Actions of parabolic subgroups of $GL(V)$ on certain unipotent subgroups and quasi-hereditary algebras

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Abstract

A parabolic subgroup P of the general linear group over an infinite field acts by conjugation on its unipotent radical P_u and the members $P_u^{(l)}$ of the descending central series of P_u. Let $\ell(P_u)$ denote the nilpotency class of P_u. We show that P acts with a finite number of orbits on $P_u^{(l)}$ precisely when $\ell(P_u) \leq 4$ for $l = 0$, or $\ell(P_u) \leq 5 + 2l$ for $l \geq 1$. This result is obtained by interpreting the orbits of P on $P_u^{(l)}$ as isomorphism classes of good modules over certain quasi-hereditary algebras.

1 Introduction

(1.1) Let G be a reductive algebraic group defined over an algebraically closed field k. Let P be a parabolic subgroup of G. It acts by conjugation on its unipotent radical P_u. By a theorem of R.W. Richardson, P admits a dense orbit on P_u, see [Ri]. Recently, L.Hille and G.Röhrle classified in [HR1] and [HR2] those parabolic subgroups of classical groups which act with a finite number of orbits on their unipotent radical.

The strategy to obtain this classification is the following: At first, the cases where G is of type B, C or D are reduced to those of type A by interpreting the classical groups $SO_m(k)$ and $Sp_m(k)$ as fixed point subgroups of $SL_m(k)$ under a suitable automorphism. Secondly, if G is a simple algebraic group of type A, the orbits of P in P_u are identified with certain isomorphism classes of modules over some finite-dimensional algebra. The purpose of this paper is to extend the second step, the translation from orbits of a parabolic subgroup of a general linear group to modules over algebras in more general situations.

(1.2) Let V be a finite-dimensional k-vector space endowed with a flag of subspaces $F : \{0\} = V_0 \subseteq \cdots \subseteq V_t = V$ of length t. The stabilizer P of F in the general linear group of V is a parabolic subgroup of $GL(V)$. The members of the descending central series of P_u are defined by $P_u^{(l)} := [P_u, P_u^{(l-1)}]$, where $P_u^{(0)} := P_u$ and $l \in \mathbb{N}$. If the flag F is *proper*, i.e. $V_{i-1} \neq V_i$ for $i = 1, \ldots, t$, then the nilpotency class of P_u is $\ell(P_u) = t - 1$.

Note that instead of the action of P on $P_u^{(l)}$ we may as well consider the adjoint action of P on the corresponding Lie algebras $\mathfrak{p}_u^{(l)}$. We identify $\mathfrak{p}_u^{(l)}$ with the space of all endomorphisms f of V such that $f(V_i) \subseteq V_{i-l-1}$ for $i = l+1, \ldots, t$.

The main result of this paper is the following:

Theorem. *Let k be an infinite field and let $P = Stab(F)$ be a parabolic subgroup of a general linear group, realized as stabilizer of a proper flag F of length t. Then the number of orbits of P on $\mathfrak{p}_u^{(l)}$ is finite precisely when $t \leq 5$ for $l = 0$, or $t \leq 6 + 2l$ for $l \geq 1$.*

Note that the result for $l = 0$ is shown in [HR1] and [HR2].

(1.3) The proof of this Theorem generalizes the methods of [HR1]. In a first step we consider simultaneously the pairs (F, f) where F is a flag of fixed length t with associated parabolic subgroup $P = \mathrm{Stab}(F)$ and $f \in \mathfrak{p}_u^{(l)}$ for some fixed $l \in \mathbb{N}$. These pairs can be naturally viewed as objects of a category $\mathcal{F}(t, l)$ whose isoclasses for a fixed flag F correspond bijectively to the orbits of P on $\mathfrak{p}_u^{(l)}$. The advantage of this approach is that $\mathcal{F}(t, l)$ is an additive category, hence we are reduced to the question: When does $\mathcal{F}(t, l)$ have finitely many indecomposable objects?

The second step follows an idea of Gabriel: We show that $\mathcal{F}(t, l)$ is equivalent to the category of Δ-good modules over a quasi-hereditary algebra $\mathcal{A}(t, l)$. In case $l = 0$, the algebra $\mathcal{A}(t, 0)$ is the Auslander-algebra of the truncated polynomial ring $k[X]/X^t$ (see [HR1]).

