choice of representatives the same is true for ψ; consequently, $\Gamma_\chi = \ker(\psi)$ is a normal subgroup of index at most $(H : N)$ in Γ and is independent of the system of representatives used in its definition. If $\Gamma' = \Gamma$ then $\Gamma_\chi = \chi^{-1}(N)$, while for an abelian group Γ and every homomorphism $\chi : \Gamma' \to H$ with $(\Gamma : \Gamma') < \infty$

$$\Gamma_\chi = \big\{\gamma \in \Gamma : \chi(\gamma^{(\Gamma:\Gamma')}) \in N\big\}.$$

Combining (3) with the defining property (iv) of $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$ it follows that the homomorphism χ_τ constructed to $\tau \in \mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$ satisfies $\Gamma_{\chi_\tau} = \Gamma_N$. Hence, the assignment $\tau \mapsto \chi_\tau$ induces a map

$$\Psi^{\Gamma_N}_{\Gamma_\Pi}(\Omega, \omega_0) : \mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)/U(\Omega, \omega_0) \longrightarrow \bigcup_{\Gamma' \in \mathcal{S}(|\Omega|, \Gamma_\Pi)} \big\{\chi \in \mathrm{Hom}(\Gamma', H) : \Gamma_\chi = \Gamma_N\big\},$$

where for a positive integer n and a normal subgroup $\Gamma_\Pi \trianglelefteq \Gamma$ we put

$$\mathcal{S}(n, \Gamma_\Pi) := \big\{\Gamma' : \quad \Gamma' \leq \Gamma,\ (\Gamma : \Gamma') = n,\ \Gamma' \cap \Sigma = \emptyset,\ \tau^{-1}_{\Gamma'}(\Pi(\Gamma'\backslash\Gamma)) = \Gamma_\Pi\big\}.$$

Given n and two normal subgroups $\Gamma_\Pi, \Gamma_N \trianglelefteq \Gamma$ define $s(n, \Gamma_\Pi, \Gamma_N)$ to be the cardinal number

$$s(n, \Gamma_\Pi, \Gamma_N) := \sum_{\Gamma' \in \mathcal{S}(n, \Gamma_\Pi)} \big|\big\{\chi \in \mathrm{Hom}(\Gamma', H) : \quad \Gamma_\chi = \Gamma_N\big\}\big|. \tag{4}$$

The key observation in our context, to be proved in the next subsection, is that the map $\Psi^{\Gamma_N}_{\Gamma_\Pi}(\Omega, \omega_0)$ introduced above is in fact a bijection. Presupposing this result for the moment we can summarize our discussion of the sets $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$ as follows.

Proposition 1. *Suppose that for some choice of data $\Gamma, H, \Sigma, N, \Pi(-), \Omega, \Gamma_\Pi$, and Γ_N as described the set $\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)$ is finite. Then the cardinal number $s(|\Omega|, \Gamma_\Pi, \Gamma_N)$ is finite and we have*

$$|\mathcal{R}^{\Gamma_N}_{\Gamma_\Pi}(\Omega)| = |H|^{|\Omega|-1}(|\Omega| - 1)!\ s(|\Omega|, \Gamma_\Pi, \Gamma_N). \tag{5}$$

1.2. **Bijectivity of the map $\Psi^{\Gamma_N}_{\Gamma_\Pi}(\Omega, \omega_0)$.** The purpose of this subsection is to complete the proof of Proposition 1 just stated by establishing the following.

Lemma. *Given $\Gamma, H, \Sigma, N, \Pi(-), \Omega, \Gamma_\Pi$, and Γ_N as described the map $\Psi^{\Gamma_N}_{\Gamma_\Pi}(\Omega, \omega_0)$ associated with these data is a bijection.*

Proof. (a) *Injectivity.* Let $\tau_1, \tau_2 \in \mathcal{R}_{\Gamma_\Pi}^{\Gamma_N}(\Omega)$ and suppose that $\chi_{\tau_1} = \chi_{\tau_2}$, i.e.,

(i) $\operatorname{stab}_{\epsilon\tau_1}(\omega_0) = \Gamma' = \operatorname{stab}_{\epsilon\tau_2}(\omega_0)$
(ii) $f_\gamma^{(1)}(\omega_0) = f_\gamma^{(2)}(\omega_0) \quad (\gamma \in \Gamma')$,

where for $j = 1, 2$ and $\gamma \in \Gamma$ we put $\tau_j(\gamma) = (f_\gamma^{(j)}, \pi_\gamma^{(j)})$. We have to show that τ_1 and τ_2 are equivalent under the action of $U(\Omega, \omega_0)$. Consider the map $\Delta : \Gamma \to H$ given by $\Delta(\gamma) := (f_\gamma^{(1)}(\omega_0))^{-1} f_\gamma^{(2)}(\omega_0)$. By (ii) we have

$$\Delta(\gamma) = 1, \quad \gamma \in \Gamma'. \tag{6}$$

Also, an immediate calculation shows that

$$\Delta(\gamma_1\gamma_2) = \left(f_{\gamma_2}^{(1)}(\pi_{\gamma_1}^{(1)}(\omega_0))\right)^{-1} \Delta(\gamma_1)\, f_{\gamma_2}^{(2)}(\pi_{\gamma_1}^{(2)}(\omega_0)) \quad (\gamma_1, \gamma_2 \in \Gamma). \tag{7}$$

It follows now from (6), (7), and (i) that Δ is constant on the right cosets of Γ modulo Γ', i.e.,

$$\Delta(\gamma'\gamma) = \Delta(\gamma) \quad (\gamma \in \Gamma, \gamma' \in \Gamma'). \tag{8}$$

Choose elements $\gamma_\omega \in \Gamma$ for $\omega \in \Omega$ such that

$$\epsilon\tau_1(\gamma_\omega)(\omega_0) = \omega, \quad \omega \in \Omega,$$

and define a map $f_x : \Omega \to H$ via $f_x(\omega) := \Delta(\gamma_\omega)$; in particular $f_x(\omega_0) = 1$ by (6). Moreover, we define a permutation $\pi_x \in S(\Omega)$ with $\pi_x(\omega_0) = \omega_0$ by setting $\pi_x(\omega) := \pi_{\gamma_\omega}^{(2)}(\omega_0)$. We claim that the element $x := (f_x, \pi_x) \in U(\Omega, \omega_0)$ satisfies $\tau_1 \cdot x = \tau_2$. Proving this is equivalent to verifying the equations

$$\pi_x(\pi_\gamma^{(1)}(\omega)) = \pi_\gamma^{(2)}(\pi_x(\omega)) \tag{9}$$

$$f_\gamma^{(1)}(\omega)\, f_x(\pi_\gamma^{(1)}(\omega)) = f_x(\omega)\, f_\gamma^{(2)}(\pi_x(\omega)) \tag{10}$$

for $\omega \in \Omega$ and all $\gamma \in \Gamma$. This is done in two steps. We first observe that (9) and (10) hold in fact for $\omega = \omega_0$ and all $\gamma \in \Gamma$. This is clear for equation (9) in view of the definition of π_x and assumption (i). For equation (10) this follows from the definitions of f_x and Δ, and the fact, expressed in (8), that Δ is a class function:

$$\begin{aligned} f_\gamma^{(1)}(\omega_0)\, f_x(\pi_\gamma^{(1)}(\omega_0)) &= f_\gamma^{(1)}(\omega_0)\, \Delta(\gamma_\omega) \\ &= f_\gamma^{(1)}(\omega_0)\, \Delta(\gamma) \\ &= f_\gamma^{(1)}(\omega_0)\, (f_\gamma^{(1)}(\omega_0))^{-1}\, f_\gamma^{(2)}(\omega_0) \\ &= f_\gamma^{(2)}(\omega_0), \end{aligned}$$

where $\Gamma'\gamma = \Gamma'\gamma_\omega$. Using this we can now complete the proof of (9) and (10). Concerning (9) we have

$$\begin{aligned} \pi_x(\pi_\gamma^{(1)}(\omega)) &= \pi_x(\pi_\gamma^{(1)}(\pi_{\gamma_\omega}^{(1)}(\omega_0))) \\ &= \pi_x(\pi_{\gamma_\omega\gamma}^{(1)}(\omega_0)) \\ &= \pi_{\gamma_\omega\gamma}^{(2)}(\omega_0) \\ &= \pi_\gamma^{(2)}(\pi_{\gamma_\omega}^{(2)}(\omega_0)) \\ &= \pi_\gamma^{(2)}(\pi_x(\omega)). \end{aligned}$$

Similarly, using (7) and the special case of (10) already proved, we have

$$\begin{aligned} f_\gamma^{(1)}(\omega)\, f_x(\pi_\gamma^{(1)}(\omega)) &= f_\gamma^{(1)}(\omega)\, f_x(\pi_\gamma^{(1)}(\pi_{\gamma_\omega}^{(1)}(\omega_0))) \\ &= f_\gamma^{(1)}(\omega)\, f_x(\pi_{\gamma_\omega\gamma}^{(1)}(\omega_0)) \\ &= f_\gamma^{(1)}(\omega)\, \Delta(\gamma_\omega\gamma) \\ &= f_\gamma^{(1)}(\omega)\, \big(f_\gamma^{(1)}(\pi_{\gamma_\omega}^{(1)}(\omega_0))\big)^{-1}\, \Delta(\gamma_\omega)\, f_\gamma^{(2)}(\pi_{\gamma_\omega}^{(2)}(\omega_0)) \\ &= f_x(\omega)\, f_\gamma^{(2)}(\pi_x(\omega)), \end{aligned}$$

and the injectivity of $\Psi_{\Gamma_\Pi}^{\Gamma_N}(\Omega, \omega_0)$ is established.

(b) *Surjectivity.* Given a subgroup $\Gamma' \in \mathcal{S}(|\Omega|, \Gamma_\Pi)$ and a homomorphism $\chi : \Gamma' \to H$ satisfying $\Gamma_\chi = \Gamma_N$ we have to exhibit a representation $\tau \in \mathcal{R}_{\Gamma_\Pi}^{\Gamma_N}(\Omega)$ with $\chi_\tau = \chi$. This is equivalent to finding a transitive and Σ-free permutation representation $\tau_1 : \Gamma \to S(\Omega)$ satisfying $\tau_1^{-1}(\Pi(\Omega)) = \Gamma_\Pi$ and $\mathrm{stab}_{\tau_1}(\omega_0) = \Gamma'$ and a map $f : \Gamma \times \Omega \to H$ such that

$$f(\gamma_1\gamma_2, \omega) = f(\gamma_1, \omega)\, f(\gamma_2, \tau_1(\gamma_1)(\omega)) \quad (\gamma_1, \gamma_2 \in \Gamma,\ \omega \in \Omega) \tag{11}$$

$$f(\gamma, \omega_0) = \chi(\gamma) \quad (\gamma \in \Gamma') \tag{12}$$

$$\prod_\omega f(\gamma, \omega) \in N \Leftrightarrow \gamma \in \Gamma_N \quad (\gamma \in \Gamma). \tag{13}$$

To define τ_1 we look at the action of Γ by right multiplication on the set $\Gamma'\backslash\Gamma$ of right cosets and identify the set $\Gamma'\backslash\Gamma$ with Ω by means of a bijection $\alpha : \Gamma'\backslash\Gamma \to \Omega$ such that $\alpha(\Gamma' \cdot 1) = \omega_0$. In this way we obtain a permutation representation $\tau_1 : \Gamma \to S(\Omega)$ satisfying our requirements. By the transitivity of τ_1 we can choose elements $\gamma_\omega \in \Gamma$ for $\omega \in \Omega$ such that

$$\begin{gathered} \tau_1(\gamma_\omega)(\omega_0) = \omega, \quad \omega \in \Omega \\ \gamma_{\omega_0} = 1. \end{gathered}$$

