ON ISOMETRIES OF THE HYPERBOLIC FIBONACCI MANIFOLDS *

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Abstract

The closed orientable hyperbolic three-dimensional manifolds uniformized by the Fibonacci groups were constructed by H.Helling, A.Kim and J.Mennicke. These manifolds can be obtained as the cyclic branched coverings of the figure-eight knot. In the present paper the isometry groups of these manifolds are described. The quotient orbifolds arising under the isometries actions are discussed.
Keywords: isometry group, hyperbolic 3-manifold, cyclic covering.

1. Introduction

In the present paper we study the isometry groups of the three-dimensional orientable closed hyperbolic manifolds uniformized by the Fibonacci groups. Recall, that *the Fibonacci groups* $F(2,m)$, $m \geq 3$, were introduced by Conway [3] and are defined by the following presentation:

$$F(2,m) = \langle x_1, \ldots, x_m \mid x_i \, x_{i+1} = x_{i+2} \quad i = 1, \ldots, m \rangle,$$

where all suffices are taken by mod m.

The review of some results about finite and arithmetic properties of the Fibonacci groups and of their generalizations can be found in [24]. The using of topological and geometrical methods for investigation of the Fibonacci group was started in the paper [9] of H. Helling, A. Kim and J. Mennicke. They constructed a family of three-dimensional manifolds M_n, $n \geq 2$, such that $\pi_1(M_n) \cong F(2,2n)$. Moreover, it was shown in [9] that manifolds M_n have geometric structures of the constant curvature. More exactly, the group

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$F(2,4) \cong \mathbb{Z}_5$ acts by isometries on the spherical space $\mathbb{S}^3$ and the manifold M_2 is the lens space $L(5,2)$. The group $F(2,6)$ is isomorphic to the Euclidean crystallographic group acting in the space $\mathbb{E}^3$ and the manifold M_3 is the Euclidean Hanzsche-Wendt manifold [29]. For $n \geq 4$ the group $F(2,2n)$ is isomorphic to a discrete cocompact subgroup of the group $\mathrm{PSL}(2,\mathbb{C})$ acting in the Lobachevsky space $\mathbb{H}^3$ without fixed points. Manifolds M_n, $n \geq 4$, are *hyperbolic*, i.e. have a constant negative curvature. Manifolds M_n introduced in [9] are referred to *Fibonacci manifolds*.

Recall some geometric and topological properties of the manifolds M_n. In [26] the exact formulae for volumes of the hyperbolic Fibonacci manifolds were obtained in terms of the Lobachevsky function. It was shown in [11] that for $n \geq 2$ the manifold M_n is the n-fold cyclic covering of the three-dimensional sphere branched over the *figure-eight knot*. By [27] for $n \geq 2$ the manifold M_n is the two-fold covering of the three-dimensional sphere, branched over the closed three-strings braid $(\sigma_1 \sigma_2^{-1})^n$. In particular, M_2 is the two-fold covering branched over the figure-eight knot and M_3 is the two-fold covering branched over the three components link *Borromean rings*. In [14] the Fibonacci manifolds were obtained by $1/0$ - Dehn surgery on the manifold $\mathcal{M}(\phi)$ that is the punctured torus bundle. In this case the homeomorphism ϕ is induced by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2$.

In the present paper we develop the geometric approach to the studding of the Fibonacci groups. We investigate the isometry groups of the hyperbolic Fibonacci manifolds. In virtue of the rigidity theorem [1], the isometry group $Isom(M_n)$ of a hyperbolic manifold M_n is isomorphic to the group $Out(F(2,2n))$ of outer automorphisms of its fundamental group

$$F(2,2n) = \langle x_1, \ldots, x_{2n} \mid x_i \, x_{i+1} = x_{i+2} \quad i = 1, \ldots, 2n \rangle.$$

The obvious automorphism $x_i \longrightarrow x_{i+1}$, $i = 1, \ldots, 2n$, generates a cyclic subgroup $\mathbb{Z}_{2n}$ of the isometry group $Isom(M_n)$. In [13] N. Kuiper asked: is the group $Isom(M_n)$ isomorphic to the cyclic group of order $2n$? The negative answer follows from [27] where it is shown that the manifold M_n admits an involution which does not belong to the group $\mathbb{Z}_{2n}$. Moreover [14], all foliation preserving homeomorphisms of $\mathcal{M}(\phi)$ induce homeomorphisms of a Fibonacci manifold. The group of foliation preserving homeomorphisms of the manifold $\mathcal{M}(\phi)$ can be found in [16] and it is of order $8n$.

The main aim of the present paper is to describe the isometry group $Isom(M_n)$. This is given in the following theorem.

Theorem 3.2. *The isometry group $Isom(M_n)$ of the hyperbolic Fibonacci manifold M_n, $n \geq 4$, consists of $8n$ elements and has the following presentation:*

$$Isom(M_n) = \langle x, y \,|\, x^{2n} = y^4 = (yx)^2 = (y^{-1}x)^2 = 1 \rangle.$$

Recall [9], [11], that for $n = 4, 5, 6, 8, 12$ a manifold M_n is arithmetic (in the sense of [2]). For such values of n the theorem was proven in [14] by compearing volumes of the Fibonacci manifolds calculated in [26] and the estimations of volumes for arithmetic orbifolds over a given field [2]. In the present paper the proof of the theorem for all $n \geq 6$ is given. We use geometric methods based on the results from [20] and [7] and ideas appeared in [17], [18] and [19]. Remark that the group $Isom(M_n)$ is the group $(4,\, 2n \,|\, 2,\, 2)$ according to notations of [4].

We recall that in the case $n = 3$ the group $Out(F(2,6))$ of outer automorphisms of the Fibonacci group $F(2,6)$ was found by B. Zimmermann [29]. There is an exact sequence

$$1 \longrightarrow (\mathbb{Z}_2)^3 \longrightarrow Out(F(2,6)) \longrightarrow S_3 \oplus \mathbb{Z}_2 \longrightarrow 1,$$

which does not split.

The structure of the paper is the following. In §2 we recall classical results of M. Dehn [6] and W. Magnus [15] on symmetries of the figure-eight knot. We will show that these symmetries induce isometries of the Fibonacci manifolds (Theorem 2.1). In §3 discrete extensions of the group $F(2,2n)$ will be studied. We will describe the normalizer of $F(2,2n)$ in the full group of isometries of the Lobachevsky space (Theorem 3.1). It will admit us to find the isometry group $Isom(M_n)$ (Theorem 3.2). In §4 we will discuss the quotient orbifolds arising under the isometries actions on the Fibonacci manifolds.

