A NOTE ON NORMAL ABELIAN SUBGROUPS IN PARABOLIC GROUPS

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Abstract. Let G be a reductive algebraic group, P a parabolic subgroup of G with unipotent radical P_u, and A a closed connected unipotent subgroup of P_u which is normalized by P. We show that P acts on A with finitely many orbits provided A is abelian. This generalizes a well-known finiteness result, namely the case when A is central in P_u. Also, we obtain an analogous result for the adjoint action of P on invariant linear subspaces which are abelian as subalgebras of the Lie algebra of P_u under some mild characteristic restrictions.

1. Introduction

Throughout, G denotes a (connected) reductive algebraic group defined over an algebraically closed field k of characteristic $p \geq 0$ and P is a parabolic subgroup of G with unipotent radical P_u. The aim of this note is the following result.

Theorem 1.1. *Let G be a reductive algebraic group, P a parabolic subgroup of G, and A a closed connected normal subgroup of P in P_u. If A is abelian, then P has finitely many orbits on A.*

The particular case when A is in the center of P_u is well-known. Then the action factors through a Levi factor of P. Here, the finiteness is due to Vinberg [35, §2] in characteristic 0 (see also Kac [14]) and follows in general from work of Richardson [24, §3]. For a detailed account of the orbit structure in this situation, see [20] and [25, §2, §5]. Although, many further instances of Theorem 1.1 are surely known, as stated here, it does not seem to appear in the literature.

Observe that for abelian P-invariant subfactors in P_u, the analogous statement of the theorem is false in general. Indeed, this fact is the basis for constructing entire families of parabolic subgroups which admit an infinite number of orbits on the unipotent radical, or its Lie algebra, e.g., see [21], [22], [26], and [27]. Examples in this context also show that a parabolic subgroup may already have an infinite number of orbits on a normal subgroup of nilpotency class two, cf. [9], [15].

There are several recent articles related to this subject. For instance, all parabolic subgroups P of classical algebraic groups with a finite number of orbits on P_u are determined in [11] and [12]. Similar results for exceptional groups are obtained in [13]. In [8] all such P in $GL_n(k)$ are classified with a finite number of orbits on a given term of the lower central series of P_u. More generally, in [21] and [28] the number of parameters on which the action of P on P_u depends upon, i.e. the modality of this action, is investigated.

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Thanks to Brundan's recent extension of the classification of spherical Levi subgroups of reductive groups to arbitrary characteristic [7, §4] (see Section 3), we immediately obtain a proof of Theorem 1.1 for all cases where A is contained in the unipotent radical of a parabolic subgroup whose Levi factor is spherical in G. In particular, this includes all possibilities when G is classical (see Section 4). In the other instances where we cannot appeal to spherical Levi subgroups directly, a construction from [1] allows us to apply these results partially.

It would be very desirable to have a uniform proof of Theorem 1.1 free of case analysis, even for the expense of some characteristic restrictions.

2. Notation and Preliminaries

We denote the Lie algebra of G by $\operatorname{Lie} G$ or $\mathfrak{g}$ and the identity element of G by e; likewise for subgroups of G. Let T be a fixed maximal torus in G and $\Psi = \Psi(G)$ the set of roots of G with respect to T. Fix a Borel subgroup B of G containing T and let $\Sigma = \{\sigma_1, \sigma_2, \dots\}$ be the set of simple roots of Ψ defined by B such that the positive integral span of Σ in Ψ is $\Psi(B)$. The highest (long) root in Ψ is denoted by ϱ and, in the presence of two root lengths, the highest short one by ϱ_s. For a root β of G we denote by U_β the corresponding one-parameter unipotent subgroup of G normalized by T, and the root subspace $\operatorname{Lie} U_\beta$ by $\mathfrak{u}_\beta$. The members of U_β are called *root elements*.

Suppose that G is simple (over its center). A prime is said to be *bad* for G if it divides the coefficient of a simple root in ϱ, else it is called *good* for G, see [31, §I,4]. Furthermore, we say that a prime is *very bad* for G if it divides a structure constant of the Chevalley commutator relations for G. Thus, if $\operatorname{char} k$ is very bad for G, there are degeneracies in these relations. This only occurs if $p = 2$ and G is of type B_r, C_r, F_4, or G_2, or $p = 3$ and G is of type G_2. The same notions apply to reductive groups by means of simple components [33, 3.6].