The last step extends the techniques developed in [DR] for the case $l = 0$: The category of good $\mathcal{A}(t, l)$-modules is shown to be almost equivalent to the category of modules over some algebra $\mathcal{B}(t, l)$. This allows us to apply the classification results for finite-dimensional algebras, which in turn yields the main result.

(1.4) In the last section of this paper, we discuss two generalizations of the main theorem. At first, we determine which of the representation-infinite algebras $\mathcal{B}(t, l)$ are tame and which are wild. Secondly, we study the P-orbits on arbitrary ideals of $\mathfrak{p}$ in $\mathfrak{p}_u$. We refer to a computer program which allows us to solve the finiteness question for these ideals and their quotients when the nilpotency degree of P_u is sufficiently small. In the finite cases, the program computes representatives of all orbits. In fact, these representatives are independent of the field k, even if k is a finite field.

We refer to [R2] and [DR] for an introduction to quasi-hereditary algebras. Concerning the theory and the classification results for finite-dimensional algebras we refer to [HV],[R1],[ARS] and [GR].

2 From orbits of parabolic subgroups of the general linear group to Krull-Schmidt categories

Let k be a field. All vector spaces are defined over k.

The first step to prove the Theorem is to collect all orbits of all P in $\mathfrak{p}_u^{(l)}$ for fixed $l, t \in \mathbb{N}$ into one additive category $\mathcal{F}(t, l)$. The objects of $\mathcal{F}(t, l)$ are the pairs (F, f), where $F : \{0\} = V_0 \subseteq \cdots \subseteq V_t = V$ is a flag of finite-dimensional k-vector spaces of length t and f is an element of $\mathfrak{p}_u^{(l)}$. A morphism $\phi : (F, f) \to (F', f')$ is a linear map $\phi : V \to V'$ which respects the flags, that is $\phi(V_i) \subseteq V_i'$, and commutes with the endomorphisms: $\phi f = f' \phi$.

Given a flag $F : \{0\} = V_0 \subseteq \cdots \subseteq V_t = V$, we define the dimension vector $d = \dim(F) \in \mathbb{N}^t$ of F as $d = (\dim V_1, \dim V_2 - \dim V_1, \ldots, \dim V_t - \dim V_{t-1})$. From the definition of morphisms in $\mathcal{F}(t,l)$ we obtain the following result.

Lemma 1. *For a fixed flag F, the isomorphism classes of objects (F,f) in $\mathcal{F}(t,l)$ are in natural bijection to the orbits of $P = Stab(F)$ on $\mathfrak{p}_u^{(l)}$.*

Moreover, the category $\mathcal{F}(t,l)$ has the nice feature of being an aggregate (cf. [GR,3.5]) resp. a Krull-Schmidt category (cf. [R,2.2]):

Lemma 2. *$\mathcal{F}(t,l)$ is a Krull-Schmidt category.*

Proof. The interpretation of the Lie algebra $\mathfrak{p}_u^{(l)}$ given in (1.2) allows us to view $\mathfrak{p}_u^{(l)}$ as a bifunctor on the category of flags: To each pair (F,F') of flags of length t we associate the vector space $\mathfrak{p}_u^{(l)}(F,F')$ consisting of all linear maps f from V to V' which satisfy $f(V_i) \subseteq V'_{i-l-1}$ for $i = l+1, \ldots, t$. Then the category $\mathcal{F}(t,l)$ introduced above is just the category of matrices over the bifunctor $\mathfrak{p}_u^{(l)}$. These categories are well-known to be Krull-Schmidt (cf. [C, 2]). $\square$

Recall that a Krull-Schmidt category is an additive category where the decomposition of an object into indecomposable objects is unique up to permutation and isomorphism. Therefore, we obtain representatives of all orbits of $P = \mathrm{Stab}(F)$ on $\mathfrak{p}_u^{(l)}$ just by forming direct sums of indecomposable objects in $\mathcal{F}(t,l)$ such that the dimension vectors of their flags add up to $d = \dim(F)$.
This suggests to change the original problem where we asked for the finiteness of the action of a *fixed* parabolic subgroup P into one where we just fix the parameters t, l and ask: Does $\mathcal{F}(t,l)$ have finitely many indecomposable objects? If this is the case, we know that *each* parabolic subgroup P with $\ell(P_u) = t-1$ acts with a finite number of orbits on $\mathfrak{p}_u^{(l)}$. Otherwise, we obtain a k-family of non-isomorphic objects in $\mathcal{F}(t,l)$, and this shows that the corresponding parabolic subgroup P acts with an infinite number of orbits on $\mathfrak{p}_u^{(l)}$ provided the field k is infinite (see (4.2)).

