Circular Biextensions of Tame Concealed Algebras

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1 Introduction and main result

Let k be an algebraically closed field. Associated with the extended Dynkin diagrams $\widetilde{\mathbb{A}}_n$ $(n \geq 1)$, $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ there are the tame concealed algebras. They are defined as the endomorphism algebras of basic preprojective tilting modules of hereditary algebras $k\vec{\Delta}$ where the underlying graph Δ of the quiver $\vec{\Delta}$ is one of the extended Dynkin diagrams. If Δ is different from $\widetilde{\mathbb{A}}_n$, then all quivers $\vec{\Delta}$ having Δ as underlying graph yield the same tame concealed algebras B. On the other hand, for $\Delta = \widetilde{\mathbb{A}}_n$ one obtains various classes depending on the number p of clockwise and q of counterclockwise arrows in $\vec{\Delta}$. Hence we define the *extended Dynkin type* of a tame concealed algebra B as Δ if B is the endomorphism algebra of a basic preprojective tilting module over $k\vec{\Delta}$ where Δ is one of the diagrams $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$. Moreover, the extended Dynkin type is defined as $\widetilde{\mathbb{A}}_{p,q}$ if B is the endomorphism algebra of a basic preprojective tilting module over $k\vec{\Delta}$ where Δ is $\widetilde{\mathbb{A}}_n$ and in $\vec{\Delta}$ there are p clockwise and q counterclockwise arrows. If we require $p \geq q$, then the extended Dynkin type is uniquely determined by B.

The tame concealed algebras furnish the simplest class of tame algebras which are not of finite representation type. Recall that a finite-dimensional k-algebra is said to be tame if for any non-negative integer d the indecomposable (left) A-modules of dimension d lie up to isomorphism on finitely many 1-parameter families.

Besides a preprojective and a preinjective component the Auslander-Reiten quiver of a tame concealed algebra B consists of a family $(\mathcal{T}_\lambda)_{\lambda \in \mathbb{P}}$ of standard stable tubes $\mathcal{T}_\lambda$ where the index set $\mathbb{P} = \mathbb{P}^1(k)$ is the projective line over k. Each of the tubes $\mathcal{T}_\lambda$ is an abelian length category having only finitely many simple objects. The number $n(\lambda)$ of these simple objects is called the rank of the tube $\mathcal{T}_\lambda$. Almost each of the tubes $\mathcal{T}_\lambda$ is homogeneous, i.e. $n(\lambda) = 1$.

One-point extensions and more general branch enlargements by simple modules in tubes $\mathcal{T}_\lambda$ led to many examples of tame algebras (see e.g. [Ri2]) and even to the complete classification of strongly simply connected algebras which are tame of polynomial growth (see [Sk]). It is well-known that one-point extensions by modules of length 2 in tubes $\mathcal{T}_\lambda$ lead into the area of tame algebras of non-polynomial growth whereas one-point extensions by modules of length ≥ 3 always are wild (see [Ri1]). Multiple one-point extensions by modules of length 2 (even more general derived cases) were investigated in [DrS]. In some proofs in this paper a new class of tame algebras arose. The aim of this paper is to introduce this class of algebras systematically and to analyze the occurring representation types.

Let us introduce the necessary notation. The objects of length l in a tube $\mathcal{T}_\lambda$ form an orbit under the Auslander-Reiten translation τ of B. Let us fix a module R^λ of length 2 in $\mathcal{T}_\lambda$. For all $i = 1, \dots, n(\lambda)$ we put $R_i^\lambda := \tau^i R^\lambda$. Since $R^\lambda_{n(\lambda)} = R^\lambda$, we can assume that the index set for i is $\mathbb{Z}/n(\lambda)$. For each $i \in \mathbb{Z}/n(\lambda)$ there is a non-zero homomorphism $f_i : R^\lambda_{i+1} \to R^\lambda_i$ which is unique up to a multiple in k and satisfies $f_i f_{i+1} = 0$ for all $i \in \mathbb{Z}/n(\lambda)$.

Let us define for $n \in \mathbb{N}$ the finite-dimensional algebra C_n as the algebra kQ_n/I where Q_n is the quiver in Figure 1 whose vertex set is identified with $\mathbb{Z}/n$. The ideal I of the path algebra kQ_n is generated by the paths $X_{i+1}X_i$ for all $i \in \mathbb{Z}/n$.