Moreover, we choose elements $h_\omega \in H$ for $\omega \in \Omega$, $h_{\omega_0} = 1$, and define a function $f : \Gamma \times \Omega \to H$ via

$$f(\gamma, \omega) := h_\omega^{-1}\, \chi(\gamma')\, h_{\omega'},$$

where $\gamma_\omega\gamma = \gamma'\gamma_{\omega'}$ with $\gamma' \in \Gamma'$ and $\omega' \in \Omega$. If $\gamma' \in \Gamma'$ then the trivial equation $\gamma_{\omega_0}\gamma' = \gamma'\gamma_{\omega_0}$ shows that

$$f(\gamma', \omega_0) = h_{\omega_0}^{-1}\, \chi(\gamma')\, h_{\omega_0} = \chi(\gamma'),$$

i.e., f satisfies (12). Next, we show that the pair (τ_1, f) also satisfies equation (11). Again this is done in two steps starting with the special case of (11) where $\omega = \omega_0$. Let $\gamma_1, \gamma_2 \in \Gamma$ and suppose that $\gamma_1 = \gamma_1'\gamma_\omega$, i.e., $\tau_1(\gamma_1)(\omega_0) = \omega$, and that $\gamma_\omega\gamma_2 = \gamma_2'\gamma_{\omega'}$, where $\gamma_1', \gamma_2' \in \Gamma'$ and $\omega, \omega' \in \Omega$. Then we have

$$\gamma_1\gamma_2 = \gamma_1'\gamma_\omega\gamma_\omega^{-1}\gamma_2'\gamma_{\omega'} = \gamma_1'\gamma_2'\gamma_{\omega'},$$

and since χ is a homomorphism we find that

$$\begin{aligned} f(\gamma_1\gamma_2, \omega_0) &= \chi(\gamma_1'\gamma_2')\, h_{\omega'} \\ &= \chi(\gamma_1')\, h_\omega\, h_\omega^{-1}\, \chi(\gamma_2')\, h_{\omega'} \\ &= f(\gamma_1, \omega_0)\, f(\gamma_2, \tau_1(\gamma_1)(\omega_0)). \end{aligned}$$

Now let $\gamma_1, \gamma_2 \in \Gamma$ and $\omega \in \Omega$. Put $\gamma_\omega\gamma_1 = \gamma_1'\gamma_{\omega'}$ and $\gamma_{\omega'}\gamma_2 = \gamma_2'\gamma_{\omega''}$ with $\gamma_1', \gamma_2' \in \Gamma'$ and $\omega', \omega'' \in \Omega$. Then

$$\gamma_\omega\gamma_1\gamma_2 = \gamma_1'\gamma_{\omega'}\gamma_{\omega'}^{-1}\gamma_2'\gamma_{\omega''} = \gamma_1'\gamma_2'\gamma_{\omega''}$$

and we find that

$$\begin{aligned} f(\gamma_1\gamma_2, \omega) &= h_\omega^{-1}\, \chi(\gamma_1'\gamma_2')\, h_{\omega''} \\ &= h_\omega^{-1}\, \chi(\gamma_1')\, h_{\omega'}\, \big(\chi(\gamma_1')\, h_{\omega'}\big)^{-1}\, \chi(\gamma_1'\gamma_2')\, h_{\omega''} \\ &= f(\gamma_1, \omega)\, \big(f(\gamma_\omega\gamma_1, \omega_0)\big)^{-1}\, f(\gamma_\omega\gamma_1\gamma_2, \omega_0) \\ &= f(\gamma_1, \omega)\, f(\gamma_2, \tau_1(\gamma_\omega\gamma_1)(\omega_0)) \\ &= f(\gamma_1, \omega)\, f(\gamma_2, \tau_1(\gamma_1)(\omega)), \end{aligned}$$

as was claimed. It remains to verify that f satisfies (13); but this follows from what we have already proved in (b), the fact that $\Gamma_\chi = \Gamma_N$, and formula (3) (whose proof does not depend upon the defining property (iv) of $\mathcal{R}_{\Gamma_\Pi}^{\Gamma_N}(\Omega)$), whence the lemma. $\square$

2. Enumerating representations in $(H \wr \Pi_n)_N$

2.1. **Admissibility.** As before let Γ and H be groups, $|H| < \infty$, and let $\Sigma \subseteq \Gamma - \{1\}$ be a normal subset of Γ. Moreover, fix a non-empty set $\Lambda \subseteq \mathbb{N}$ of positive integers and collect all these data into a quadruple $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$. Given $\hat{Q}$ and a finite set Ω we denote by $\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega))$ the set of all homomorphisms $\tau : \Gamma \to H \wr S(\Omega)$ such that the action of Γ on Ω induced by τ is Σ-free and decomposes Ω into orbits whose lengths are contained in Λ. The elements of $\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega))$ will be called (Σ, Λ)-*admissible* representations of Γ in $H \wr S(\Omega)$. A quadruple $\hat{Q}$ is called *admissible* if the sets $\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S_n)$ are finite for all $n \geq 0$. If, for instance, the group Γ is finitely generated or of finite (subgroup) rank, then $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ is admissible for each normal set $\Sigma \subseteq \Gamma - \{1\}$, every finite group H, and every set Λ of positive integers. Observe that if $\hat{Q}$ is admissible the construction $\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(-))$ provides a covariant functor F on the category Ens of finite sets and bijections. Indeed, a bijection $\alpha : \Omega \to \Omega'$ between two finite sets Ω and Ω' induces an isomorphism

$$\begin{aligned} J_\alpha : H \wr S(\Omega) &\to H \wr S(\Omega') \\ (f, \pi) &\mapsto (f\alpha^{-1}, \alpha\pi\alpha^{-1}), \end{aligned}$$

and for a representation $\tau : \Gamma \to H \wr S(\Omega)$ and a bijection $\alpha : \Omega \to \Omega'$ we have a commutative diagram

$$\begin{tikzcd} \Gamma \arrow[r, "\tau"] \arrow[d, "\mathrm{id}"'] & H \wr S(\Omega) \arrow[r, "\epsilon"] \arrow[d, "J_\alpha"] & S(\Omega) \arrow[d, "\iota_\alpha"] \\ \Gamma \arrow[r, "\tau'"'] & H \wr S(\Omega') \arrow[r, "\epsilon"'] & S(\Omega'). \end{tikzcd}$$

It follows that

$$\epsilon\, \tau' = \epsilon\, J_\alpha\, \tau = \iota_\alpha\, \epsilon\, \tau,$$

i.e., the permutation representations $\epsilon\tau$ and $\epsilon\tau'$ induced by τ respectively τ' are similar. Hence, if τ is (Σ, Λ)-admissible then so is τ' and the assignment $\tau \mapsto \tau'$ defines a map

$$F(\alpha) : \mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega)) \to \mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega')).$$

Replacing α by α^{-1} yields a map

$$\mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega')) \to \mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega))$$

which inverts $F(\alpha)$ since $J_{\alpha^{-1}} = J_\alpha^{-1}$. It follows now that for an admissible quadruple $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ and every finite set Ω the set $F(\Omega) = \mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega))$ is again finite.

Let $N \leq H$ be a subgroup of index $(H : N) \leq 2$ and let $\{\Pi_n\}_0^\infty$ denote either the sequence $\{A_n\}_0^\infty$ of alternating groups or the sequence $\{S_n\}_0^\infty$ of symmetric groups. Moreover, let $\Pi(-)$ be the normal functor generated by the sequence $\{\Pi_n\}_0^\infty$ and put

$$\alpha_\Lambda(\Pi) := \max_{n \in \Lambda} (S_n : \Pi_n).$$

Assume from now on that $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ is admissible and form the sets $\mathcal{R}_{\Gamma_\Pi}^{\Gamma_N}(\Omega)$, defined in the previous paragraph, relative to the data Γ, H, Σ, N, and $\Pi(-)$ just introduced for every choice of normal subgroups $\Gamma_\Pi, \Gamma_N \trianglelefteq \Gamma$ and each non-empty finite set Ω. Since $\hat{Q}$ is admissible the inclusion

$$\mathcal{R}_{\Gamma_\Pi}^{\Gamma_N}(\Omega) \subseteq \mathrm{Hom}_\Sigma^\Lambda(\Gamma, H \wr S(\Omega)), \quad |\Omega| \in \Lambda$$

shows that $\mathcal{R}_{\Gamma_\Pi}^{\Gamma_N}(\Omega)$ is finite for every set Ω with $|\Omega| \in \Lambda$ and all normal subgroups $\Gamma_\Pi, \Gamma_N \trianglelefteq \Gamma$. Hence, by Proposition 1, the cardinal number $s(n, \Gamma_\Pi, \Gamma_N)$ defined in (4) is finite whenever $n \in \Lambda$ and we have

$$|\mathcal{R}_{\Gamma_\Pi}^{\Gamma_N}(\Omega)| = |H|^{|\Omega|-1}\, (|\Omega| - 1)!\, s(|\Omega|, \Gamma_\Pi, \Gamma_N), \quad |\Omega| \in \Lambda. \tag{14}$$

For a set $M \subseteq \mathbb{N}_0$ of non-negative integers and a finite set Ω we denote by

$$\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma, (H \wr \Pi(\Omega))_N) \tag{15}$$

the set of those representations $\tau : \Gamma \to (H \wr \Pi(\Omega))_N$ which are (Σ, Λ)-admissible and have the property that the number of orbits into which Ω decomposes under the Γ-action induced by τ is contained in the set M. Since $\hat{Q}$ is admissible the set (15) is finite for each finite set Ω and every set M of non-negative integers. Our first main result computes the exponential generating function of the sequence $\left\{\left|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma, (H \wr \Pi_n)_N)\right|\right\}_{n=0}^\infty$. Before describing and proving this result (in Subsection 2.3) we take a closer look at one of its key ingredients.

2.2. **The function Φ_Γ.** Assume that Γ contains only finitely many subgroups of index $k_0 := \max\big(\alpha_\Lambda(\Pi), (H:N)\big)$. Consider the system of subgroup pairs

$$\mathcal{U}_\Gamma := \big\{(\Gamma_\Pi, \Gamma_N):\ \ \Gamma_\Pi, \Gamma_N \leq \Gamma,\ (\Gamma:\Gamma_\Pi) \leq \alpha_\Lambda(\Pi),\ (\Gamma:\Gamma_N) \leq (H:N)\big\}$$

and fix a set $E \subseteq \Gamma$ which generates Γ as a normal subgroup. With each pair $(\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma$ we associate a (formal) variable $z_{(\Gamma_\Pi,\Gamma_N)}$ and a discrete variable $\nu(\Gamma_\Pi, \Gamma_N)$ taking non-negative integral values and form the power series

$$\Phi_\Gamma\big(z_{(\Gamma_\Pi,\Gamma_N)};\ (\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma\big) := \sum \prod_{(\Gamma_\Pi,\Gamma_N)} z_{(\Gamma_\Pi,\Gamma_N)}^{\nu(\Gamma_\Pi,\Gamma_N)} / \nu(\Gamma_\Pi, \Gamma_N)!,$$

where the right-hand sum is extended over those maps $\nu : \mathcal{U}_\Gamma \to \mathbb{N}_0$ satisfying the following $2|E|+1$ conditions:

$$\left\{\begin{array}{l} \sum\limits_{(\Gamma_\Pi,\Gamma_N)} \nu(\Gamma_\Pi, \Gamma_N) \in M \\ \sum\limits_{e \notin \Gamma_\Pi} \nu(\Gamma_\Pi, \Gamma_N) \equiv 0 \mod \alpha_\Lambda(\Pi), \quad e \in E \\ \sum\limits_{e \notin \Gamma_N} \nu(\Gamma_\Pi, \Gamma_N) \equiv 0 \mod (H:N), \quad e \in E. \end{array}\right. \tag{16}$$

Denote by $\mathcal{J}_\Gamma$ the intersection of all subgroups of index k_0 in Γ. The quotient $\bar{\Gamma} = \Gamma/\mathcal{J}_\Gamma$ is an elementary abelian 2-group of rank $r = \log_2\big(\sum_{k \leq k_0} s_\Gamma(k)\big)$.

