Stochastic calculus of generalized Dirichlet forms

Probabilités/Probability Theory

Gerald Trutnau
Fakultät für Mathematik, Universität Bielefeld, D–33501 Bielefeld, Germany

Abstract – We show Fukushima's decomposition of AF's in the frame work of quasi–regular generalized semi–Dirichlet forms considered in [8],[9]. The only condition assumed, is that the co-resolvent is sub–Markovian. No dual process is needed. The key-point for the proof of the decomposition is an integral representation theorem for coexcessive functions and a resulting description of $\mathcal{E}$–exceptional sets by an appropriate class of measures. We also derive an Itô-type formula for the transformation of the martingale part of the decomposition.

Calcul stochastique pour les formes de Dirichlet généralisées

Résumé – La décomposition de Fukushima est donnée dans le cadre des formes semi–Dirichlet géneralisées qui ont été considérées dans [8],[9]. Nous supposons seulement la co–résolvente sous–Markovienne, mais non l'éxistence d'un processus en dualité. Le point décisif est un théorème de représentation intégrale pour les fonctions co–excessives et, par conséquent, une description des boréliens $\mathcal{E}$–exceptionels au moyen d'une classe de mesures appropriée. Nous décrivons aussi une formule de type Itô pour la transformation de la partie martingale de la décomposition.

Version française abrégée – Les notations utilisées ici seront celles de [8],[9]. Soit alors $\mathcal{E}$ une forme semi–Dirichlet généralisée quasi–régulière, mais on ne supposera pas la condition D3 de [8],[9]. La quasi–régularité de $\mathcal{E}$ implique que tout élément de $\mathcal{F}$ admet un représentant $\mathcal{E}$–quasi–continu ($\mathcal{E}$–q.c.). On notera $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) l'ensemble des fonctions 1–excessives (resp. 1–coexcessives) dans $\mathcal{V}$. Pour un ensemble $\mathcal{C}$ de fonctions de E dans $\overline{\mathrm{R}}$ on définit $\mathcal{P}_{\mathcal{C}} := \{u \in \mathcal{P} \mid \exists f \in \mathcal{C},\ u \leq f\}$, ainsi que $\hat{\mathcal{P}}_{\mathcal{C}} := \{\hat{u} \in \hat{\mathcal{P}} \mid \exists \hat{f} \in \mathcal{C},\ \hat{u} \leq \hat{f}\}$. Pour un sous–ensemble $\mathcal{G}$ de $\mathcal{V}$ on

pose $\widetilde{\mathcal{G}}$ pour l'ensemble des représentants $\mathcal{E}$–q.c. d'éléments de $\mathcal{G}$. Alors $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ est un e.v. non vide (puisque $\widetilde{\mathcal{P}}_{\mathcal{F}} \supset \widetilde{\mathcal{F} \cap \mathcal{P}}$) et stable par troncation avec tout $\alpha \geq 0$, autrement dit $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ est un treillis de Stone. Toutefois $\widetilde{f}$ désignera un représentant $\mathcal{E}$–q.c. de f et inversement f sera la m–classe représentée par $\widetilde{f}$.
Théorème 1 Soit $\hat{u} \in \hat{\mathcal{P}}_{\hat{\mathcal{F}}}$. Il existe alors une unique mesure $\mu_{\hat{u}}$ σ–finie et positive sur $(E, B(E))$, ne chargeant pas les ensembles $\mathcal{E}$–exceptionels, tel que

$$\int \widetilde{f}\, d\mu_{\hat{u}} = \lim_{\alpha \to \infty} \mathcal{E}_1(f, \alpha \hat{G}_{\alpha+1}\hat{u}) \quad \forall \widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}} \ .$$

Fixons maintenant un processus standard spécial et m–tendu $\mathbb{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ dont la résolvente $R_\alpha f$ est un représentant $\mathcal{E}$–q.c. de $G_\alpha f$ pour tout $\alpha > 0$, $f \in \mathcal{H} \cap B_b(E)$. On trouvera dans [8],[9] des conditions nécessaires et suffisantes pour l'éxistence d'un tel processus. Posons

$$\hat{S}_{00} := \{\mu_{\hat{u}} \mid \ \hat{u} \in \hat{\mathcal{P}}_{\hat{G}_1\mathcal{H}_b^+} \text{ et } \mu_{\hat{u}}(E) < \infty\}$$

où $\mathcal{H}_b^+$ désigne les éléments positifs et bornés de $\mathcal{H}$.