2. Symmetries of the figure-eight knot

From the presentation of the Fibonacci group

$$F(2,2n) = \langle x_1, \ldots, x_{2n} \;|\; x_i x_{i+1} = x_{i+2}, \quad i = 1, \ldots, 2n \rangle$$

we see an automorphism ρ of order n:

$$\rho \, : \, x_i \longrightarrow x_{i+2}, \quad i = 1, \ldots, 2n,$$

where all suffices are taken by module $2n$. Consider an extension Γ_n of the group $F(2,2n)$ by the automorphism ρ. We will get the semidirect product of the Fibonacci group $F(2,2n)$ and of the cyclic group generated by ρ. Thus

$$\Gamma_n \;\cong\; F(2,2n) \, \lambda \, \langle \rho \rangle$$

$$\cong \; \langle x_1, \ldots, x_{2n}, \rho \; | \; \rho^n \, = \, 1, \, \rho^{-1} \, x_i \, \rho \, = \, x_{i+2}, \, x_i \, x_{i+1} \, = \, x_{i+2}, \, i = 1, \ldots, 2n \rangle$$

$$\cong \; \langle x_1, x_2, \rho \; | \; \rho^n \, = \, 1, \quad \rho^{-1} \, x_1 \, \rho \, = \, x_1 \, x_2, \quad \rho^{-1} \, x_2 \, \rho \, = \, x_2 \, x_1 \, x_2 \rangle.$$

In particular, we have the relation

$$\rho^{-1} \, x_2 \, \rho \; = \; x_2 \, \rho^{-1} \, x_1 \, \rho,$$

from which one can eliminate x_1:

$$x_1 \; = \; \rho \, x_2^{-1} \, \rho^{-1} \, x_2.$$

Therefore,

$$\Gamma_n \; \cong \; \langle x_2, \rho \; | \; \rho^n \; = \; 1, \quad x_2^{-1} \, \rho^{-1} \, x_2 \, \rho \; = \; \rho \, x_2^{-1} \, \rho^{-1} \, x_2 \, x_2 \rangle.$$

Considering b such that $x_2 \; = \; b \, \rho$, we can rewrite the last relation in the form

$$\rho^{-1} \, b^{-1} \, \rho^{-1} \, b \, \rho \; = \; b^{-1} \, \rho^{-1} \, b \, \rho \, b.$$

Hence, we get the presentation

$$\Gamma_n \; \cong \; \langle b, \rho \; | \; \rho^n \; = \; b^n \; = \; 1, \quad \rho^{-1} \, [b, \rho] \; = \; [b, \rho] \, b \rangle,$$

where $[b, \rho] \; = \; b^{-1} \rho^{-1} b \rho$. It is easy to check directly by the Wirtinger algorithm [5] that the group

$$\Gamma \; \cong \; \langle b, \rho \; | \; \rho^{-1} \, [b, \rho] \; = \; [b, \rho] \, b \rangle$$

is the group of the figure-eight knot. This knot is pictured in Fig. 1, where the generators b and ρ are marked. Hence [8], the group Γ_n is isomorphic to the group of an orbifold $\mathcal{O}(n)$ whose underlying space is the three-dimensional sphere and whose singular set is the figure-eight knot with the singularity index n.

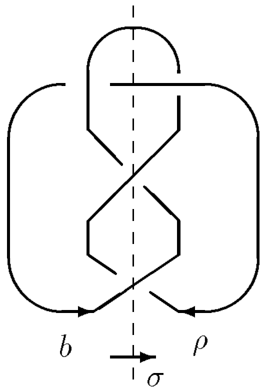

Fig. 1. The figure-eight knot.

There are following connections between generators $x_1, \ldots, x_{2n}$ of the Fibonacci group $F(2, 2n) \cong \pi_1(M_n)$ and generators b and ρ of the group $\Gamma_n \cong \pi_1(\mathcal{O}(n))$:

$$x_2 = b\,\rho,$$

$$x_1 = \rho\, x_2^{-1}\, \rho^{-1}\, x_2 = b^{-1}\, \rho^{-1}\, b\, \rho = [b, \rho],$$

and by $\rho^{-1} x_i \rho = x_{i+2}$,

$$x_{2i+2} = \rho^{-i}\, (b\rho)\, \rho^i,$$

$$x_{2i+1} = \rho^{-i}\, [b, \rho]\, \rho^i,$$

where $i = 0, \ldots, n-1$.

We recall that symmetries of the figure-eight knot were investigated by M. Dehn [6]. He demonstrated that the group Γ has eight outer automorphisms which form the dihedral group

$$\mathbb{D}_8 = \langle \sigma, \tau \mid \sigma^2 = \tau^4 = (\sigma\tau)^2 = 1 \rangle.$$

The action of generators σ and τ on the group Γ is given by the following rules:

$$\sigma(b) = \rho, \qquad \sigma(\rho) = b,$$
$$\tau(b) = b^{-1}\, \rho\, b, \qquad \tau(\rho) = \rho\, b\, \rho^{-1}.$$

The investigations were continued by W. Magnus [15] who proved that there are no other outer automorphisms of the group Γ. We point out that the automorphism σ induces an involution of the figure-eight knot with the axe pictured by dotted line in Fig. 1. The automorphism τ induces an orientation reversing symmetry of order four which visualization can be found in [12, p. 51]. The action of automorphisms from the group $\mathbb{D}_8$ on generators b and ρ of the group Γ is given in Table 1. The action of the group $\mathbb{D}_8$ in other system of generators was given in [10].

	b	ρ
σ	ρ	b
τ	$b^{-1}\rho\, b$	$\rho\, b\, \rho^{-1}$
τ^2	ρ^{-1}	b^{-1}
τ^3	$\rho\, b^{-1}\rho^{-1}$	$b^{-1}\rho^{-1}\, b$
$\sigma\,\tau$	$\rho\, b\, \rho^{-1}$	$b^{-1}\rho\, b$
$\sigma\,\tau^2$	b^{-1}	ρ^{-1}
$\sigma\,\tau^3$	$b^{-1}\rho^{-1}\, b$	$\rho\, b^{-1}\rho^{-1}$

Table 1.

Comparing presentations of groups Γ and Γ_n, one can see that σ and τ induce automorphisms of the group Γ_n and, consequently, induce symmetries of the orbifold $\mathcal{O}(n)$. The connection between symmetries of the figure-eight knot and isometries of the Fibonacci manifolds is described in the following statement.