We may assume that each parabolic subgroup P of G considered contains B.

Let N be a closed connected normal subgroup of P in P_u. Since N is normalized by $T \subset P$, the root spaces of $\mathfrak{n}$ relative to T are also root spaces of $\mathfrak{g}$ relative to T, and the set of roots of N with respect to T, $\Psi(N)$, is a subset of Ψ. Suppose that $\Psi(N)$ is closed under addition in Ψ. Note that this is automatically satisfied provided $\operatorname{char} k$ is not very bad for G. Then $\mathfrak{n} = \oplus \mathfrak{u}_\beta$ $(\beta \in \Psi(N))$ and consequently, $N = \prod U_\beta$, where the product is taken in some fixed order over $\Psi(N)$. The *support* of an element x in N, denoted by $\operatorname{supp} x$, consists of all roots β for which the projection $N \to U_\beta$ is nontrivial when evaluated on x. By the *shape of a root* $\beta = \sum n_\sigma(\beta)\sigma$ $(\sigma \in \Sigma)$ *relative to* P, we mean the sub-sum over the elements of $\Sigma(P_u) = \Psi(P_u) \cap \Sigma$, see [1].

The descending central series of P_u is defined as usual by $\mathcal{C}^0 P_u := P_u$ and $\mathcal{C}^{i+1} P_u := (\mathcal{C}^i P_u, P_u)$ for $i \geq 0$. Since P_u is nilpotent, the smallest integer m such that $\mathcal{C}^m P_u = \{e\}$ is the class of nilpotency of P_u, i.e., the *length* of this series, and is also denoted by $\ell(P_u)$.

Throughout, we use the labeling of the Dynkin diagram of G (i.e. of Σ) as well as the notation for roots in systems of exceptional type as in [3, Planches I - IX]. Our general reference for algebraic groups is Borel's book [2].

3. Spherical Levi Subgroups

The following basic result is due to Brion [4] and Vinberg [36] in characteristic 0 and in arbitrary characteristic by Knop [16, 2.6].

Theorem 3.1. *Let G be a reductive algebraic group and B a Borel subgroup of G. Let X be an irreducible G-variety admitting a dense B-orbit. Then B has finitely many orbits on X.*

A closed subgroup H of G is called *spherical* if H has a dense orbit on G/B, or equivalently, if there is a dense B-orbit on G/H, or equivalently, by Theorem 3.1, if there is a finite number of B-orbits on G/H. Concerning recent results on spherical subgroups in positive characteristic, consult [7]. The associated *spherical varieties* G/H have been studied extensively in the literature; see [6] for a survey.

For $\operatorname{char} k = 0$ all reductive spherical subgroups have been classified in [17], [5], and [19]. When $\operatorname{char} k \neq 2$ many instances are known as centralizers of involutions by a result of Springer [30]. Brundan's method of "integral embeddings" [7, §2] and the representation theoretic fact that Levi subgroups of G are good filtration subgroups (see [10], [18]) enable him to extend the classification of spherical Levi subgroups from characteristic zero [17] to arbitrary characteristic [7, Theorem 4.1]:

Theorem 3.2. *Let L be a Levi subgroup of G. Let $G = \prod_{i=1}^{r} G_i$ as a commuting product of simple factors and $L_i := L \cap G_i$. Then L is spherical if and only if, for each i, either $L_i = G_i$, or (G_i, L_i') is one of $(A_r, A_m A_{r-m-1})$, (B_r, B_{r-1}), (B_r, A_{r-1}), (C_r, C_{r-1}), (C_r, A_{r-1}), (D_r, D_{r-1}), (D_r, A_{r-1}), (E_6, D_5), or (E_7, E_6).*

Theorem 3.2 and the next result [7, Lemma 4.2] yield a classification of all parabolic subgroups $P = LP_u$ of G for which a Borel subgroup B_L of L has a dense orbit on P_u, whence finitely many orbits thanks to Theorem 3.1.