Théorème 2 Pour $B \in B(E)$ les propriétés suivantes sont equivalentes:
(i) B est $\mathcal{E}$–exceptionel
(ii) $\mu(B) = 0\ \forall \mu \in \hat{S}_{00}$.

On définit les fonctionelles additives (abrégé FA), l'equivalence de FA et les espaces $\mathcal{M}$, $\overset{\circ}{\mathcal{M}}$, $\mathcal{N}_c$ comme dans [1]. Pour $\widetilde{u} \in \widetilde{\mathcal{V}}$ on pose $A_t^{[u]} = \widetilde{u}(Y_t) - \widetilde{u}(Y_0)$. La FA $A_t^{[u]}$ est alors indépendente du choix de $\widetilde{u}$. Dorénavent nous allons supposer la co–résolvente sous–Markovienne.

En utilisant le Théorème 2 on démontre comme dans [1], p.182, p.203 le Lemme et le Théorème suivant.
Lemme 3 Soit $(\widetilde{u}_n)_{n\in\mathbb{N}}$ une suite de représentants $\mathcal{E}$–q.c. d'élements de $\mathcal{F}$ tel que $(u_n)_{n\in\mathbb{N}}$ soit une suite de Cauchy dans $\mathcal{F}$. Il existe alors une sous–suite $(\widetilde{u}_{n_k})_{k\in\mathbb{N}}$ tel que pour $\mathcal{E}$–q.t. $z \in E$

$$P_z(\widetilde{u}_{n_k}(X_t) \text{ converge uniformement en } t \text{ sur tout compact de } [0,\infty)) = 1.$$

Théorème 4 Soit $(M^n)_{n\in\mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$ e–Cauchy. Il existe alors une sous–suite $(n_k)_{k\in\mathbb{N}}$ et un unique $M \in \overset{\circ}{\mathcal{M}}$, tel que $\lim_{n\to\infty} e(M^n - M) = 0$ et tel que pour $\mathcal{E}$–q.t. $z \in E$

$$P_z(\lim_{k\to\infty} M_t^{n_k} = M_t \text{ uniformement en } t \text{ sur tout compact de } [0,\infty)) = 1.$$

Nous allons maintenant énonçer notre résultat principal.

Théorème 5 Soit $u \in \mathcal{F}$. Il existe $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ et $N^{[u]} \in \mathcal{N}_c$ uniques, tel que

$$A^{[u]} = M^{[u]} + N^{[u]}. \tag{1}$$

Supposons que la co–forme $\hat{\mathcal{E}}$ soit aussi quasi–régulière, et que $\mathbb{M}$ soit une diffusion (jusqu'à $+\infty$). Alors on a

Théorème 6 Soit $f = (f_1, \ldots, f_n)$ un n–tuplet d'éléments bornés de $\mathcal{F}$. Si $A^{[\Phi(f)]}$ admet de manière unique la décomposition (1) pour tout $\Phi \in C^1(\mathbb{R}^n)$, nulle en 0, alors

$$M^{[\Phi(f_1,\ldots,f_n)]} = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\widetilde{f}_1, \ldots, \widetilde{f}_n) \ \bullet \ M^{[f_i]}$$

pour chaque tel Φ. Ici $f \bullet M$ pour $f \in B_b(E)$, $M \in \overset{\circ}{\mathcal{M}}$ est par définition l'unique élément de $\overset{\circ}{\mathcal{M}}$ tel que

$$\frac{1}{2} \int f \ d\mu_{\langle M,L \rangle} = e(f \bullet M, L) \qquad \forall L \in \overset{\circ}{\mathcal{M}} \ .$$

Les démonstrations de ces résultats ainsi que des applications seront données dans [10].