Proposition 2. (i) *The function Φ_Γ does not depend on the choice of the normal generating system E.*

(ii) *The functions Φ_Γ and $\Phi_{\bar{\Gamma}}$, formed with respect to the same data $\{\Pi_n\}_0^\infty, H, N, \Lambda$, and M, coincide up to a canonical identification of variables*

$$z_{(\Gamma_\Pi,\Gamma_N)} \mapsto z_{(\Gamma_\Pi/\mathcal{J}_\Gamma, \Gamma_N/\mathcal{J}_\Gamma)}, \quad (\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma.$$

Proof.† (i) Consider the linear space $V_\Gamma = \mathbf{F}_2^{\mathcal{U}_\Gamma}$ of maps $\mathcal{U}_\Gamma \to \mathbf{F}_2$ over the field $\mathbf{F}_2$ with two elements and denote by $\nu(\Gamma_\Pi, \Gamma_N)$ the value of a vector $\underline{\nu} \in V_\Gamma$ at (Γ_Π, Γ_N). We give a description for the subspace

$$W_\Gamma := \Big\{\underline{\nu} \in V_\Gamma:\ \sum_{e \notin \Gamma_\Pi} \nu(\Gamma_\Pi, \Gamma_N) = 0\ \ (e \in E)\Big\},$$

which is independent of the choice of E. For $\gamma \in \Gamma$ and $\underline{\nu} \in V_\Gamma$ define a set

$$S_{\underline{\nu}}(\gamma) := \big\{(\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma:\ \gamma \notin \Gamma_\Pi \text{ and } \nu(\Gamma_\Pi, \Gamma_N) = 1\big\}.$$

Observe that a vector $\underline{\nu} \in V_\Gamma$ is in W_Γ if and only if $|S_{\underline{\nu}}(\gamma)| \equiv 0\ (2)$ for all $\gamma \in E$. Also, given $\underline{\nu} \in V_\Gamma$ and $\gamma_1, \gamma_2 \in \Gamma$ note that

$$S_{\underline{\nu}}(\gamma_1\gamma_2) = S_{\underline{\nu}}(\gamma_1)\ \Delta\ S_{\underline{\nu}}(\gamma_2),$$

where $A \Delta B$ denotes the symmetric difference of the sets A and B. Since

$$|S_{\underline{\nu}}(\gamma_1)\ \Delta\ S_{\underline{\nu}}(\gamma_2)| \equiv |S_{\underline{\nu}}(\gamma_1)| + |S_{\underline{\nu}}(\gamma_2)| \mod 2$$

it follows that the map $\psi_{\underline{\nu}} : \Gamma \to (\mathbf{F}_2, +)$ given by $\gamma \mapsto |S_{\underline{\nu}}(\gamma)| \mod 2$ is a homomorphism. Clearly,

$$W_\Gamma = \big\{\underline{\nu} \in V_\Gamma:\ \ker(\psi_{\underline{\nu}}) = \Gamma\big\}.$$

† The proof of Proposition 2 given here arose from a discussion with John Shareshian.

Since the definition of $\psi_{\underline{\nu}}$ does not involve E, we conclude that the system of congruences

$$\sum_{e\notin\Gamma_\Pi} \nu(\Gamma_\Pi,\Gamma_N) \equiv 0 \mod \alpha_\Lambda(\Pi),\quad e\in E$$

has a set of solutions over $\mathbb{N}_0$ which is independent of the choice of E. A completely analogous argument yields the same conclusion for the system of congruences

$$\sum_{e\notin\Gamma_N} \nu(\Gamma_\Pi,\Gamma_N) \equiv 0 \mod (H:N),\quad e\in E,$$

whence the invariance of Φ_Γ.

(ii) Pick a set of generators $\{x_i : i\in I\}$ for $\mathcal{J}_\Gamma$, choose a minimal generating system $\bar{E}=\{\bar{y}_1,\dots,\bar{y}_r\}$ for $\bar{\Gamma}$, and lift each $\bar{y}_k$ to an element $y_k\in\Gamma$. Think of the functions Φ_Γ and $\Phi_{\bar{\Gamma}}$ as being calculated with respect to the generating systems $E:=\{x_i : i\in I\}\cup\{y_1,\dots,y_r\}$ and $\bar{E}$, respectively. The terms occurring in the series Φ_Γ correspond in a one-to-one fashion to maps $\nu:\mathcal{U}_\Gamma\to\mathbb{N}_0$ satisfying the conditions

$$\begin{aligned}
&\sum_{(\Gamma_\Pi,\Gamma_N)} \nu(\Gamma_\Pi,\Gamma_N) \ \in M\\
&\sum_{y_k\notin\Gamma_\Pi} \nu(\Gamma_\Pi,\Gamma_N) \equiv 0 \mod \alpha_\Lambda(\Pi),\quad 1\le k\le r\\
&\sum_{y_k\notin\Gamma_N} \nu(\Gamma_\Pi,\Gamma_N) \equiv 0 \mod (H:N),\quad 1\le k\le r.
\end{aligned}$$

But, since for a subgroup Γ' of index at most 2 in Γ and $k\in[r]$ we have $y_k\in\Gamma'$ if and only if $\bar{y}_k\in\bar{\Gamma}'=\Gamma'/\mathcal{J}_\Gamma$, these maps $\nu:\mathcal{U}_\Gamma\to\mathbb{N}_0$ are precisely those obtained by composing the solutions $\mathcal{U}_{\bar{\Gamma}}\to\mathbb{N}_0$ of the corresponding system of conditions for the series $\Phi_{\bar{\Gamma}}$ with the canonical bijection $\mathcal{U}_\Gamma\to\mathcal{U}_{\bar{\Gamma}}$ given by $(\Gamma_\Pi,\Gamma_N)\mapsto(\Gamma_\Pi/\mathcal{J}_\Gamma,\Gamma_N/\mathcal{J}_\Gamma)$, whence (ii). □

2.3. **The main counting result.** For each $(\Gamma_\Pi,\Gamma_N)\in\mathcal{U}_\Gamma$ introduce the formal power series

$$\Theta_{\Gamma_\Pi}^{\Gamma_N}(z) := \sum_{n\in\Lambda} \frac{|H|^{n-1}s(n,\Gamma_\Pi,\Gamma_N)}{n} z^n$$

with $s(n,\Gamma_\Pi,\Gamma_N)$ as in (4).

Theorem 1. *Let $\hat{Q}=(\Gamma,H,\Sigma,\Lambda)$ be an admissible quadruple, $N\le H$ a subgroup of index $(H:N)\le 2$, $M\subseteq\mathbb{N}_0$ a set of non-negative integers, and let $\{\Pi_n\}_0^\infty$ denote either the sequence $\{A_n\}_0^\infty$ of alternating groups or the sequence $\{S_n\}_0^\infty$ of symmetric groups. Assume that Γ contains only finitely many subgroups of index* $\max(\alpha_\Lambda(\Pi),(H:N))$. *Then*

$$\sum_{n\ge 0} \frac{|Hom_\Sigma^{\Lambda,M}(\Gamma,(H\wr\Pi_n)_N)|}{n!} z^n = \Phi_\Gamma\big(\Theta_{\Gamma_\Pi}^{\Gamma_N}(z);\ (\Gamma_\Pi,\Gamma_N)\in\mathcal{U}_\Gamma\big), \tag{17}$$

i.e., the exponential generating function of the sequence $\big\{|Hom_\Sigma^{\Lambda,M}(\Gamma,(H\wr\Pi_n)_N)|\big\}_{n=0}^\infty$ *is obtained from the series Φ_Γ by replacing each variable $z_{(\Gamma_\Pi,\Gamma_N)}$ with the corresponding power series $\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)$.*

Proof. Decomposing the set $\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma,(H\wr\Pi_n)_N)$ according to the orbit decomposition of $[n]$ associated with its elements τ, decomposing each representation τ into the disjoint sum of the representations induced by τ on its orbits, and applying (14) we have for $n\geq 0$

$$
\begin{aligned}
&|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma,(H\wr\Pi_n)_N)| = \\
&\sum_{\mu\in M}\sum_{\substack{\lambda\vdash n\\ |\lambda|=\mu\\ \lambda_j=0(j\notin\Lambda)}}\Big(\prod_j\lambda_j!\Big)^{-1}\binom{n}{\underbrace{1,\dots,1}_{\lambda_1},\dots,\underbrace{n,\dots,n}_{\lambda_n}}\sum_{\varphi_\lambda}\prod_{(j,k_j)}\Big\{|H|^{j-1}(j-1)!\\
&\qquad\qquad\times s(j,\Gamma_\Pi(j,k_j),\Gamma_N(j,k_j))\Big\} = \\
&\sum_{\substack{\lambda\vdash n\\ |\lambda|\in M\\ \lambda_j=0(j\notin\Lambda)}}\frac{n!}{\prod_j(j^{\lambda_j}\lambda_j!)}\sum_{\varphi_\lambda}\prod_{(j,k_j)}\Big\{|H|^{j-1}s(j,\Gamma_\Pi(j,k_j),\Gamma_N(j,k_j))\Big\},
\end{aligned}
$$

where φ_λ runs through the system of maps

$$
\begin{aligned}
\varphi_\lambda : \dot{\bigcup_{j=1}^{\infty}}\big(\{j\}\times[\lambda_j]\big) &\to \mathcal{U}_\Gamma\\
(j,k_j) &\mapsto (\Gamma_\Pi(j,k_j),\Gamma_N(j,k_j))
\end{aligned}
$$

satisfying

$$
\begin{aligned}
|\{(j,k_j):\ e\notin\Gamma_\Pi(j,k_j)\}| &\equiv 0 \mod \alpha_\Lambda(\Pi),\quad e\in E\\
|\{(j,k_j):\ e\notin\Gamma_N(j,k_j)\}| &\equiv 0 \mod (H:N),\quad e\in E.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{n\geq 0}\frac{|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma,(H\wr\Pi_n)_N)|}{n!}z^n = \\
&\sum_{n\geq 0}\sum_{\substack{\lambda\vdash n\\ |\lambda|\in M\\ \lambda_j=0(j\notin\Lambda)}}\prod_j\big(j^{\lambda_j}\lambda_j!\big)^{-1}\sum_{\varphi_\lambda}\prod_{(j,k_j)}\Big\{|H|^{j-1}s(j,\Gamma_\Pi(j,k_j),\Gamma_N(j,k_j))\Big\}z^n = \\
&\sum_{\mu\in M}\frac{1}{\mu!}\sum_{\substack{|\lambda|=\mu\\ \lambda_j=0(j\notin\Lambda)}}\binom{\mu}{\lambda_1,\lambda_2,\dots}\sum_{\varphi_\lambda}\prod_{(j,k_j)}\Big\{\frac{|H|^{j-1}s(j,\Gamma_\Pi(j,k_j),\Gamma_N(j,k_j))}{j}z^j\Big\} = \\
&\sum_{\mu\in M}\frac{1}{\mu!}\sum_{n_1,\dots,n_\mu\in\Lambda}\sum_{(*)}\prod_{i=1}^{\mu}\Big\{\frac{|H|^{n_i-1}s(n_i,\Gamma_\Pi(i),\Gamma_N(i))}{n_i}z^{n_i}\Big\},
\end{aligned}
$$

where $\sum_{(*)}$ denotes the sum over all partitions of $[\mu]$ into pairwise disjoint (possibly empty) sets $E(\Gamma_\Pi, \Gamma_N)$ indexed by the elements of $\mathcal{U}_\Gamma$ subject to the conditions

$$\sum_{e \notin \Gamma_\Pi} |E(\Gamma_\Pi, \Gamma_N)| \equiv 0 \mod \alpha_\Lambda(\Pi), \quad e \in E$$

$$\sum_{e \notin \Gamma_N} |E(\Gamma_\Pi, \Gamma_N)| \equiv 0 \mod (H : N), \quad e \in E,$$

and $(\Gamma_\Pi(i), \Gamma_N(i))$ is the unique element in $\mathcal{U}_\Gamma$ such that $i \in E(\Gamma_\Pi(i), \Gamma_N(i))$. Interchanging the summations $\sum_{n_1,\ldots,n_\mu \in \Lambda}$ and $\sum_{(*)}$, combining the two summations $\sum_{\mu \in M} \sum_{(*)}$ into one summation over the corresponding (more general) type of set systems $\{E(\Gamma_\Pi, \Gamma_N)\}$, and introducing the cardinalities $\nu(\Gamma_\Pi, \Gamma_N)$ of the sets $E(\Gamma_\Pi, \Gamma_N)$ we find that this last expression equals

$$\sum_{(16)} \prod_{(\Gamma_\Pi, \Gamma_N)} \left(\Theta_{\Gamma_\Pi}^{\Gamma_N}(z)\right)^{\nu(\Gamma_\Pi, \Gamma_N)} / \nu(\Gamma_\Pi, \Gamma_N)! = \Phi_\Gamma\left(\Theta_{\Gamma_\Pi}^{\Gamma_N}(z);\ (\Gamma_\Pi, \Gamma_N) \in \mathcal{U}_\Gamma\right)$$

as claimed. □

3. Examples

We consider three special cases of Theorem 1 related to (i) symmetric wreath products, (ii) alternating wreath products, and (iii) the Weyl groups of type D_n, respectively. Each case is illustrated with some specific examples.