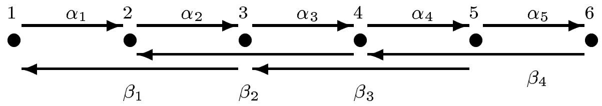
3 The objects of $\mathcal{F}(t,l)$ as modules over quasi-hereditary algebras

(3.1) In this section we show that $\mathcal{F}(t,l)$ is equivalent to the category of Δ-filtered modules over a quasi-hereditary algebra $\mathcal{A}(t,l)$. In case $l = 0$, the algebra $\mathcal{A}(t,l)$ turns out to be the Auslander-algebra of the truncated polynomial ring $k[X]/X^t$.

Definitions. We describe algebras by quivers with relations. A *quiver* $Q = (Q_0, Q_1)$ is an oriented graph with set of vertices Q_0 and set of arrows Q_1. The *path algebra* kQ of a quiver Q is the k-vector space whose basis consists of the paths in Q, including one path of length 0 for each vertex of Q. The multiplication for this basis is defined by the composition of paths, if this is possible, and zero otherwise. An ideal in kQ is called *admissible* if it is generated by linear combinations of paths of length at least two and contains all paths of length n for some sufficiently large n. Recall that the isomorphism classes of simple modules over a quotient kQ/I of a path algebra by an admissible ideal I are in one-to-one correspondence to the vertices of the quiver Q.

By $I(i)$ and $P(i)$ we denote the injective hull and the projective cover of the simple module corresponding to a vertex i. These modules represent the isomorphism classes of indecomposable injective and projective modules. For an introduction to path algebras and their properties we refer to [R1] and [GR]. Recall that any left kQ-module is determined by a set of vector spaces $V_i, i \in Q_0$ and a linear map $V_\alpha : V_i \to V_j$ for each arrow $\alpha : i \to j$ in Q_1.

We define the algebra $\mathcal{A}(t,l)$ as follows. First we introduce a quiver $Q = Q(t,l)$. Let $Q_0 = \{1, \ldots, t\}$ be the set of vertices of Q. Moreover, we define arrows $i \xrightarrow{\alpha_i} i+1$ for $i = 1, \ldots, t-1$ and $i+l+1 \xrightarrow{\beta_i} i$ for $i = 1, \ldots, t-l-1$. Then we consider the ideal I in the path algebra of Q, generated by the relations $\beta_1\alpha_{l+1} = 0$ and $\beta_{i-l}\alpha_i = \alpha_{i-l-1}\beta_{i-l-1}$ for $l+1 < i < t$. Finally, we define $\mathcal{A}(t,l) := kQ/I$. We give an example of the quiver $Q(6,1)$:

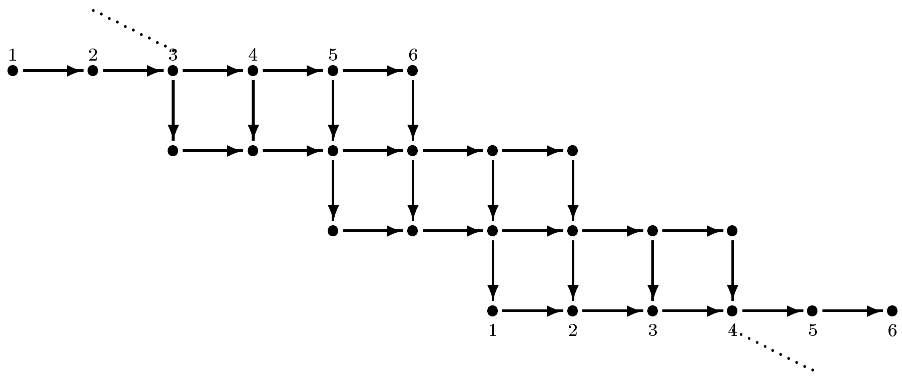


Let $\mathcal{F}(\Delta)$ denote the category consisting of all $\mathcal{A}(t,l)$-modules Y such that $Y(\alpha_i)$ is injective for $i = 1, \ldots, t-1$.