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Let E be a Hausdorff space such that its Borel σ–algebra $B(E)$ is generated by the set $C(E)$ of all continuous functions on E. Let m be a σ–finite measure on $(E, B(E))$ such that $\mathcal{H} = L^2(E, m)$ is a separable (real) Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$. Let $(A, \mathcal{V})$ be a real valued coercive closed form on $\mathcal{H}$. Then $\mathcal{V}$ equipped with inner product $\widetilde{\mathcal{A}}_1(u,v) := \frac{1}{2}(\mathcal{A}(u,v) + \mathcal{A}(v,u)) + (u,v)_{\mathcal{H}}$ is again a separable real Hilbert space. Let $\|\cdot\|_{\mathcal{V}}$ be the corresponding norm. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$, we have that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ densely and continuously.

Let Λ with domain $D(\Lambda, \mathcal{V}')$ be a linear operator on $\mathcal{V}'$ satisfying the following conditions:

D1 (i) $(\Lambda, D(\Lambda, \mathcal{V}'))$ generates a strongly continuous semigroup $(U_t)_{t\geq 0}$ of bounded linear operators on $\mathcal{V}'$.

(ii) $(U_t)_{t\geq 0}$ can be restricted both to a strongly continuous **contraction** semigroup on $\mathcal{H}$ and to a strongly continuous semigroup of bounded linear operators on $\mathcal{V}$.

Denote by $(\Lambda, D(\Lambda, \mathcal{H}))$ and $(\Lambda, D(\Lambda, \mathcal{V}))$ the generators corresponding to the restricted semigroups.

The dual operator $(\hat{\Lambda}, D(\Lambda, \mathcal{V}'))$ of $(\Lambda, D(\Lambda, \mathcal{V}))$ also satisfies condition D1. We can now define the Hilbert spaces $\mathcal{F} := D(\Lambda, \mathcal{V}') \cap \mathcal{V}$ with norm $\|u\|^2_{\mathcal{F}} := \|u\|^2_{\mathcal{V}} + \|\Lambda u\|^2_{\mathcal{V}'}$ and $\hat{\mathcal{F}} := D(\hat{\Lambda}, \mathcal{V}') \cap \mathcal{V}$ with norm $\|u\|^2_{\hat{\mathcal{F}}} := \|u\|^2_{\mathcal{V}} + \|\hat{\Lambda} u\|^2_{\mathcal{V}'}$.

Let the form $\mathcal{E}$ be given by

$$\mathcal{E}(u,v) := \begin{cases} \mathcal{A}(u,v) - \langle \Lambda u, v\rangle, & \text{for } u \in \mathcal{F},\ v \in \mathcal{V} \\ \mathcal{A}(u,v) - \langle \hat{\Lambda} v, u\rangle, & \text{for } u \in \mathcal{V}, v \in \hat{\mathcal{F}}, \end{cases} \tag{1}$$

and $\mathcal{E}_\alpha(u,v) := \mathcal{E}(u,v) + \alpha(u,v)_{\mathcal{H}}$ for $\alpha.0$.

Here, $\langle \cdot,\cdot \rangle$ denotes the dualization between $\mathcal{V}$ and $\mathcal{V}'$. Note that $\mathcal{E}$ is well-defined. We can also define the co–form $\hat{\mathcal{E}}$ of $\mathcal{E}$ given by

$$\hat{\mathcal{E}}(u,v) := \begin{cases} \mathcal{A}(v,u) - \langle \hat{\Lambda} u, v\rangle, & \text{for } u \in \hat{\mathcal{F}},\ v \in \mathcal{V} \\ \mathcal{A}(v,u) - \langle \hat{\Lambda} v, u\rangle, & \text{for } u \in \mathcal{V}, v \in \mathcal{F}, \end{cases}$$

It follows, from [4], p.248, Théorème 1.1, that for all $\alpha > 0$ there exist continuous, linear bijections $W_\alpha : \mathcal{V}' \to \mathcal{F}$ and $\hat{W}_\alpha : \mathcal{V}' \to \hat{\mathcal{F}}$ such that $\mathcal{E}_\alpha(W_\alpha f, u) = \langle f, u\rangle = \mathcal{E}_\alpha(u, \hat{W}_\alpha f)$, $\forall f \in \mathcal{V}'$, $u \in \mathcal{V}$. Restricting W_α and $\hat{W}_\alpha$ to $\mathcal{H}$ we get strongly continuous contraction resolvents $(G_\alpha)_{\alpha>0}$ and $(\hat{G}_\alpha)_{\alpha>0}$ on $\mathcal{H}$ satisfying $\lim_{\alpha\to\infty} \alpha G_\alpha f = f$ (resp. $\lim_{\alpha\to\infty} \alpha \hat{G}_\alpha f = f$) both in $\mathcal{V}$ for $f \in \mathcal{V}$ and in $\mathcal{F}$ for $f \in \mathcal{F}$ (resp. in $\hat{\mathcal{F}}$ for $f \in \hat{\mathcal{F}}$).
We assume the following condition:
D2 $u \in \mathcal{F} \Rightarrow u^+ \wedge 1 \in \mathcal{V}$ and $\mathcal{E}(u, u - u^+ \wedge 1) \geq 0$.
The following definition is a modified version of the one given in [8],[9].

