AN ALTERNATIVE PROOF OF SCHEIDERER'S THEOREM ON THE HASSE PRINCIPLE FOR PRINCIPAL HOMOGENEOUS SPACES

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1. Introduction

Let Y be a smooth irreducible projective curve defined over the real number field $\mathbb{R}$ and $k = \mathbb{R}(Y)$ be the field of $\mathbb{R}$-rational functions on Y. For a point $P \in Y(\mathbb{R})$ we denote the completion of k at the point P by k_P. The present paper is devoted to the Hasse principle for the existence of a rational point on principal homogeneous spaces of a connected linear algebraic group G defined over k. It was Colliot-Thélène who conjectured ([CT], Conjecture 2.9) that for any such space X the Hasse principle holds relative to all local fields k_P, $P \in Y(\mathbb{R})$, i.e. $X(k) \neq \emptyset$ iff $X(k_P) \neq \emptyset$ for each $P \in Y(\mathbb{R})$. Since principal homogeneous spaces of G are in natural one-to-one correspondence with elements of the set $H^1(k, G)$ the latter statement is equivalent to the following: the natural map of pointed sets

$$H^1(k, G) \longrightarrow \prod_{P \in Y(\mathbb{R})} H^1(k_P, G) \tag{1}$$

has trivial kernel ([S]).

In [CT] Colliot-Thélène proved the Hasse principle for algebraic k-tori and reduced the general case to that of a simple simply connected algebraic group G. The case of an arbitrary connected k-group G has been studied by Scheiderer ([Sch1]). To prove the Hasse principle he first made an important observation (which eventually turned out to be crucial); namely that local objects k_P can be replaced by real closures k_ξ of k, $\xi \in \Omega_k$, where Ω_k denotes the set of all orderings of k. Indeed, using the description of orderings of k and the so-called Artin-Lang homomorphism theorem ([Srl], Theorem 3.1) it is easy to show that the condition $X(k_P) \neq \emptyset$ for each real point P on Y implies $X(k_\xi) \neq \emptyset$ for each ordering ξ of k and hence the triviality of the kernel of (1) follows immediately from the triviality of the kernel of

$$\theta : \ H^1(k, G) \longrightarrow \prod_{\xi \in \Omega_k} H^1(k_\xi, G) \tag{2}$$

The question whether θ is injective makes sense not only for the function fields of projective curves but also for an arbitrary field k and it turned out that θ is indeed injective if k has virtual cohomological dimension (vcd) at

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most 1, i.e. $\operatorname{cd} k(\sqrt{-1}) \leq 1$ (recall that the function fields in one variable over $\mathbb{R}$ are such). We have even more.

Theorem 1. (Scheiderer, [Sch1]) *Let K be any field of virtual cohomological dimension ≤ 1. Then the Hasse principle holds for any homogeneous K-space X of a connected linear algebraic K-group G.*

Scheiderer's proof can be divided into two parts. In the first one it is proved that for X as in the theorem (here G may even be not connected) there exists a principal homogeneous space Z which is everywhere locally trivial and dominates X. The strategy of the proof in this part going back to Springer ([S], [Sp]) consists of replacing X by a homogeneous space which dominates X and has a smaller stabilizer. It is worth mentioning that in this part most arguments do not use specific properties of K and so most of them are valid over an arbitrary perfect field.

The second part of Scheiderer's proof is devoted to the case of a principal homogeneous space. To treat such a space Scheiderer first constructs a locally constant sheaf of sets $\mathcal{H}^1(G)$ on Ω_K whose stalks are just the sets $H^1(K_\xi, G)$. Then he shows that there exists the natural bijection between the set of global sections of $\mathcal{H}^1(G)$ and $H^1(K, G)$. As a whole the proof in this part is quite long and complicated. It is based on using étale machinery and, in particular, strongly relies on results of the book [Sch2]. The aim of this paper is to provide a simpler and shorter self-contained proof which is based only on the Bruhat-Tits decomposition in semisimple algebraic groups and the so-called strong approximation property (SAP) of fields (see § 3). We show that in fact the Hasse principle follows immediately modulo two facts. Informally speaking one of them says that the kernel of the natural map $H^1(K, T) \to H^1(K, G)$, where G is an (absolutely) simple simply connected linear K-group and T is a K-torus splitting over $K(\sqrt{-1})$, can be parametrised by "good" rational functions (see § 2) and the other says that any field of virtual cohomological dimension ≤ 1 is an SAP field.

2. Algebraic groups splitting over quadratic extensions

Let K be a field and G be an (absolutely) simple simply connected algebraic group of rank n defined over K and splitting over its quadratic extension $L = K(\sqrt{d})$. For simplicity we will also assume in this section that G is K-anisotropic. Consider a Borel L-subgroup B and let

$$\Theta = \operatorname{Gal}(L/K) = < \tau \mid \tau^2 = 1 > .$$

Clearly, $T = B \cap \tau(B)$ is a maximal K-defined torus splitting over L. Then the sequence of L-points

$$1 \longrightarrow T(L) \longrightarrow G(L) \longrightarrow (G/T)(L) \longrightarrow 1$$

is exact and hence it induces the exact sequence of pointed cohomology sets

$$(G/T)(K) \xrightarrow{\eta_1} H^1(\Theta, T(L)) \xrightarrow{\eta_2} H^1(\Theta, G(L))$$

Identify the set of cocycles $Z^1(\Theta, T(L))$ with $K^* \times \cdots \times K^*$ and let

$$\eta_3 : K^* \times \cdots \times K^* \longrightarrow H^1(\Theta, T(L)) = K^*/N_{L/K}(L^*) \times \cdots \times K^*/N_{L/K}(L^*)$$

be the canonical map. Denote $s = \dim G - n$.

Theorem 2. *There exists a K-defined dominant rational map $\eta_4 : K^s \to (K^*)^n$ such that $\mathrm{Im}\,(\eta_3 \circ \eta_4) = \mathrm{Ker}\,\eta_2$.*

The proof of this statement can be extracted from the author's paper [Ch]. However it is written in Russian and the translation made by the AMS is unreadable and contains a lot of misprints. So for the sake of expository completeness and the reader's convenience we include here a proof.

