On the number of cells of a cellular algebra

by

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1 Introduction

Cellular algebras have been introduced by Graham and Lehrer in [2] in order to discuss the structure of group algebras of symmetric groups and related algebras like certain Hecke algebras (for instance, of type A or B), Brauer algebras, Temperley–Lieb algebras and many others. The definition of a cellular algebra is by the existence of a socalled cell chain of certain ideals (see [3] and section 2 below). Suppose $0 = J_0 \subseteq J_1 \subseteq ... \subseteq J_n = A$ is such a cell chain of ideals of a cellular algebra A. We consider the following two questions:

(1) Is this cell chain a maximal chain, or can it be refined to a longer cell chain?

(2) Is the length n of a maximal cell chain an invariant of the given cellular algebra?

The class of quasi-hereditary algebras (as defined in [1]) has a large intersection with the class of cellular algebras. Thus we recall first that in this case, each heredity chain can be refined to a maximal heredity chain and the length of any maximal heredity chain is an invariant of the algebra, since it is equal to the number of isomorphism classes of simple A–modules. For cellular algebras, a directly analogous statement is the following: The number of J_j occuring in the cell chain with $J_j^2 \not\subseteq J_{j-1}$ is an invariant (see [3]), since again it equals the number of isomorphism classes of simple A–modules.

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The purpose of this note is to answer the above questions and in doing so to provide some further information on cell ideals. More precisely, we show: First, each cell chain is maximal (that is, it cannot be refined any more). However, secondly, the length of a cell chain is not an invariant of a cellular algebra (we provide a counterexample).

2 Cellular algebras

Let us first recall the original definition of cellular algebras in [2] and the equivalent one given in [3].

Definition 2.1 *(Graham and Lehrer [2]) Let R be a commutative Noetherian integral domain. An associative R–algebra A is called a* **cellular algebra** *with cell datum (I, M, C, i) if the following conditions are satisfied:*

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R–basis $C^{\lambda}_{S,T}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R–linear anti–automorphism of A with $i^2 = id$ which sends $C^{\lambda}_{S,T}$ to $C^{\lambda}_{T,S}$.

(C3) For each $\lambda \in I$ and $S, T \in M(\lambda)$ and each $a \in A$ the product $aC^{\lambda}_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S) C^{\lambda}_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in R$ do not depend on T.

In the following, an R–linear anti–automorphism i of A with $i^2 = id$ will be called an **involution**.

Definition 2.2 *(see [3]) Let A be an R–algebra where R is a commutative Noetherian integral domain. Assume there is an involution i on A. A two–sided ideal J in A is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A–bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i–image of Δ) making the following diagram commutative:*

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta) \end{tikzcd}$$

The algebra A (with the involution i) is called **cellular** *if and only if there is an R–module decomposition $A = J_1' \oplus J_2' \oplus \ldots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two–sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.*

We call the above chain of ideals defining a cellular algebra a **cell chain**. If a chain of ideals in A cannot be refined to a longer one, we say that such a cell chain is **maximal**.

3 Cell chains

From now on, we always will assume that the ring R = k is an (arbitrary) field.

To check whether an ideal J is a cell ideal, the following necessary condition is sometimes useful.

Lemma 3.1 *(a) If J is an n^2-dimensional cell ideal in A, then the k–dimension of $Fix_i(J) := \{x \in J \mid i(x) = x\}$ satisfies*

$$dim_k(Fix_i(J)) = n(n+1)/2.$$

(b) If $0 \subset J_1 \subset J_2 \subset \ldots \subset J_m = A$ is a cell chain with the corresponding cell ideals having k–dimensions $n_1^2, n_2^2, \ldots, n_m^2$, then the k–dimension of $Fix_i(A)$ satisfies

$$dim_k(Fix_i(A)) = n_1(n_1+1)/2 + n_2(n_2+1)/2 + \ldots + n_m(n_m+1)/2.$$

The proof of (a) is straightforward from linear algebra since the symmetric $n \times n$–matrices form a vector space of dimension $n(n+1)/2$.