Definition 1 *The form $\mathcal{E}$ with domain $\mathcal{F} \times \mathcal{V} \cup \mathcal{V} \times \hat{\mathcal{F}}$ is called a* **generalized semi–Dirichlet form** *if D1 and D2 hold and a* **generalized Dirichlet form** *if both $\mathcal{E}$ and its co–form $\hat{\mathcal{E}}$ are generalized semi–Dirichlet forms.*

Let $\mathcal{P}$ (resp. $\hat{\mathcal{P}}$) denote the 1–excessive (resp. 1–co–excessive) elements for $(G_\beta)_{\beta>0}$ (resp. $(\hat{G}_\beta)_{\beta>0}$) in $\mathcal{V}$. An increasing sequence of closed subsets $(F_k)_{k\geq 1}$ is called an $\mathcal{E}$–nest, if for every function $u \in \mathcal{P} \cap \mathcal{F}$ it follows that $u_{F_k^c} \to 0$ in $\mathcal{H}$ and weakly in $\mathcal{V}$. A subset $N \subset E$ is called $\mathcal{E}$–exceptional if there is an $\mathcal{E}$–nest $(F_k)_{k\geq 1}$ such that $N \subset \cap_{k\geq 1} E - F_k$. $\mathcal{E}$–quasi–everywhere ($\mathcal{E}$–q.e.) and $\mathcal{E}$–quasi–continuous ($\mathcal{E}$–q.c.) are also defined like in [8],[9]. At any time $\widetilde{f}$ stands for an $\mathcal{E}$–q.c. m–version of f, conversely f denotes the m–class represented by an $\mathcal{E}$–q.c. m–version $\widetilde{f}$ of f.

From now on we assume that our given generalized semi–Dirichlet form is quasi–regular [8],[9]. We remark that then every element in $\mathcal{F}$ admits an $\mathcal{E}$–q.c. m-version. For a set $\mathcal{C}$ of $\overline{\mathrm{R}}$–valued functions on E we define $\mathcal{P}_{\mathcal{C}} := \{u \in \mathcal{P} \mid \exists f \in \mathcal{C},\ u \leq f\}$ and $\hat{\mathcal{P}}_{\mathcal{C}} := \{u \in \hat{\mathcal{P}} \mid \exists \hat{f} \in \mathcal{C},\ \hat{u} \leq \hat{f}\}$. For a subset $\mathcal{G} \subset \mathcal{H}$ denote by $\widetilde{\mathcal{G}}$ all the $\mathcal{E}$–q.c. m–versions of elements in $\mathcal{G}$, then $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ is a non–empty (since $\widetilde{\mathcal{P}}_{\mathcal{F}} \supset \widetilde{\mathcal{F} \cap \mathcal{P}}$) linear lattice, that is $\widetilde{u} \wedge \alpha \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ for all $\alpha \geq 0$ and all $\widetilde{u} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. We emphasize that in general not every element in $\mathcal{P}_{\mathcal{F}}$ admits an $\mathcal{E}$–q.c. m–version.

We are now in the situation to state an integral representation theorem for elements in $\hat{\mathcal{P}}_{\hat{\mathcal{F}}}$.