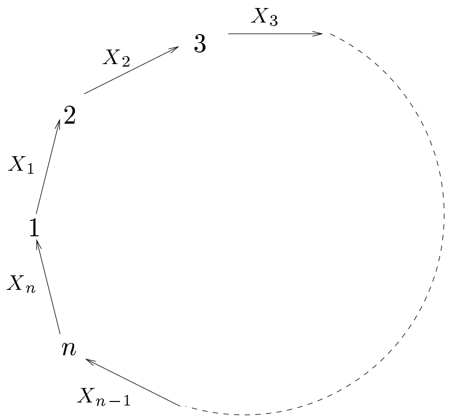

Figure 1: Circular quiver

For $\lambda \in \mathbb{P}$ we obtain a contravariant representation $R(\lambda)$ of $C_{n(\lambda)}$ in $B-$mod by mapping each vertex i of $Q_{n(\lambda)}$ to R_i^λ and each arrow X_i to f_i. But a representation of $C_{n(\lambda)}$ in $B-$mod is nothing but a B-$C_{n(\lambda)}$-bimodule. More general, for any finite subset $T = \{\lambda_1, \dots, \lambda_t\}$ of $\mathbb{P}$ we define the algebra $C(T)$ as the product $C_{n(\lambda_1)} \times \cdots \times C_{n(\lambda_t)}$ and the B-$C(T)$-bimodule $R(T)$ as $\oplus_{i=1}^t R(\lambda_i)$. Given a second finite subset S of $\mathbb{P}$ the *circular biextension* $\boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ of B by the sets S, T is defined to be the matrix algebra

$$\begin{pmatrix} C(T) & 0 & 0 \\ R(T) & B & 0 \\ \mathrm{D}\, R(S)) \otimes_B R(T) & \mathrm{D}\, R(S) & C(S) \end{pmatrix}$$

where the multiplication is given by the usual matrix multiplication and $\mathrm{D} = \mathrm{Hom}_k(-, k)$ is the usual duality. We denote by $\boxed{\mathrm{S}}B$ (resp. $B\boxed{\mathrm{T}}$) the truncated versions where T (resp. S) is the empty set. The algebra $B\boxed{\mathrm{T}}$ will be called *circular extension* of B by T. Dually $\boxed{\mathrm{S}}B$ will be called *circular coextension* of B by S. Now we can formulate the first result of our paper:

Theorem (A). *Let B be a tame concealed algebra and S, T be two finite subsets of $\mathbb{P}$. Then the circular biextension $\boxed{S}B\boxed{T}$ is tame in the following cases:*

(i) The extended Dynkin type of B is $\widetilde{\mathbb{D}}_n$ and $T = \{\lambda\}$, $S = \{\mu\}$ such that $n(\lambda) = n(\mu) = n - 2$.

(ii) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{p,q}$ and $T = \{\lambda_1, \lambda_2\}$, $S = \{\mu_1, \mu_2\}$ such that $n(\lambda_1) = n(\mu_1) = p$ and $n(\lambda_2) = n(\mu_2) = q$.

(iii) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{2,2}$ and $T = \{\lambda\}$, $S = \{\mu\}$ such that $n(\lambda) = n(\mu) = 1$.

(iv) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{2,2}$ and $T = \{\lambda\}$, $S = \{\mu_1, \mu_2\}$ such that $n(\lambda) = 1$, $n(\mu_1) = n(\mu_2) = 2$ or $T = \{\lambda_1, \lambda_2\}$, $S = \{\mu\}$ such that $n(\lambda_1) = n(\lambda_2) = 2$, $n(\mu) = 1$.

Clearly, one obtains tame biextensions $\boxed{\text{S'}}B\boxed{\text{T'}}$ if S' is a subset of S and T' a subset of T for sets T, S as in Theorem A. The question, whether the list of tame circular biextensins of tame concealed algebras obtained in this way is complete, will be answered positively by our second result:

Theorem (B). *Suppose that the circular extension $B\boxed{T}$ of the tame concealed algebra B is tame where T is a non-empty finite subset of $\mathbb{P}$. Then one of the following three assertions is valid:*

(i) The extended Dynkin type of B is $\widetilde{\mathbb{D}}_n$ and $T = \{\lambda\}$ such that $n(\lambda) = n - 2$.