In order to also prove (b) we use the following observation: If a vector space V can be decomposed as $U \oplus W$ in such a way that a given involution i acting on V sends both U and W into itself, then the k–dimension of the space of fixed points is additive, since in fact the spaces of fixed points add up: $Fix_i(V) = Fix_i(U) \oplus Fix_i(W)$. Thus (b) follows by noting that A has a cell basis $C^{\lambda}_{S,T}$, hence can be written as a direct sum of spaces $V^{\lambda}_{S,T}$, each of them generated by the one or two basis elements $C^{\lambda}_{S,T}$ and $C^{\lambda}_{T,S}$ and each $V^{\lambda}_{S,T}$ being fixed under i. ■

This result implies that for a cellular algebra A of k–dimension smaller than or equal to eleven, the length of the cell chain is an invariant. In fact, in this case, only cell ideals J of k–dimension one or four or nine can occur, and then the subspaces $Fix_i(J)$ have k–dimension one or three or six, respectively. Adding up these numbers in various ways, one arrives at the uniqueness assertion. For instance, if $dim_k(A)$ equals eleven, then a cell chain can contain one subquotient of dimension nine (plus two subqoutients of dimension one) or two subquotients of dimension four (plus three subquotients of dimension one) or one of dimension four (plus seven subquotients of dimension one). Then the dimensions of the fixed points add up to eight or nine or ten, respectively. Hence the dimension of $Fix_i(A)$ determines in which of these cases we are. The other cases are dealt with in a similar manner.

Proposition 3.2 *Each cell chain of a cellular algebra is maximal.*

Proof. The Proposition follows by induction on the minimal number of ideals in a cell chain from the definition of cellular algebras and the following lemma. ■

Lemma 3.3 *Let J be a cell ideal in a cellular algebra A with respect to an involution i. Suppose J_1 is another cell ideal such that there is an inclusion $0 \subseteq J_1 \subseteq J$ and that J/J_1 is filtered by a chain of ideals with subquotients being cell ideals. Then $J_1 = J$.*

Proof. Denote the k–dimension of J by n^2. Denote the chain of ideals filtering J by $J_1 \subseteq J_2 \ldots \subseteq J_n = J$ for some $n \geq 2$ and the corresponding k–dimensions of cell ideals by $n_1^2, n_2^2, \ldots, n_m^2$. We have (by the argument which proved lemma 3.1) the following equalities of dimensions:

$$dim_k(J) = n^2 = n_1^2 + n_2^2 + \ldots + n_m^2,$$

$$dim_k(Fix_i(J)) = n(n+1)/2 = n_1(n_1+1)/2+n_2(n_2+1)/2+\ldots+n_m(n_m+1)/2.$$

This implies another equation

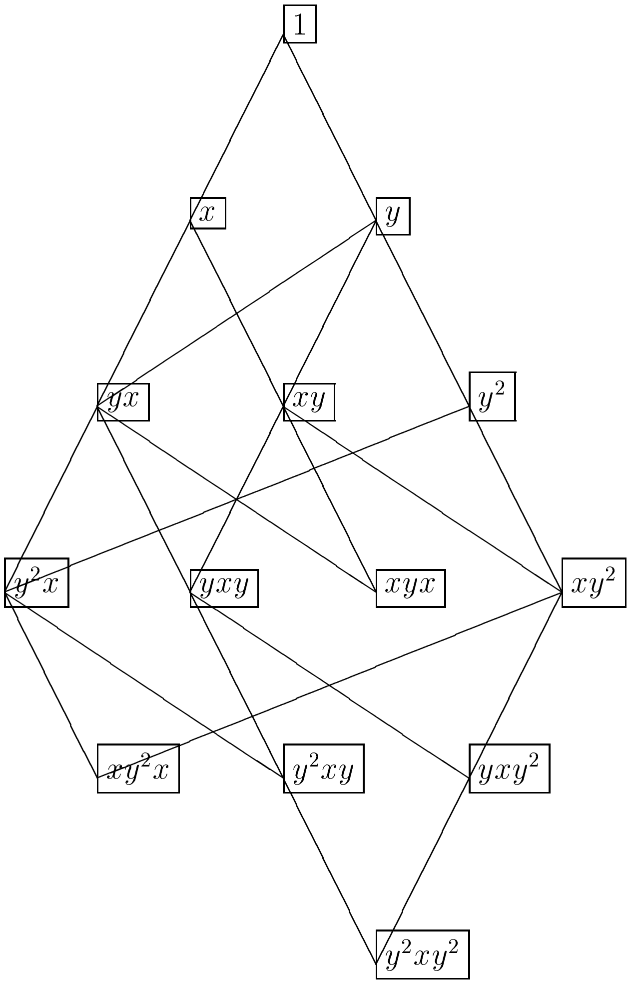
$$n = n_1 + n_2 + \ldots + n_m,$$

which together with the first equation implies the desired equality $m = 1$. This finishes the proof. ■

At this point, it might be tempting to conjecture that each cell chain of a given cellular algebra has the same length. Unfortunately, this is not the case as our example will show.