Theorem 2 *Let $\hat{u} \in \hat{\mathcal{P}}_{\hat{\mathcal{F}}}$. Then there exists a unique σ–finite and positive measure $\mu_{\hat{u}}$ on $(E, B(E))$ charging no $\mathcal{E}$–exceptional set, such that*

$$\int \widetilde{f}\, d\mu_{\hat{u}} = \lim_{\alpha\to\infty} \mathcal{E}_1(f, \alpha\hat{G}_{\alpha+1}\hat{u}) \quad \forall \widetilde{f} \in \widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}} \ .$$

Remark 3 *If $\hat{u} \in \hat{\mathcal{F}} \cap \hat{\mathcal{P}}$ then $\lim_{\alpha\to\infty} \mathcal{E}_1(f, \alpha\hat{G}_{\alpha+1}\hat{u}) = \mathcal{E}_1(f, \hat{u})$ for all f in $\mathcal{P}_{\mathcal{F}} - \mathcal{P}_{\mathcal{F}}$.*

Sketch of the proof: Define a positive linear functional $I_{\hat{u}}$ on $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$ by $I_{\hat{u}}(\widetilde{f}) := \lim_{\alpha\to\infty} \mathcal{E}_1(f, \alpha\hat{G}_{\alpha+1}\hat{u})$. Similar to the proof of theorem 1 in [2] one can show that $I_{\hat{u}}$ is monotonely continuous on $\widetilde{\mathcal{P}}_{\mathcal{F}} - \widetilde{\mathcal{P}}_{\mathcal{F}}$. Hence the result follows by the theorem of Daniell-Stone.

We fix an m–tight special standard process $\mathbb{M} = (\Omega, \mathcal{M}, (\mathcal{M}_t)_{t\geq 0}, (Y_t)_{t\geq 0}, (P_z)_{z\in E_\Delta})$ such that the resolvent $R_\alpha f$ of $\mathbb{M}$ is an $\mathcal{E}$–q.c. m–version of $G_\alpha f$ for all $\alpha > 0$, $f \in \mathcal{H} \cap B_b(E)$. Necessary and sufficient conditions for the existence of such a process are given in [8],[9].

As a generalization of [1], p.78, we introduce the following class of measures $\hat{S}_{00} := \{\mu_{\hat{u}} \mid \hat{u} \in \hat{\mathcal{P}}_{\hat{G}_1\mathcal{H}_b^+}$ and $\mu_{\hat{u}}(E) < \infty\}$ where $\mathcal{H}_b^+$ are the positive and bounded elements in $\mathcal{H}$. Then we have

Theorem 4 *For $B \in B(E)$ the following conditions are equivalent:*
(i) B is $\mathcal{E}$–exceptional
(ii) $\mu(B) = 0$ $\forall \mu \in \hat{S}_{00}$

Remark 5 (i) Similar to [1] it is possible to define the measures of finite (co–) energy integral and to show that these measures have properties similar to those in [1].
(ii) Note that if $(\hat{G}_\alpha)_{\alpha>0}$ is sub-Markovian we may replace $\hat{S}_{00}$ by the larger class $\{\mu_{\hat{u}} \mid \parallel \hat{u} \parallel_\infty < \infty$ and $\mu_{\hat{u}}(E) < \infty\}$ and then our definition coincides with the one of [1], p.78.

As in [1] we define the equivalence of additive functionals (AF's) and the spaces $\mathcal{M}$, $\overset{\circ}{\mathcal{M}}$, $\mathcal{N}_c$. The energy of an additive functional A of $\mathbb{M}$ is given by

$$e(A) = \frac{1}{2} \lim_{\alpha\to\infty} \alpha^2\ E_m \left(\int_0^\infty e^{-\alpha t} A_t^2\ dt \right)$$

whenever this limit exists. We will set $\overline{e}(A)$ for the same expression but with $\overline{\lim}$ instead of lim.

From now on let us assume that the co–resolvent $(\hat{G}_\alpha)_{\alpha>0}$ is sub–Markovian. To every positive continuous AF A of $\mathbb{M}$ we can associate a σ–finite positive measure μ_A on $(E, B(E))$, called the *Revuz measure of* A [7], namely

$$\int f\ d\mu_A := \lim_{\alpha\to\infty} \alpha\ E_m\left(\int_0^\infty e^{-\alpha t} f(Y_t)\ dA_t\right) \quad f \in B^+(E).$$

It then follows that one half of the total mass of the Revuz measure $\mu_{\langle M\rangle}$ associated to the quadratic variation of $M \in \mathcal{M}$ is equal to the energy of M, i.e.

$$\frac{1}{2}\mu_{\langle M\rangle}(E) = e(M), \tag{2}$$

If we set $A_t^{[u]} = \widetilde{u}(Y_t) - \widetilde{u}(Y_0)$ for $\widetilde{u} \in \widetilde{\mathcal{V}}$, it can be shown that $A_t^{[u]}$ is independent of the special choice $\widetilde{u}$ and that

$$\overline{e}(A^{[u]}) \leq \overline{\lim_{\alpha\to\infty}}\ \alpha(u - \alpha G_\alpha u, u) \tag{3}$$

where the right hand side is equal to $\mathcal{E}(u, u)$ if $u \in \mathcal{F}$.