(ii) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{p,q}$ and T is a subset of $\{\lambda_1, \lambda_2\}$ where $n(\lambda_1) = p$ and $n(\lambda_2) = q$.

(iii) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{2,2}$ and $T = \{\lambda\}$ such that $n(\lambda) = 1$.

Let as finish the introduction by an example of a tame circular biextension $\boxed{\text{S}}B\boxed{\text{T}}$ where B is of extended Dynkin type $\widetilde{\mathbb{D}}_6$. It is given by the quiver in Figure 2 bound by the relations $\nu_1\alpha_1$, $\nu_1\alpha_2$, $\gamma_1\mu_3$, $\gamma_2\mu_3$, $\alpha_1\sigma_1+\alpha_2\sigma_2+\mu_1 X_1$, $\tau_1\gamma_1+\tau_2\gamma_2+Y_3\nu_3$, $\gamma_i\mu_3+\gamma_i\beta_2\beta_1\alpha_1\sigma_1 X_4$ $(i = 1, 2)$, $\nu_1\alpha_i+Y_4\tau_1\gamma_1\beta_2\beta_1\alpha_i$ $(i = 1, 2)$, $\beta_i\mu_i+\mu_{i+1}X_{i+1}$ $(i = 1, 2)$, $Y_i\nu_i+\nu_{i+1}\beta_i$ $(i = 1, 2)$, $X_{i+1}X_i$ $(i = 1, \ldots, 4)$, $Y_{i+1}Y_i$ $(i = 1, \ldots, 4)$.

2 The associated bimodule problem

We recall some notation from [GR]. An aggregate $\mathcal{A}$ is a skeletally small k-additive category with splitting idempotents and finite-dimensional morphism spaces. The category $A-$mod of finite-dimensional left A-modules over a finite-dimensional algebra A is an aggregate.

A bimodule M over the two aggregates $\mathcal{A}$, $\mathcal{B}$ is a k-linear bifunctor $\mathcal{A} \times \mathcal{B} \to k-$mod which is contravariant in the first and covariant in the second argument (see [Dd]). A representation of M is a triple (a, m, b) where a is an object of $\mathcal{A}$, b is an object of $\mathcal{B}$ and m is an element of $M(a, b)$. A morphism from a representation (a, m, b) to another representation (a', m', b') of M is a pair (f, g) such that $f : a \to a'$ in $\mathcal{A}$, $g : b \to b'$ in $\mathcal{B}$ and $M(\mathrm{id}_a, g)(m) = M(f, \mathrm{id}_b)(m')$. Defining the composition of morphisms of representations of M in the obvious way, we obtain the category $\operatorname{rep} M$ which is again an aggregate.

Let us return to algebras $\boxed{\text{S}}B\boxed{\text{T}}$ where B is tame concealed and S, T are finite subsets of $\mathbb{P}$. It is well-known that the modules over such a biextension algebra can