Example. Let k be any field (or even any commutative ring) and let A be the monomial quotient algebra of the free algebra $k < x, y >$ modulo the ideal generated by $x^2, y^3, xyxy, yxyx, xy^2xy$ and yxy^2x. Then A is a $14-$dimensional algebra and admits an involution i which fixes x and y.

In the following picture we visualize the structure of this algebra as follows: Basis elements appear in boxes which are connected by lines if the corresponding basis elements are related by (left or right) multiplication (which goes from top to bottom of the picture).



Now let us exhibit two different cell chains of A by giving a basis in square form in each case (according to Proposition 3.4 below).

The first cell chain $J_1 \subseteq J_2 \subseteq J_3 \subseteq J_4 \subseteq J_5 \subseteq J_6 = A$ is defined by

$$J_1 : xy^2x; \qquad\qquad J_2/J_1 : xyx;$$

$$J_3/J_2: \begin{array}{ccc} x, & yx, & y^2x, \\ xy, & yxy, & y^2xy, \\ xy^2, & yxy^2, & y^2xy^2; \end{array}$$

$$J_4/J_3 : y^2; \qquad J_5/J_4 : y; \qquad J_6/J_5 : 1$$

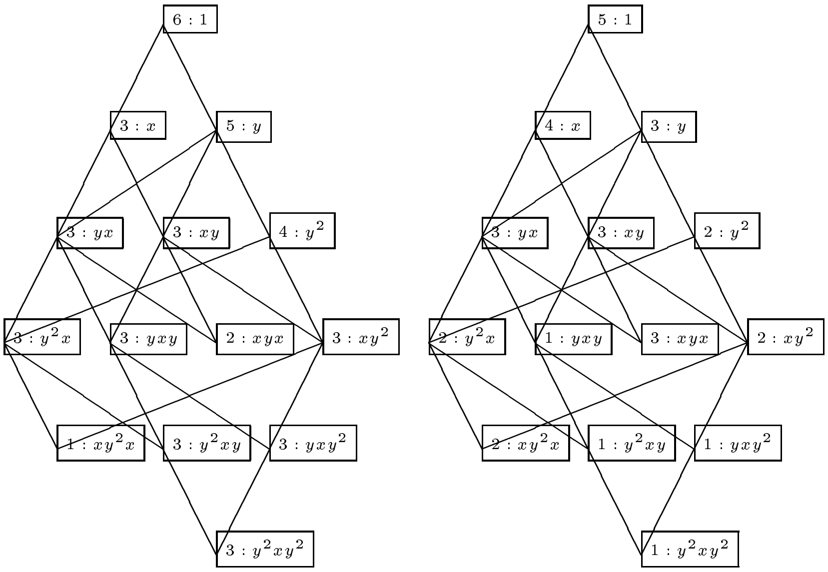
The second cell chain $J_1 \subseteq J_2 \subseteq J_3 \subseteq J_4 \subseteq J_5 = A$ is as follows:

$$J_1: \begin{array}{cc} yxy, & y^2xy, \\ yxy^2, & y^2xy^2; \end{array} \qquad J_2/J_1: \begin{array}{cc} y^2, & xy^2, \\ y^2x, & xy^2x; \end{array}$$

$$J_3/J_2: \begin{array}{cc} y, & xy, \\ yx, & xyx; \end{array}$$

$$J_4/J_3 : x; \qquad J_5/J_4 : 1$$

We visualize the two cell chains in the following picture, where each basis element gets a label indicating in which subquotient in the cell chain it occurs.



Notice that in the first chain the sequence of dimensions of Δ's is $(1, 1, 3, 1,1, 1)$ and in the second one it is $(2, 2, 2, 1, 1)$. Thus in one cell chain there occur Δ's of a k–dimension not occuring in the other cell chain, hence even the k–dimensions of the modules Δ are not invariants of the cellular algebra. Finally, we remark that we may use the results in [4] to construct (using the above example) many more algebras having two cell chains of different lengths.

