A characteristic free approach to Brauer algebras

by

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1 Introduction

Schur–Weyl duality relates the representation theory of the infinite group $GL_n(k)$ with that of the symmetric group Σ_r via the mutually centralising actions of the two groups on the space $(k^n)^{\otimes r}$. Brauer defined the algebras which are now called 'Brauer algebras' (or, slightly incorrect, 'Brauer centraliser algebras') by an analogous situation where GL_n is replaced by either an orthogonal or a symplectic group and the group algebra of the symmetric group is replaced by a Brauer algebra. More precisely, for a fixed integer r and a given base field k, a whole family of Brauer algebras $B_k(r, \delta)$ is defined, depending on a parameter $\delta \in k$, which has to be specialised to certain integers to cover the situation Brauer was interested in.

More recently Brauer algebras and their generalisations, especially the BMW algebras, have been looked at in the context of quantum groups and low–dimensional topology (see [6]).

The classical case of k being the complex numbers (or just having characteristic zero) has been clarified to a large extent by results of Brown, Hanlon

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and Wales, and Wenzl: If δ is not an integer of small absolute value, then $B_k(r, \delta)$ is semisimple and its simple representations are known. In the other case of δ being 'singular', Wenzl also obtained a good description of the simple modules. Another such description in the semisimple case has been announced by Kerov. The ring structure of $B_k(r, \delta)$, however, seems to be still unknown even in the characteristic zero case.

Recently, Graham and Lehrer have shown that over any field k, and for any δ, the Brauer algebra $B_k(r, \delta)$ is a cellular algebra. In this way they got a parametrization (which in fact does not depend on k) of the isomorphism classes of simple modules in general. Considering cellular structures thus opens a way for a characteristic free structure theory of Brauer algebras.

The aim of this paper is to work out such a characteristic free approach and to avoid any complicated computation in doing so by using structural properties instead. We start from the scratch and do not use any of the results mentioned above. We first prove that Brauer algebras are cellular (our proof is different from and less computational than that of Graham and Lehrer). More precisely we exhibit in this algebra a structure which we call 'iterated inflation'. As a consequence we do not only get a new proof of the result of Graham and Lehrer, but also of some results of Hanlon and Wales.

Moreover we consider block decompositions of Brauer algebras. Assuming that δ is not singular (but k is arbitrary) we prove in theorem 6.3 that the Brauer algebra is Morita equivalent to a direct product of group algebras of symmetric groups (which correspond to the layers in the iterated inflation). Hence this case is equivalent to the type A case. As a corollary (6.5) we classify under this assumption the blocks of finite representation type of Brauer algebras – these turn out to be Brauer tree algebras.

We organize the paper as follows: In section two we recall the definition of Brauer algebras and the axiomatics of cellular algebras. In sections three and four we consider inflations, which in general are algebras without unit. They occur as pieces of the iterated inflation as which we write the Brauer algebra in section five. More precisely theorem 5.6 tells us that any Brauer algebra can be written as iterated inflation of certain group algebras of symmetric groups (which are explicitly given). This implies the results of Graham and Lehrer. In section six we consider Brauer algebras over a field with nonsingular parameter. In this case, we determine the blocks of the Brauer algebra, and among them those of finite representation type.

2 Definitions

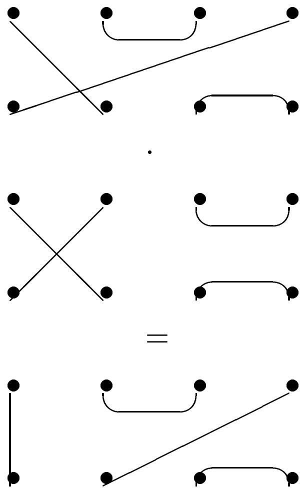
In this section we recall the definition of Brauer algebras and also that of a cellular algebra, which is our main tool.

2.1 Definition of Brauer algebras

Let k be any field and n a natural number. Let V be the vector space k^n on which the group $GL_n(k)$ acts naturally, say on the left. Then $Gl_n(k)$ also acts (diagonally) on the r–fold tensor product $V^{\otimes r}$ for any natural number r. On this space, also the symmetric group Σ_r acts on the right, by place permutations. The two actions centralise each other. In particular, the endomorphism ring $\mathrm{End}_{kGl_n(k)}(V^{\otimes r})$ is a quotient of the group algebra $k\Sigma_r$. This setup is called Schur–Weyl duality. Richard Brauer's starting point for defining 'Brauer algebras' was the following question: Which algebra shall replace $k\Sigma_r$ in this setup if we replace $GL_n(k)$ by either its orthogonal or its symplectic subgroup (in the latter case, of course, n has to be even)? He defined such an algebra by generators and relations, and he also gave an equivalent definition via diagrams. This definition is a special case of the following one, where the parameter δ has to be chosen as a positive or negative integer (for orthogonal or symplectic groups, respectively).