First recall some basic facts about the structure of the group $G(L)$ (for details see [St]). Let $\Sigma = R(T, G)$ be the root system of G relative to T. The Borel subgroup B determines an ordering on the set Σ and hence a system of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$. If $\{X_\alpha, \alpha \in \Sigma; H_{\alpha_1}, \dots, H_{\alpha_n}\}$ is a Chevalley basis of the Lie algebra of G, then $G(L)$ is generated by the corresponding root subgroups $G_\alpha = < x_{\pm\alpha}(t) \mid t \in L >$, where

$$x_\alpha(t) = \sum_{n=0}^{\infty} t^n X_\alpha^n / n! \ ,$$

and the torus T is generated by $T_\alpha = T \cap G_\alpha = < h_\alpha(t) >$, where $h_\alpha(t) = w_\alpha(t) w_\alpha(1)^{-1}$ and $w_\alpha(t) = x_\alpha(t) x_{-\alpha}(-t^{-1}) x_\alpha(t)$.

Furthermore, since G is simply connected the following relations hold in G (cf. [St]):

A) $T = < h_{\alpha_1}(t_1) > \times \cdots \times < h_{\alpha_n}(t_n) >$ and for $\alpha \in \Sigma$ we have

$$h_\alpha(t) = \prod_{i=1}^{n} h_{\alpha_i}(t)^{n_i}, \qquad \text{where } H_\alpha = \sum_{i=1}^{n} n_i H_{\alpha_i}; \tag{3}$$

B) For $\alpha, \beta \in \Sigma$ let $< \beta, \alpha > = 2\,(\beta, \alpha)/(\alpha, \alpha)$. Then we have

$$h_\alpha(t) x_\beta(u) h_\alpha(t)^{-1} = x_\beta(t^{<\beta,\alpha>} u) \tag{4}$$

C) For all $u, v \in L$ such that $1 + uv \neq 0$ we have

$$x_{-\alpha}(u) x_\alpha(v) = x_\alpha(v(1+uv)^{-1}) h_\alpha(1+uv)^{-1} x_{-\alpha}(u(1+uv)^{-1}) \tag{5}$$

D) For all $\alpha, \beta \in \Sigma$, $\beta \neq -\alpha$, we have

$$x_\alpha(v) x_\beta(u) x_\alpha(v)^{-1} x_\beta(u)^{-1} = \prod_{i,j>0} x_{i\alpha+j\beta}(c_{i,j} v^i u^j) \tag{6}$$

where the product on the right hand side is taken over all roots of the form $i\alpha + j\beta$ and the $c_{i,j}$ are integers which depend on α, β and on the chosen ordering of the roots but do not depend on v and u.

Since T is K-defined the automorphism τ acts in a natural way on the root system Σ. More exactly, for any $\alpha \in \Sigma$ the character $\alpha + \tau(\alpha)$ is K-defined and hence is zero, i.e. $\tau(\alpha) = -\alpha$, since, by assumption, T is K-anisotropic. It follows that there exist constants $c_\alpha \in L$ such that $\tau(X_\alpha) = c_\alpha X_{-\alpha}$; in particular, for any subset $I \subset \Sigma$ the subgroup $G_I =< x_{\pm\alpha}(u) \mid \alpha \in I, u \in \overline{K} >$ is K-defined.

The constants c_α actually belong to the field K and $c_{-\alpha} = c_\alpha^{-1}$. Indeed, for rank one groups, i.e. for groups of the form $\mathrm{SL}(1, D)$, where D is a quaternion algebra over K, this fact can be verified directly. The general case easily reduces to the rank one case since $G_\alpha =< x_{\pm\alpha}(u) \mid u \in \overline{K} >$ is a simple simply connected group of rank 1 defined over K. Thus, we have

Lemma 1. *There exist constants $c_\alpha \in K$ such that for any $u \in L$ one has $\tau(x_\alpha(u)) = x_{-\alpha}(c_\alpha \tau(u))$.*

We need also the following

Lemma 2. *The positive roots $\Sigma^+ = \{\beta_1, \dots, \beta_m\}$ can be ordered in such a way that the following two properties hold:*
1) for any pair of roots β_i, β_j, for which $i < j$ and $\beta_i + \beta_j = \beta_k \in \Sigma^+$, the root β_k is between β_i and β_j, i.e. $i < k < j$;
2) if Σ is a root system of type either A_{2n-1} or D_n or E_6 then for any root $\beta_i \in \Sigma^+$ the roots β_i and $\sigma(\beta_i)$ are neighbours, where σ is the outer automorphism of Σ induced by the non-trivial automorphism of the corresponding Dynkin diagram.

Proof. Consider three cases separately.

a) Σ is a root system of type A_{2n-1}. Let $\alpha_1, \dots, \alpha_{2n-1}$ be a basis of Σ and Σ_1 is the subsystem generated by the roots $\alpha_2, \dots, \alpha_{2n-2}$. By induction, we can pick an ordering $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$ with the required properties. Let us number the remaining roots

$$\Sigma^+ \setminus \{\Sigma_1^+ \cup \alpha_1 + \dots + \alpha_{2n-1}\} = \{\beta_{k+1}, \dots, \beta_{m-1}\}$$

in the order of decreasing of their heights. Let β_i denote the last root among $\{\beta_{k+1}, \dots, \beta_{m-1}\}$ such that $\mathrm{ht}(\beta_i) \geq n$. Then one can easily verify that the ordering

$$\Sigma^+ = \{\beta_1, \dots, \beta_k, \beta_{k+1}, \dots, \beta_i, \alpha_1 + \dots + \alpha_{2n-1}, \beta_{i+1}, \dots, \beta_{m-1}\}$$

is as required.

b) Σ is a root system of type $A_{2n}, B_n, C_n, D_n, E_7$. It follows from the description of root systems of the above types that there exists a subsystem Σ_1 generated by $n-1$ simple roots, say $\alpha_1, \dots, \alpha_{n-1}$, such that any root $\beta \in \Sigma^+ \setminus \Sigma_1^+$ can be written as a sum $\beta = m_1\alpha_1 + \dots + m_{n-1}\alpha_{n-1} + \alpha_n$. If Σ is of type D_n we may assume in addition that the set $\{\alpha_1, \dots, \alpha_{n-1}\}$ is stable under σ. The root system Σ_1 has rank $n-1$ and so by induction, there exists an ordering of the required type on the set $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$. We number the remaining roots $\Sigma^+ \setminus \Sigma_1^+ = \{\beta_{k+1}, \dots, \beta_m\}$ in the order of decreasing of their heights. Then the ordering $\{\beta_1, \dots, \beta_m\}$ has the desired properties.