Theorem 2.1. *For every hyperbolic Fibonacci manifold M_n each symmetry of the figure-eight knot induces an isometry of M_n.*

Proof. Consider generators σ and τ of the symmetry group $\mathbb{D}_8$ which act on b and ρ as above. Because $F(2,2n)$ is a normal subgroup of the group Γ_n, images

$$\sigma(x_2) \;=\; \sigma(b\,\rho) \;=\; \rho\, b \;=\; x_2\, x_1^{-1},$$

$$\tau(x_2) \;=\; \tau(b\,\rho) \;=\; b^{-1}\rho\, b\, \rho\, b\, \rho^{-1} \;=\; \rho\,(x_1^{-1}\, x_2\, x_1^{-1})\,\rho^{-1}$$

belong to the group $F(2,2n)$. Since

$$x_{2i+2} \;=\; \rho^{-i}\, x_2\, \rho^{i},$$

$$x_{2i+1} \;=\; \rho^{-i}\,(\rho\; x_2^{-1}\,\rho^{-1}\, x_2)\,\rho^{i},$$

hold for all $i = 0, \dots, n-1$, images

$$\sigma(x_{2i+2}) \; = \; b^{-i}\, \sigma(x_2)\, b^{i},$$

$$\sigma(x_{2i+1}) \; = \; b^{-i}\, (b\, \sigma^{-1}(x_2)\, b^{-1}\, \sigma(x_2))\, b^{i},$$

$$\tau(x_{2i+2}) \; = \; (\rho b \rho^{-1})^{-1}\, \tau(x_2)\, (\rho b \rho^{-1}),$$

$$\tau(x_{2i+1}) \; = \; (\rho b \rho^{-1})^{-i}\, (\, (\rho b \rho^{-1})\, \tau^{-1}(x_2)\, (\rho b \rho^{-1})^{-1}\, \tau(x_2)\,)\, (\rho b \rho^{-1})^{i},$$

which can be obtained by conjugating of elements of the group $F(2, 2n)$ by elements of the group Γ_n. So, these images are in $F(2, 2n)$ again. Therefore, σ and τ are automorphisms of the group $F(2, 2n)$ and they generate the group $\mathbb{D}_8$ of isometries of the manifold M_n.

$\square$

To simplify notations, furthermore we will be use the same symbols for automorphisms of groups and for symmetries of manifolds and orbifolds induced by these automorphisms.

Consider the natural extension of the group Γ_n by the above automorphisms:

$$\Omega(n) \; = \; \Gamma_n \leftthreetimes \mathbb{D}_8 \; \cong \; F(2, 2n) \leftthreetimes \langle \rho, \sigma, \tau \rangle.$$

The group $\Omega(n)$ can be described as a semidirect product of groups. The importance of the group $\Omega(n)$ will be shown in the next section.

3. The maximality of the group $\Omega(n)$

Let $Norm(F(2, 2n))$, $n \geq 4$, be a normalizer of the Fibonacci group $F(2, 2n)$ in the group $Isom(\mathbb{H}^3)$ of all isometries of the Lobachevsky space $\mathbb{H}^3$. Denote by $Isom(M_n)$ the isometry group of a hyperbolic Fibonacci manifold M_n.

Recall [1] that the isometry group of a three-dimensional hyperbolic manifold is isomorphic to the group of outer automorphisms of its fundamental group. Therefore,

$$Isom(M_n) \; \cong \; Norm(F(2, 2n)) \,/\, F(2, 2n).$$

It was shown in section 2 that symmetries of the figure-eight knot induce isometries of the Fibonacci manifolds, and

$$Norm(F(2, 2n)) \; \supset \; \Omega(n) \; \cong \; F(2, 2n) \leftthreetimes \langle \rho,\, \sigma,\, \tau \rangle.$$

Theorem 3.1 *Let $n \geq 6$ and Δ be a discrete group of isometries of $\mathbb{H}^3$ such that $\Gamma_n \subset \Delta$. Then $\Delta \subset \Omega(n)$.*

Proof. Recall that the group Γ_n is the fundamental group of the orbifold $\mathcal{O}(n)$ and consider its fundamental set. The singular set of the orbifold $\mathcal{O}(n)$ is the figure-eight knot which also can be described as the two-bridge 5/2-knot. By [21] as a fundamental polyhedron for the orbifold $\mathcal{O}(n)$ we can take a polyhedron $\mathcal{P}_n$, pictured in Fig. 2. It was shown in [20] that for $n \geq 4$ the polyhedron $\mathcal{P}_n$ can be realized in the Lobachevsky space $\mathbb{H}^3$ as a non-convex fundamental polyhedron for the group Γ_n.

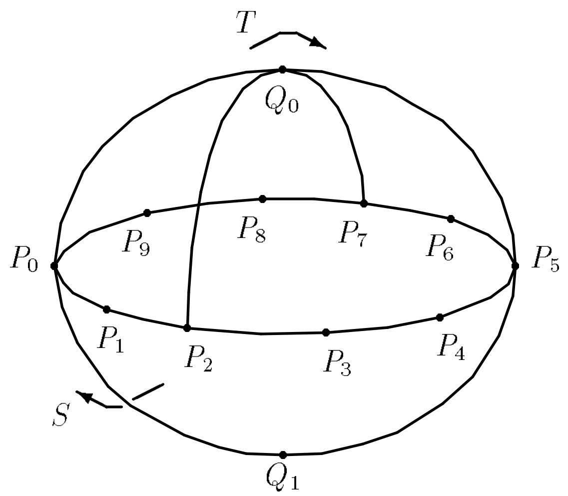

Fig. 2. The polyhedron $\mathcal{P}_n$.

Denote its vertices by $P_0, P_1, \ldots, P_9$. The polyhedron $\mathcal{P}_n$ has four faces each of which is non-flat:

$$P_0P_1P_2P_3P_4P_5, \qquad P_0P_9P_8P_7P_6P_5, \qquad P_2P_3P_4P_5P_6P_7, \qquad P_2P_1P_0P_9P_8P_7,$$

and edges P_0P_5, P_2P_7 and P_iP_{i+1}, $i = 0, \ldots, 9$, where all suffices are taken by module 10.

The group Γ_n is generated by two isometries S and T. The isometry S is a rotation of order n about the edge P_0P_5. This isometry identifies faces $P_0P_1P_2P_3P_4P_5$ and $P_0P_9P_8P_7P_6P_5$. The isometry T is a rotation of order n about the edge P_2P_7. This isometry identifies faces $P_2P_3P_4P_5P_6P_7$ and

$P_2P_1P_0P_9P_8P_7$. Thus, by the Poincare theorem [23] we get

$$\Gamma_n \cong \langle S,\, T \mid S^n = T^n, \quad T^{-1}\,[S,T] = [S,T]\,S \rangle,$$

where $[S,T] = S^{-1}T^{-1}ST$. It is easy to check that vertices $P_0, \dots, P_9$ of the polyhedron $\mathcal{P}_n$ are divided in two classes of equivalence in respect to the group Γ_n action:

$$\{P_0, P_6 = S^{-1}(P_0), P_4 = T^{-1}S^{-1}(P_0), P_2 = ST^{-1}S^{-1}(P_0), P_8 = TST^{-1}S^{-1}(P_0)\},$$

and

$$\{P_5, P_1 = S(P_5), P_9 = TS(P_5), P_7 = S^{-1}TS(P_5), P_3 = T^{-1}S^{-1}TS(P_5)\}.$$

Therefore, for each of points $P_0, \dots, P_9$ there is an axe of an elliptic element of order n from the group Γ_n whose axe does pass throughout a given point.