Lemma 3.3. *Let $P = LP_u$ be a parabolic subgroup of G and B_L a Borel subgroup of L. Then L is spherical in G if and only if there is a dense B_L-orbit on P_u.*

Remark 3.4. Let $P = LP_u$ be parabolic in G. Then L is spherical in G if and only if either P_u is abelian, or $p \neq 2$ and one of the pairs (G_i, L_i) as in Theorem 3.2 equals (B_r, A_{r-1}) or (C_r, C_{r-1}). This is immediate from Theorem 3.2 and the well-known instances when P_u is abelian, e.g., see [25, Remark 2.3] for p not very bad for G. In this latter case, a direct proof of the finiteness of the number of B_L-orbits on P_u is given in [34, §4].

4. Proof of Theorem 1.1

Clearly, we may assume that G is simple (over its center) and that A is a maximal closed connected abelian subgroup of B_u normalized by B. All possibilities for A are compiled in Table 1 in Section 6. Here we simply list the roots α such that A is the normal closure in B of the root subgroups U_α. The fact that A is abelian follows either from the observation that the sum of two roots in $\Psi(A)$ is not a root, because it exceeds ϱ in some coefficient, and thus, by the commutator relations, A is commutative, or else because p is a very bad prime for G leading to commutation degeneracies. As indicated in the table, some extra cases do occur for the prime 2.

It is easy to check that each group A recorded in Table 1 is maximal among closed connected normal abelian subgroups of B and that our list is complete.

Since A is B-invariant, $N_G(A)$, the normalizer of A in G, is parabolic in G.

4.1. First we discuss the entries on the left hand side of Table 1 - in particular, this covers all classical occurrences. Here the desired finiteness statements follow immediately from the results in Section 3, as in each of these cases A is contained in the unipotent radical of a parabolic subgroup $P = LP_u$ of G, where L is spherical in G, simply by inspection of the list in Theorem 3.2. Hence, by Lemma 3.3 and Theorem 3.1, B_L and thus B acts on P_u and thus on A with a finite number of orbits. In all but one of these instances $N_G(A) = P$ and $A = P_u$. Namely, if G is of type B_r, $r \geq 3$, and $p \neq 2$, the normal closure in B of $\{U_\beta, U_{\varrho_s}\}$, where $\beta = \alpha_{r-1} + 2\alpha_r$, is a maximal normal abelian subgroup of B which is properly contained in the unipotent radical of the maximal parabolic subgroup P of G corresponding to α_r, i.e., whose Levi factor is of type A_{r-1}. Here $N_G(A)$ is of semisimple corank 1 in P. This corresponds to the third case listed in Theorem 3.2 which equally applies when $p = 2$ and then, P_u itself is abelian. This leads to the third entry in this table for B_r. Similarly, for $p = 2$ there is an additional incident on our list for C_r.

Now we turn to the right side of Table 1. Here we cannot appeal directly to Section 3. However, a construction from [1] allows us to apply these results in part.

4.2. First, we focus on those cases where $N_G(A)$ is a maximal parabolic; here we write $P = LP_u$ for $N_G(A)$. This happens precisely when A is the normal closure in B of a single root subgroup U_α, i.e., we consider all events for E_6, E_7, E_8, and the first ones for F_4 and G_2. Observe that 2α exceeds ϱ in the unique simple root in $\Sigma(P_u)$, and $A = \mathcal{C}^s P_u$ for $s = 1$ or $s = 2$. The latter occurs only once, for E_8.