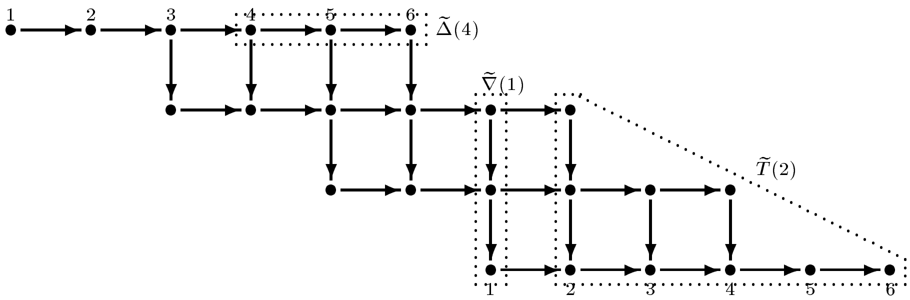
Lemma 3. *The categories $\mathcal{F}(t,l)$ and $\mathcal{F}(\Delta) \subset \mathcal{A}(t,l)$–mod are equivalent.*

Proof. The images of the maps $Y_1 \xrightarrow{\alpha_1} Y_2 \ldots \xrightarrow{\alpha_{t-2}} Y_{t-1} \xrightarrow{\alpha_{t-1}} Y_t$ define a flag in Y_t. Moreover, the map $Y(\beta_{t-l-1})$ defines an endomorphism of this flag, satisfying the condition for an object in $\mathcal{F}(t,l)$. The commutativity relations in $\mathcal{A}(t,l)$ force that $Y(\beta_i)$ is uniquely determined by $Y(\beta_{t-l-1})$ for $i = 1, \ldots, t-l-2$. Thus we obtain an equivalence $\mathcal{F}(\Delta) \to \mathcal{F}(t,l)$. $\square$

(3.2) We are going to show that the algebra $\mathcal{A}(t,l)$ is quasi-hereditary. For properties and definitions of such algebras we refer to the work of Dlab and Ringel, in particular to [DR] and [R2].
Below we consider several $\mathcal{A}(t,l)$-modules which we want to introduce beforehand. A convenient method is to work in the $\mathbb{Z}$-covering $\widetilde{\mathcal{A}}(t,l)$ of the algebra $\mathcal{A}(t,l)$. We refer to [BG] and [B] for the concept of coverings and the push-down functor $\widetilde{\mathcal{A}}(t,l)$–mod $\to \mathcal{A}(t,l)$–mod, $\widetilde{M} \mapsto M$. In the previous example $t = 6$ and $l = 1$ the $\mathbb{Z}$-covering is shown in the figure below. Here we draw the arrows α from left to right and the arrows β pointing downwards.



We introduce certain $\widetilde{\mathcal{A}}(t,l)$-modules by specifying full subquivers Γ of $\widetilde{\mathcal{A}}(t,l)$. The corresponding module is then obtained by placing the vector space k at each vertex of the subquiver and taking identity maps on each arrow of Γ. The modules $\widetilde{\Delta}(i)$ are concentrated on a row with starting point i and end point t. The modules $\widetilde{\nabla}(j)$ lie on a column consisting of all points lying above j. The module $\widetilde{T}(i)$ is concentrated on the triangle consisting of all points lying on the right and above from i. We illustrate this by an example.



The push-down functor $\widetilde{\mathcal{A}}(t,l)$–mod $\to \mathcal{A}(t,l)$–mod yields the corresponding $\mathcal{A}(t,l)$-modules $\Delta(i)$, $\nabla(i)$ and $T(i)$. The module $\Delta(i)$ for instance is one-dimensional in all vertices j with $i \leq j \leq t$. For simplification in the following lemma we set $P(i) = 0 = I(i)$, whenever $i \leq 0$.