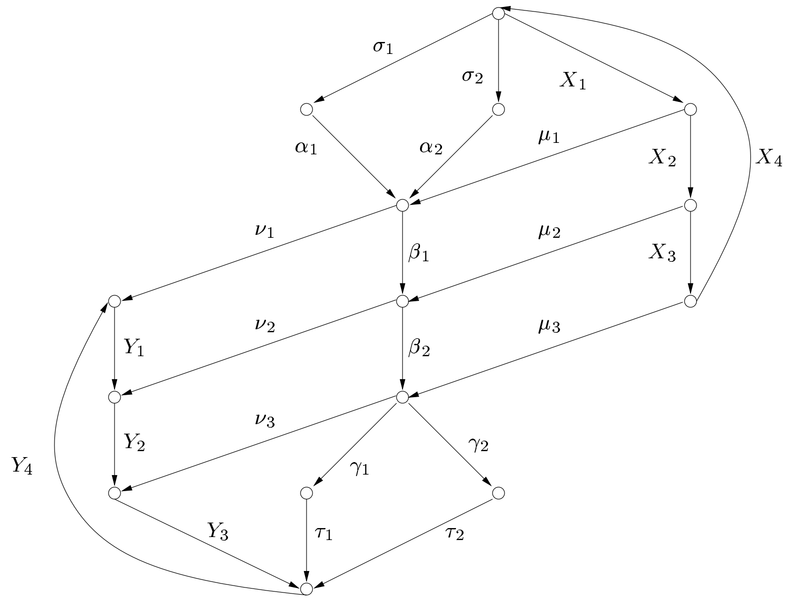

Figure 2: Example 1

be identified with quintuples $(U, \gamma, X, \delta, V)$ where $U \in C(T)-\text{mod}$, $X \in B-\text{mod}$, $V \in C(S)-\text{mod}$, $\gamma \in \operatorname{Hom}_{C(T)}(U, \operatorname{Hom}_B(R(T), X)$, and $\delta \in \operatorname{Hom}_{C(S)}(\mathrm{D}\, R(S) \otimes_B X, V)$. By mapping the quintuple $(U, \gamma, X, \delta, V)$ to the triple $(U \oplus V, \gamma \oplus \mathrm{D}\,\delta, X)$, we obtain an equivalence $\boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ mod $\to \operatorname{rep} M(S, B, T)$ where $M(S, B, T)$ is the bimodule $(C(T) \times C(S))-\text{mod} \times B-\text{mod} \to k-\text{mod}$ which maps $U \oplus V$ in $(C(T) \times C(S))-\text{mod}$ and X in $B-\text{mod}$ to $\operatorname{Hom}_{C(T)}(U, \operatorname{Hom}_B(R(T), X) \oplus \mathrm{D} \operatorname{Hom}_{C(S)}(\mathrm{D}\, R(S) \otimes_B X, V)$.

We want to characterize the tameness of $\boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ via $M(S, B, T)$ and invoke [Dr][Theorem 3.3] which can be generalized immediately from vector space categories to our bimodule situation because the category $(C(T) \times C(S))-\text{mod}$ has up to isomorphism only finitely many indecomposable objects. Let us define the category $\mathcal{B}$ as the factor category of $B-\text{mod}$ by the ideal given by the intersection of the kernel of the functors $\operatorname{Hom}_B(R(T), -) : B-\text{mod} \to k-\text{mod}$ and the kernel of the functor $\mathrm{D}\, R(S) \otimes_B - : B-\text{mod} \to k-\text{mod}$. We denote by $\mathcal{M}(S, B, T)$ the induced bimodule which acts on $(C(T) \times C(S))-\text{mod} \times \mathcal{B}$.

Lemma. *Let B be a tame concealed algebra and S, T be finite subsets of $\mathbb{P}$. Then the circular biextension algebra $\boxed{S}B\boxed{T}$ is tame if and only if the bimodule $\mathcal{M}(S, B, T)$ is tame.*

Proof. Following [Dr][Theorem 3.3] the tameness of $\boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ is characterized by three conditions. Fortunately, two of them are obviously satisfied. Namely, first the algebra B is tame and next the finite collection of tubes $\mathcal{T}_\lambda$ such that $\lambda \in S \cup T$ together with the preprojective and the preinjective component contain only finitely many indecomposable B-modules whose dimension is bounded by a fixed integer. The third condition is just the tameness of $\mathcal{M}(S, B, T)$. $\square$

Let B and B' be two tame concealed algebras of the same extended Dynkin type. Then there exists a preprojective tilting module X in $B-\text{mod}$ whose endomorphism algebra is

B'. That X is preprojective implies that the associated tilting functor $F = \mathrm{Hom}_B(X, -)$ induces an equivalence of cofinite subcategories of $B-$mod and $B'-$mod. In particular, the respective families of tubes are mapped to each other. Therefore we will identify the two families. Using this convention the lemma above allows us to prove:

Proposition. *Let B and B' two tame concealed algebras of the same extended Dynkin type and S, T be two finite subsets of $\mathbb{P}$. Then the algebras $\boxed{S}B\boxed{T}$ and $\boxed{S}B'\boxed{T}$ have the same representation type.*

Proof. By [Ha] the tilting functor F extends to a triangle equivalence $D^b(B) \to D^b(B')$ of derived categories of bounded complexes which we will consider as an identification. Via the canonical embedding we obtain $B-$mod and $B'-$mod as full subcategories of $D^b(B)$. We observe the isomorphism $\mathrm{D}\, R(S) \otimes_B - \cong \mathrm{D}\,\mathrm{Hom}_B(-, R(S)) : B-\text{mod} \to k-\text{mod}$ and call $\mathcal{D}$ the factor category of $D^b(B)$ by the ideal given by the intersection of the kernel of the functor $\mathrm{Hom}_{D^b(B)}(R(T), -) : D^b(B) \to k-\text{mod}$ and the kernel of the functor $\mathrm{D}\,\mathrm{Hom}_{D^b(B)}(-, R(S)) : D^b(B) \to k-\text{mod}$. Thus $\mathcal{B}$ and $\mathcal{B}'$ become full subcategories of $\mathcal{D}$.