Now we will describe geometric parameters of the polyhedron $\mathcal{P}_n$. Denote by Q_0 and Q_1 the midpoints of edges P_0P_5 and P_2P_7, respectively. Let us draw additional edges Q_0P_i and Q_1P_i for $i = 0, \dots, 9$. In this case the polyhedron $\mathcal{P}_n$ can be represented as the union of ten tetrahedra $\mathcal{T}_i = Q_0Q_1P_iP_{i+1}$, $i = 0, \dots, 9$. It was shown in [20], that these tetrahedra can be divided in two classes of pairwise isometric: $\{\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_4, \mathcal{T}_5, \mathcal{T}_6, \mathcal{T}_7, \mathcal{T}_9, \mathcal{T}_0\}$ and $\{\mathcal{T}_3, \mathcal{T}_8\}$, which are pictured in Fig. 3.

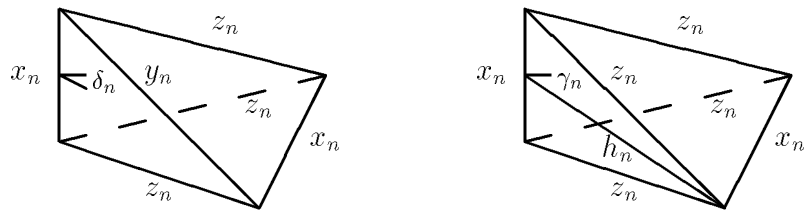

Fig. 3.

Using formulae of non-Euclidean trigonometry one can get the following system of equations [20] for lengths of edges and for dihedral angles of the tetrahedra pictured in Fig. 3 :

$$\begin{cases} \sinh\, h_n\; \sin(\gamma_n/2) = \sinh\,(x_n/2), \\ \sinh\, h_n\; \sin\delta_n = \sinh\, x_n\; \sin(\pi/n), \\ \cosh\, h_n\, \cosh\,(x_n/2) = \cosh\, z_n, \\ \coth\, y_n\, \tanh\,(x_n/2) = \cot\,(\pi/n)\, \tan\,\delta_n, \\ \cosh\, z_n = \cosh\, x_n\, \cosh\, y_n, \\ \gamma_n + 4\,\delta_n = \pi. \end{cases}$$

Introducing the parameter

$$\alpha_n = \sqrt{4\sin^2(\pi/n) + 1},$$

one can write solutions of the system in the following form:

$$\cosh x_n = \frac{1}{\alpha_n - 1},$$

$$\cosh h_n = \sqrt{\frac{2\alpha_n}{\alpha_n - 1}} \cos(\pi/n),$$

$$\cosh z_n = \frac{\alpha_n}{\alpha_n - 1} \cos(\pi/n),$$

$$\cosh y_n = \alpha_n \cos(\pi/n),$$

$$\cos 2\delta_n = \frac{1}{\alpha_n - 1},$$

$$\gamma_n = \pi - 4\delta_n.$$

As we see from Fig. 3, faces of tetrahedra can be divided in two classes of pairwise isometric triangles:

$$\begin{aligned} \mathcal{W}_n^1 = \{ \quad & Q_0P_1P_2, Q_0P_2P_3, Q_0P_3P_4, Q_0P_6P_7, Q_0P_7P_8, Q_0P_8P_9 \\ & Q_1P_3P_4, Q_1P_4P_5, Q_1P_5P_6, Q_1P_8P_9, Q_1P_9P_0, Q_1P_0P_1 \}, \end{aligned}$$

with edge lengths x_n, z_n and z_n, and

$$\mathcal{W}_n^2 = \{Q_0P_0P_1, Q_0P_9P_0, Q_0P_4P_5, Q_0P_5P_6, Q_1P_1P_2, Q_1P_2P_3, Q_1P_6P_7, Q_1P_7P_8\},$$

with edge lengths x_n, y_n and z_n. The union of two these classes is a surface of the polyhedron $\mathcal{P}_n$.

Let us assume that the group Γ_n is extended by some isometry $h \in Isom(\mathbb{H}^3)$ such that the group $\langle \Gamma_n, h \rangle$ is discrete. Recall that $S \in \Gamma_n$ is an elliptic element of order n and denote its axe by l. Since $\mathcal{P}_n$ is the fundamental polyhedron for the group Γ_n we can assume without loss of generality that $h(l) \cap \mathcal{P}_n \neq \emptyset$. In this case two possibilities holds: either $h(l)$ does pass through a vertex of some triangle and is disjoint with interior points of triangles from $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$, or $h(l)$ intersects some triangle from the set $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$ in an interior point. We will show that the second possibility cannot take a place.

Indeed, $h(l)$ is an axe of the element of order n which is conjugated to the element S. As it was pointed out above, each of vertices of tetrahedra $\mathcal{T}_0, \ldots, \mathcal{T}_9$ belongs to the axe of some element of order n. Obviously, the same is true for vertices of triangles from the set $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$. Estimations of distances between axes of elliptic elements in a discrete group were obtained by F. Gehring and G. Martin in [7]. More exactly, for a distance ρ_6 between two axes of elements of order six the following estimation holds:

$$\sinh \rho_6 \geq \sqrt{3}.$$

For a distance ρ_n between two axes of elements of order n, $n \geq 7$, the estimation is given by

$$\sinh \rho_n \geq \frac{\sqrt{2\cos(2\pi/n) - 1}}{2\sin^2 \pi/n}.$$

Proposition 3.1. *Let $n \geq 6$ and T be a triangle from the set $\mathcal{W}_n^1$. Then for any point of the triangle T there exists its vertex such that the distance between these two points is less then ρ_n.*

Proof. Let us rewrite Gehring-Martin estimations in the following form:

$$\cosh \rho_6 \geq c_6 = 2$$

and

$$\cosh \rho_n \geq c_n = \frac{1}{2\sin^2(\pi/n)} - 1,$$

for $n \geq 7$. One can see that the following asymptotic holds:

$$c_n \sim \frac{n^2}{2\pi^2}, \qquad n \to \infty.$$

Consider a triangle $T = ABC$ from the set $\mathcal{W}_n^1$ with edges $BC = x_n$, $AB = z_n$ and $AC = z_n$ pictured in Fig. 4. Denote by R_n^1 the radius of the circle described around it with the center at the point O. Then $OA = OB = R_n^1$. From the point O we drop perpendiculars OD and OE to edges AB and BC respectively. Denote by φ_n the value of the angle DAO.

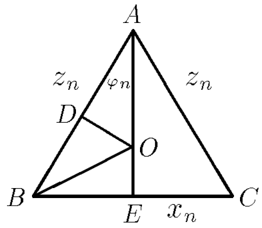

Fig. 4.

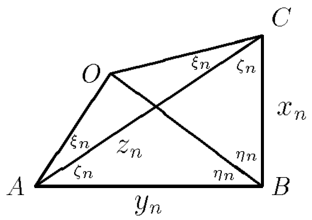

Fig. 5.