3.1. **The wreath products $H \wr S_n$.** Putting $N = H$ and $\Pi_n = S_n$ $(n \geq 0)$ we have $(H \wr \Pi_n)_N = H \wr S_n$, $\alpha_\Lambda(\Pi) = 1$, and $\mathcal{U}_\Gamma = \{(\Gamma, \Gamma)\}$. Dropping the subscript (Γ, Γ) the exterior function Φ_Γ becomes

$$\Phi_\Gamma(z) = \sum_{\nu \in M} z^\nu / \nu! =: e_M(z)$$

and Theorem 1 yields the following.

Corollary 1. *Let $\hat{Q} = (\Gamma, H, \Sigma, \Lambda)$ be an admissible quadruple and $M \subseteq \mathbb{N}_0$ a set of non-negative integers. Then*

$$\sum_{n \geq 0} \frac{|Hom_\Sigma^{\Lambda, M}(\Gamma, H \wr S_n)|}{n!} z^n = e_M\left(\sum_{n \in \Lambda} \frac{|H|^{n-1} s_{\Gamma,\Sigma}^H(n)}{n} z^n\right), \tag{18}$$

where $s_{\Gamma,\Sigma}^H(n) := \sum_{\substack{(\Gamma:\Gamma')=n \\ \Gamma' \cap \Sigma = \emptyset}} |Hom(\Gamma', H)|$.

Example 1. Let Γ be a finitely generated group and put $\Sigma = \emptyset$, $\Lambda = \mathbb{N}$, $M = \mathbb{N}_0$, and $H = 1$. Then the function $s_{\Gamma,\Sigma}^H(n) = s_\Gamma(n)$ counts the subgroups of index n in Γ, and formula (18) relates these subgroup numbers to the number $|\mathrm{Hom}(\Gamma, S_n)|$ of Γ-actions on an n-set via

$$\sum_{n \geq 0} \frac{|\mathrm{Hom}(\Gamma, S_n)|}{n!} z^n = \exp\left(\sum_{n \geq 1} \frac{s_\Gamma(n)}{n} z^n\right) \tag{19}$$

or, equivalently,

$$nh_\Gamma(n) = \sum_{k=1}^{n} h_\Gamma(n-k)s_\Gamma(k), \quad n \geq 1, \tag{20}$$

where $h_\Gamma(n) := |\mathrm{Hom}(\Gamma, S_n)|/(n!)$. A particularly simple instance of (20) is Hall's recursion formula ([8, Theorem 5.2])

$$s_{F_r}(n) = n(n!)^{r-1} - \sum_{k=1}^{n-1} \big((n-k)!\big)^{r-1} s_{F_r}(k)$$

for the number of index n subgroups in the free group F_r of (finite) rank r. Dey's corresponding formula [4, Theorem 6.10] for (non-trivial) free products is another special case of (20). In retrospective, it appears that Hall's formula together with its background on Schreier systems and their associated subgroups in free groups developed in [9] marks the beginning of what has in recent years become known as the theory of subgroup growth; cf. for example [10] and [11]. For $\Gamma = G$ a finite group (19) takes the form

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, S_n)|}{n!} z^n = \exp\left(\sum_{d|m} \frac{s_G(d)}{d} z^d\right), \quad m = |G|. \tag{21}$$

This formula, which exhibits the exponential generating function of the sequence $\{|\mathrm{Hom}(G, S_n)|\}_0^\infty$ as a particular type of entire function, was a starting point in [14] for the asymptotic enumeration of finite group actions; cf. [14, Theorem 5]. For $G = C_m$ a cyclic group (21) was proved already in [3].

Example 2. How many fixed-point-free $SL_2(\mathbb{Z})$-actions on a 10-set are there having exactly 4 orbits? Call this number N. In Corollary 1 we put $\Gamma = SL_2(\mathbb{Z})$, $H = 1$, $\Sigma = \emptyset$, $M = \{4\}$, and $\Lambda = \{2, 3, 4\}$. By (18) we have †

$$N = \frac{10!}{4!}\left\langle z^{10}, \left(\sum_{n=2}^{4} \frac{s_\Gamma(n)}{n} z^n\right)^4 \right\rangle. \tag{22}$$

Using the presentation $\Gamma \cong \langle a, b \mid a^4 = b^6 = b^3 a^{-2} = 1\rangle$ we find that

$$\begin{aligned} |\mathrm{Hom}(\Gamma, S_1)| &= 1,\\ |\mathrm{Hom}(\Gamma, S_2)| &= 2,\\ |\mathrm{Hom}(\Gamma, S_3)| &= 12,\\ |\mathrm{Hom}(\Gamma, S_4)| &= 96. \end{aligned}$$

Plugging this information into (19) and taking log gives

$$\sum_{n\geq 1} \frac{s_\Gamma(n)}{n} z^n = \sum_{k\geq 1} (-1)^{k-1} \frac{1}{k} \left(z + z^2 + 2z^3 + 4z^4 + \cdots\right)^k,$$

from which we read off that $s_\Gamma(2) = 1$, $s_\Gamma(3) = 4$, and $s_\Gamma(4) = 9$. Using these values in (22) gives $N = 573300$.

Example 3. Take Γ to be a finitely generated virtually free group, $\Sigma = \mathrm{tor}(\Gamma) - \{1\}$ as the set of non-trivial torsion elements in Γ, and put $\Lambda = \mathbb{N}$, $M = \mathbb{N}_0$, and

† $\langle z^\nu, f(z)\rangle$ denotes the coefficient of z^ν in the formal power series $f(z)$.

$H = 1$. In this case $s^H_{\Gamma,\Sigma}(n)$ is the number of free‡ subgroups of index n in Γ and $|\mathrm{Hom}^{\Lambda,M}_\Sigma(\Gamma, H \wr S_n)| = |\mathrm{Hom}_\Sigma(\Gamma, S_n)|$ counts the *torsion-free* Γ-actions on an n-set, i.e., those Γ-actions on $[n]$ which are free when restricted to finite subgroups. Denoting by m_Γ the least common multiple of the orders of the finite subgroups in Γ we have

$$|\mathrm{Hom}_\Sigma(\Gamma, S_n)| = s^H_{\Gamma,\Sigma}(n) = 0, \quad m_\Gamma \nmid n.$$

Putting $a_\Gamma(\lambda) := |\mathrm{Hom}_\Sigma(\Gamma, S_{m_\Gamma\lambda})|/(m_\Gamma\lambda)!$ and $b_\Gamma(\lambda) := s^H_{\Gamma,\Sigma}(m_\Gamma\lambda)$ we recover from (18) the identity ([12, formula (3)])

$$\sum_{\mu=1}^{\lambda} a_\Gamma(\lambda-\mu)\, b_\Gamma(\mu) = m_\Gamma\, \lambda\, a_\Gamma(\lambda), \quad \lambda \geq 1.$$

This relation has been the starting point for a detailed analysis of the growth behaviour and the asymptotics of the function $b_\Gamma(\lambda)$ attached to a finitely generated virtually free group; cf. [12] and [13].

Example 4. Let $\Gamma = G$ and H be arbitrary finite groups and put $\Lambda = \mathbb{N}$, $M = \mathbb{N}_0$, and $\Sigma = \emptyset$. Then we find from Corollary 1 that

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, H\wr S_n)|}{n!} z^n = \exp\left(\sum_{d|m} \frac{|H|^{d-1}\, s^H_G(d)}{d} z^d\right), \quad m = |G|, \tag{23}$$

where $s^H_G(d) := s^H_{G,\emptyset}(d) = \sum_{(G:U)=d} |\mathrm{Hom}(U,H)|$. For $G = C_m$ (23) specializes to Chigira's result ([1, Theorem 2])

$$\sum_{n=0}^{\infty} \frac{\beta_m(H\wr S_n)}{n!} z^n = \exp\left(\sum_{d|m} \frac{|H|^{d-1}\, \beta_{m/d}(H)}{d} z^d\right), \tag{24}$$

where $\beta_m(B)$ is the number of solutions of the equation $x^m = 1$ in a finite group B. For $G = C_p^r$ an elementary abelian p-group (23) takes the form

$$\sum_{n=0}^{\infty} \frac{\beta_{p,r}(H\wr S_n)}{n!} z^n = \exp\left(\sum_{\rho=0}^{r} \frac{|H|^{p^\rho-1} \binom{r}{\rho}_p \beta_{p,r-\rho}(H)}{p^\rho} z^{p^\rho}\right), \tag{25}$$

where, for a finite group B, $\beta_{p,r}(B)$ denotes the number of solutions in B for the system of equations

$$x_i^p = [x_i, x_j] = 1, \quad 1 \leq i, j \leq r,$$

and the $\binom{r}{\rho}_p$ are Gaussian coefficients. We will come back to (23) in the next section when investigating the asymptotic behaviour of the function $|\mathrm{Hom}(G, H\wr S_n)|$.

Remarks. 1. Corollary 1 can also be proved by applying the exponential principle established in [7] to the functor $F_{\hat{Q}} = \mathrm{Hom}^\Lambda_\Sigma(\Gamma, H\wr S(-))$ on Ens and then using the special case of Proposition 1 where $N = H, \Pi(-) = S(-)$, and $\Gamma_\Pi = \Gamma_N = \Gamma$ to analyze the cardinalities of the sets $(F_{\hat{Q}})_\eta([n])$ for $n \in \Lambda$.

‡Recall that every torsion-free subgroup of a virtually free group is free; cf. for example [6, I.4.5 & IV.1.6].

2. For $M = \mathbb{N}_0$ formula (18) is equivalent to the recursion formula

$$n\, h^H_{\Gamma,\Sigma}(n,\Lambda) = \sum_{\substack{1\le k\le n\\ k\in\Lambda}} |H|^{k-1}\, s^H_{\Gamma,\Sigma}(k)\, h^H_{\Gamma,\Sigma}(n-k,\Lambda), \quad n\ge 1,$$

where $h^H_{\Gamma,\Sigma}(n,\Lambda) := |\mathrm{Hom}^\Lambda_\Sigma(\Gamma, H\wr S_n)|/(n!)$.

3. For a permutation $\pi\in S(\Omega)$ on a finite set Ω we denote by $a_j(\pi)$ the number of j-cycles occurring in the cycle decomposition of π, and by $a(\pi)=\sum a_j(\pi)$ the total number of cycles of π. Given sets $\Lambda\subseteq\mathbb{N}$, $M\subseteq\mathbb{N}_0$, and a permutation group $\Pi\le S(\Omega)$ on Ω define the (generalized) cycle index $\mathrm{Cyc}^{\Lambda,M}_\Pi(x_1,\dots,x_{|\Omega|})$ of Π with respect to the sets Λ and M as

$$\mathrm{Cyc}^{\Lambda,M}_\Pi(x_1,\dots,x_{|\Omega|}) := \frac{1}{|\Pi|}\sum_{\substack{\pi\in\Pi\\ a(\pi)\in M\\ a_j(\pi)=0 (j\notin\Lambda)}} \prod_{j=1}^{|\Omega|} x_j^{a_j(\pi)} \quad \in \mathbb{Q}[x_1,\dots,x_{|\Omega|}]$$

(the ordinary cycle index $\mathrm{Cyc}_\Pi(x_1,\dots,x_{|\Omega|})$ of Π corresponds to the case where $\Lambda=\mathbb{N}$ and $M=\mathbb{N}_0$). Then we have for an admissible quadruple $\hat{Q}=(\Gamma,H,\Sigma,\Lambda)$

$$|\mathrm{Hom}^{\Lambda,M}_\Sigma(\Gamma,H\wr S_n)| = n!\,\mathrm{Cyc}^{\Lambda,M}_{S_n}(0!\,|H|^0 s^H_{\Gamma,\Sigma}(1),\dots,(n-1)!\,|H|^{n-1} s^H_{\Gamma,\Sigma}(n)),$$

i.e., the number $|\mathrm{Hom}^{\Lambda,M}_\Sigma(\Gamma,H\wr S_n)|$ of (Σ,Λ)-admissible representations $\tau:\Gamma\to H\wr S_n$ having the property that the number of orbits into which $[n]$ decomposes under the Γ-action induced by τ is contained in the set M is obtained from the integral polynomial $n!\,\mathrm{Cyc}^{\Lambda,M}_{S_n}(x_1,\dots,x_n)$ by the substitutions $x_j = (j-1)!\,|H|^{j-1} s^H_{\Gamma,\Sigma}(j)$, $1\le j\le n$.