To establish our claim it is enough to find an autoequivalence of $\mathcal{D}$ which maps $\mathcal{B}$ into $\mathcal{B}'$. We denote by $\mathcal{D}_0$ the full subcategory of $\mathcal{B}$ whose objects are direct sums of indecomposables in the tubes $\mathcal{T}_\lambda$. The full subcategory $\mathcal{D}_\leq$ (resp. $\mathcal{D}_\geq$) of $\mathcal{D}$ has as objects direct sums of indecomposables which are predecessors (resp. successors) of objects in $\mathcal{D}_0$. Clearly, $\mathcal{D}_\leq \cup \mathcal{D}_\geq$ additively generates $\mathcal{D}$ and $\mathcal{D}_\leq \cap \mathcal{D}_\geq = \mathcal{D}_0$. Moreover, since the family $(\mathcal{T}_\lambda)_{\lambda \in \mathcal{P}}$ is separating in $\mathrm{D}^b(B)$, there is no non-zero map from $\mathcal{D}_\leq \setminus \mathcal{D}_0$ to $\mathcal{D}_\geq \setminus \mathcal{D}_0$. This shows that we can extend an autoequivalence $\phi_\leq$ of $\mathcal{D}_\leq$ and an autoequivalence $\phi_\geq$ of $\mathcal{D}_\geq$ to an autoequivalence ϕ of $\mathcal{D}$ provided that $\phi_\leq$ and $\phi_\geq$ coincide on $\mathcal{D}_0$.

To construct our desired autoequivalences we proceed as follows. Let N be a common multiple of all the ranks $n(\lambda)$ of the tubes $\mathcal{T}_\lambda$. It τ is the Auslander-Reiten translation in $D^b(B)$ then τ^{-N} induces an autoequivalence $\phi_\leq$ of $\mathcal{D}_\leq$ which is the identity on $\mathcal{D}_0$. Dually τ^N acts as an autoequivalence $\phi_\geq$ on $\mathcal{D}_\geq$ which leaves $\mathcal{D}_0$ invariant. Clearly, a sufficiently large power of the induced autoequivalence ϕ of $\mathcal{D}$ will map $\mathcal{B}$ into $\mathcal{B}'$ and also $\mathcal{B}'$ into $\mathcal{B}$. $\square$

3 Proof of Theorem A

Let B be a tame concealed algebra and S, T be finite subsets of $\mathbb{P}$. If $S \cap T = \emptyset$, then there does not exist any indecomposable B-module X satisfying $\mathrm{Hom}_B(R(T), X) \neq 0$ and $\mathrm{Hom}_B(X, R(S)) \neq 0$. Consequently, every indecomposable representation of $M(S, B, T)$ is either a representation of $M(\emptyset, B, T)$ or a representation of $M(S, B, \emptyset)$. Thus we proved the following 'splitting lemma':

Lemma. *If B is a tame concealed algebra and S, T are finite subsets of $\mathbb{P}$ such that $S \cap T = \emptyset$, then any indecomposable module over $\boxed{S}B\boxed{T}$ is either a module over $\boxed{S}B$ or a module over $B\boxed{T}$. In particular, $\boxed{S}B\boxed{T}$ is tame provided the algebras $\boxed{S}B$ and $B\boxed{T}$ are tame.*

We are now going to proof the tameness of $\boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ in the four cases of Theorem A and start with case (i). Thus the extended Dynkin type of B is $\widetilde{\mathbb{D}}_n$ for some $n \geq 4$ and