According to above notations for the rectangular triangle ABE we have

$$\sin\varphi_n \sinh z_n = \sinh(x_n/2),$$

and for the rectangular triangle AOD we have

$$\tanh R_n^1 \cos\varphi_n = \tanh(z_n/2).$$

Thus,

$$\tanh R_n^1 = \frac{\cosh z_n - 1}{\sqrt{\cosh^2 z_n - \frac{1}{2}\cosh x_n + \frac{3}{2}}}.$$

Using above expressions for $\cosh z_n$ and $\cosh x_n$ in terms of α_n, after obvious simplifications we get

$$\cosh R_n^1 = \frac{\sqrt{\alpha_n}\cdot\sqrt{2\,\alpha_n\cos^2(\pi/n) - \alpha_n + 1}}{\sqrt{\alpha_n - 1}\cdot\sqrt{-3\,\alpha_n + 4\,\alpha_n\cos(\pi/n) + 2}}.$$

From this formula one can see the following asymptotic:

$$\cosh R_n^1 \sim \frac{n}{\sqrt{3}\,\pi}, \qquad n \to \infty.$$

Therefore, the function c_n of n is of quadratic growth, and the function $\cosh R_n^1$ of n is of linear growth. Computing values of functions c_n and $\cosh R_n^1$ for small n and compearing their graphs, we get that for all $n \geq 6$ the following inequality holds: $\cosh R_n^1 < c_n$. In particular, values of c_n and $\cosh R_n^1$ for $n = 6, \ldots, 16$ are given in Table 2.

n	c_n	$\cosh\ R_n^1$	$\cosh\ R_n^2$
6	2	$1.4812\ldots$	$1.4362\ldots$
7	$1.6559\ldots$	$1.6323\ldots$	$1.5782\ldots$
8	$2.4142\ldots$	$1.7867\ldots$	$1.7245\ldots$
9	$3.2743\ldots$	$1.9447\ldots$	$1.8748\ldots$
10	$4.2360\ldots$	$2.1060\ldots$	$2.0287\ldots$
11	$5.2993\ldots$	$2.2700\ldots$	$2.1855\ldots$
12	$6.4641\ldots$	$2.4365\ldots$	$2,3449\ldots$
13	$7.7302\ldots$	$2.6050\ldots$	$2.5063\ldots$
14	$9.0978\ldots$	$2.7753\ldots$	$2.6696\ldots$
15	$10.5667\ldots$	$2.9471\ldots$	$2.8343\ldots$
16	$12.1370\ldots$	$3.1201\ldots$	$3.0003\ldots$

Table 2.

Hence, Proposition 3.1 is proven.

□

Proposition 3.2. *Let $n \geq 6$ and T be a triangle from the set $\mathcal{W}_n^1$. Then for any point of the triangle T there exists its vertex such that the distance between these two points is less then ρ_n.*

Proof. To find the radius of a circle described about a triangle T we will use the following property.

Lemma 3.1. *Let ABC be a triangle in $\mathbb{H}^2$ with angles α, β, and γ at vertices A, B, and C respectively. Assume that the length of the edge opposite to the angle β is equal to b. Then for the radius R of the described circle the following formula holds:*

$$\tanh\ R = \frac{\cosh\ b\ -\ 1}{\sinh\ b\ \cos(\frac{\alpha-\beta+\gamma}{2})}.$$

Proof. Denote the center of the described circle by O and consider triangles ABO, BCO and CAO. Let us introduce notations $\xi = \angle OAC = \angle OCA$, $\eta = \angle OBC = \angle OCB$ and $\zeta = \angle OAB = \angle OBA$. Assume that the point O is outside the triangle ABC (see Fig. 5). In this case we get equalities

$$\begin{cases} -\xi + \zeta = \alpha, \\ \eta + \zeta = \beta, \\ -\xi + \eta = \gamma, \end{cases}$$

from which one can eliminate ξ, η and ζ through α, β and γ. In virtue of cosine theorem for hyperbolic triangles [23], we get

$$\cosh\, R = \cosh\, R \cosh\, b - \sinh\, R \sinh\, b\, \cos \xi,$$

hence

$$\tanh\, R \;=\; \frac{\cosh\, b \;-\; 1}{\sinh\, b\, \cos \xi},$$

where

$$\xi \;=\; \frac{-\alpha + \beta - \gamma}{2}.$$

The case when the point O is inside of the triangle ABC can be considered similar. □

Let us return to the proof of Proposition 3.2. Consider triangle $T = ABC$ from the set $\mathcal{W}_n^2$ with edges $AB = y_n$, $AC = z_n$ and $BC = x_n$, pictured in Fig. 5. Denote by R_n^2 the radius of the described circle. In virtue of Lemma 3.1

$$\tanh\, R_n^2 \;=\; \frac{\cosh\, z_n \;-\; 1}{\sinh\, z_n\, \cos(\frac{\angle A + \angle C - \pi/2}{2})}.$$

By formulae of the hyperbolic trigonometry [23], from expressions for x_n, y_n and z_n in terms of α_n, we get

$$\cosh\, R_n^2 \;=\; \frac{\sqrt{\alpha_n}\,\sqrt{\alpha_n\, \cos(\pi/n) \,+\, 1}}{\sqrt{\alpha_n^2\, \cos(\pi/n) \,+\, 3\,\alpha_n \,-\, 2 \,-\, 2\,\alpha_n\, \cos(\pi/n)}}.$$

From the above formula the following asymptotic holds:

$$\cosh\, R_n^2 \;\sim\; \frac{n}{\sqrt{3}\,\pi}, \qquad n \to \infty.$$

Thus, the function c_n of n is of quadratic growth, and the function $\cosh\, R_n^2$ of n is of linear growth. Computing values of c_n and $\cosh\, R_n^1$ for small n and compearing their graphs, we get that for all $n \geq 6$ the following inequality holds: cosh $R_n^2 < c_n$. In particular, values of c_n and cosh R_n^1 for $n = 6, \ldots, 16$ are given in Table 2.

□

We recall that for any triangle from the set $\mathcal{W}_n^1 \cup \mathcal{W}_n^2$ each its vertex belongs to an axe of some elliptic element of order n. Therefore, in virtue of Propositions 3.1 and 3.2, the axe $h(l)$ cannot intersect interior points of triangles.