For each $i \in \mathbb{N}$ define $V_i := \mathcal{C}^{s+i-1}P_u/\mathcal{C}^{s+i}P_u$. Because A is abelian, we may regard each V_i as a subgroup of A. Let $t := \ell(P_u) - s$. Then $V_t = Z(P_u)$ is the last term in the descending central series of P_u. For each $i = 1, \dots, t$ there is a unique root α_i in $\Psi(V_i)$ of minimal height; let S_i be its shape relative to P. Note that $\alpha_1 = \alpha$. And observe that V_i is the product of all root subgroups of roots of shape S_i. Denote by Ψ_i the set of all roots whose shapes are integral multiples of S_i. This is a closed semisimple subsystem of Ψ, and it contains $\Psi(L)$ (the set of roots of shape 0) as a subsystem of corank 1, since its elements are all multiples of S_i modulo the integral span of $\Psi(L)$. Thus the positive simple system for Ψ_i consists of $\Sigma(L)$ and α_i. Since $2S_i$ is not the shape of any root in Ψ, the union of $\pm\Psi(V_i)$ and $\Psi(L)$ equals Ψ_i. Furthermore, since $\Psi(L)$ has corank 1 in Ψ, the subsystem Ψ_i is of maximal rank in Ψ. Therefore, if G_i is the connected semisimple subgroup of G corresponding to Ψ_i (i.e. $\Psi(G_i) = \Psi_i$), then $T \subset G_i$, and $P_i := LV_i$ is the maximal parabolic subgroup of G_i corresponding to the simple root α_i with unipotent radical V_i. The group G_i need not be simple (over its center), but the simple component containing α_i contains all of $\Psi(V_i)$ and hence also the group V_i, while the other simple components occur passively. Observe that each V_i naturally carries the structure of an L-module, cf. [1]. Since V_i is abelian, L is a spherical Levi subgroup of G_i by Remark 3.4. Thus, the previous results applied to $P_i = LV_i$ in G_i yield that B_L has a finite number of orbits on V_i for each $i = 1, \dots, t$. Therefore, since $A = V_1 \cdots V_t$ and $B = B_L P_u$, we are able to conclude our desired finiteness statement, once we have established that there are only finitely many P_u-orbits in each coset of the form $vV_{i+1} \cdots V_t$, where v is in $V_i \setminus \{e\}$, for $1 \leq i < t$. We discuss some examples in detail below.

We now concentrate on the situation when $t = 2$ (as $t = 3$ only for third E_8 entry in Table 1) and consider the following condition:

(†) Suppose that for every pair of roots (β, γ) in $\Psi(V_1) \times \Psi(V_2)$ the difference $\gamma - \beta$ is again a root in Ψ (i.e. in $\Psi(P_u)$).

If (†) is satisfied and char k is not very bad for G, then every B-orbit passing through an element of $A = V_1V_2$ with nontrivial support in V_1 already has a representative in V_1, as every V_2-factor can be removed using conjugation by suitable root elements in P_u. Then, since B_L has a finite number of orbits on V_1 and on V_2, there is a finite number of B-orbits on all of A.

4.3. One checks that (†) is satisfied precisely when G is of type E_6, as well as in the first entries for E_7, E_8, F_4, and G_2. Thus, our finiteness result follows in these instances from the last paragraph provided $p \neq 2$ if G is of type F_4 or G_2. Note that for G_2 only the prime 2 leads to an obstruction here. But in the first G_2 entry in Table 1 we require that $p \neq 2$. Here $\dim V_2 = 2$ except for G_2, where $\dim V_2 = 1$.

4.4. Next we consider the first F_4 entry when $p = 2$. As before, $A = \mathcal{C}^1 P_u$, where $P = N_G(A)$. Although (†) is satisfied, the commutator relations in F_4 are degenerate when $p = 2$ for short root subgroups spanning a subsystem of Ψ of type B_2. Let v_1 be in $V_1 \setminus \{e\}$. We consider the set of B-orbits passing through v_1V_2. Since the two roots in $\Psi(V_2)$ are long and there are no degeneracies for the commutators of long root subgroups in $\Psi(B)$, we can argue as above, provided there is a long root in $\operatorname{supp} v_1$. Consequently, we only need to consider the case when v_1 is supported entirely on short roots. Since any two short roots in $\Psi(V_1)$ span a subsystem of Dynkin type A_2 (and the structure constants in the corresponding group are ± 1), we may assume (possibly after using the action of B_L) that v_1 is a single short root element. Thus, any element in $v_1V_2 = v_1U_{1342}U_{2342}$ is supported by at most three linearly independent roots, and we can use the action of the torus T in B to obtain a finite set of orbit representatives in this case as well.