3.2. **The wreath products $H\wr A_n$.** In Theorem 1 put $N=H$ and $\Pi_n=A_n$ $(n\ge 0)$. Then $(H\wr\Pi_n)_N = H\wr A_n$ and, assuming that $\Lambda\ne\{1\}$, the set $\mathcal{U}_\Gamma$ can be identified with the system of all subgroups Γ_Π of index at most 2 in Γ. Following the procedure of our theorem we associate with each such subgroup Γ_Π a formal variable z_{Γ_Π} and a discrete variable $\nu(\Gamma_\Pi)$ taking non-negative integral values and form the power series

$$\Phi_\Gamma\big(z_{\Gamma_\Pi};\ (\Gamma:\Gamma_\Pi)\le 2\big) = \sum \prod_{(\Gamma:\Gamma_\Pi)\le 2} z_{\Gamma_\Pi}^{\nu(\Gamma_\Pi)}/\nu(\Gamma_\Pi)!,$$

where the right-hand sum is extended over the system of variables $\nu(\Gamma_\Pi)$ subject to the conditions

$$\begin{cases} \sum\limits_{(\Gamma:\Gamma_\Pi)\le 2} \nu(\Gamma_\Pi)\in M \\ \sum\limits_{e\notin\Gamma_\Pi} \nu(\Gamma_\Pi)\equiv 0 \mod 2, \quad e\in E \end{cases}$$

(recall that E is a fixed normal generating system for Γ). Moreover, for each subgroup $\Gamma_\Pi\le\Gamma$ of index at most 2 we form the power series

$$\Theta_{\Gamma_\Pi}(z) = \sum_{n\in\Lambda} \frac{|H|^{n-1} s(n,\Gamma_\Pi)}{n} z^n$$

with

$$s(n,\Gamma_\Pi)=s(n,\Gamma_\Pi,\Gamma)=\sum_{\substack{(\Gamma:\Gamma')=n\\ \Gamma'\cap\Sigma=\emptyset\\ \tau_{\Gamma'}^{-1}(A(\Gamma'\backslash\Gamma))=\Gamma_\Pi}}|\mathrm{Hom}(\Gamma',H)|.$$

The exponential generating function of the sequence $\left\{|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma,H\wr A_n)|\right\}_{n=0}^\infty$ is then obtained from Φ_Γ by replacing each variable z_{Γ_Π} with the corresponding series $\Theta_{\Gamma_\Pi}(z)$.

Corollary 2. *Let $\hat{Q}=(\Gamma,H,\Sigma,\Lambda)$ be an admissible quadruple with $\Lambda\neq\{1\}$ and $M\subseteq\mathbb{N}_0$ a set of non-negative integers. Assume that Γ contains only finitely many subgroups of index 2. Then we have*

$$\sum_{n\geq 0}\frac{|Hom_\Sigma^{\Lambda,M}(\Gamma,H\wr A_n)|}{n!}z^n=\Phi_\Gamma\big(\Theta_{\Gamma_\Pi}(z);\ (\Gamma:\Gamma_\Pi)\leq 2\big) \tag{26}$$

with Φ_Γ and Θ_{Γ_Π} as described above.

These formulas simplify somewhat further if we restrict to the case when $\Gamma=G$ is a finite group, $\Sigma=\emptyset$, $\Lambda=\mathbb{N}$, and $M=\mathbb{N}_0$:

$$s(d,G_\Pi)=\sum_{\substack{(G:U)=d\\ \tau_U^{-1}(A(U\backslash G))=G_\Pi}}|\mathrm{Hom}(U,H)|,$$

$$\Theta_{G_\Pi}(z)=\sum_{d|m}\frac{|H|^{d-1}s(d,G_\Pi)}{d}z^d,\quad m=|G|,$$

$$\Phi_G\big(z_{G_\Pi};\ (G:G_\Pi)\leq 2\big)=\sum_{\substack{\sum\limits_{e\notin G_\Pi}\nu(G_\Pi)\equiv 0(2)\\ (e\in E)}}\ \prod_{(G:G_\Pi)\leq 2}z_{G_\Pi}^{\nu(G_\Pi)}/\nu(G_\Pi)!,$$

$$\sum_{n=0}^\infty\frac{|\mathrm{Hom}\,(G,H\wr A_n)\,|}{n!}z^n=\Phi_G\big(\Theta_{G_\Pi}(z);\ (G:G_\Pi)\leq 2\big).$$

Our next two examples focus on this situation. Observe by the way that if G contains no subgroup of index 2 then $|\mathrm{Hom}(G,H\wr A_n)|=|\mathrm{Hom}(G,H\wr S_n)|$ for each finite group H and every $n\geq 0$.

Example 5. Let $G=C_m=\langle\zeta\rangle$ be a cyclic group of order m with m even. Then there are two subgroups of index at most 2 in G, $G_1=G$ and $G_2=\langle\zeta^2\rangle$, and we denote by z_1, ν_1 respectively z_2, ν_2 their associated variables. Put $E=\{\zeta\}$. Our exterior function Φ_G becomes

$$\Phi_G(z_1,z_2)=\sum_{\substack{\nu_1,\nu_2\geq 0\\ \nu_2\equiv 0(2)}}\frac{z_1^{\nu_1}z_2^{\nu_2}}{\nu_1!\nu_2!}=\exp(z_1)\cosh(z_2)=\frac{1}{2}\left\{e^{z_1+z_2}+e^{z_1-z_2}\right\}.$$

For $d\mid m$ consider the subgroup $U=\left\langle\zeta^d\right\rangle$ of index d in G. The generator ζ acts on the coset space $U\backslash G$ as the d-cycle $\tau_U(\zeta)=(U\cdot 1,U\cdot\zeta,\ldots,U\cdot\zeta^{d-1})$.

Consequently, $\tau_U(G) \subseteq A(U\backslash G)$ if and only if d is odd. From this we see that

$$s(d,G_1) = \begin{cases} \beta_{m/d}(H), & \text{d odd} \\ 0, & \text{d even} \end{cases} \quad \text{and} \quad s(d,G_2) = \begin{cases} \beta_{m/d}(H), & \text{d even} \\ 0, & \text{d odd.} \end{cases}$$

It follows that

$$\begin{aligned} \Theta_1(z) \quad &:= \quad \Theta_{G_1}(z) = \sum_{\substack{d\mid m \\ d\equiv 1(2)}} \frac{|H|^{d-1}\beta_{m/d}(H)}{d} z^d, \\ \Theta_2(z) \quad &:= \quad \Theta_{G_2}(z) = \sum_{\substack{d\mid m \\ d\equiv 0(2)}} \frac{|H|^{d-1}\beta_{m/d}(H)}{d} z^d, \end{aligned}$$

and hence

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{\beta_m(H \wr A_n)}{n!} z^n = \frac{1}{2}\Bigg\{ &\exp\left(\sum_{d\mid m} \frac{|H|^{d-1}\beta_{m/d}(H)}{d} z^d\right) \\ &+ \exp\left(\sum_{d\mid m} \frac{(-|H|)^{d-1}\beta_{m/d}(H)}{d} z^d\right)\Bigg\}. \end{aligned} \tag{27}$$

For odd m this equation coincides with (24). Thus we have obtained Chigira's result [1, Theorem 4]. The special case where $H = 1$ and $m = 2$ is already found in [18, Sect. 5].

Example 6. Consider $G = C_2 \times C_2 = \langle \zeta, \eta \rangle$. There are four subgroups in G of index at most 2: $G_0 = G$, $G_\zeta = \langle \zeta \rangle$, $G_\eta = \langle \eta \rangle$, and $G_{\zeta\eta} = \langle \zeta\eta \rangle$. With these subgroups we associate the variables $z_0, \nu_0, z_\zeta, \nu_\zeta, z_\eta, \nu_\eta, z_{\zeta\eta}, \nu_{\zeta\eta}$, respectively. Put $E = \{\zeta, \eta\}$. We have

$$\begin{aligned} \Phi_G(z_0, z_\zeta, z_\eta, z_{\zeta\eta}) &= \sum_{\substack{\nu_0,\nu_\zeta,\nu_\eta,\nu_{\zeta\eta}\geq 0 \\ \nu_\zeta+\nu_{\zeta\eta}\equiv 0(2) \\ \nu_\eta+\nu_{\zeta\eta}\equiv 0(2)}} \frac{z_0^{\nu_0} z_\zeta^{\nu_\zeta} z_\eta^{\nu_\eta} z_{\zeta\eta}^{\nu_{\zeta\eta}}}{\nu_0!\nu_\zeta!\nu_\eta!\nu_{\zeta\eta}!} \\ &= \exp(z_0)\big\{\cosh(z_\zeta)\cosh(z_\eta)\cosh(z_{\zeta\eta}) \ + \ \sinh(z_\zeta)\sinh(z_\eta)\sinh(z_{\zeta\eta})\big\} \\ &= \frac{e^{z_0}}{4}\left\{e^{z_\zeta+z_\eta+z_{\zeta\eta}} + e^{z_\zeta-z_\eta-z_{\zeta\eta}} + e^{z_\eta-z_\zeta-z_{\zeta\eta}} + e^{z_{\zeta\eta}-z_\zeta-z_\eta}\right\}. \end{aligned}$$

Moreover,

$$s(d,G_0) = \begin{cases} \beta_{2,2}(H), & \text{d} = 1 \\ 0, & \text{d} = 2 \\ 1, & \text{d} = 4 \end{cases}$$

and

$$s(d,G_\zeta) = s(d,G_\eta) = s(d,G_{\zeta\eta}) = \begin{cases} \beta_2(H), & \text{d} = 2 \\ 0, & \text{otherwise.} \end{cases}$$

This gives

$$\begin{aligned} \Theta_0(z) &= \beta_{2,2}(H)z + \frac{|H|^3}{4}z^4, \\ \Theta_\zeta(z) &= \Theta_\eta(z) = \Theta_{\zeta\eta}(z) = \frac{|H|\,\beta_2(H)}{2}z^2, \end{aligned}$$

and hence

(28)

$$\sum_{n=0}^{\infty} \frac{\beta_{2,2}(H \wr A_n)}{n!} z^n = \frac{1}{4}\left(\sum_{n=0}^{\infty} \frac{\beta_{2,2}(H \wr S_n)}{n!} z^n\right)\Big\{1 + 3\exp(-2\,|H|\,\beta_2(H)\,z^2)\Big\},$$

where

$$\sum_{n=0}^{\infty} \frac{\beta_{2,2}(H \wr S_n)}{n!} z^n = \exp\left(\beta_{2,2}(H)z + \frac{3\,|H|\,\beta_2(H)}{2}z^2 + \frac{|H|^3}{4}z^4\right);$$

cf. formula (25).

3.3. **The Weyl groups W_n.** If we put $H = C_2$, $N = 1$, and $\Pi_n = S_n$ $(n \geq 0)$ then $(H \wr \Pi_n)_N = W_n$ is the Weyl group of type D_n. In this case the set $\mathcal{U}_\Gamma$ can again be identified with the collection of subgroups of index at most 2 in Γ, and, after this identification, the exterior function Φ_Γ coincides with that calculated in the case where $N = H$, $\Pi_n = A_n$ $(n \geq 0)$, and $\Lambda \neq \{1\}$. For each subgroup $\Gamma_N \leq \Gamma$ of index at most 2 we form the power series

$$\Theta^{\Gamma_N}(z) := \sum_{n \in \Lambda} \frac{2^{n-1}\, s(n, \Gamma, \Gamma_N)}{n} z^n$$

with

$$s(n, \Gamma, \Gamma_N) = \sum_{\substack{(\Gamma:\Gamma')=n \\ \Gamma' \cap \Sigma = \emptyset}} \left|\big\{\chi \in \mathrm{Hom}\,(\Gamma', C_2) : \ \ \Gamma_\chi = \Gamma_N\big\}\right|$$

and

$$\Gamma_\chi = \Big\{\gamma \in \Gamma : \ \prod_{j=1}^{n} \chi(\gamma_j \gamma \overline{\gamma_j \gamma}^{-1}) = 1\Big\},$$

$\{\gamma_1, \ldots, \gamma_n\}$ being a right transversal for Γ' in Γ and $^- : \Gamma \to \{\gamma_1, \ldots, \gamma_n\}$ the corresponding representative function (Γ_χ is a subgroup of index at most 2 in Γ). By Theorem 1 the exponential generating function of the sequence $\big\{|\mathrm{Hom}_\Sigma^{\Lambda,M}(\Gamma, W_n)|\big\}_{n=0}^{\infty}$ is obtained from Φ_Γ by replacing each variable z_{Γ_N} with the corresponding series $\Theta^{\Gamma_N}(z)$.