c) Σ is a root system of type E_6, E_8, F_4, G_2. Here one can argue as in case a). Namely, there exists a subsystem Σ_1 generated by simple roots $\alpha_1, \dots, \alpha_{n-1}$ such that any root $\beta \in \Sigma^+ \setminus \Sigma_1^+$ except for the maximal root $\tilde{\alpha}$ is of the form $\beta = m_1\alpha_1 + \dots + m_{n-1}\alpha_{n-1} + \alpha_n$ and the maximal root is of the form $\tilde{\alpha} = m_1\alpha_1 + \dots + m_{n-1}\alpha_{n-1} + 2\alpha_n$. Let $b = \mathrm{ht}(\tilde{\alpha})$ be the height of $\tilde{\alpha}$. Again, applying induction we can find an ordering $\Sigma_1^+ = \{\beta_1, \dots, \beta_k\}$ with the desired properties and then number the roots $\Sigma^+ \setminus \{\Sigma_1^+ \cup \tilde{\alpha}\} = \{\beta_{k+1}, \dots, \beta_{m-1}\}$ in the order of decreasing of their heights. If Σ has type

E_6 we may assume in addition that β and $\sigma(\beta)$ are neighbours for all $\beta \in \Sigma^+$. If β_i is the last root among $\{\beta_{k+1}, \dots, \beta_{m-1}\}$ such that $\mathrm{ht}\,(\beta_i) \geq b/2$, then the ordering

$$\Sigma^+ = \{\beta_1, \dots, \beta_k, \beta_{k+1}, \dots, \beta_i, \tilde{\alpha}, \beta_{i+1}, \dots, \beta_{m-1}\}$$

has the desired properties. Indeed, if $\beta_j = \beta_s + \beta_t$, where $s < t$ and $j \in \{k+1, \dots, m-1\}$, then clearly β_s belongs to Σ_1^+. Since $\mathrm{ht}\,(\beta_j) \geq \mathrm{ht}\,(\beta_s), \mathrm{ht}\,(\beta_t)$, the root β_j lies between β_s and β_t. Now let $\tilde{\alpha} = \beta_s + \beta_t$, $s < t$,. Then $s, t \in \{k+1, \dots, m-1\}$ and $\mathrm{ht}\,(\beta_s) \geq b/2$, $\mathrm{ht}\,(\beta_t) < b/2$ (we use that $\mathrm{ht}(\tilde{\alpha})$ is odd), implying $\tilde{\alpha}$ is also between β_s and β_t. □

Corollary 1. *Let β_i and β_j be any two positive roots and $j < i$. Then for any positive root β_k of the form $\beta_k = r\beta_j - l\beta_i$, $r, l > 0$, one has $k < j$. Analogously, for any negative root of the form $-\beta_k = r\beta_j - l\beta_i$, $r, l > 0$, one has $k > i$.*

Proof. We distinguish three cases.

a) $< \beta_i, \beta_j >_{\mathbb{Q}} \cap \Sigma$ has type A_2. Then $r = l = 1$ and hence if $\beta_k = \beta_j - \beta_i$ is a positive root then $\beta_k + \beta_i = \beta_j$, implying $k < j < i$. Analogously, if $\beta_j - \beta_i = -\beta_k$ then we have $j < i < k$.

b) $< \beta_i, \beta_j >_{\mathbb{Q}} \cap \Sigma$ has type B_2. Then either $r = l = 1$ or $r = 1$ and $l = 2$ or $r = 2$ and $l = 1$. The case $r = l = 1$ was already handled in part a). Now let $\beta_k = \beta_j - 2\beta_i$. Then $\beta_j - \beta_i = \beta_s$ is also a positive root implying $s < j$. Futhermore, $\beta_k = \beta_s - \beta_i$ and $s < j < i$. So again we have $k < s < j$. The remaining cases can be handled in a similar way.

c) $< \beta_i, \beta_j >_{\mathbb{Q}} \cap \Sigma$ has type G_2. Here the proof is analogous to that of case b) and so we omit it. □

Proposition 1. *Fix an order in Σ^+ as in Lemma 2. Then the rational map*

$$\omega : \overline{K}^{2m+n} \longrightarrow G\,, \qquad (t_1, \dots, t_n, u_1, \dots, u_m, v_1, \dots, v_m) \longrightarrow$$

$$\prod_{i=1}^{n} h_{\alpha_i}(t_i)\, x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_m}(u_m) x_{\beta_m}(v_m)$$

is birational over L.

Proof. Both sides have the same dimension and hence it suffices to prove the injectivity of ω on some Zariski open subset. To do this we first show that for any integer i the element

$$A_i = x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_i}(u_i) x_{\beta_i}(v_i)$$

of the group G can be written in the form

$$A_i = \prod_{k=1}^{n} h_{\alpha_k}(f_k) \prod_{j=1}^{m} x_{-\beta_j}(r_j) \prod_{j=1}^{i-1} x_{\beta_j}(s_j)\, x_{\beta_i}(v_i)$$

where the parameters $u_1, \dots, u_i$ and $v_1, \dots, v_i$ belong to some Zariski open subset and f_k, r_j and s_j are rational functions depending on $u_1, \dots, u_i$ and $v_1, \dots, v_{i-1}$.