Lemma 4. *With the notation introduced above, we have exact sequences of $\mathcal{A}(t,l)$-modules*

$$0 \longrightarrow P(i-l-1) \longrightarrow P(i) \longrightarrow \Delta(i) \longrightarrow 0 \quad \textit{and}$$

$$0 \longrightarrow \nabla(j) \longrightarrow I(j) \longrightarrow I(j-1) \longrightarrow 0.$$

Proof. Recall that each arrow $i \longrightarrow j$ in Q defines a homomorphism $P(j) \to P(i)$ and a homomorphism $I(j) \to I(i)$ between projective and injective modules respectively. It is straightforward to check that the arrows $\beta_i \in Q_1$ define injective maps between the projectives with cokernel $\Delta(i)$. Analogously, the arrows $\alpha_j \in Q_1$ yield surjective maps between the injectives and thus the second of the exact sequences above . $\square$

Next one may check directly, that the image of $P(j)$ in $P(i)$ for any $j < i$ and any homomorphism $P(j) \to P(i)$ is contained in the image of $P(i-l-1)$ in $P(i)$. Thus the $\Delta(i)$ are the standard modules in the notation of [DR, section 1] and the algebra $\mathcal{A}(t,l)$ is quasi-hereditary. An analogous statement holds for the injective modules and the costandard modules $\nabla(j)$. Moreover, the $\mathcal{A}(t,l)$-modules with a filtration whose factors belong to $\Delta = (\Delta(1),\ldots,\Delta(t))$ are exactly those modules Y with all $Y(\alpha_i)$ injective. Thus $\mathcal{F}(\Delta)$ is the category of modules with Δ-filtration.
We collect these observations in the following proposition.

Proposition 1. *For any $t,l \in \mathbb{N}$, the algebra $\mathcal{A}(t,l)$ is quasi-hereditary. Each standard module is of projective dimension at most one and each costandard module is of injective dimension at most one. The category $\mathcal{F}(t,l)$ is equivalent to the category $\mathcal{F}(\Delta)$ of $\mathcal{A}(t,l)$-modules with Δ-filtration.*

(3.3) Following [DR], we now investigate the category $\mathcal{F}(\Delta)$ in more detail. Recall that there exists a unique $\mathcal{A}(t,l)$-module T, called the *characteristic module*, which admits both a filtration by standard and a filtration by costandard modules and has no isomorphic direct summands. The module T decomposes as $T = \oplus_{i \in Q_0} T(i)$ with $T(i)$ defined in (3.2). The modules $T(i)$ have a short projective and injective resolution [DR,4]. We need a more precise statement:

Lemma 5. *Let $T(i), i \in Q_0$ denote the direct summands of the characteristic module T of $\mathcal{A}(t,l)$. Then there exist short exact sequences*

$$0 \to P(i-l-1) \to P(\xi(i)) \to T(i) \to 0,$$

where $\xi(i)$ is determined by $\xi(i) \equiv i \bmod l$ and $t-l \leq \xi(i) \leq t$. In particular, $T(i)$ is projective when $1 \leq i \leq l+1$.

This is easily seen from the definition of the $T(i)$'s in (3.2).
By [DR,4.1], the characteristic module T is a tilting module, thus it defines a torsion pair $(\mathcal{G}(T), \mathcal{H}(T))$ where

$$\mathcal{G}(T) = \{Y \in \mathcal{A}(t,l)\text{–mod} \mid \operatorname{Ext}^1(T,Y) = 0\} \text{ and}$$

$$\mathcal{H}(T) = \{Y \in \mathcal{A}(t,l)\text{–mod} \mid \operatorname{Hom}(T,Y) = 0\}.$$

Given an $\mathcal{A}(t,l)$-module M, we denote by $\eta_T M$ the *trace of T in M*, i.e. the image under the canonical map $\operatorname{Hom}(T,M) \otimes T \to M$.

Corollary 1. *The functor $\mathcal{F}(\Delta)/\langle T \rangle \to \mathcal{H}(T), M \mapsto \eta_T M$ is an equivalence.*

Proof. This is an immediate consequence of Proposition 1 and [DR, Theorem 3].