4.5. Now we discuss the remaining E_7 and E_8 cases. Here (†) fails. In the second instances for E_7 and E_8 we have $\dim V_1 = 15$, $\dim V_2 = 5$ and $\dim V_1 = 28$, $\dim V_2 = 8$, respectively. From our discussion in (4.2) we know that B_L has a finite number of orbits on V_1 and on V_2. Now let $v = v_1v_2$ be a B-orbit representative in $A = V_1V_2$, where $v_i \in V_i \setminus \{e\}$ for $i = 1, 2$. One checks that for a fixed $\beta \in \Psi(V_1)$ there is precisely one $\gamma \in \Psi(V_2)$ such that $\gamma - \beta$ is not a root in the E_7 instance, while there are two such roots γ for the E_8 case. Let δ be a root in the support of v_1 of minimal height. Thus, we may assume that all roots of the form $\delta + \eta$ with $\eta \in \Psi(B_L)$ have been removed from the support of v_1 and that v_2 consists of a single root element in the E_7 case and of at most two root elements in the E_8 event. If there are no further roots in the support of v_1, then v is supported by two, respectively at most three linearly independent roots and we use the maximal torus T in B to scale the coefficients in the various root elements to equal 1 and obtain a single orbit representative. Furthermore, one checks that if there are any extra roots in $\operatorname{supp} v_1$, then we can remove all the root element factors in v_2, by conjugating with suitable root elements from P_u, without reintroducing any root factors which have already been removed and we are left with an orbit representative in V_1. Consequently, we have a finite number of B-orbits on $A = V_1V_2$ in each of these two cases.

Finally, we address the third E_8 case. Here $\dim V_1 = 20$, $\dim V_2 = 10$, and $\dim V_3 = 4$. Note that here the condition in (†) holds for V_2 and V_3 in place of V_1 and V_2, respectively. Thus B has a finite number of orbits on V_2V_3, by the same argument as above. Let $v = v_1v_2v_3$ be a B-orbit representative in $A = V_1V_2V_3$, where $v_i \in V_i$ for $i = 1, 2, 3$ and $v_1 \neq e \neq v_2v_3$. One checks that for a fixed $\beta \in \Psi(V_1)$ there are precisely four roots $\gamma \in \Psi(V_2)$ such that $\gamma - \beta$ is not a root and there is precisely one root $\delta \in \Psi(V_3)$ such that $\delta - \beta$ is not a root in Ψ. Let τ be a root in the support of v_1 of minimal height. Thus, we may assume that all roots of the form $\tau + \eta$ with $\eta \in \Psi(B_L)$ have been removed from the support of v_1 (using the action of B_L) and that v_2 consists of at most four root elements and that either $v_3 = e$, or v_3 is a single root element (using the action of P_u). If there are no further roots in the support of v_1, then v is supported by at most six linearly independent roots and we can use the maximal torus T in B to scale the coefficients in the remaining root elements of v to equal 1 and obtain a single orbit representative in this instance. Furthermore, if there are any additional roots in $\operatorname{supp} v_1$, then (using the action of P_u) we can remove all the root element factors in v_2 and v_3 without reintroducing any new root elements or ones which have already been removed. Thus, we are left with an orbit representative in V_1. Consequently, we have a finite number of B-orbits on all of $A = V_1V_2V_3$.

4.6. Next we turn to the remaining F_4 cases. It is advantageous to first consider a particular normal abelian subgroup of B which, although not maximal, still fits the setting of (4.2). Namely, let V be the normal closure in B of U_δ, where $\delta = 0122$. Let $P = N_G(V)$. Then $P = LP_u$ is the maximal parabolic subgroup of G of type B_3, $V = Z(P_u)$, and $\dim V = 7$. We can apply the same construction as in (4.2). Let Ψ_1 be the closed semisimple subsystem of Ψ formed by $\Psi(L)$ together with $\pm\Psi(V)$ and positive simple system consisting of $\Sigma(L)$ and δ. Let G_1 be the semisimple subgroup of G corresponding to Ψ_1. Then G_1 is of type B_4 and $P_1 = LV$ is the maximal parabolic subgroup of G_1 corresponding to the simple root δ. Since V is abelian, L is a spherical Levi subgroup of G_1 by Remark 3.4. Hence, B_L and thus B has a finite number of orbits on V. Note this is valid also for $p = 2$.

Now in the second F_4 incident on our list $A = U_{1221}U_{1231}V$. Thus, by the last paragraph, it suffices to consider orbit representatives whose support contains at least one of 1221 or 1231. One easily checks that there is only a finite number of such B-orbits passing through A.

In the third case for F_4 we have $A = U_{0121}U_{1121}U_{1221}U_{1231}V$. As above, we only need to consider elements whose support involves at least one of the first four roots shown. Keeping in mind that $p = 2$ in this instance, one checks directly that there is only a finite number of such B-orbits in A.

The commutative groups from the last two F_4 entries (both for $p = 2$ only) share the 10-dimensional normal subgroup of B which is generated by U_{1121} and U_{0122}. Thus, by the previous result, in the fourth case we only need to show that there are finitely many B-orbits passing through elements in A whose support involves 1111. This is easily verified.