By induction, the element

$$A_{i-1} = x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_{i-1}}(u_{i-1}) x_{\beta_{i-1}}(v_{i-1})$$

can be written in the form

$$\prod_{k=1}^{n} h_{\alpha_k}(f_k) \prod_{j=1}^{m} x_{-\beta_j}(r_j) \prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1}).$$

To write the element $A_i = A_{i-1}\, x_{-\beta_i}(u_i)\, x_{\beta_i}(v_i)$ in the same form we have to transpose $x_{-\beta_i}(u_i)$ with each factor in the product $\prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1})$. By (6), every time doing so we obtain additional factors of the form either $x_{\beta_s}(\,)$ or $x_{-\beta_s}(\,)$. According to Corollary 1 in the first case we get $s < i-1$ and in the second case we get $s > i$. Collecting together all these factors corresponding to negative roots we can write the element $\prod_{j=1}^{i-2} x_{\beta_j}(s_j)\, x_{\beta_{i-1}}(v_{i-1})\, x_{-\beta_i}(u_i)$ in the form

$$\prod_{k=1}^{n} h_{\alpha_i}(\tilde{f}_k) \prod_{j=1}^{m} x_{-\beta_j}(\tilde{r}_j) \prod_{j=1}^{i-1} x_{\beta_j}(\tilde{s}_j)$$

and hence our claim follows.

Now we are ready to prove the injectivity of ω. Suppose that

$$\omega(t_1, \dots, t_n,\, u_1, \dots, u_m,\, v_1, \dots, v_m) = \omega(\tilde{t}_1, \dots, \tilde{t}_n,\, \tilde{u}_1, \dots, \tilde{v}_m) \tag{7}$$

From the above argument and the Bruhat-Tits decomposition we get immediately $v_m = \tilde{v}_m$. To show that $u_m = \tilde{u}_m$ we may apply (4), (5). Namely, it follows from (4), (5) that the element

$$\omega(t_1, \dots, t_n,\, u_1, \dots, u_m,\, v_1, \dots, v_m)$$

can be written in the form

$$\prod_{i=1}^{n} h_{\alpha_i}(f_i)\, x_{\beta_1}(s_1)\, x_{-\beta_1}(r_1) \cdots x_{\beta_{m-1}}(s_{m-1})\, x_{-\beta_{m-1}}(r_{m-1})$$

$$x_{\beta_m}[v_m(1+u_m v_m)]\, x_{-\beta_m}[u_m(1+u_m v_m)^{-1}]\,,$$

where $f_1, \dots, f_n,\, s_1, \dots, s_{m-1},\, r_1, \dots, r_{m-1}$ are some rational functions. Writing the element $\omega(\tilde{t}_1, \dots, \tilde{t}_n,\, \tilde{u}_1, \dots, \tilde{v}_m)$ in the same form and arguing as above we get

$$u_m(1+u_m v_m)^{-1} = \tilde{u}_m(1+\tilde{u}_m \tilde{v}_m)^{-1}$$

and hence $u_m = \tilde{u}_m$. Canceling the factor $x_{-\beta_m}(u_m)\, x_{\beta_m}(v_m)$ in (7) and repeating the same argument we obtain consecutively that $v_{m-1} = \tilde{v}_{m-1}$, $u_{m-1} = \tilde{u}_{m-1}$ and so on. □

Let us come back to Theorem 2. If $(t_\tau) \in Z^1(\Theta, T(L))$ is a cocycle trivial in $Z^1(\Theta, G(L))$ then there exists an element $g \in G(L)$ such that $g^{1-\tau} = t_\tau$. Since $G(K)$ is Zariski dense in $G(L)$ and for any $x \in G(K)$ one has $g^{1-\tau} = (gx)^{1-\tau}$, we may assume without loss of generality that $g \in \operatorname{Im}\omega$. Thus we arrive at the problem of finding conditions on the parameters of an element $g \in \operatorname{Im}\omega$ ensuring that $g^{1-\tau} \in T(L)$ is satisfied.

Let $g = \omega(t_1, \dots, t_n,\, u_1, \dots, u_n,\, v_1, \dots, v_n)$ be any element of $\operatorname{Im}\omega$. By virtue of Lemma 1, one has

$$\tau(g) = \prod_{i=1}^{n} h_{\alpha_i}(\tau(t_i^{-1}))x_{\beta_1}(c_{\beta_1}^{-1}\tau(u_1))x_{-\beta_1}(c_{\beta_1}\tau(v_1)) \cdots$$

$$x_{\beta_m}(c_{\beta_m}^{-1}\tau(u_m))x_{-\beta_m}(c_{\beta_m}\tau(v_m)).$$

Furthermore, applying relation (5) we have

$$x_{\beta_i}(c_{\beta_i}^{-1}\tau(u_i))x_{-\beta_i}(c_{\beta_i}\tau(v_i)) =$$

$$h_{\beta_i}(1+\tau(u_iv_i))x_{-\beta_i}[c_{\beta_i}\tau(v_i)(1+\tau(u_iv_i))]x_{\beta_i}[\tau(u_i)/c_{\beta_i}(1+\tau(u_iv_I))]$$

and therefore, $\tau(g)$ can be written in the form

$$\tau(g) = \prod_{i=1}^{n} h_{\alpha_i}(\tau(t_i^{-1})) \prod_{i=1}^{m} h_{\beta_i}(1+\tau(u_iv_i))\cdot$$

$$x_{-\beta_1}[\overline{c}_{\beta_1}\tau(v_1)(1+\tau(u_1v_1))]x_{\beta_1}[\tau(u_1)/\overline{c}_{\beta_1}(1+\tau(u_1v_1))]\cdots$$

$$x_{-\beta_m}[\overline{c}_{\beta_m}\tau(v_m)(1+\tau(u_mv_m))]x_{\beta_m}[\tau(u_m)/\overline{c}_{\beta_m}(1+\tau(u_mv_m))],$$

where

$$\overline{c}_{\beta_m} = c_{\beta_m}, \qquad \overline{c}_{\beta_{m-1}} = c_{\beta_{m-1}}(1+\tau(u_mv_m))^{<\beta_{m-1},\beta_m>},$$

$$\overline{c}_{\beta_{m-2}} = c_{\beta_{m-2}}(1+\tau(u_mv_m))^{<\beta_{m-2},\beta_m>}(1+\tau(u_{m-1}v_{m-1}))^{<\beta_{m-2},\beta_{m-1}>}, \dots,$$

$$\overline{c}_{\beta_1} = c_{\beta_1}(1+\tau(u_mv_m))^{<\beta_1,\beta_m>}\cdots(1+\tau(u_2v_2))^{<\beta_1,\beta_2>}.$$