4 The algebra $\mathcal{B}(t,l)$

(4.1) We finally show that the category $\mathcal{H}(T)$ is equivalent to the category of modules over an algebra $\mathcal{B}(t,l)$.
We define the algebra $\mathcal{B}(t,l)$ as quotient of the algebra $\mathcal{A}(t-l-1,l)$ by the ideal generated by the relation $\alpha_{t-2l-2}\beta_{t-2l-2} = 0$. Note that we can view $\mathcal{B}(t,l)$ as a quotient $\mathcal{A}(t,l)/I$ where the ideal I is generated by the idempotent elements of $\mathcal{A}(t,l)$ corresponding to the vertices $t-l, \ldots, t$. Therefore, we may consider $\mathcal{B}(t,l)$–mod as a full subcategory of $\mathcal{A}(t,l)$–mod formed by those modules Y which satisfy $\operatorname{Hom}(P(i), Y) = 0$ for $i = l+1, \ldots, t$.

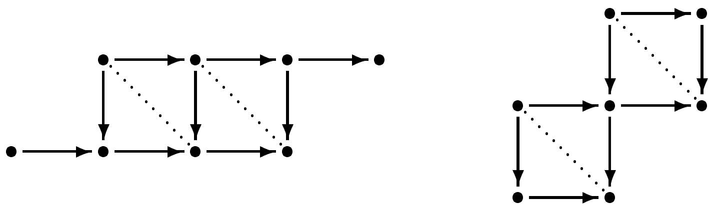
Proposition 2. *The category $\mathcal{H}(T)$ coincides with $\mathcal{B}(t,l)$–mod.*

Proof. We know by Lemma 5 that the modules $T(i)$ for $i = 1, \ldots, l+1$ are precisely the projective modules $P(j)$ for $j = t-l, \ldots, t$. Consequently, a module in $\mathcal{H}(T)$ is a $\mathcal{B}(t,l)$-module. Moreover, the short exact sequence of Lemma 5 shows that any $\mathcal{B}(t,l)$-module Y also satisfies $\operatorname{Hom}(T(i), Y) = 0$ for all i, hence is in $\mathcal{H}(T)$. □

(4.2) The aim of the present part is to show that the algebra $\mathcal{B}(t,l)$ is representation finite precisely when the condition of the main Theorem holds.

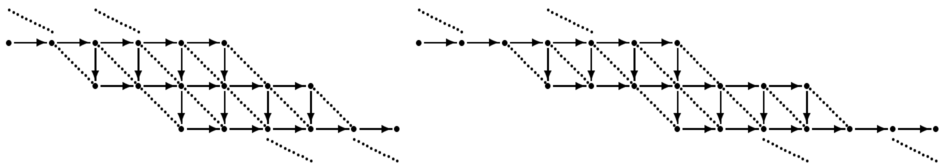
Lemma 6. *The algebra $\mathcal{B}(t,l)$ is of finite representation type if and only if (a) $t \leq 5$ for $l = 0$ or (b) $t - 2l \leq 6$ for $l \geq 1$.*

Proof. First we note, that the two path algebras with relations drawn below are in the list of Happel and Vossieck ([HV]), thus of infinite representation type. At least one of them is a full convex subalgebra of the $\mathbb{Z}$-covering $\widetilde{\mathcal{B}}(t,l)$ of $\mathcal{B}(t,l)$ if neither of the conditions (a) and (b) holds. Therefore, $\mathcal{B}(t,l)$ is of infinite representation type in this case.



On the other hand it is easy to see that under condition (a) or (b) none of the algebras in the list of [HV] appears as a full convex subalgebra of $\widetilde{\mathcal{B}}(t,l)$. Thus $\widetilde{\mathcal{B}}(t,l)$ is representation-finite and using covering theory (cf. [BG] or [B]) the finiteness result for $\mathcal{B}(t,l)$ follows immediately. $\square$