Now we consider the case when $h(l)$ passes through a vertex of a polyhedron $\mathcal{P}_n$. We recall that vertices $P_0, \ldots, P_9$ are divided in two classes under the group Γ_n action: either equivalent to P_0, or equivalent to P_1. Therefore, using $\tilde{h} = hh'$, where $h' \in \Gamma_n$, we can suppose that $\tilde{h}(Q_0)$ coincides with one of four following points: Q_0, Q_1, P_0 or P_1. We remark that there exists an isometry of the space $\mathbb{H}^3$ induced by the automorphism from the group $\Gamma_n \leftthreetimes \langle \tau \rangle$, such that the image of Q_0 is P_1 and the image of Q_1 is P_0. Indeed, according to the action of τ on the group Γ_n, we have

$$\tau(S) = S^{-1}TS, \qquad \tau(T) = TST^{-1}.$$

Denote by g an automorphism which is obtained from τ by the conjugation by TS^{-1}. Then

$$g(S) = ST^{-1}(S^{-1}TS)TS^{-1} = T^{-1}S^{-1}T = (T^{-1}ST)^{-1},$$

$$g(T) = ST^{-1}(TST^{-1})TS^{-1} = S.$$

Therefore, g induces an isometry of the space $\mathbb{H}^3$ such that the image of the axe of S is the axe of $T^{-1}ST$, and the image of the axe T is the axe of S. We remark [20] that the segment $[Q_0, Q_1]$ is the common perpendicular to the axe of S (at the point Q_0) and to the axe of T (at the point Q_1). The segment $[P_0, P_1]$ is a common perpendicular to the axe of $T^{-1}ST$ (at the point P_1) and to the axe of S (at the point P_0). Thus, $g(Q_0) = P_1$ and $g(Q_1) = P_0$. Hence, using $\tilde{h} = hh''$, $h'' \in \Gamma_n \lambda \langle \tau \rangle$, we can suppose that $\tilde{h}(Q_0)$ equals either Q_0 or Q_1. We recall that the group Γ_n has an automorphism σ such that

$$\sigma(S) = T, \qquad \sigma(T) = S.$$

This automorphism induces an isometry of the space $\mathbb{H}^3$ which sends the axe of S to the axe of T and otherwise. Since the segment $[Q_0, Q_1]$ is a common perpendicular to these axes, we get $\sigma(Q_0) = Q_1$ and $\sigma(Q_1) = Q_0$.

Therefore, using $\tilde{h} = hh'''$, $h''' \in \Gamma_n \lambda \langle \sigma, \tau \rangle = \Omega(n)$, we can suppose that $\tilde{h}(Q_0) = Q_0$. Thus, axes l and $\tilde{h}(l)$ of orders $n \geq 6$ have a common point Q_0 and so, coincide by [17]. Under the action of $\tilde{h}$ the axe of the element S passes to itself. Multiplying $\tilde{h}$ on some degree of the element S, without loss of generality, we can assume that $\tilde{h}(Q_1) \in \mathcal{P}_n$. Hence $\tilde{h}(Q_1) = Q_1$ and under the action of the isometry $\tilde{h}$ the axe of the element T passes to itself. Thus, the isometry $\tilde{h}$ has two invariant skew lines and a common perpendicular $[Q_0, Q_1]$ for them is fixed. Multiplying $\tilde{h}$, if it is necessary, on a symmetry from $\mathbb{D}_8$, we get that $\tilde{h} = id$. Hence $h \in \Omega(n)$. The proof is complete.

□

We recall that a discrete group is said to be a *maximally discrete group* if it is not a proper subgroup of an other discrete group.

Corollary 3.1. *For $n \geq 6$ a group $\Omega(n)$ is maximally discrete.*

Proof. Let us assume that Δ is a discrete group such that $\Omega(n) \subset \Delta$. Then $\Gamma_n \subset \Omega(n) \subset \Delta$. At the same time in virtue of Theorem 3.1 we have $\Delta \subset \Omega(n)$. Therefore $\Omega(n)$ is a maximally discrete group.

□

Remark. For $n = 4, 5$ groups $\Omega(n)$ are maximally discrete groups also [14]. This fact follows from their arithmeticity and from compearing of hyperbolic volumes. We remark that the fundamental group $F(2, 10)$ is commensurable with the Coxeter group with the following Coxeter scheme:

This Coxeter group is generated by reflections in faces of a compact hyperbolic tetrahedron.

Theorem 3.2. *The isometry group $Isom(M_n)$ of a hyperbolic Fibonacci manifold M_n, $n \geq 4$, consists of $8n$ elements and has the following presentation*

$$Isom(M_n) = \langle x, y \,|\, x^{2n} = y^4 = (yx)^2 = (y^{-1}x)^2 = 1 \rangle.$$

Proof. For cases $n = 4, 5, 6, 8, 12$, when the Fibonacci group is arithmetic, the statement was proven by C. Maclachlan and A. Reid [14]. We will prove the statement for all $n \geq 6$. We recall that

$$Isom(M_n) \cong Norm(F(2, 2n)) \,/\, F(2, 2n),$$

where $Norm(F(2,2n))$ is the normalizer of the group $F(2,2n)$ in the isometry group of the hyperbolic space $\mathbb{H}^3$. From the one side, by Theorem 2.1 we have

$$Norm(F(2,2n)) \supset \Omega(n).$$

From the other side, the group $Norm(F(2,2n))$ is discrete as a finite index extension of the group Γ_n. Therefore, by Corollary 3.1 we get

$$Norm(F(2,2n)) \subset \Omega(n),$$

Thus,

$$Norm(F(2,2n)) = \Omega(n) \cong F(2,2n)\lambda\langle\rho,\sigma,\tau\rangle,$$

and so,

$$Norm(F(2,2n))/F(2,2n) \cong \langle\rho,\sigma,\tau\rangle.$$

The given above presentation of the group with two generators can be obtained due to [16].

□

4. Quotient orbifolds

In this section we will discuss some quotient orbifolds which arise under isometries actions on the hyperbolic Fibonacci manifolds. As we remarked above, the quotient space of the manifold M_n by the action of the isometry ρ is an orbifold $\mathcal{O}(n)$ whose underlying space is the three-dimensional sphere and whose singular set is the figure-eight knot with singularity index n. Thus, $\mathcal{O}(n) = M_n/\langle\rho\rangle$.

Let us consider the involution $\tau^2 \in \mathbb{D}_8$. The quotient space of the orbifold $\mathcal{O}(n)$ by the action of this involution was discussed, for example, in [27] and [25].

Proposition 4.1. *The quotient space $M_n/\langle\rho,\tau^2\rangle$ of the hyperbolic Fibonacci manifold M_n, $n \geq 4$, by the action of the group of isometries generated by ρ and τ^2 is an orbifold whose underlying space is the three-dimensional sphere and whose singular set is the two-component link 6_2^2, pictured in Fig. 6 with singularity indices 2 and n on its components.*

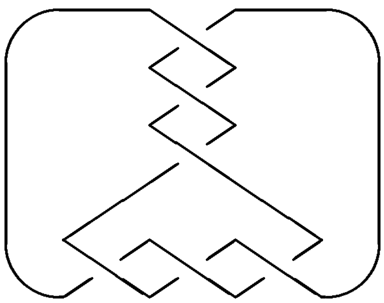

Fig. 6. The singular set of $M_n/\langle \rho, \tau^2 \rangle$.

Proof. It follows from the fact that $M_n/\langle \rho, \tau^2 \rangle \cong \mathcal{O}(n)/\langle \tau^2 \rangle$.

$\square$

Let us consider the involution $\sigma \in \mathbb{D}_8$, which induces the symmetry of the figure-eight knot whose axe is pictured by dotted line in Fig. 1.