This means that $g^{1-\tau} \in T(L)$ iff

$$\begin{cases} u_m = \overline{c}_{\beta_m}\tau(v_m)(1+\tau(u_mv_m)), & v_m = \tau(u_m)/\overline{c}_{\beta_m}(1+\tau(u_mv_m)), \\ \cdots & \cdots \\ u_1 = \overline{c}_{\beta_1}\tau(v_1)(1+\tau(u_1v_1)), & v_1 = \tau(u_1)/\overline{c}_{\beta_1}(1+\tau(u_1v_1)) \end{cases}$$

or equivalently

$$(8) \qquad \begin{cases} u_m = \overline{c}_{\beta_m}\tau(v_m)(1-\overline{c}_{\beta_m}v_m\tau(v_m))^{-1} \\ \cdots \qquad\qquad \cdots \\ u_1 = \overline{c}_{\beta_1}\tau(v_1)(1-\overline{c}_{\beta_1}v_1\tau(v_1))^{-1} \end{cases}$$

Thus, we showed that if $g^{1-\tau} \in T(L)$ then (8) holds and

$$g^{1-\tau} = \prod_{i=1}^{n} h_{\alpha_i}(t_i\tau(t_i)) \prod_{i=1}^{m} h_{\beta_i}(1+\tau(u_iv_i)).$$

Substituting (8) into the last equation and using (3) we obtain that, up to equivalence in $Z^1(\Theta, T(L))$, the cocycle $g^{1-\tau}$ can be written in the form

$$(9) \qquad \prod_{i=1}^{m} h_{\beta_i}(1+\tau(u_iv_i)) = \prod_{i=1}^{m} h_{\beta_i}((1-\overline{c}_{\beta_i}\tau(v_i)v_i)^{-1}) = \prod_{i=1}^{n} h_{\alpha_i}(\varphi_i),$$

where φ_i is a K-rational function on the K-coordinates of $v_1, \dots, v_m$ which looks like a product of "quadratic forms".

Summarizing, we finally get that the K-rational map

$$\eta_4 : K^s \to K^{*n}, \qquad (v_1, \dots, v_m) \longrightarrow (\varphi_1, \dots, \varphi_n)$$

has the required properties. Thus Theorem 2 is proved. □

In § 4 we will deal with simple simply connected K-anisotropic algebraic groups which are either split or quasi-split over a quadratic extension L/K

and hence for a case of K-anisotropic L-quasi-split group G we also need to describe elements $g \in G(L)$ for which $g^{1-\tau} \in T(L)$. For such a case the arguments are entirely analogous to the previous ones and so we formulate only the final answer.

Type ${}^2A_n, {}^2D_n, {}^2E_6$. First let us describe L-points of G. Denote by E the minimal extension of K over which G is an inner form and let $F = E \cdot L$. Let τ (resp. σ) be an automorphism of F over K whose restriction to L (resp. E) is non-trivial. Consider an L-Borel subgroup B of G. Then $T = B \cap \tau(B)$ is a maximal K-torus of G. Clearly, σ induces a non-trivial outer automorphism on the root system $\Sigma = \mathrm{R}\,(T, G)$ which will be denoted by the same letter.

Lemma 3. *Let us fix an order of Σ^+ as in Lemma 2. Then an element*

$$g = \prod_{i=1}^{n} h_{\alpha_i}(t_i)\, x_{-\beta_1}(u_1) x_{\beta_1}(v_1) \cdots x_{-\beta_m}(u_m) x_{\beta_m}(v_m)$$

where t_i, u_i, $v_i \in F$, belongs to $G(L)$ if and only if

a) u_i, $v_i, t_i \in L$ if $\sigma(\beta_i) = \beta_i$, $\sigma(\alpha_i) = \alpha_i$;
b) $v_{i+1} = \sigma(v_i)$, $u_{i+1} = \sigma(u_i)$, $t_i = \sigma(t_j)$ if $\sigma(\beta_i) = \beta_{i+1}$, $\sigma(\alpha_j) = \alpha_i$.

Proposition 2. *Keep the above notation. Then $g^{1-\tau} \in T(L)$ if and only if*

a) $u_i = \overline{c}_{\beta_i} \tau\sigma(v_i)(1 - \overline{c}_{\beta_i} v_i \tau\sigma(v_i))^{-1}$, if $\sigma(\beta_i) = \beta_{i+1}$;
b) $u_i = \overline{c}_{\beta_i} \tau(v_i)(1 - \overline{c}_{\beta_i} v_i \tau(v_i))^{-1}$, if $\sigma(\beta_i) = \beta_i$.

Here $\overline{c}_{\beta_i}$ are some constants defined inductively just as in the split case.

Proof. It suffices to note that τ acts on the root system Σ as $-\sigma$. The rest of the proof works exactly as in the L-split case. □

Type ${}^{3,6}D_4$. Here the arguments are entirely analogous to the previous ones and so we leave the details to the reader.

3. The case of commutative groups

From now on we assume that $\mathrm{vcd}\,(K) \leq 1$ and we let $L = K(\sqrt{-1}\,)$. We also assume that the set Ω_K of all orderings on K is non-empty; this means, in particular, that $\mathrm{char}\, K = 0$. Recall ([Srl]) that there is a canonical topology on Ω_K under which Ω_K is compact and totally disconnected.

Remark 1. If $\Omega_K = \emptyset$ then -1 is a sum of squares in K and so $\mathrm{cd}\,(K) = \mathrm{cd}\,(K(\sqrt{-1}\,) \leq 1$ ([S], Ch. 2, Prop. 10′). Therefore, if $\Omega_K = \emptyset$ then by the Steinberg theorem one has $H^1(K, G) = 1$ for any connected linear algebraic K-group G.

To reduce the proof of the Hasse principle to the case of simply connected semisimple groups we need two auxiliary cohomological statements (Propositions 3 and 5 below) which are very particular cases of the general Theorem 12.13 in [Sch2]. Since we do not need to consider such a generality as in [Sch2] we include here the straightforward proofs of these statements.