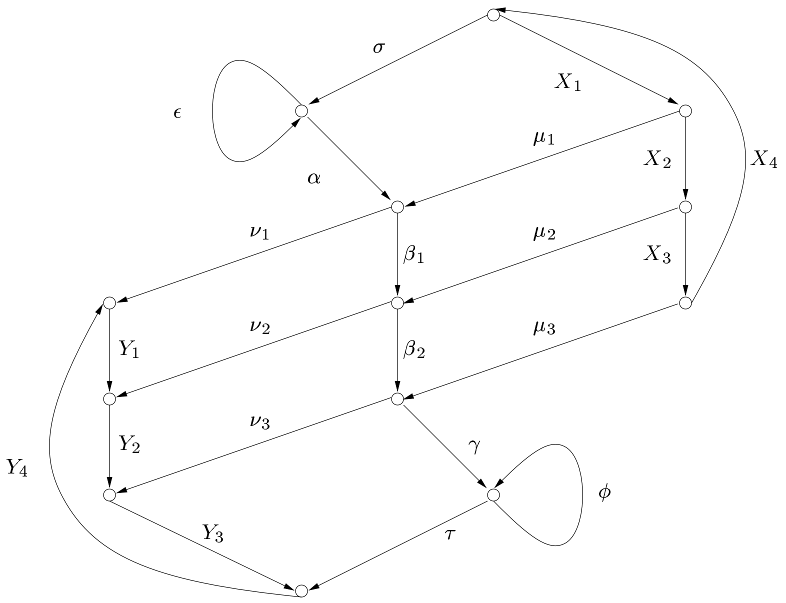

Figure 3: Clannish algebra 1

$T = \{\lambda\}$, $S = \{\mu\}$ such that $n(\lambda) = n(\mu) = n - 2$. If $n > 4$ this implies $S = T$. Only for $n = 4$ the case $S \neq T$ is possible. But then the tameness of $\boxed{\text{S}}B\boxed{\text{T}}$ follows from the tameness of $\boxed{\text{S}}B$ and $B\boxed{\text{T}}$ by the above splitting lemma. On the other hand the tameness of $\boxed{\text{S}}B$ (resp. $B\boxed{\text{T}}$) follows from the tameness of $\boxed{\text{S}}B\boxed{\text{S}}$ (resp. $\boxed{\text{T}}B\boxed{\text{T}}$). Thus we only have to deal with the case $S = T$.

By the Proposition from section 2 we can chose $B = k\vec{\Delta}$ hereditary such that the underlying graph Δ is $\widetilde{\mathbb{D}}_n$. We can even choose a convenient orientation and select the one from [DR] because this enables us to use the tables in this paper. Then $\boxed{\text{S}}B\boxed{\text{T}}$ has exactly the shape as in the example at the end of section one. Of course this is only true for $n = 6$ but the arguments for arbitrary n are completely the same.

By replacing all X_i (resp. Y_i) in the relations by tX_i (resp. tY_i) we obtain a family of algebras $(A_t)_{t\in k}$ with the same dimension. Since $A_t \cong \boxed{\text{S}}B\boxed{\text{T}}$ for all $t \neq 0$, we obtain that A_0 is a degeneration of $\boxed{\text{S}}B\boxed{\text{T}}$. Thus by [Ge] the tameness of A_0 implies the tameness of $\boxed{\text{S}}B\boxed{\text{T}}$. But A_0 is a clannish algebra whose tameness is known by [C-B]. It is given by the quiver in Figure 3 together with the relations $\nu_1\alpha$, $\gamma\mu_3$, $\alpha\sigma$, $\tau\gamma$, $\beta_i\mu_i$ $(i = 1, 2)$, $\nu_{i+1}\beta_i$ $(i = 1, 2)$, $X_{i+1}X_i$ $(i = 1, \dots, 4)$, $Y_{i+1}Y_i$ $(i = 1, \dots, 4)$, $\epsilon(\epsilon - 1)$, $\phi(\phi - 1)$.

Because the proof of tameness in case (ii) is very similar, we skip it and mention only some special features. First, it is not too surprising that the degeneration algebra A_0 where we finally arrive at in this case is actually special biserial. Moreover, there are some more special cases to handle. Namely, for $q \geq 2$ one knows again that $S = T$. But for $p > 1$ and $q = 1$ the intersection $S \cap T$ may have 1 or 2 elements. Both cases have to be settled separately, but the arguments are parallel. For $p = q = 1$, which means that B is the Kronecker algebra, in addition $S \cap T = \emptyset$ is possible. Fortunately, this is settled by the splitting lemma and the other two cases work as for $p > 1$.