Corollary 3. *Let $\hat{Q} = (\Gamma, C_2, \Sigma, \Lambda)$ be an admissible quadruple and $M \subseteq \mathbb{N}_0$ a set of non-negative integers. Assume that Γ contains only finitely many subgroups of index 2. Then we have*

$$\sum_{n \geq 0} \frac{|Hom_\Sigma^{\Lambda,M}(\Gamma, W_n)|}{n!} z^n = \Phi_\Gamma\big(\Theta^{\Gamma_N}(z);\ (\Gamma : \Gamma_N) \leq 2\big) \tag{29}$$

with Φ_Γ and $\Theta^{\Gamma_N}(z)$ as described above.

Again, these formulas simplify somewhat further if we restrict our attention to the case when $\Gamma = G$ is a finite group, $\Sigma = \emptyset$, $\Lambda = \mathbb{N}$, and $M = \mathbb{N}_0$:

$$s(d, G, G_N) = \sum_{(G:U)=d} \left|\left\{\chi \in \mathrm{Hom}(U, C_2) : \ G_\chi = G_N\right\}\right|$$

with

$$G_\chi = \left\{g \in G : \prod_{i=1}^{d} \chi(g_i g \overline{g_i g}^{-1}) = 1\right\}, \quad (G : G_\chi) \leq 2,$$

$$\Theta^{G_N}(z) = \sum_{d|m} \frac{2^{d-1}\, s(d, G, G_N)}{d} z^d, \quad m = |G|,$$

$$\Phi_G\big(z_{G_N};\ (G : G_N) \leq 2\big) = \sum_{\substack{\sum_{e \notin G_N} \nu(G_N) \equiv 0(2) \\ (e \in E)}} \ \prod_{(G:G_N)\leq 2} z_{G_N}^{\nu(G_N)} / \nu(G_N)!,$$

$$\sum_{n=0}^{\infty} \frac{|\mathrm{Hom}(G, W_n)|}{n!} z^n = \Phi_G\big(\Theta^{G_N}(z);\ (G : G_N) \leq 2\big).$$

Our last two examples will focus on this situation. Note that, similar to the case of alternating wreath products, if G is such that $s_G(2) = 0$ then $|\mathrm{Hom}(G, W_n)| = |\mathrm{Hom}(G, C_2 \wr S_n)|$ for all $n \geq 0$.

Example 7. We consider again the cyclic group $G = C_m$ of order m with m even, using the notation of Example 5. We have

$$\Phi_G(z_1, z_2) = \frac{1}{2}\left\{e^{z_1+z_2} + e^{z_1-z_2}\right\}.$$

For $d \mid m$ consider the subgroup $U = \langle \zeta^d \rangle$ of index d in G and a homomorphism $\chi : U \to C_2$. If m/d is odd then $\chi(u) = 1$ for all $u \in U$ and $G_\chi = G_1$. If on the other hand m/d is even then there are two homomorphisms $\chi : U \to C_2$: the trivial homomorphism $\chi_0 \equiv 1$ with $G_{\chi_0} = G_1$ and a non-trivial homomorphism χ_1. In the latter case

$$\prod_{i=0}^{d-1} \chi_1(\zeta^i \zeta \overline{\zeta^{i+1}}^{-1}) = \chi_1(\zeta^d) \neq 1,$$

i.e., $\zeta \notin G_{\chi_1}$ and hence $G_{\chi_1} = G_2$. We conclude that

$$s(d, G, G_1) = 1 \quad (d \mid m) \quad \text{and} \quad s(d, G, G_2) = \begin{cases} 1, & \text{m/d even} \\ 0, & \text{m/d odd.} \end{cases}$$

It follows that

$$\Theta^{G_1}(z) = \sum_{d|m} \frac{2^{d-1}}{d} z^d \quad \text{and} \quad \Theta^{G_2}(z) = \sum_{\substack{d|m \\ m/d \equiv 0(2)}} \frac{2^{d-1}}{d} z^d,$$

and hence

$$\sum_{n=0}^{\infty} \frac{\beta_m(W_n)}{n!} z^n = \frac{1}{2} \exp\left(\sum_{\substack{d|m \\ m/d\equiv 1(2)}} \frac{2^{d-1}}{d} z^d\right)\left\{1 + \exp\left(\sum_{\substack{d|m \\ m/d \equiv 0(2)}} \frac{2^d}{d} z^d\right)\right\}. \tag{30}$$

Since this formula also holds for odd m (in virtue of (24)) we have obtained Chigira's result [1, Theorem 5].

Example 8. We consider the Klein 4-group $G = C_2 \times C_2$, using the notation of Example 6. Again, the crux is to identify the invariant G_χ attached to a homomorphism $\chi : U \to C_2$, where U is a subgroup of G. One finds that:

(i) If $U = 1$ then $\chi \equiv 1$ and $G_\chi = G_0$.
(ii) If $(G : U) = 2$ then there are two homomorphisms $\chi : U \to C_2$, and $G_\chi = G_0$ in both cases.
(iii) If $U = G$ then there are four homomorphisms $\chi : U \to C_2$:
$\chi_0 \equiv 1 :\quad G_{\chi_0} = G_0$,
χ_ζ with $\chi_\zeta(\zeta) \neq 1$, $\chi_\zeta(\eta) = 1$: $G_{\chi_\zeta} = G_\eta$,
χ_η with $\chi_\eta(\zeta) = 1$, $\chi_\eta(\eta) \neq 1$: $G_{\chi_\eta} = G_\zeta$,
$\chi_{\zeta,\eta}$ with $\chi_{\zeta,\eta}(\zeta) \neq 1 \neq \chi_{\zeta,\eta}(\eta)$: $G_{\chi_{\zeta,\eta}} = G_{\zeta\eta}$.

It follows from these observations that

$$s(d, G, G_0) = \begin{cases} 6, & \mathrm{d} = 2 \\ 0, & \text{otherwise,} \end{cases}$$

and

$$s(d, G, G_\zeta) = s(d, G, G_\eta) = s(d, G, G_{\zeta\eta}) = \begin{cases} 1, & \mathrm{d} = 1 \\ 0, & \text{otherwise,} \end{cases}$$

and consequently

$$\Theta^{G_0}(z) = z + 6z^2 + 2z^4, \quad \Theta^{G_\zeta}(z) = \Theta^{G_\eta}(z) = \Theta^{G_{\zeta\eta}}(z) = z.$$

Plugging these expressions into the function Φ_G of Example 6 gives

$$\sum_{n=0}^{\infty} \frac{\beta_{2,2}(W_n)}{n!} z^n = \frac{1}{4} \left(\sum_{n=0}^{\infty} \frac{\beta_{2,2}(C_2 \wr S_n)}{n!} z^n \right) \left\{ 1 + 3\, e^{-4z} \right\} \tag{31}$$

with

$$\sum_{n=0}^{\infty} \frac{\beta_{2,2}(C_2 \wr S_n)}{n!} z^n = \exp\left(4z + 6z^2 + 2z^4\right);$$

cf. (25).

4. Asymptotics of the function $|Hom(G, H \wr S_n)|$

In [1, Sect. 2] Chigira indicates how Wilf's argument in [20] leading to an asymptotic formula for $\beta_m(S_n)$ with m fixed and $n \to \infty$ can be modified to yield a similar asymptotic formula for the function $\beta_m(H \wr S_n) = |\mathrm{Hom}(C_m, H \wr S_n)|$. Chigira's result is that for fixed m, as $n \to \infty$

$$\beta_m(H \wr S_n)/(n!) \sim \frac{\tau^n}{\sqrt{2\pi mn}} \exp\left(\sum_{d|m} \frac{|H|^{d-1}\, \beta_{m/d}(H)}{d\tau^d} \right),$$

where

$$\tau = \tau(m,n) := \big(n|H|\big)^{-1/m}\Bigg\{|H| + \frac{1}{mn}\sum_{\substack{d|m\\ d<m}}\big(n|H|\big)^{d/m}\,\beta_{m/d}(H) + \varepsilon_{m,n}\Bigg\}$$

and

$$\varepsilon_{m,n} := \begin{cases} \frac{(\beta_2(H))^2}{2m^2 n|H|}, & m \text{ even}\\ 0, & m \text{ odd};\end{cases}$$

cf. [1, Theorem 3]. Here, we combine formula (23) with some machinery developed in [14] to obtain a complete asymptotic expansion of the function $|\mathrm{Hom}(G, H\wr S_n)|$ for arbitrary finite groups G and H in terms of $m = |G|, |H|$, and the numbers $s_G^H(d) = \sum_{(G:U)=d} |\mathrm{Hom}(U,H)|$ for $d \mid m$. Its main term yields the asymptotic formula

(32)

$$|\mathrm{Hom}(G, H\wr S_n)| \sim K_G^H\,\big(n|H|\big)^{(1-1/m)n}\,\exp\Bigg(-\frac{m-1}{m}n + \frac{1}{|H|}\sum_{\substack{d|m\\ d<m}}\frac{s_G^H(d)}{d}\big(n|H|\big)^{d/m}\Bigg)$$

with

$$K_G^H := \begin{cases} m^{-1/2}, & m \text{ odd}\\ m^{-1/2}\exp\Big(-\frac{(s_G(m/2)\beta_2(H))^2}{2m|H|}\Big), & m \text{ even},\end{cases} \tag{33}$$

and the quotient

$$|\mathrm{Hom}(G, H\wr S_n)|\big/\big\{\text{right-hand side of (32)}\big\}$$

is expanded as a Poincaré series in $(n|H|)^{-1/m}$ with coefficients given explicitly in terms of m, $|H|$, and the $s_G^H(d)$. For $H = 1$ Theorems 4 and 5 below reduce to our earlier results [14, Theorems 5 – 6] on the asymptotic enumeration of finite group actions.[§]

4.1. **Asymptotic expansion of $e^{P(z)}$.** Consider the expansion

$$n! \approx (2\pi)^{1/2}\, n^{n+1/2} e^{-n}\Bigg\{1 + \sum_{\nu=1}^{\infty}\mathfrak{c}_\nu n^{-\nu}\Bigg\}\quad (n\to\infty) \tag{34}$$

of factorials derived from Stirling's asymptotic expansion of $\log(\Gamma(z))$. The coefficients $\mathfrak{c}_\nu$ in (34) can be expressed in terms of Bernoulli numbers via the formula

$$\exp\Bigg(\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}z^{-(2k-1)}\Bigg) = 1 + \sum_{\nu=1}^{\infty}\mathfrak{c}_\nu z^{-\nu}; \tag{35}$$

cf. for example [21, Sect. 12.33] or [5, Sect. 3.10].[¶] Formally, (34) can be interpreted as pertaining to the coefficients of the exponential function e^z expanded around the origin. In [14] an analogue of Stirling's expansion (34) is established

[§]Comments on the history of the latter problem can be found in the introduction of [14].