Let A be a discrete $\mathrm{Gal}\,(\overline{K}/K)$-module. Clearly, any cocycle from $Z^1(K_\xi, A)$ is determined by the single element $a \in A$ such that $a\tau_\xi(a) = 1$,

where τ_ξ is the non-trivial element of $\mathrm{Gal}\,(\overline{K}_\xi/K_\xi)$. We will say that an element $\{a_\xi\}_{\xi\in\Omega_K} \in \prod_{\xi\in\Omega_K} H^1(K_\xi, A)$ is determined locally by global elements if there are a decomposition $\Omega_K = U_1 \cup \ldots \cup U_l$ into disjoint clopen (= open and closed) sets and elements $\{a_1, \ldots, a_l\}$ of A such that for any $\xi \in U_i$ a cocycle representing a_ξ is determined by a_i. Analogously, for all integers $i \geq 1$ one defines the subset of elements in $\prod_{\xi\in\Omega_K} H^i(K_\xi, A)$ which are determined locally by global elements. We denote this subset by $\left(\prod_{\xi\in\Omega_K} H^i(K_\xi, A)\right)^{lg}$. Clearly, one has the canonical map

$$\varphi_i : H^i(K, A) \longrightarrow \left(\prod_{\xi\in\Omega_K} H^i(K_\xi, A)\right)^{lg} \subset \prod_{\xi\in\Omega_K} H^i(K_\xi, A)$$

induced by the restriction map.

Proposition 3. *If A is a finite discrete $\mathrm{Gal}\,(\overline{K}/K)$-module then the maps φ_i are injective for all integers $i \geq 2$.*

Proof. Since $H^i(L, A) = 1$, $i \geq 2$, the *"res-cores"* argument shows that the group $H^i(K, A)$ has exponent 2 and hence replacing A, if necessary, by its 2-Sylow subgroup we may assume that A is a 2-group. First examine the case $A = \mathbb{Z}/2\mathbb{Z}$.

Lemma 4. *Let $A = \mathbb{Z}/2\mathbb{Z}$. Then φ_i is surjective if $i \geq 1$ and injective if $i \geq 2$.*

Proof. Recall (see [L], §17) that a field F is said to be an SAP field (strong approximation property) if for any two disjoint closed subsets $A, B \subset \Omega_F$ there exists an element $f \in F$ such that f is positive at all orderings in A, but negative at all orderings in B.

Proposition 4. *If $\mathrm{vcd}\,(K) \leq 1$ then K is a SAP field.*

Proof: see [L], Theorem 17.9.

Surjectivity of φ_i, $i \geq 1$. In view of the periodicity of $H^i(K_\xi, \mathbb{Z}/2\mathbb{Z})$ it suffices to consider the cases $i = 1, 2$. If $i = 1$ then one has $H^1(K, \mathbb{Z}/2\mathbb{Z}) = K^*/K^{*2}$, hence the surjectivity of φ_1 follows immediately from Proposition 4. Furthermore, any element from $H^2(K, \mathbb{Z}/2\mathbb{Z})$ splits over L and so can be represented by a quaternion algebra having L as a maximal subfield. Then clearly, the surjectivity of φ_2 again follows from Proposition 4.
Injectivity of φ_i, $i \geq 2$. The proof is similar to that of [B.-P], Lemma 2.3. Namely, by Arason's theorem ([A1], Satz 3), local triviality of $\zeta \in H^i(K, \mathbb{Z}/2\mathbb{Z})$ implies that $\zeta \cup (-1)^r = 0$ for some integer r, where $\cup$ denotes the cup product. On the other hand from the exact sequence ([A2], Corollary 4.6)

$$H^i(L, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{cor} H^i(K, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\cup(-1)} H^{i+1}(K, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{res} H^{i+1}(L, \mathbb{Z}/2\mathbb{Z})$$

and from the equalities

$$H^i(L, \mathbb{Z}/2\mathbb{Z}) = H^{i+1}(L, \mathbb{Z}/2\mathbb{Z}) = 1, \quad i \geq 2$$

we conclude that the product $\cup(-1)$ is an isomorphism. Therefore, $\zeta = 1$, as required. Lemma 4 is proved. $\square$

Now let us come back to the case of an arbitrary module A. Let Γ_2 be a Sylow 2-subgroup of $\mathrm{Gal}\,(\overline{K}/K)$. The restriction map $H^i(K, A) \to H^i(\Gamma_2, A)$ is injective and so after replacing $\mathrm{Gal}\,(\overline{K}/K)$ by Γ_2, if necessary, we may assume that that $\mathrm{Gal}\,(\overline{K}/K)$ is a profinite 2-group. Then any irreducible $\mathrm{Gal}\,(\overline{K}/K)$-module is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ ([S], §4, Proposition 20) and therefore there exists a submodule $A' \subset A$ such that $A/A' = \mathbb{Z}/2\mathbb{Z}$. It induces the commutative diagram

$$\begin{array}{ccccc} H^i(K, \mathbb{Z}/2\mathbb{Z}) & \longrightarrow & H^{i+1}(K, A') & \longrightarrow \\ \downarrow \theta_1 & & \downarrow \theta_2 & \\ \left(\prod_{\xi \in \Omega_K} H^i(K_\xi, \mathbb{Z}/2\mathbb{Z})\right)^{lg} & \longrightarrow & \left(\prod_{\xi \in \Omega_K} H^{i+1}(K_\xi, A')\right)^{lg} & \longrightarrow \end{array}$$

$$\begin{array}{ccc} H^{i+1}(K, A) & \longrightarrow & H^{i+1}(K, \mathbb{Z}/2\mathbb{Z}) \\ \downarrow \theta_3 & & \downarrow \theta_4 \\ \left(\prod_{\xi \in \Omega_K} H^{i+1}(K_\xi, A)\right)^{lg} & \longrightarrow & \left(\prod_{\xi \in \Omega_K} H^{i+1}(K_\xi, \mathbb{Z}/2\mathbb{Z})\right)^{lg} \end{array}$$

By what has been proved above, θ_1 (resp. θ_4) is surjective (resp. injective) and by induction, θ_2 is injective. It follows that θ_3 is injective as well. Proposition 3 is proved. $\square$

Proposition 5. *For any finite discrete* $\mathrm{Gal}\,(\overline{K}/K)$*-module* A *the map* φ_1 *is surjective.*

Proof. Since φ_i, $i \geq 2$, are injective maps, one can easily verify that if the statement holds both for a submodule $A' \subset A$ and the quotient A/A' then it also holds for A. So we may assume, if necessary, that A is irreducible; in particular, by Lemma 4, there is nothing to prove if $\mathrm{Gal}\,(\overline{K}/K)$ is a 2-profinite group. Clearly, it suffices to prove that for a given $\xi \in \Omega_K$ and an element $a \in A$ for which $a\tau_\xi(a) = 1$, where as above τ_ξ is the non-trivial element of $\mathrm{Gal}\,(\overline{K}_\xi/K_\xi)$, there exist a small clopen neighbourhood $U \subset \Omega_K$ of ξ and a cocycle $\zeta \in Z^1(K, A)$ such that $\mathrm{res}_{K_{\xi'}}(\zeta)$ is determined by a if $\xi' \in U$ and is trivial otherwise.