Let us give again a complete proof of the tameness of $\boxed{\text{S}}B\boxed{\text{T}}$ in case (iii). Since the extended Dynkin type of B is $\widetilde{\mathbb{A}}_{2,2}$, we may choose B to be of the shape $k\vec{\Delta}$ where

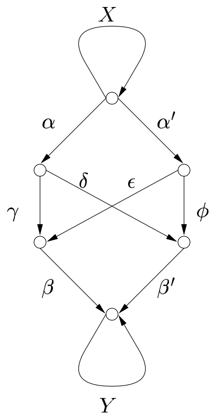

Figure 4: Example 2

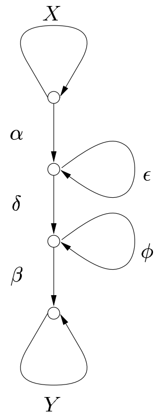

Figure 5: Clannish algebra 2

the underlying graph Δ is $\widetilde{\mathbb{A}}_3$ and $\vec{\Delta}$ has two sources and two sinks. By the splitting lemma we may assume that $S = T$. After possibly applying an automorphism of B, we arrive at the algebra $\boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ given by the quiver of Figure 4 with the relations $\epsilon\alpha + \phi\alpha'$, $\gamma\alpha X + \epsilon\alpha' + \gamma\alpha$, $\beta\text{‘}\phi + \beta\epsilon$, $Y\beta\gamma + \beta'\delta + \beta\gamma$, X^2, Y^2.

Replacing X by tX and Y by tY we again obtain a family $(A_t)_{t\in k}$ of algebras of the same dimension where $A_t \cong \boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ for all $t \neq 0$. As A_0 is again a clannish algebra given by the quiver in Figure 5 and the relations $\delta\alpha$, $\beta\delta$, X^2, Y^2, $\epsilon(\epsilon-1)$, $\phi(\phi-1)$, tameness follows by [C-B] as in case (i).

Finally, in case (iv) the tameness of $\boxed{\mathrm{S}}B\boxed{\mathrm{T}}$ follows from the cases (ii) and (iii) via the splitting lemma.

4 Proof of Theorem B

The circular extension $B\boxed{\mathrm{T}}$ is wild provided one finds a subset T' of T such that $B\boxed{\mathrm{T'}}$ is wild. The reason is that the second algebra appears as a convex subcategory of the first one. Therefore, in order to prove Theorem B, it is sufficient to prove the wildness of the circular extension $B\boxed{\mathrm{T}}$ where B is a tame concealed algebra and T is a subset of $\mathbb{P}$ in the following cases:

(a) The extended Dynkin type of B is $\widetilde{\mathbb{E}}_n$ ($n = 6, 7, 8$) and $T = \{\lambda\}$ for λ in $\mathbb{P}$.

(b) The extended Dynkin type of B is $\widetilde{\mathbb{D}}_n$ ($n \geq 4$) and $T = \{\lambda\}$ for λ in $\mathbb{P}$ such that $n(\lambda) < n - 2$.

(c) The extended Dynkin type of B is $\widetilde{\mathbb{D}}_4$ and $T = \{\lambda, \mu\}$ for λ, μ in $\mathbb{P}$ such that $n(\lambda) = n(\mu) = 2$.

(d) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{p,q}$ ($p \geq 3$, $q \geq 2$) and $T = \{\lambda\}$ for λ in $\mathbb{P}$ such that $n(\lambda) = 1$.

(e) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{p,1}$ ($p \geq 2$) and $T = \{\lambda, \mu\}$ for λ, μ in $\mathbb{P}$ such that $n(\lambda) = n(\mu) = 1$.

(f) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{1,1}$ and $T = \{\lambda, \mu, \nu\}$ for λ, μ, ν in $\mathbb{P}$ such that $n(\lambda) = n(\mu) = n(\nu) = 1$.

(g) The extended Dynkin type of B is $\widetilde{\mathbb{A}}_{2,2}$ and $T = \{\lambda, \mu\}$ for λ, μ in $\mathbb{P}$ such that $n(\lambda) = 2$ and $n(\mu) = 1$.

Before going through these various cases, we need some auxiliary results. Note that for λ in $\mathbb{P}$ the module R^λ is of length 2 in the tube $\mathcal{T}_\lambda$ which is an abelian length category. Hence it has a socle which we denote by V^λ and a top which we denote by W^λ. Consequently, $R^\lambda / V^\lambda \cong W^\lambda$ holds and $V^\lambda = W^\lambda$ if $n(\lambda) = 1$.