[¶]Note that the k^{th} Bernoulli number in the sense of Whittaker and Watson is $(-1)^{k-1}B_{2k}$ in de Bruijn's and our notation.

for the coefficients of entire functions of the form $f(z) = e^{P(z)}$ with $P(z) \in \mathbb{R}[z]$ a polynomial of degree at least 2 satisfying certain technical conditions. We also obtain there a (fairly complicated) analogue of the identity (35). In order to state this result we first have to give some definitions. Let $P(z) = \sum_{\mu=1}^{m} c_\mu z^\mu \in \mathbb{R}[z]$ be a polynomial of degree m. Define a sequence $\gamma_1, \gamma_2, \dots$ by

$$\gamma_\mu := \beta_1^{-(\mu+1)} \sum_{\substack{\alpha \geq 0, \nu \geq 1 \\ \alpha m + \nu = \mu + 1}} \binom{-\nu/m}{\alpha} \frac{\beta_\nu}{\nu},$$

where $\beta_\nu := \left\langle z^{\nu-1},\ \left(\sum_{\mu=1}^{m} \mu c_\mu z^{m-\mu}\right)^{\nu/m} \right\rangle$. Next, we introduce a sequence $\mathcal{A}_1, \mathcal{A}_2, \dots$ via the following hierarchy of definitions:

(i)

$$\psi_{\alpha,\lambda} := \left(m^2 c_m\right)^{-\alpha} \sum_{\substack{\mu_1 + \cdots + \mu_\alpha = \alpha m - \lambda \\ 1 \leq \mu_1, \dots, \mu_\alpha \leq m/2}} \mu_1^2 \dots \mu_\alpha^2 c_{\mu_1} \dots c_{\mu_\alpha},$$

(ii)

$$\chi_{\mu,k,\kappa} := \left(\frac{m^2 c_m}{2}\right)^{-(k+\kappa)} \sum_{\substack{j_1 + \cdots + j_\kappa = 2k \\ j_1, \dots, j_\kappa \geq 1}} \sum_{\substack{\mu_1 + \cdots + \mu_\kappa = \mu \\ 1 \leq \mu_1, \dots, \mu_\kappa \leq m}} \prod_{\mathfrak{l}=1}^{\kappa} \left[\frac{\mu_\mathfrak{l} c_{\mu_\mathfrak{l}}}{j_\mathfrak{l} + 2} \left((-1)^{j_\mathfrak{l}} + \binom{\mu_\mathfrak{l} - 1}{j_\mathfrak{l} + 1}\right)\right],$$

(iii)

$$\tilde{\chi}_{\nu,k,\kappa} := \chi_{m(k+\kappa)-\nu,k,\kappa} + \sum_{\substack{\lambda, \mu \geq 1 \\ m(k+\kappa)+\lambda = \mu + \nu}} \sum_{\alpha \geq 1} (-1)^\alpha \binom{k + \kappa + \alpha - 1}{\alpha} \psi_{\alpha,\lambda}\, \chi_{\mu,k,\kappa},$$

(iv)

$$\chi_\nu := \sum_{k \geq 1} \sum_{\kappa=1}^{2k} (-1)^{k+\kappa}\ 2^{1-2(k+\kappa)} \frac{(2k + 2\kappa - 1)!}{\kappa!(k + \kappa - 1)!} \tilde{\chi}_{\nu,k,\kappa},$$

(v)

$$\tilde{\chi}_\nu := \chi_\nu + \sum_{\beta \geq 1} \sum_{\lambda \geq 1} \binom{\beta}{\lambda} \chi_\beta \sum_{\substack{\nu_1 + \cdots + \nu_\lambda = \nu - \beta \\ \nu_1, \dots, \nu_\lambda \geq m/2}} \gamma_{\nu_1} \dots \gamma_{\nu_\lambda},$$

(vi)

$$\mathcal{A}_\nu := \sum_{\lambda \geq 1} (-1)^{\lambda-1} \frac{1}{\lambda} \sum_{\substack{\nu_1 + \cdots + \nu_\lambda = \nu \\ \nu_1, \dots, \nu_\lambda \geq 1}} \tilde{\chi}_{\nu_1} \cdots \tilde{\chi}_{\nu_\lambda}.$$

Finally, we define a sequence $\mathcal{B}_1, \mathcal{B}_2, \ldots$ by

(36)

$$\mathcal{B}_\nu = \mathcal{B}_\nu(P) :=$$

$$\mathcal{A}_\nu + mc_m \sum_{\lambda\geq 1}(-1)^\lambda \frac{1}{\lambda}\left(\frac{\psi_{\lambda,\nu}}{2mc_m} - \sum_{\substack{\nu_1+\cdots+\nu_\lambda=m+\nu\\ \nu_1,\ldots,\nu_\lambda\geq m/2}} \gamma_{\nu_1}\cdots\gamma_{\nu_\lambda}\right)$$

$$+\sum_{1\leq\mu\leq m}\sum_{\lambda\geq 1}(-1)^\lambda\binom{\mu+\lambda-1}{\lambda} c_\mu \sum_{\substack{\nu_1+\cdots+\nu_\lambda=\mu+\nu\\ \nu_1,\ldots,\nu_\lambda\geq m/2}} \gamma_{\nu_1}\cdots\gamma_{\nu_\lambda}$$

$$+\sum_{\mu\geq 0}\sum_{\alpha\geq 1}\sum_{\lambda\geq 1}(-1)^{\alpha+\lambda}\frac{1}{2\lambda}\big(m^2c_m\big)^{-\lambda} \sum_{\substack{\nu_1+\cdots+\nu_\alpha=\nu-\mu\\ \nu_1,\ldots,\nu_\alpha\geq m/2}} \gamma_{\nu_1}\cdots\gamma_{\nu_\alpha} \sum\prod_{\mathfrak{l}=1}^{\lambda}\left[\mu_{\mathfrak{l}}^2 c_{\mu_{\mathfrak{l}}}\binom{\mu_{\mathfrak{l}}+\alpha_{\mathfrak{l}}-1}{\alpha_{\mathfrak{l}}}\right],$$

the last sum being extended over those (2λ)-tuples $(\mu_1,\ldots,\mu_\lambda,\alpha_1,\ldots,\alpha_\lambda)$ for which $\mu_1+\cdots+\mu_\lambda=\lambda m-\mu$, $\alpha_1+\cdots+\alpha_\lambda=\alpha$, $1\leq\mu_1,\ldots,\mu_\lambda\leq m$, $\alpha_1,\ldots,\alpha_\lambda\geq 0$, and $(\mu_1,\alpha_1),\ldots,(\mu_\lambda,\alpha_\lambda)\neq(m,0)$.

Having given these definitions we can now state the result mentioned above.

Theorem 2. ([14, Theorem 2]). *Suppose that the polynomial $P(z)=\sum_{\mu=1}^m c_\mu z^\mu\in\mathbb{R}[z]$ has degree $m\geq 2$ and meets the conditions*

($\mathcal{P}1$) *$c_\mu\geq 0$ for $1\leq\mu\leq m$,*
($\mathcal{P}2$) *$c_1\neq 0$,*
($\mathcal{P}3$) *$c_\mu=0$ for $m/2<\mu<m$.*

Then there exist constants $\mathcal{C}_\nu=\mathcal{C}_\nu(P)$ such that the coefficients α_n of the entire function $\exp(P(z))=\sum_{n=0}^\infty\alpha_n z^n$ have the asymptotic expansion

$$\alpha_n \approx \frac{K}{\sqrt{2\pi n}}\, n_0^{-n/m}\exp\Big(P\big(n_0^{1/m}\big)\Big)\left\{1+\sum_{\nu=1}^\infty \mathcal{C}_\nu n_0^{-\nu/m}\right\}\quad (n\to\infty), \tag{37}$$

where $n_0:=n/(mc_m)$ and

$$K=K(P):=\begin{cases} m^{-1/2}, & m \text{ odd}\\ m^{-1/2}\exp\Big(-\frac{c_{m/2}^2}{8c_m}\Big), & m \text{ even.}\end{cases}$$

Moreover, the $\mathcal{C}_\nu$'s satisfy the formal identity

$$1+\sum_{\nu=1}^\infty \mathcal{C}_\nu z^{-\nu}=\exp\left(\sum_{\nu=1}^\infty \mathcal{B}_\nu z^{-\nu}\right), \tag{38}$$

where the $\mathcal{B}_\nu=\mathcal{B}_\nu(P)$ are given by (36).

From Theorem 2 one also deduces an asymptotic expansion for the quotient α_n/α_{n-1} of successive coefficients of the function $\exp(P(z))$.

Theorem 3. ([14, Theorem 3]). *In the notation and under the hypotheses of Theorem 2 there exist constants $\tilde{Q}_\nu = \tilde{Q}_\nu(P)$ such that α_n/α_{n-1} has the asymptotic expansion*

$$\alpha_n/\alpha_{n-1} \approx n_0^{-1/m}\Big\{1 + \sum_{\nu=1}^{\infty} \tilde{Q}_\nu n_0^{-\nu/m}\Big\} \quad (n \to \infty). \tag{39}$$

Moreover, the $\tilde{Q}_\nu$ satisfy the identity

$$1 + \sum_{\nu=1}^{\infty} \tilde{Q}_\nu z^{-\nu} = \exp\Big(\sum_{\nu=1}^{\infty} Q_\nu z^{-\nu}\Big), \tag{40}$$

where

$$\begin{aligned} Q_\nu = Q_\nu(P) := &- \sum_{1 \leq \lambda < \nu/m} \binom{\nu/m - 1}{\lambda} (mc_m)^{-\lambda} \mathcal{B}_{\nu - m\lambda}(P) \\ &+ \begin{cases} -\frac{\nu+m-2}{2\nu(\nu+m)m^{\nu/m-1}c_m^{\nu/m}}; & \nu \equiv 0 \quad (m) \\ 0; & \nu \equiv i \quad (m),\ 1 \leq i < m/2 \\ (-1)^{(\nu-i)/m}\binom{1-i/m}{1+(\nu-i)/m}(mc_m)^{-1-(\nu-i)/m}c_{m-i}; & \nu \equiv i \quad (m),\ m/2 \leq i < m. \end{cases} \end{aligned}$$

4.2. **The function $|\mathbf{Hom}(G, H \wr S_n)|$.** With each pair (G, H) of finite groups we associate the polynomial

$$P_G^H(z) := \sum_{d|m} \frac{|H|^{d-1} s_G^H(d)}{d} z^d, \quad m = |G|,$$

where $s_G^H(d) = \sum_{(G:U)=d} |\mathrm{Hom}(U, H)|$. By (23) we have

$$|\mathrm{Hom}(G, H \wr S_n)| = n!\,\big\langle z^n, \exp(P_G^H(z))\big\rangle, \quad n \geq 0,$$

and if $G \neq 1$ the polynomial $P_G^H(z)$ satisfies the hypotheses of Theorem 2. Hence, Theorem 2 combined with (34) and (35) yields the following.

Theorem 4. *Let G and H be finite groups, $|G| = m \geq 2$. Then there exist constants $\tilde{\mathcal{C}}_\nu = \tilde{\mathcal{C}}_\nu(G, H)$ such that the function $|Hom(G, H \wr S_n)|$ has the asymptotic expansion*

(41)

$$\begin{aligned} |Hom(G, H \wr S_n)| \approx K_G^H \big(n|H|\big)^{(1-1/m)n} \exp\Bigg(-\frac{m-1}{m}n + \frac{1}{|H|}\sum_{\substack{d|m\\ d<m}} \frac{s_G^H(d)}{d}\big(n|H|\big)^{d/m}\Bigg) \\ \times \Big\{1 + \sum_{\nu=1}^{\infty} \tilde{\mathcal{C}}_\nu(G, H)\big(n|H|\big)^{-\nu/m}\Big\} \quad (n \to \infty), \end{aligned}$$

with K_G^H as defined in (33). *Moreover, the $\tilde{\mathcal{C}}_\nu$'s satisfy the identity*

$$1 + \sum_{\nu=1}^{\infty} \tilde{\mathcal{C}}_\nu(G, H) z^{-\nu} = \exp\Big(\sum_{\nu=1}^{\infty} \tilde{\mathcal{B}}_\nu(G, H) z^{-\nu}\Big), \tag{42}$$

where

$$\tilde{\mathcal{B}}_\nu(G,H) := |H|^\nu\,\mathcal{B}_\nu(P_G^H) + \begin{cases} \frac{m^2|H|^{\nu/m}B_{\nu/m+1}}{\nu(\nu+m)}, & \nu\equiv m \quad (2m)\\ 0, & \textit{otherwise},\end{cases}$$

with $B_{\nu/m+1}$ *a Bernoulli number and* $\mathcal{B}_\nu(P_G^H)$ *as given by* (36).

Corollary 4. *Let G and H be finite groups, $|G| = m$. Then the function $|Hom(G, H\wr S_n)|$ satisfies the asymptotic formula* (32).

This is trivial for $m = 1$ and follows from Theorem 4 otherwise. An alternative approach to Corollary 4 is provided by [14, Theorem 1].