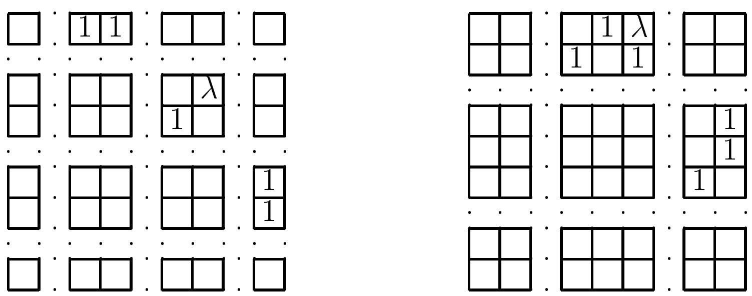
In the next figure we present two representation finite cases for the algebra $\widetilde{\mathcal{B}}(t,l)$, namely the cases $\widetilde{\mathcal{B}}(8,1)$ and $\widetilde{\mathcal{B}}(10,2)$.



(4.3) **Proof of the main result.** We suppose here that k is an infinite field. Let P be a parabolic subgroup of some $GL(V)$, realized as the stabilizer of a proper flag F of length t, and let $l \in \mathbb{N}$. We know by the considerations in section 2 and Lemma 3, Corollary 1 and Lemma 6 that the number of orbits is finite in the cases indicated in Theorem 1.

Finally, we show that in the remaining cases there is an infinite number of orbits. We simply describe here a k-family of pairwise non-isomorphic objects in $\mathcal{F}(t,l)$ with $t = 6$ for $l = 0$, or $t = 7 + 2l$ for $l \geq 1$ (see the following figure). They lie in the Lie-algebra of the Borel subgroup, that is the dimension vector d equals $d = (1, \ldots, 1)$. Therefore *any* parabolic $P = \mathrm{Stab}(F)$ where F is a proper flag of length at least t has infinitely many orbits. One can check directly, that the one-parameter families below consist of non-isomorphic objects, which finishes the proof.

Of course, we obtained these two families from the two critical algebras shown in (4.2). For simplicity we indicate only those entries in the matrix below which are non-zero. Moreover we should explain the dotted lines. We assume that the dotted columns and rows are all of the same size, say s. Then the size of the first matrix is $t = 6 + 3s$ (and of the second $t = 7 + 2s$) and $l = s$. For $l = 0, 1$ this shows the infiniteness for the tuples $(t, l) = (6, 0)$ and $(9, 1)$ (consider the first matrix) and for $l \geq 1$ this shows the infiniteness for the tuples $(9, 1)$, $(11, 2), \ldots, (7 + 2l, l)$ (consider the second matrix). $\square$



5 Some generalizations

(5.1) Tame actions.

The class of finite-dimensional algebras over an algebraically closed field k is divided into two disjoint classes. The algebras of the first class are called *tame*. Their indecomposable modules can be parametrized in each dimension by a finite number of discrete and a finite number of one-parameter families. If no one-parameter family occurs, the algebra is of finite representation type.
The second class is formed by the wild algebras. The classification problem of square matrices under simultaneous conjugacy (a well-known unsolved problem) occurs as a subproblem when one studies modules over a wild algebra up to isomorphism. We refer to [PG] for more information on tame and wild algebras.

In the preceding sections we constructed algebras $\mathcal{B}(t, l)$ whose isoclasses of modules correspond to orbits of certain parabolic subgroups. When proving the main theorem, we classified those algebras $\mathcal{B}(t, l)$ which are of finite representation type. It is now natural to ask when the algebras $\mathcal{B}(t, l)$ are tame or wild, and what the different representation types mean for the corresponding parabolic subgroups.

Let $P = \mathrm{Stab}(F)$ be a parabolic subgroup of $GL(V)$, where $F : \{0\} = V_0 \subseteq \cdots \subseteq V_t = V$ is a flag of k-vector spaces of length t. Then we define for each $n \in \mathbb{N}$ a parabolic subgroup $P(n)$ of $GL(V \otimes k^n)$ by $P(n) = \mathrm{Stab}(F(n))$, where $F(n)$ is the t-flag $F \otimes k^n : \{0\} = V_0 \otimes k^n \subseteq \cdots \subseteq V_t \otimes k^n = V \otimes k^n$.