Lemma. *Let B be a tame concealed algebra and λ in $\mathbb{P}$ such that $T = \{\lambda\}$. Then the circular extension $B\boxed{T}$ is wild if one of the following assertions holds:*

(i) $n(\lambda) > 1$ and the one-point extension $B[R^\lambda]$ is wild.

(ii) $n(\lambda) = 1$ and the one-point extension $(B \times k)[W^\lambda \oplus k]$ is wild.

Proof. For $n(\lambda) > 1$ the algebra $B[R^\lambda]$ is a subcategory of $B\boxed{\mathrm{T}}$. This proves (i). To see (ii), we observe that $\begin{pmatrix} 0 & 0 \\ V^\lambda & 0 \end{pmatrix}$ is an ideal of $B\boxed{\mathrm{T}}$ such that the associated factor algebra is isomorphic to $\begin{pmatrix} C(T) & 0 \\ W^\lambda & B \end{pmatrix}$ where X_1 acts trivially on W^λ. In the canonical $\mathbb{Z}$-covering of this algebra obtained by stretching the loop X_1 to a line, the algebra $(B \times k)[W^\lambda \oplus k]$ appears as convex subcategory. By [DoS] the wildness of $(B \times k)[W^\lambda \oplus k]$ implies the wildness of $B\boxed{\mathrm{T}}$. □

Note, that $(B \times k)[W^\lambda \oplus k]$ is a branch extension of B by the simple regular module W^λ where the added branch is of length 2 (see [Ri2]). Hence the algebra $(B \times k)[W^\lambda \oplus k]$ is wild provided its tubular type is neither a Dynkin nor an extended Dynkin diagram (see [Pe]).

Using the lemma, cases (a) and (b) are settled easily. Namely, for $n(\lambda) > 1$ the wildness of $B[R^\lambda]$ is known by [Ri1] and for $n(\lambda) = 1$ the tubular type of $(B \times k)[W^\lambda \oplus k]$

is neither Dynkin nor extended Dynkin. In case (c) the two-point extension $B[R^\lambda, R^\mu]$ is a wild subcategory of $B\boxed{\mathrm{T}}$.

To solve the remaining four cases dealing with B of extended Dynkin type $\widetilde{\mathbb{A}}_{p,q}$, we recall that by the Proposition in section 2 we can choose $B = k\vec{\Delta}$ where $\vec{\Delta}$ is the quiver from Figure 6.

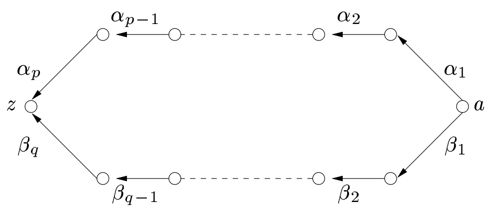

Figure 6: $\widetilde{\mathbb{A}}_{p,q}$

In the cases (d), (e), (f) we form the factor algebras of $B\boxed{\mathrm{T}}$ by the idempotents associated with z and possibly some predecessors of z, obtaining one of the algebras whose quivers appear in Figure 7 whereas the relations are X^2 in case (d), X^2, Y^2 in case (e) and X^2, Y^2, Z^2 in case (f). It is easy to see that in the universal Galois covering of each of these algebras there appears a convex subcategory which is a hereditary algebra associated with a hyperextended Dynkin diagram. Consequently, the coverings and therefore (see [DoS]) also the factor algebras are wild. Case (g) is eventually settled by factoring out from $B\boxed{\mathrm{T}}$ the ideal generated by the idempotents belonging to z and one suitable predecessor. The factor algebra is the same as for case (e).

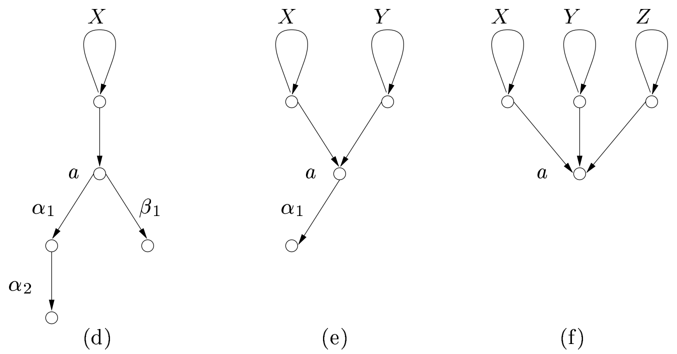

Figure 7: Factor algebras

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