Denote by T_n the number of C_2-actions on an n-set. In [2] Chowla et al. showed among other things that the quotient T_n/T_{n-1} is asymptotically equal to $n^{1/2}$, [2, Theorem 3], and conjectured that, more precisely, an asymptotic expansion of the form

$$T_n/T_{n-1} \approx n^{1/2}\left\{1+\sum_{\nu=1}^{\infty}\tilde{Q}_\nu n^{-\nu/2}\right\} \quad (n\to\infty)$$

holds with appropriate constants $\tilde{Q}_\nu$. This was first proved by Moser and Wyman, [18, 2.12], who also obtained the first two coefficients explicitly as $\tilde{Q}_1 = 1/2$ and $\tilde{Q}_2 = -1/8$. Later, the conjecture was extended to arbitrary cyclic groups to the effect that the quotient $\beta_m(S_n)/\beta_m(S_{n-1})$ should have an asymptotic expansion for fixed m as $n\to\infty$ of the form $n^{1-1/m}$ times a Poincaré series in $n^{-1/m}$ whose coefficients depend only on m. In [14] an asymptotic expansion of this form was deduced for the quotient $|\mathrm{Hom}(G,S_n)|/|\mathrm{Hom}(G,S_{n-1})|$ associated with an arbitrary finite group G, thereby in particular establishing Chowla's conjecture; cf. [14, Theorem 6]. Combining Theorem 3 with formula (23) now yields the following generalization of this result.

Theorem 5. *Let G and H be finite groups, $|G| = m \geq 2$. Then there exist constants $\tilde{Q}_\nu = \tilde{Q}_\nu(G,H)$ such that as $n\to\infty$*

(43)

$$|Hom(G,H\wr S_n)|/|Hom(G,H\wr S_{n-1})| \approx \big(n|H|\big)^{1-1/m}\left\{1+\sum_{\nu=1}^{\infty}\tilde{Q}_\nu(G,H)\big(n|H|\big)^{-\nu/m}\right\}.$$

Moreover, the $\tilde{Q}_\nu$'s satisfy the formal identity

$$1+\sum_{\nu=1}^{\infty}\tilde{Q}_\nu(G,H)z^{-\nu} = \exp\left(\sum_{\nu=1}^{\infty}Q_\nu(G,H)z^{-\nu}\right), \tag{44}$$

where

$$Q_\nu = Q_\nu(G,H) := |H|^\nu Q_\nu(P_G^H) = -\sum_{1\leq\lambda<\nu/m}\binom{\nu/m-1}{\lambda}|H|^{\nu-\lambda(m-1)}\mathcal{B}_{\nu-m\lambda}(P_G^H)$$
$$+\begin{cases} -\frac{m(\nu+m-2)|H|^{\nu/m}}{2\nu(\nu+m)}; & \nu\equiv 0 \quad (m)\\ \binom{d/m}{(\nu+d)/m}(-|H|)^{(\nu-m+d)/m}\,\frac{s_G^H(d)}{d}; & \nu\equiv m-d \quad (m),\ d\mid m,\ d<m\\ 0; & \textit{otherwise.}\end{cases}$$

Apart from their intrinsic interest, Theorems 4 and 5 also lead to a substantial improvement over the results obtained in [15] concerning the subgroup growth of virtually free groups. This will be explained in a separate paper [16].

4.3. **The functions $|\mathbf{Hom}(G, H \wr A_n)|$ and $|\mathbf{Hom}(G, W_n)|$.** Having obtained a precise asymptotic estimate for the function $|\mathrm{Hom}(G, H\wr S_n)|$ associated with a pair (G,H) of finite groups one may ask about the asymptotic behaviour of closely related arithmetic functions such as $|\mathrm{Hom}(G, H\wr A_n)|$ or $|\mathrm{Hom}(G, W_n)|$. Here we have the following.

Conjecture. (a) *For each finite group G there exists a constant $A_1 = A_1(G) > 1$ such that*

$$|\mathrm{Hom}(G, W_n)| = (1+s_G(2))^{-1}|\mathrm{Hom}(G, C_2\wr S_n)|\big\{1+\mathcal{O}\big(A_1^{-n^{1/m}}\big)\big\},\quad m=|G|.$$

(b) *For each pair (G,H) of finite groups there exists a constant $A_2 = A_2(G,H) > 1$ such that*

$$|\mathrm{Hom}(G, H\wr A_n)| = (1+s_G(2))^{-1}|\mathrm{Hom}(G, H\wr S_n)|\big\{1+\mathcal{O}\big(A_2^{-n^{1/m}}\big)\big\},\quad m=|G|.$$

In particular, the asymptotic expansions provided by Theorem 4 *for* $|\mathrm{Hom}(G, C_2\wr S_n)|$ and $|\mathrm{Hom}(G, H\wr S_n)|$ *carry over without change to* $(1+s_G(2))|\mathrm{Hom}(G, W_n)|$ *respectively* $(1+s_G(2))|\mathrm{Hom}(G, H\wr A_n)|$.

Our final result provides some evidence for this conjecture.

Proposition 3. (i) *For each $m\in 2\mathbb{N}$ and every real number ε satisfying $0<\varepsilon<2$ we have*

$$\beta_m(W_n) = \frac{\beta_m(C_2\wr S_n)}{2}\Big\{1+\mathcal{O}\big(A_1^{-\sqrt{n}}\big)\Big\} \tag{45}$$

with $A_1 = A_1(m,\varepsilon) := \begin{cases} e^{(2-\varepsilon)/\sqrt{2}}, & m=2\\ e^{2\sqrt{2}/m}, & m>2.\end{cases}$

(ii) *For each real number ε with $0<\varepsilon<4$ we have*

$$\beta_{2,2}(W_n) = \frac{\beta_{2,2}(C_2\wr S_n)}{4}\Big\{1+\mathcal{O}\big(e^{-(4-\varepsilon)(n/8)^{1/4}}\big)\Big\}. \tag{46}$$

(iii) *We have*

$$\beta_2(H\wr A_n) = \frac{\beta_2(H\wr S_n)}{2}\Big\{1+\mathcal{O}\big(e^{-\beta_2(H)\sqrt{n/|H|}}\big)\Big\}. \tag{47}$$

Proof. (i) By formula (30) we have

$$\beta_m(W_n) = \frac{\beta_m(C_2 \wr S_n)}{2}\big\{1 + q_n\big\}$$

with

$$q_n := \Big\langle z^n,\ \exp\Bigg(\sum_{\substack{d\mid m\\ m/d\equiv 1(2)}} \frac{2^{d-1}}{d} z^d\Bigg)\Big\rangle \Big/ \Big\langle z^n, \exp\big(P_{C_m}^{C_2}(z)\big)\Big\rangle$$

and

$$P_{C_m}^{C_2}(z) = \sum_{\substack{d\mid m\\ m/d\equiv 1(2)}} \frac{2^{d-1}}{d} z^d \ + \sum_{\substack{d\mid m\\ m/d\equiv 0(2)}} \frac{2^{d}}{d} z^d.$$

Since $P_{C_m}^{C_2}(z)$ satisfies the hypotheses of Theorem 2 we find in particular that for $n \geq 1$

$$\begin{aligned}&\big\langle z^n,\ \exp(P_{C_m}^{C_2}(z))\big\rangle \gg \\ &\quad n^{-1/2}\big(\frac{n}{2^{m-1}}\big)^{-n/m} \exp\Bigg(\sum_{\substack{d\mid m\\ m/d\equiv 1(2)}} \frac{2^{d-1}}{d}\big(\frac{n}{2^{m-1}}\big)^{d/m} + \sum_{\substack{d\mid m\\ m/d\equiv 0(2)}} \frac{2^{d}}{d}\big(\frac{n}{2^{m-1}}\big)^{d/m}\Bigg).\end{aligned}$$

As m is even, the polynomial occurring in the numerator of q_n does not satisfy condition $(\mathcal{P}2)$ of Theorem 2, but this can be remedied by adding an extra term εz and applying Theorem 2 to the resulting polynomial. Fixing a real number ε with $0 < \varepsilon < 2$ we find in this way that

$$\begin{aligned}0 &\leq \big\langle z^n,\ \exp\Bigg(\sum_{\substack{d\mid m\\ m/d\equiv 1(2)}} \frac{2^{d-1}}{d} z^d\Bigg)\big\rangle \\ &\leq \big\langle z^n,\ \exp\Bigg(\varepsilon z + \sum_{\substack{d\mid m\\ m/d\equiv 1(2)}} \frac{2^{d-1}}{d} z^d\Bigg)\big\rangle \\ &\ll n^{-1/2}\big(\frac{n}{2^{m-1}}\big)^{-n/m} \exp\Bigg(\varepsilon\big(\frac{n}{2^{m-1}}\big)^{1/m} + \sum_{\substack{d\mid m\\ m/d\equiv 1(2)}} \frac{2^{d-1}}{d}\big(\frac{n}{2^{m-1}}\big)^{d/m}\Bigg).\end{aligned}$$

From these estimates we conclude that

$$q_n \ll \exp\Bigg(-(2-\varepsilon)\big(\frac{n}{2^{m-1}}\big)^{1/m} - \sum_{\substack{1<d\mid m\\ m/d\equiv 0(2)}} \frac{2^{d}}{d}\big(\frac{n}{2^{m-1}}\big)^{d/m}\Bigg) \leq A_1^{-\sqrt{n}}$$

with $A_1 = A_1(m, \varepsilon)$ as defined in the proposition, whence (45).

(ii) This follows by a similar argument using formula (31) instead of (30).

(iii) By formula (27) we have

$$\beta_2(H \wr A_n) = \frac{\beta_2(H \wr S_n)}{2}\{1 + q_n\}$$

with $q_n := \langle z^n, \exp\big(\beta_2(H)z - \frac{|H|}{2}z^2\big)\rangle / \langle z^n, \exp(P_{C_2}^H(z))\rangle$ and $P_{C_2}^H(z) = \beta_2(H)z + \frac{|H|}{2}z^2$. By Theorem 2

$$\langle z^n, \exp\big(P_{C_2}^H(z)\big)\rangle \gg n^{-1/2}\big(\frac{n}{|H|}\big)^{-n/2} \exp\Big(\beta_2(H)\big(\frac{n}{|H|}\big)^{1/2} + \frac{n}{2}\Big), \quad n \geq 1. \tag{48}$$

The coefficients of $\exp\big(\beta_2(H)z - \frac{|H|}{2}z^2\big)$ cannot be estimated in the same way; as it turns out however they are related to values of the Hermite polynomials. Put

$$V_n := n!\,\langle z^n, \exp\Big(\beta_2(H)z - \frac{|H|}{2}z^2\Big)\rangle, \quad n \geq 0.$$

From the definition we see that V_n satisfies the recursion

$$\begin{aligned} &V_n = \beta_2(H)V_{n-1} - (n-1)|H|V_{n-2}, \quad n \geq 2 \\ &V_0 = 1, \quad V_1 = \beta_2(H). \end{aligned}$$

Comparing this recursion with the recurrence

$$\begin{aligned} &H_n(x) = 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x), \quad n \geq 2 \\ &H_0(x) = 1, \quad H_1(x) = 2x \end{aligned}$$

for the Hermite polynomials one finds by induction on n that

$$V_n = \big(\frac{|H|}{2}\big)^{n/2} H_n\big(\beta_2(H)/\sqrt{2|H|}\big), \quad n \geq 0. \tag{49}$$

Combining (49) with [19, 8.22.8] we obtain the asymptotic relation

$$\begin{aligned} V_n = \big(\frac{|H|}{2}\big)^{n/2} \frac{\Gamma(n+1)}{\Gamma(n/2+1)} &\exp\big(\frac{(\beta_2(H))^2}{4|H|}\big) \\ &\times \big\{\cos\big(\sqrt{2n+1}\beta_2(H)/\sqrt{2|H|} - n\pi/2\big) + o(1)\big\}, \end{aligned}$$

which together with the asymptotic formula for the gamma function implies that

$$\langle z^n, \exp\Big(\beta_2(H)z - \frac{|H|}{2}z^2\Big)\rangle \ll n^{-1/2}\big(\frac{e|H|}{n}\big)^{n/2}. \tag{50}$$

From (48) and (50) we find that $q_n \ll e^{-\beta_2(H)\sqrt{n/|H|}}$, whence (47). □

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Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, 33501 Bielefeld, Germany
E-mail address: thomue@mathematik.uni-bielefeld.de