We need the following two simple properties of orderings of K (see [Srl]):

a) if F/K *is an extension of odd degree then for any ordering* $\xi \in \Omega_K$ *there is an extension of* ξ *to* F*; moreover, the restriction map* $\phi : \Omega_F \to \Omega_K$ *is a local homeomorphism;*
b) there is a natural one-to-one correspondence between points of the set Ω_K *and conjugacy classes of involutions (= elements of order* 2*) in the group* $\mathrm{Gal}\,(\overline{K}/K)$.

Let E be a finite Galois extension of K over which A is a trivial module and let $F \subset E$ be the subfield corresponding to a Sylow 2-subgroup of $\mathrm{Gal}\,(E/K)$. Put $G = \mathrm{Gal}\,(\overline{K}/K)$, $H = \mathrm{Gal}\,(\overline{K}/F)$. Let $\phi^{-1}(\xi) =$

$\{\xi_1, \ldots, \xi_t\} \subset \Omega_F$, where, as above, $\phi : \Omega_F \to \Omega_K$ is the restriction map. Denote by $[\tau_{\xi_1}], \ldots, [\tau_{\xi_t}]$ the conjugacy classes in H of involutions $\tau_{\xi_1}, \ldots, \tau_{\xi_t}$ corresponding to $\xi_1, \ldots, \xi_t$.

By construction, $\phi(\xi_i) = \xi$, hence there exist elements $g_1 = 1, g_2, \ldots, g_t$ in G such that $g_i\,[\tau_{\xi_1}]\,g_i^{-1} = [\tau_{\xi_i}]$, $i = 1, \ldots, t$. Clearly, we can pick a small clopen neighbourhood $U_1 \subset \Omega_F$ of ξ_1 such that for $U = \phi(U_1)$ the restriction map $\phi|_{U_1} : U_1 \to U$ is a homeomorphism, the neighbourhoods $U_1, U_2 = g_2(U_1), \ldots, U_t = g_t(U_1)$ of $\xi_1, \ldots, \xi_t$ are disjoint and $\phi^{-1}(U) = U_1 \cup \ldots \cup U_t$. Moreover, since every involution τ in G is self-centralizing (i.e. $C_G(\tau) = <\tau>$), it is easy to see that for any $\xi' \in U_1$ and for any corresponding involution $\tau_{\xi'} \in H$ the elements $g_1, \ldots, g_t$ are representatives of those double cosets $M_{\xi'}\,gH$ in G, where $M_{\xi'} = <\tau_{\xi'}>$, for which $g^{-1}M_{\xi'}g \subset H$.

To construct the desired cocycle ζ we will use

Proposition 6. ([Br], Ch. III, Proposition 9.5) *Let G be a group, $M \subset H \subset G$ be subgroups and A be a G-module. If $[G : H] < \infty$ and $z \in H^*(H, A)$ then we have*

$$\mathrm{res}\,^G_M \circ \mathrm{cor}\,^G_H\,(z) = \sum_{g \in \Delta} \mathrm{cor}\,^M_{M \cap gHg^{-1}} \circ \mathrm{res}\,^{gHg^{-1}}_{M \cap gHg^{-1}}\,(\hat{g}(z)),$$

where Δ is a set of representatives of double cosets MgH and

$$\hat{g} : H^*(H, A) \to H^*(gHg^{-1}, A)$$

is the natural map induced by $\mathrm{int}(g^{-1})$.

We have already pointed out the map φ_1 is surjective for the field F (since $\mathrm{Gal}\,(E/F)$ is a 2-group). Therefore, we can pick a cocycle $\zeta' \in Z^1(F, A)$ such that $\mathrm{res}_{F_{\xi'}}(\zeta')$ is determined by the element a if $\xi' \in U_1$ and is trivial otherwise. We claim that the cocycle $\zeta = \mathrm{cor}\,^F_K\,(\zeta')$ has the same property. Indeed, first take $\xi' \in U$ and let $\tau_{\xi'}$ be an involution corresponding to ξ' and belonging to H. Then applying Proposition 6 we have

$$\mathrm{res}_{K_{\xi'}}\,(\zeta) = \sum \mathrm{res}\,^{gHg^{-1}}_{M_{\xi'} \cap gHg^{-1}}\,\left(\hat{g}(\zeta')\right) = \sum_{i=1}^{t} \mathrm{res}\,^H_{g_i^{-1}M_{\xi'}\,g_i}(\zeta') = \mathrm{res}\,^H_{M_{\xi'}}(\zeta')$$

where $M_{\xi'} = <\tau_{\xi'}>$, implying that $\mathrm{res}_{K_{\xi'}}\,(\zeta)$ is defined by the element a. Analogously, one shows that $\mathrm{res}_{K_{\xi'}}\,(\zeta)$ is trivial if $\xi' \notin U$. Thus Proposition 5 is proved. □

Corollary 2. *For any commutative connected linear algebraic K-group A the natural map*

$$H^i(K, A) \longrightarrow \prod_{\xi \in \Omega_K} H^i(K_\xi, A)$$

is injective for all integers $i \geq 2$.

