On a conjecture of Kierstead *

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Abstract

The aim of this note is to disprove a conjecture of H. Kierstead [5].

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1 Introduction

For a natural n let $[n] = \{1, 2, \ldots, n\}$ and $P = (X, \leq)$ be a finite partially ordered set. A linear extension of P is an injection $\sigma : X \to [|X|]$ such that $x \leq y$ implies $\sigma(x) \leq \sigma(y)$. The dimension dim P of P is the least t such that there is a collection Σ of t linear extensions such that not $x \leq y$ implies $\sigma(y) < \sigma(x)$, for some $\sigma \in \Sigma$.

Let B_n denote the Boolean lattice obtained by ordering the power set of $[n]$ by inclusion. Let $B_n(j, k)$ be the restriction of B_n to subsets of size j and k, i.e. $B_n(j, k)$ consists of the j's and k's level of B_n and the respective ordering by inclusion. Let $\dim(j, k; n) = \dim B_n(j, k)$. The important special case which appeared in many papers is $\dim(1, k; n)$. To understand better why this is important we need some definitions.

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The comparability graph G of a partially ordered set P is defined to be $G = (X, E)$, where $E = \{xy : x, y \in X and\ x < y \text{ or } y < x\}$. For each positive integer Δ, let $g(\Delta)$ denote the maximum value of the dimension of an ordered set whose comparability graph has maximum degree Δ. The main result in [3] is the following upper bound of $g(\Delta)$:

For all Δ,

$$g(\Delta) = O(\Delta \log^2 \Delta). \tag{1}$$

To prove this result Fűredi and Kahn proved that if P is an ordered set whose comparability graph has maximum degree Δ, then

$$\dim P = O(\frac{\Delta}{\log \Delta} \dim(1, \log \Delta; \Delta \log \Delta)).$$

Erdős, Kierstead and Trotter [2] used probabilistic methods to show that

$$g(\Delta) = \Omega((\Delta \log \Delta).$$

Thus there has been considerable interest in the correct value of $\dim(1, \log n; n)$. In [6] it was proved that

$$\dim(1, \log n; n) \leq \log^3 n.$$

Kierstead [5] proved that

$$\dim(1, \log n; n) = \Omega(\log^3 n / \log \log n).$$

To prove this he used well known combinatorial objects, so-called (n, k, l)-packings.

A family F of k-subsets of $[n]$ is a (n, k, l)-packing iff the intersection of any two distinct elements has cardinality less than l. Let $m(n, k, l)$ be the greatest integer M such that there exists a (n, k, l)-packing of cardinality M.

2 A conjecture

In [5] the following conjecture was stated.

Conjecture. There exists a constant $c > 0$ such that with the following notations

$$N = \frac{c \log^3 n}{\log\log n}, \quad k = \frac{\log^2 n}{\log\log n}, \quad l = \frac{\log n}{\log\log n}$$

the inequality

$$m(N, k, l) \geq n$$

holds for all n.

It was noted in [5] that if this conjecture is true than the Fűredi-Kahn upper bound for $g(\Delta)$ may be improved. Our aim here is to disprove this conjecture and to prove the following theorem.

Theorem. With the above notations

$$m(N, k, l) = o(n).$$

3 Proof of the theorem

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To prove the above Theorem we will use the folowing inequality obtained in [1]

$$m(N, k, l) \leq \frac{\binom{N}{l-d-1}}{\binom{k}{l-d-1}} \frac{1}{1 - \frac{(d+1)(N-k)}{(k-l+1)(k-l+d+2)}}, \tag{2}$$

where d, $d \leq l - 2$, is a parameter such that

$$(k - l + d + 2)(k - l + 1) > (d + 1)(N - l + d + 1). \tag{3}$$

We choose

$$d = \lfloor \frac{k^2}{c_1 N} \rfloor = \lfloor \frac{\log n}{c_1 c \log\log n} \rfloor = \lfloor \frac{l}{c_1 c} \rfloor,$$

where $c_1 > 1$ is some constant. As the function $m(N, k, l)$ is monotone in N we may assume also that $c > 1$.

It is not difficult to verify that condition (3) is valid for our choice of parameters. Indeed, the asymptotics of the left side of (3) is $\frac{\log^4 n}{(\log\log n)^2}$ and the asymptotics of the right side of (3) is $\frac{\log^4 n}{c_1 (\log\log n)^2}$. Taking into account that $c_1 > 1$ we obtain that (3) holds for sufficiently large n and we may apply the inequality (2).

Note that with our notations we have $l^2 = o(N)$ and $l^2 = o(k)$. Thus we may estimate the asymptotics of the first term of the right side of (2) as follows

$$\frac{\binom{N}{l-d-1}}{\binom{k}{l-d-1}} \sim (\frac{N}{k})^{l-d-1}.$$

Then (2) implies

$$m(N,k,l) \leq \frac{1+o(1)}{1-\frac{1}{c_1}}(\frac{N}{k})^{l(1-\frac{1}{cc_1})}O((c\log n)^{\frac{\log n}{\log\log n}(1-\frac{1}{cc_1})}) =$$

$$= 2^{(1+o(1))(1-\frac{1}{cc_1})\log n} = n^{(1+o(1))(1-\frac{1}{cc_1})} = o(n),$$

because c_1 and c are constants such that $cc_1 > 1$. Thus, the Theorem is proved and the conjecture is false.

In order to keep our proof independent of [1] we give a proof of inequality (2). We will use two well-known inequalities for $m(n,k,l)$. The first one is the recurrence inequality

$$m(n,k,l) \leq \frac{n}{k}m(n-1,k-1,l-1). \tag{4}$$

The second one is Johnson's inequality [4]

$$m(n,k,l) \leq \frac{n(k-l+1)}{n(k-l+1)-k(n-k)}, \tag{5}$$

which holds when the denominator is positive. Inequality (4) may be applied $l-d-2$ times, obtaining

$$m(n,k,l) \leq \frac{(n)_{l-d-2}}{(k)_{l-d-2}}m(n-l+d+2,k-l+d+2,d+2). \tag{6}$$

Then we apply to the last term inequality (5) and obtain

$$m(n-l+d+2,k-l+d+2,d+2) \leq$$

$$\leq \frac{(n-l+d+2)(k-l+1)}{(n-l+d+2)(k-l+1)-(k-l+d+2)(n-k)} =$$

$$= \frac{n-l+d+2}{k-l+d+2} \, \frac{(k-l+d+2)(k-l+1)}{(k-l+d+2)(k-l+1)-(d+1)(n-k)}.$$

Combining this with (6) we obtain inequality (2).

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