Recall that the modality of the action of P on $\mathfrak{p}_u^{(l)}$, $\mathrm{mod}(P : \mathfrak{p}_u^{(l)})$, is the maximal number of parameters a family of P-orbits on $\mathfrak{p}_u^{(l)}$ depends on (cf. [V]).
With these notions we are able to express the three possible representation types of the algebras $\mathcal{B}(t, l)$ in terms of their corresponding parabolic subgroups for each $l, t \in \mathbb{N}$:

$\mathcal{B}(t,l)$ is representation-finite if and only if $\mathrm{mod}(P : \mathfrak{p}_u^{(l)}) = 0$ for each proper t-flag F and $P = \mathrm{Stab}(F)$,
$\mathcal{B}(t,l)$ is tame precisely when the number $\mathrm{mod}(P(n) : \mathfrak{p}(n)_u^{(l)})$ grows linearly in $n \in \mathbb{N}$ for each proper t-flag F and $P = \mathrm{Stab}(F)$ and finally
$\mathcal{B}(t,l)$ is wild if and only if there is a proper t-flag F with $P = \mathrm{Stab}(F)$ such that $\mathrm{mod}(P(n) : \mathfrak{p}(n)_u^{(l)})$ grows quadratically in $n \in \mathbb{N}$.

In the following diagram we describe the representation types of the algebras $\mathcal{B}(t,l)$. Note that the case $l = 0$ is treated in [DR].

representation type of $\mathcal{B}(t,l)$	if $l = 0$	if $l \geq 1$
finite	$t \leq 5$	$t \leq 2l + 6$
tame infinite	$t = 6$	$t = 2l + 7$
wild	$t \geq 7$	$t \geq 2l + 8$

We plan to study the tame algebras $\mathcal{B}(t,l)$ in more detail in a forthcoming paper.

(5.2) Actions on subquotients of $\mathfrak{p}_u$.
Let $P = \mathrm{Stab}(F)$ be a parabolic subgroup of some $GL(V)$ as above. In this section we consider more generally the question which $\mathfrak{p}$-ideals of $\mathfrak{p}_u$ admit only finitely many orbits under the adjoint action of P. For an infinite field k, the $\mathfrak{p}$-ideals of $\mathfrak{p}_u$ are of the form $\mathfrak{p}(\lambda) = \{f \in \mathrm{End}(V) | f(V_i) \subseteq V_{\lambda(i)}\}$ where $\lambda = (\lambda(1), \ldots, \lambda(t))$ satisfies $\lambda(i) < i$ and $\lambda(i) \leq \lambda(i+1)$.
The ideal $\mathfrak{p}_u^{(l)}$ for example is determined by the sequence $\lambda = (\lambda(1), \ldots, \lambda(t))$ with $\lambda(i) = i - l - 1$ for $i = l+1, \ldots, t$ and $\lambda(i) = 0$ for $i = 1, \ldots, l$.
Moreover, we also consider the induced action of P on quotients of the form $\mathfrak{p}(\lambda)/\mathfrak{p}(\mu)$.
As in section 2, it is possible to view the P-orbits on $\mathfrak{p}(\lambda)/\mathfrak{p}(\mu)$ in terms of a Krull-Schmidt category $\mathcal{F}(\lambda, \mu)$. We just have to interpret $\mathfrak{p}(\lambda)/\mathfrak{p}(\mu)$ as a bimodule over the category of flags as it was done in Lemma 2.
Therefore the classical bimodule reduction algorithm (cf. [C]) applies to this situation. We wrote a computer program [XPar] which implements the bimodule reduction algorithm for this particular situation. The program tests whether for given λ and μ the category $\mathcal{F}(\lambda, \mu)$ has finitely many indecomposable objects.
If the number of indecomposables is finite, the output consists of various pieces of information on the indecomposable objects: The number of indecomposable objects, the corresponding dimension vectors, the dimensions of the space of homomorphisms between the indecomposables and finally representatives of the indecomposable objects.
In case the number of indecomposables is infinite, either a 1-parameter family of indecomposables is specified or a matrix with 4 non-zero entries in a row (or column) is given (which implies that the problem is of infinite type).

Acknowledgement

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Computer program

[XPar] T. Brüstle, D. Guhe: Program xparabol which computes the orbits of an action of P on ideals in $\mathfrak{p}_u$ and their quotients. Available via ftp: Server: ftp.uni-bielefeld.de, directory: pub/math/f-d-alg/crep.

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