Proof. Since $H^i(L, A) = 1$, the group $H^i(K, A)$ has exponent 2 and hence the map $H^i(K,\, {}_2A) \to H^i(K, A)$ is surjective for all $i \geq 1$, where ${}_2A$ consists of all elements of A killed by 2. Then the injectivity of the above map follows from the injectivity of the analogous map for ${}_2A$. □

Corollary 3. *The Hasse principle holds for algebraic K-tori.*

Proof. It is known that any K-torus T can be realized as the quotient $T = S/F$ of some K-quasi-split torus S. Then from the commutative diagram

$$\begin{array}{ccccc} H^1(K,S)=1 & \longrightarrow & H^1(K,T) & \longrightarrow & H^2(K,F) \\ \downarrow\theta_1 & & \downarrow\theta_2 & & \downarrow\theta_3 \\ \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,S)=1 & \longrightarrow & \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,T) & \longrightarrow & \prod\limits_{\xi\in\Omega_K} H^2(K_\xi,F) \end{array}$$

we get that the injectivity of θ_2 follows from that of θ_3. □

4. **The Hasse principle for principle homogeneous spaces**

Let us keep the notations of § 3. In particular, we assume that K is a field with $\mathrm{vcd}\,(K) \leq 1$ and $L = K(\sqrt{-1})$. Let also τ be the non-trivial element of $\mathrm{Gal}\,(L/K)$. Using the results of the previous sections we may produce a simple proof of Theorem 1 for principle homogeneous spaces. Clearly, it suffices to prove the triviality of the kernel of (2).

a) *Reduction to semisimple groups.* Since unipotent K-groups have trivial cohomology we may assume without loss of generality that G is reductive. Then $G = T \cdot H$ is an almost direct product of the central torus T and the semisimple group $H = [G,G]$. Let $Z = T \cap H$. The exact sequence

$$1 \longrightarrow Z \longrightarrow T \times H \longrightarrow G \longrightarrow 1$$

induces the commutative diagram

$$\begin{array}{ccccccc} H^1(K,Z) & \longrightarrow & H^1(K,T\times H) & \longrightarrow & H^1(K,G) & \xrightarrow{\lambda_1} & \\ \downarrow\theta_1 & & \downarrow\theta_2 & & \downarrow\theta_3 & & \\ \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,Z) & \longrightarrow & \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,T\times H) & \longrightarrow & \prod\limits_{\xi\in\Omega_K} H^1(K_\xi,G) & \xrightarrow{\lambda_2} & \\ & & & \xrightarrow{\lambda_1} & H^2(K,Z) & & \\ & & & & \downarrow\theta_4 & & \\ & & & \xrightarrow{\lambda_2} & \prod\limits_{\xi\in\Omega_K} H^2(K_\xi,Z) & & \end{array}$$

By Proposition 3, Corollary 3 and by induction, the maps θ_4, θ_2 are injective. Then from the above diagram and Proposition 5 we get that the kernel of θ_3 is trivial as well.

b) *Reduction to simple simply connected groups.* One should argue just as above and so we omit it.

c) Let G be an (absolutely) simple simply connected K-group. Let B be an L-Borel subgroup such that $T = B \cap \tau(B)$ is a maximal K-torus of G. Since $H^1(L,G) = 1$ the map $H^1(L/K, G(L)) \to H^1(K,G)$ is surjective. By Lemma 6.28 [PR], the map $H^1(L/K, T(L)) \to H^1(L/K, G(L))$ is surjective as well. This means that any $\zeta \in H^1(K,G)$ can be represented by some cocycle $\zeta' \in Z^1(L/K, T(L))$. Denote a maximal K-split subtorus of T by S. If $S \neq 1$ then $C_G(S)$ is a proper subgroup of G and by virtue of the Bruhat-Tits decomposition, we have $\mathrm{Ker}\,(H^1(E, C_G(S)) \to H^1(E,G)) = 1$ for any extension E/K. So if in addition $\zeta \in \mathrm{Ker}\,\theta$ then for each $\xi \in \Omega_K$ the element $\mathrm{res}_{K_\xi}(\zeta')$ is trivial as an element of $H^1(K_\xi, C_G(S))$, hence the claim follows by induction.

d) *T is a K-anisotropic torus.* By the Steinberg theorem, G is either split or quasi-split over L. First examine the case of groups splitting over L. Identify $Z^1(K,T)$ with $(K^*)^n$. Arguing as in case c) we get that any element from $\operatorname{Ker}\theta$ can be represented by a cocycle $\zeta \in Z^1(L/K, T(L))$. We claim that there exist a maximal K-torus $T' \subset G$ isomorphic to T over K and a cocycle $\zeta' \in Z^1(L/K, T'(L))$ equivalent to ζ in $Z^1(L/K, G(L))$ such that ζ' is everywhere locally positive. By Corollary 3, the last would mean that the cocycle ζ' is trivial as an element of $H^1(L/K, T'(L))$, hence ζ is trivial in $H^1(L/K, G(L))$ as well.

Let $\eta : K^s \to Z^1(K,T) = (K^*)^n$ be the K-rational map constructed in Theorem 2. Recall that the image $\eta(K^s)$ consists of elements trivial in $H^1(L/K, G(L))$. Since $\zeta \in \operatorname{Ker}\theta$, we get that for each $\xi \in \Omega_K$ there exists a point $c_\xi \in K^s_\xi$ such that $\eta(c_\xi) = \zeta$. By (9), the components of the rational map η are the product of functions of the form $1 - \overline{c}_{\beta_i}\tau(v_i)v_i$, the last being clearly the reduced norm in the quaternion algebra $(-1, \overline{c}_{\beta_i})$. Then by virtue of the facts that our field K has the property SAP and the Hasse principle holds for groups of type A_1 ([B.-P], [Sch1]) we can pick a point $c \in K^s$ such that the components of $\eta(c)$ and ζ have the same signs in K_ξ for each $\xi \in \Omega_K$.

Let $g \in G(L)$ be the element corresponding to c. By construction, $g\tau(g^{-1}) = \eta(c) \in T(L)$ and hence the components of both elements $g\tau(g^{-1})$ and ζ have the same signs in K_ξ for each $\xi \in \Omega_K$. To complete the proof it remains to notice that the cocycle $\zeta' = \tau(g)^{-1}\zeta\, g$ is equivalent to ζ in $Z^1(L/K, G(L))$, lies in the K-defined torus $T' = \tau(g)^{-1}T\tau(g)$ splitting over L and ζ' is everywhere locally positive.

In view of Proposition 2 the proof for L-quasi-split groups works exactly as in the L-split case. □

Remark 2. The same argument shows that θ is still injective if we replace Ω_K by a dense set of orderings.

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