To check that the above displayed chains are cell chains, the following proposition may be a useful tool. In fact, with the help of this proposition, we can readily verify that our two chains are cell chains.

Proposition 3.4 *Let A be a k–algebra with an involution i. Suppose A is generated as algebra by $a_1, ..., a_m$. Let J be a subspace of A with a basis*

$$\begin{array}{cccc} C_{11}, & C_{12}, & \cdots, & C_{1n}, \\ C_{21}, & C_{22}, & \cdots, & C_{2n}, \\ \cdots & \cdots & \cdots & \cdots \\ C_{n1}, & C_{n2}, & \cdots, & C_{nn} \end{array}$$

such that $i(C_{lk}) = C_{kl}$ for k, l. Define $c_j := (C_{1j}, C_{2j}, ..., C_{nj})$ for $1 \le j \le n$ and $a_l c_j := (a_l C_{1j}, ..., a_l C_{nj})$. If $a_l c_j \in \sum_{t=1}^{n} k c_t$ for all l, j, then J is a cell ideal in A.

Proof. It is clear that J is an ideal in A with $i(J) = J$. Fix an index j and define $\Delta = kC_{j1} + kC_{j2} + \cdots + kC_{jn}$. Then Δ is a left ideal contained in J. We define

$$\alpha : \Delta \otimes_k i(\Delta) \longrightarrow J$$

by sending $C_{jl} \otimes C_{kj}$ to C_{kl}. Obviously, α is a k–linear bijection. We shall show that it is also a $A-A$–bimodule homomorphism. Take $a_s \in A$, consider the image of $a_s C_{jl} \otimes C_{kj}$ under the map α. Suppose $a_s C_{jl} = \sum_t \lambda_t C_{jt}$. This implies that $a_s c_l = \sum_t \lambda_t c_t$ and $a_s C_{kl} = \sum_t \lambda_t C_{kt}$. Thus we deduce that α is a bimodule homomorphism. It is clear that the diagram

$$\begin{tikzcd} J \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[l, "\alpha"'] \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J & \Delta \otimes_R i(\Delta) \arrow[l, "\alpha"'] \end{tikzcd}$$

is commutative. Hence J is a cell ideal in A. ■

Quite often, one can easily find for a given ideal a basis $\{C_{jl}\}$ with $i(C_{jl}) = C_{lj}$. But this does not ensure that one has really a cell ideal. For instance, we consider the algebra $k < x, y > /(x^2, y^2, xyx, yxy)$ with i fixing x and y. We choose the radical J of A as candidate cell ideal. Note that J has a basis

$$\begin{array}{ll} C_{11} = x, & C_{12} = xy, \\ C_{21} = yx, & C_{22} = y \end{array}$$

which satisfies $i(C_{lj}) = C_{jl}$. Since $yC_{11} = yx = r_y(2,1)C_{21}$ and $yC_{12} = 0 = r_y(1,1)C_{12} + r_y(2,1)C_{22}$, we know that $(C3)$ in Definition 2.1 is not fulfilled. Thus J is not a cell ideal. This example also shows that the condition in Proposition 3.4 can not be deleted. In fact, this condition is a sufficient and necessary condition for J to be a cell ideal (the necessity of the condition follows from Definition 2.1 of cellular algebras).

Our Example shows that the length of a cell chain is not an invariant. However, note that for two important classes of cellular algebras (which together contain most of the known examples) the cell length is an invariant.

Proposition 3.5 *(a) If A is a quasi-hereditary and cellular algebra, then all cell chains have the same length.*

(b) If A is an R–order in a split semisimple K–algebra B (where $K = frac(R)$) and A is cellular with a cell chain $J_1 \subset J_2 \subset \ldots \subset J_n = A$ such that $K \otimes_R J_1 \subset K \otimes_R J_2 \subset \ldots \subset K \otimes_R J_n = B$ is a cell chain of B, then all cell chains of A have the same length.

Proof. (a) If A is quasi–hereditary cellular algebra, then the length is equal to the number of non–isomorphic simple modules by [3].

(b) The algebra B is quasi–hereditary and cellular. Thus the statement follows from (a). ∎

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