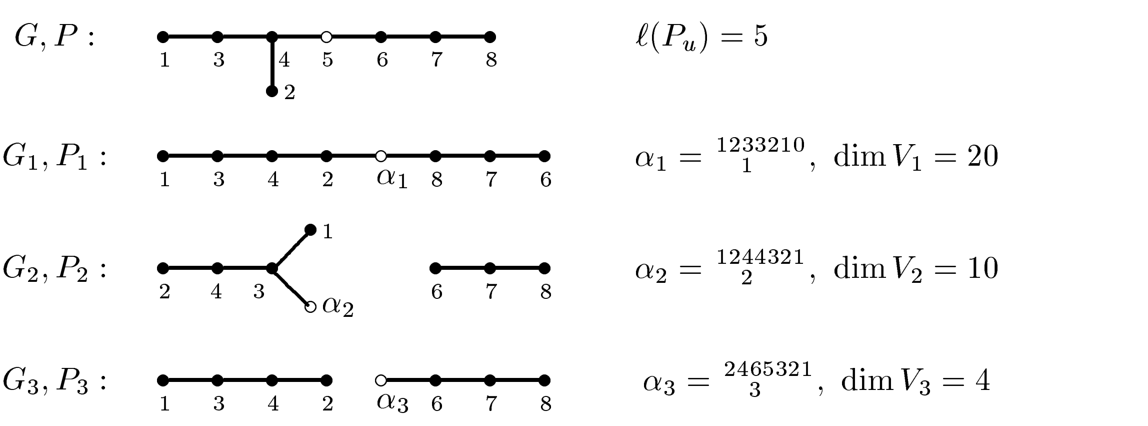
Finally, the result for the second G_2 case is readily checked directly.

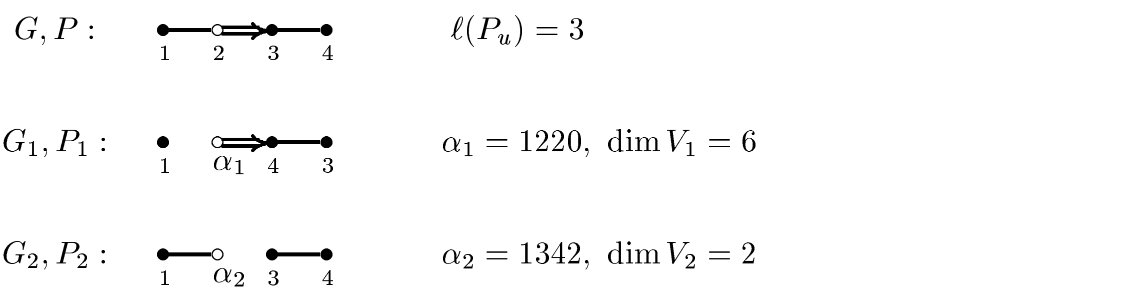
This completes the proof of Theorem 1.1.

We illustrate the construction in (4.2) with two examples.

Examples 4.7. Our first example is the third entry for E_8 from Table 1. For $P = LP_u = N_G(A)$ as in (4.2) L is of type A_3A_4, $\ell(P_u) = 5$, and $A = \mathcal{C}^2 P_u$. As shown, G_i is of type A_8, A_3D_5, or A_4A_4 for $i = 1, 2, 3$, respectively. We indicate $\Sigma(L)$ by coloring the corresponding nodes in the diagram of G, likewise for G_i. Moreover, α_i, $\dim V_i$, and the various passive components of G_i are specified.

We also present the first case for F_4. Here $P = N_G(A)$ is of type $A_1\widetilde{A}_2$, where $\widetilde{A}_2$ is the subsystem of Ψ spanned by the short simple roots, and G_1 and G_2 are of type A_1C_3 and $A_2\widetilde{A}_2$, respectively.