Proposition 4.2. *The quotient space $M_n/\langle \rho, \sigma \rangle$ of the hyperbolic Fibonacci manifold M_n, $n \geq 4$, by the action of the group of isometries generated by ρ and σ is an orbifold whose underlying space is the three-dimensional sphere and whose singular set is a spatial graph pictured in Fig. 7.*

Proof. It follows from the fact that $M_n/\langle \rho, \sigma \rangle \cong \mathcal{O}(n)/\langle \sigma \rangle$.

$\square$

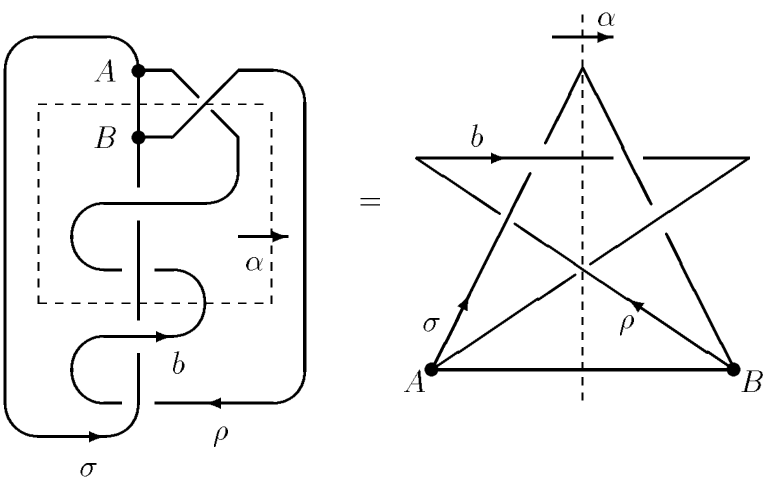

Fig. 7. The singular set of $M_n/\langle \rho, \sigma \rangle$.

In respect to the action of the symmetry σ pictured in Fig. 1, we will get a graph in Fig. 7, left. This graph has two vertices and three edges and so, is a Θ-graph in the terminology of the spatial graphs theory [28]. This graph can be also obtained as the torus knot 5_1 with a bridge AB (see Fig. 7, right). Points A and B are images of the intersection of the singular set of the orbifold $\mathcal{O}(n)$ and the axe of the involution σ. Remark that the arc AB is an image of the tunnel of the figure-eight knot in the sense of [22]. In this case two axes of the singular set of the orbifold $M_n/\langle\rho,\sigma\rangle$ are images of the axe of the involution σ and have singularity indices equal 2. The third axe is the image of the singular set of the orbifold $\mathcal{O}(n)$ and has the singularity index n. The orbifold group of the orbifold $M_n/\langle\rho,\sigma\rangle$ is a group

$$\begin{aligned}
\Delta(n) &= F(2,2n)\,\lambda\,\langle\rho,\sigma\rangle \\
&\cong \langle\rho,\, b,\, \sigma \mid \rho^n = b^n = \sigma^2 = 1, \quad \rho^{-1}\,[b,\,\rho] = [b,\,\rho]\,b, \quad \sigma\,\rho\,\sigma = b\rangle,
\end{aligned}$$

where generators ρ, b and σ correspond to Fig. 7. Indeed, the relation $\rho^{-1}\,[b,\,\rho] = [b,\,\rho]\,b$ follows from the fact that the loop around the bridge AB is of order two. The relation $\sigma\,\rho\,\sigma = b$ follows by the Wirtinger algorithm.

It is easy to see from Fig. 7 that the spatial Θ–graph has an involution α whose axe is pictured by the dotted line. The involution α induces a symmetry of the orbifold $M_n/\langle\rho,\sigma\rangle$. From Fig. 7 we see that

$$\alpha(b) = b^{-1}, \quad \alpha(\rho) = b\,\rho^{-1}\,b^{-1}.$$

Thus the involution α is conjugated to the involution $\sigma\tau^2$ (see Table 1 above). Therefore, quotient orbifolds $M_n/\langle\rho,\sigma,\alpha\rangle$ and $M_n/\langle\rho,\sigma,\tau^2\rangle$ are isometric.

Proposition 4.3. *The quotient space $M_n/\langle\rho,\sigma,\tau^2\rangle$ of the hyperbolic Fibonacci manifold M_n, $n \geq 4$, by the action of the group of isometries generated by ρ, σ and τ^2 is an orbifold whose underlying space is the three-dimensional sphere and whose singular set is a spatial graph pictured in Fig. 8.*

Proof. It follows from the fact that $M_n/\langle\rho,\sigma,\tau^2\rangle \cong (\mathcal{O}(n)/\langle\sigma\rangle)\,/\langle\tau^2\rangle$.

□

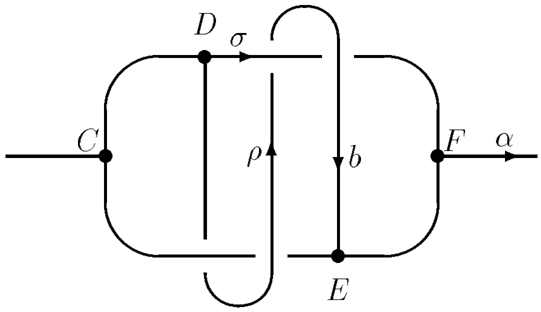

Fig. 8. The singular set of $M_n/\langle \rho, \sigma, \tau^2 \rangle$.

The graph in Fig. 8 has four vertices and six edges. As an abstract graph, this graph is isomorphic to the complete graph on four vertices. By the other hand, this graph can be obtained as the figure-eight knot with two arcs CD and EF which are its tunnels. Vertices C, E and F are images of the intersection of the singular set of the orbifold $\mathcal{O}(n)/\langle \sigma \rangle$ with the axe of the involution α. The vertex D is the image of the point A in Fig. 7. One edge of the graph has the singularity index n. All others edges have the singularity index 2. The orbifold group of the orbifold $M_n/\langle \rho, \sigma, \tau^2 \rangle$ is the group

$$\begin{aligned}
\Pi(n) &= F(2,2n) \leftthreetimes \langle \rho, \sigma, \tau^2 \rangle \\
&\cong \langle \rho,\, b,\, \sigma,\, \alpha \mid \rho^n = b^n = \sigma^2 = \alpha^2 = 1, \quad \rho^{-1}\,[b,\,\rho] = [b,\,\rho]\,b, \\
&\qquad \sigma\,\rho\,\sigma = b, \quad \alpha^{-1}\,b\,\alpha = b^{-1}, \quad \alpha^{-1}\,\rho\,\alpha = b\,\rho^{-1}\,b^{-1} \rangle,
\end{aligned}$$

where elements ρ, b, σ and α corresponds to Fig. 8. Another presentation of the group $\Pi(n)$ can be obtained by the Wirtinger algorithm:

$$\begin{aligned}
\Pi(n) \cong \langle \rho,\, \sigma,\, \alpha \mid \rho^n &= \sigma^2 = \alpha^2 = 1, \\
(\sigma\rho\sigma\rho^{-1}\sigma\alpha)^2 &= (\rho^{-1}\sigma\alpha)^2 = (\sigma\alpha\rho^{-1}\alpha^{-1})^2 = 1 \rangle.
\end{aligned}$$

As one can see from Fig. 8, the singular set of the orbifold $M_n/\langle \rho, \sigma, \tau^2 \rangle$ admits an antipodal involution χ such that $\chi(C) = F$, $\chi(D) = E$, $\chi(E) = D$ and $\chi(F) = C$. Obviously, by Corollary 3.2, $\langle \rho, \sigma, \tau^2, \chi \rangle \cong Isom(M_n)$. Because the underlying space of the orbifold $M_n/\langle \rho, \sigma, \tau^2 \rangle$ is the three-dimensional sphere, the underlying space of the quotient orbifold $M_n/Isom(M_n)$

is a pseudo-manifold P^3 which can be obtained as the union of two cones with the common base the projective plane. This pseudo-manifold and the singular set of the orbifold are pictured schematically in Fig. 9. Here one of edges singularity index is equal to n, and all other are equal to 2.