The isometry (2) and the continuity statement (3) are fundamental for the stochastic calculus related to $\mathcal{E}$.

Using Theorem 4 the proofs of the following Lemma 6 and Theorem 7 are similar to [1], p.182, p.203.

Lemma 6 *Let $(\widetilde{u}_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{E}$–q.c. functions, such that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{F}$. Then there exists a subsequence $(\widetilde{u}_{n_k})_{k\in\mathbb{N}}$, such that for $\mathcal{E}$–q.e. $z \in E$*

$$P_z(\widetilde{u}_{n_k}(X_t) \text{ converges uniformly in } t \text{ on each compact interval of} [0,\infty)) = 1.$$

Theorem 7 *Let $(M^n)_{n\in\mathbb{N}} \subset \overset{\circ}{\mathcal{M}}$ be e–Cauchy. Then there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a unique $M \in \overset{\circ}{\mathcal{M}}$, such that $\lim_{n\to\infty} e(M^n - M) = 0$ and for $\mathcal{E}$–q.e. $z \in E$*

$$P_z(\lim_{k\to\infty} M_t^{n_k} = M_t \text{ uniformly in } t \text{ on each compact interval of } [0,\infty)) = 1.$$

Now Lemma 6 and Theorem 7 together with (3) enable us to prove the following

Theorem 8 *Let $u \in \mathcal{F}$. There exists a unique $M^{[u]} \in \overset{\circ}{\mathcal{M}}$ and a unique $N^{[u]} \in \mathcal{N}_c$ such that*

$$A^{[u]} = M^{[u]} + N^{[u]}\ . \tag{4}$$

Remark 9 *In general it is not possible to find a decomposition of the additive functional* $(\widetilde{u}(Y_t) - \widetilde{u}(Y_0))_{t\geq 0}$ *for all* $u \in \mathcal{P}$ *of type* (4) *where* $N_t^{[u]}$ *is of zero energy. Here* $\widetilde{u}$ *denotes a 1–excessive regularization of* u *(cf. [9], section IV.2). As an example consider the uniform motion (to the right) on the real line and* $\widetilde{u}(x) = e^x 1_{(-\infty,0)}(x)$, $x \in \mathrm{R}$. *(In respect to this there appears to be a mistake in the proof of* [6]*, Theorem 6.4., where the contrary is claimed).*

Finally, let us assume that $\mathbb{M}$ is a diffusion (up to $+\infty$) and that the co–form $\hat{\mathcal{E}}$ is quasi–regular, too. Then we have

Theorem 10 *Let* $f = (f_1, \dots, f_n)$ *be an* n*–tuple of bounded elements in* $\mathcal{F}$. *If* $A^{[\Phi(f)]}$ *admits uniquely the decomposition* (4) *for all* $\Phi \in C^1(\mathbb{R}^n)$, *with* $\Phi(0) = 0$, *then we have*

$$M^{[\Phi(f_1,\dots,f_n)]} = \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i}(\widetilde{f}_1, \dots, \widetilde{f}_n) \bullet M^{[f_i]}$$

for all such Φ. *Here* $f \bullet M$, $f \in B_b(E)$, $M \in \overset{\circ}{\mathcal{M}}$ *is defined to be the unique element in* $\overset{\circ}{\mathcal{M}}$, *such that*

$$\frac{1}{2}\int f \; d\mu_{\langle M,L\rangle} = e(f \bullet M, L) \qquad \forall L \in \overset{\circ}{\mathcal{M}} \; .$$

Remark 11 In general we don't have that $\Phi(f) \in \mathcal{F}$, but there are many examples which satisfy the condition of Theorem 10.

All results in this paper have been presented in talks at Bielefeld University. For rigorous proofs of all theorems above we refer to [10].

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