5. Abelian ideals in the Lie Algebra of P_u

Let G, P, and A be as in the Introduction. If char k is zero or a good prime for G, we can make use of Springer's map $\varphi : \mathcal{U} \longrightarrow \mathcal{N}$ between the unipotent variety $\mathcal{U}$ of G and the nilpotent variety $\mathcal{N}$ of $\mathfrak{g}$ which is a G-equivariant bijective morphism (see [29]), and, upon "restriction" to A, we obtain a P-equivariant bijective morphism from A onto $\mathfrak{a} = \operatorname{Lie} A$, e.g., see [27, Theorem 4.1]. Note that $\mathfrak{a}$ is an abelian ideal in $\mathfrak{p}$. Consequently, we get a statement analogous to Theorem 1.1 for the adjoint action of P on P-invariant linear subspaces of $\operatorname{Lie} P_u$ which are abelian as Lie subalgebras of $\mathfrak{p}$. Using a result of Pyasetskii [23], which is also valid in positive characteristic, we obtain a similar statement for the coadjoint action of P on $\mathfrak{a}^*$.

The characteristic restrictions for the adjoint action can be removed completely in the exceptional cases in the table below by employing the computer algorithm outlined in [13]. This is valid provided char k is not very bad for G. The remaining cases when $p = 2$ for F_4 and G_2 can be verified directly.

6. The maximal normal abelian subgroups of Borel subgroups

In the subsequent table we record all maximal closed connected normal abelian subgroups A of B. Here we specify the roots α such that A is the normal closure in B of the U_α's. Recall from (4.1) that in the second B_r case $r \geq 3$ and $\beta = \alpha_{r-1} + 2\alpha_r$. Furthermore, for the cases on the right hand side of the table we indicate the Dynkin type of a Levi factor of $N_G(A)$ provided this is a maximal parabolic subgroup of G. In the first F_4 entry $\widetilde{A}_2$ refers to the A_2 component supported by the short simple roots (see the example above). In each of the other circumstances here $N_G(A)$ is not maximal parabolic. In the last three F_4 cases a Levi factor of $N_G(A)$ is of type $A_1\widetilde{A}_1$, A_2, and B_2, respectively, while in the last G_2 incident $N_G(A) = B$.

Note, if I is the ideal in $\Psi(B)$ ([32, p. 24]) generated by a positive root β, then U_γ, for some $\gamma \in I$, need not be contained in the normal closure of U_β in B, because of commutator degeneracies when char k is a very bad prime for G. This necessitates that in some of the cases which only arise for $p = 2$ we have to include an additional root here. This occurs for G_2 and in the third instance for F_4, but not for B_r or C_r, as in a subsystem of type B_2 the structure constants involved in the commutator of the two simple root groups are ± 1, cf. [32, Lemma 33].

G	A	$\dim A$	G	A	$N_G(A)$	$\dim A$
A_r	σ_i $(1 \leq i \leq r)$	$i(r-i+1)$	E_6	$\begin{smallmatrix}01210\\1\end{smallmatrix}$	$A_1A_2A_2$	11
B_r	σ_1	$2r-1$	E_7	$\begin{smallmatrix}122100\\1\end{smallmatrix}$	A_1A_5	17
	$\{\beta, \varrho_s\}$ $(p \neq 2)$	$(r^2-r+2)/2$		$\begin{smallmatrix}012210\\1\end{smallmatrix}$	A_2A_4	20
	σ_r $(p = 2)$	$(r^2+1)/2$	E_8	$\begin{smallmatrix}0122221\\1\end{smallmatrix}$	A_1E_6	29
C_r	σ_r	$(r^2+1)/2$		$\begin{smallmatrix}1232100\\2\end{smallmatrix}$	A_7	36
	σ_1 $(p = 2)$	$2r-1$		$\begin{smallmatrix}1233210\\1\end{smallmatrix}$	A_3A_4	34
D_r	σ_1	$2r+1$	F_4	1220	$A_1\widetilde{A}_2$	8
	σ_{r-1}, σ_r	$(r^2-r)/2$		$\{1221, 0122\}$ $(p \neq 2)$		9
E_6	σ_1, σ_6	16		$\{0121, 0122\}$ $(p = 2)$		11
E_7	σ_7	27		$\{1111, 0122\}$ $(p = 2)$		11
			G_2	21 $(p \neq 2)$	A_1	3
				$\{11, 21\}$ $(p = 2)$		4

Table 1: The maximal normal abelian subgroups of Borel subgroups.

Acknowledgments. The question whether a statement like Theorem 1.1 might be true was brought up by P. Neumann in connection with the results in [25], [22], and [26] during a visit by the author to Oxford in Spring of 1995. Recently, V.L. Popov also raised that question.

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