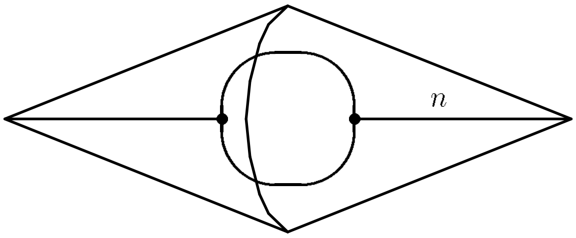

Fig. 9. The quotient orbifold $M_n/Isom(M_n)$.

Proposition 4.4. *The quotient space $M_n/Isom(M_n)$ of the hyperbolic Fibonacci manifold M_n, $n \geq 4$, by the action of the isometry group $Isom(M_n)$ is an orbifold whose underlying space is the pseudo-manifold P^3 and whose singular set is pictured in Fig. 9.*

Proof. It follows from the fact that $M_n/IsomM_n \cong (\mathcal{O}(n)/\langle\sigma, \tau^2\rangle)/\langle\chi\rangle$. $\square$

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References

[1] Benedetti, R. and Petronio, C., *Lectures on Hyperbolic Geometry*, Universitext, Springer, 1992.

[2] Borel, A., *Commensurability classes and hyperbolic volumes,* Annali Sci. Norm. Pisa **8** (1991), 1–33.

[3] Conway, J., *Advanced problem 5327,* Amer. Math. Monthly **72** (1965), 915.

[4] Coxeter, H.S.M. and Moser, W.O.J., *Generators and relations for discrete groups*, Springer, Berlin, 1965.

[5] Crowell, R.H. and Fox, R.H., *Introduction to knot theory*, Grad. Texts Math. **57**, Springer Verlag, 1963.

[6] Dehn, M., *Dei beiden Kleeblattschlingen*, Math. Ann. **75** (1914), 402–413.

[7] Gehring, F.W. and Martin, G.J., *Commutators, collars and the geometry of Möbius groups*, Journal D'Analyse Mathematique **63** (1994), 175–219.

[8] Haeflieger, A. and Quach, N.D., *Une presentation du groupe findamental d'une orbifold*, Asterisque **116** (1984), 185–192.

[9] Helling, H., Kim, A.C. and Mennicke, J., *A geometric study of Fibonacci groups,* SFB–343 Bielefeld, Diskrete Strukturen in der Mathematik, Preprint (1988).

[10] Hempel, J., *The lattice of branched covers over the figure-eight knot*, Topology and its Applications **34** (1990), 183–201.

[11] Hilden, H.M., Lozano, M.T. and Montesinos, J.M., *The arithmeticity of the figure-eight knot orbifolds,* in: Topology'90, eds. B. Apanasov, W. Neumann, A. Reid, L. Siebenmann, de Gruyter, Berlin, 1992, 169–183.

[12] Kawauchi, A., *A survey of knot theory*, Birkhauser, Basel, 1996.

[13] Kuiper, N., *Fairly symmetric hyperbolic manifolds*, Preprint IHES, 1989.

[14] Maclachlan, C. and Reid, A.W., *Generalised Fibonacci manifolds*, Transformation Groups **2** (1997), No. 2, 165–182.

[15] Magnus, W., *Untersuchungen über einige unendliche diskontinuierliche Gruppen*, Math. Ann. **105** (1931), 52–74.

[16] McCullough, D., *Automorphisms of punctured-surface bundles*, Geometry and Topology **105** (1987), 179–211.

[17] Mednykh, A.D., *Automorphism groups of three-dimensional hyperbolic manifolds*, Dokl. Akad. Nauk SSSR **285** (1985), 40–44. English transl.: Sov. Math. Dokl. **32** (1985), 633–636.

[18] Mednykh, A.D. and Vesnin, A.Yu., *On three-dimensional hyperbolic manifolds of Löbell type*, in: Complex Analysis and Applications'85, eds.: L. Iliev and I. Ramadanov, Sofia, Publ. House of Bulgarian Acad. Sci. 1986, 40–44.

[19] Mednykh, A.D., *On the isometry group of the hyperbolic space of a Seifert-Weber dodecahedron*, Sib. Mat. Zh. **28** (1987), No. 5, 134–144. English transl.: Sib. Math. J. **28** (1987), 798–806.

[20] Mednykh, A. and Rasskazov, A., *On the structure of the canonical fundamental set for the 2-bridge link orbifolds*, Preprint, 1997.

[21] Minkus, J., *The branched cyclic coverings of 2-bridge knots and links,* Memoirs of Amer. Math. Soc. **35** (1982).

[22] Morimoto, K. and Sakuma, M., *On unknotting tunnels for knots*, Math. Ann. **289** (1991), 143–167.

[23] Ratcliffe, J.G., *Foundations of hyperbolic manifolds*, Grad. Texts Math. **149**, Springer, 1994.

[24] Thomas, R.M., *The Fibonacci groups revised*, in: Groups St. Andrews 1989, edited by D.Johnson, London Math. Soc. Lecture Notes Series **160**, 445–456.

[25] Thurston, W., *The geometry and topology of 3-manifolds,* Lecture Notes, Princeton University, 1980.

[26] Vesnin, A. and Mednykh, A., *Hyperbolic volumes of the Fibonacci manifolds*, Sib. Mat. Zh. **36** (1995), No. 2, 266–277. English transl.: Sib. Math. J. **36** (1995), No. 2, 235–245.

[27] Vesnin, A. and Mednykh, A., *Fibonacci manifolds as two-fold coverings over the three-dimensional sphere and the Meyerhoff–Neumann Conjecture*, Sib. Mat. Zh. **37** (1996), No. 3, 534–542. English transl.: Sib. Math. J. **37** (1996), No. 3. 461–467.

[28] Wolcott, K., *The knotting of theta curves and other graphs in* $\mathbb{S}^3$, Geometry and Topology **105** (1987), 325–346.

[29] Zimmermann, B., *On the Hantzsche–Wendt manifold*, Monatsh. Math. **110** (1990), 321–327.