Definition 2.1 *Fix a commutative noetherian domain k, an element $\delta \in k$ and a natural number r. Then the* **Brauer algebra $\mathbf{B_k(r}, \delta)$** *is a k–vector space having a basis consisting of diagrams of the following form: a diagram contains $2n$ vertices, n of them called 'top vertices' and the other n called 'bottom vertices' such that the set of vertices is written as a disjoint union of n subsets each of them having two elements; these subsets are called 'edges'. Two diagrams x and y are multiplied by concatenation, that is, the bottom vertices of x are identified with the top vertices of y, thus giving rise to edges from the top vertices of x to the bottom vertices of y, hence defining a diagram z. Then $x \cdot y$ is defined to be $\delta^{m(x,y)}z$ where $m(x, y)$ counts those connected components of the concatenation of x and y which do not appear in z, that is, which neither contain a top vertex of x nor a bottom vertex of y.*

Let us illustrate this definition by an example. We multiply two elements in $B_k(4, \delta)$:



$\delta^1\cdot$

In the literature, the Brauer algebra sometimes is called Brauer centraliser algebra, a term which we will not use, since it is slightly misleading. In fact, in Brauer's original setup, the endomorphism algebra of $V^{\otimes r}$ in general is just a quotient of the Brauer algebra.

2.2 Cellular algebras

First, we recall the original definition of Graham and Lehrer. Then we give our equivalent definition which is convenient to use for looking at the structure of cellular algebras.

Definition 2.2 *(Graham and Lehrer, [7]) Let R be a commutative Noetherian integral domain. An associative R–algebra A is called a* **cellular algebra** *with cell datum (I, M, C, i) if the following conditions are satisfied:*

(C1) The finite set I is partially ordered. Associated with each $\lambda \in I$ there is a finite set $M(\lambda)$. The algebra A has an R–basis $C^\lambda_{S,T}$ where (S,T) runs through all elements of $M(\lambda) \times M(\lambda)$ for all $\lambda \in I$.

(C2) The map i is an R–linear anti–automorphism of A with $i^2 = id$ which sends $C^\lambda_{S,T}$ to $C^\lambda_{T,S}$.

(C3) For each $\lambda \in I$ and $S,T \in M(\lambda)$ and each $a \in A$ the product $aC^\lambda_{S,T}$ can be written as $(\sum_{U \in M(\lambda)} r_a(U,S) C^\lambda_{U,T}) + r'$ where r' is a linear combination of basis elements with upper index μ strictly smaller than λ, and where the coefficients $r_a(U,S) \in R$ do not depend on T.

The following is our equivalent definition of cellular algebra:

Definition 2.3 *[15] Let A be an R–algebra where R is a commutative Noetherian integral domain. Assume there is an anti–automorphism i on A with $i^2 = id$. A two–sided ideal J in A is called a* **cell ideal** *if and only if $i(J) = J$ and there exists a left ideal $\Delta \subset J$ such that Δ is finitely generated and free over R and that there is an isomorphism of A–bimodules $\alpha : J \simeq \Delta \otimes_R i(\Delta)$ (where $i(\Delta) \subset J$ is the i–image of Δ) making the following diagram commutative:*

$$\begin{tikzcd} J \arrow[r, "\alpha"] \arrow[d, "i"'] & \Delta \otimes_R i(\Delta) \arrow[d, "x \otimes y \mapsto i(y) \otimes i(x)"] \\ J \arrow[r, "\alpha"] & \Delta \otimes_R i(\Delta) \end{tikzcd}$$

The algebra A (with the involution i) is called **cellular** *if and only if there is an R–module decomposition $A = J_1' \oplus J_2' \oplus \ldots \oplus J_n'$ (for some n) with $i(J_j') = J_j'$ for each j and such that setting $J_j = \oplus_{l=1}^{j} J_l'$ gives a chain of two–sided ideals of A: $0 = J_0 \subset J_1 \subset J_2 \subset \ldots \subset J_n = A$ (each of them fixed by i) and for each j ($j = 1, \ldots, n$) the quotient $J_j' = J_j / J_{j-1}$ is a cell ideal (with respect to the involution induced by i on the quotient) of A/J_{j-1}.*

In the following, an R–linear anti–automorphism i of A with $i^2 = id$ will be called an involution.

Typical examples of cellular algebras are the group algebras of symmetric groups (or more generally, their Hecke algebras) and the classical Schur algebras (whose representation theory gives the polynomial representation theory of $GL_n(k)$ over an infinite field k).

3 Inflations

In order to recognise Brauer algebras as cellular algebras and also in order to study block decompositions of Brauer algebras, we will exhibit them as iterated inflations of group algebras of symmetric groups. We have introduced this concept for general cellular algebras in [16] and shown (theorem 4.1 in [16]) that actually an algebra is cellular if and only if it is such an iterated inflation. Special cases of inflations can be found in the papers of Graham and Lehrer [7], Hanlon and Wales [8] and even in the early studies of Brown [3, 4, 5], who considered 'generalised matrix algebras' which coincide in his case with our inflations.

We first recall the definition of inflations and then look at the structure of inflated algebras. This information is basic for all what follows. Thus we give the details although most of the arguments can be found in several of the above references.

In the following, all tensor products are over the base field k which we omit in the notation.

3.1 Definitions

Given a k–algebra B, a k–vector space V, and a bilinear form $\varphi : V \otimes V \to B$ with values in B, we define an associative algebra (possibly without unit) $A = A(B, V, \varphi)$ as follows: As a k–vector space, A equals $V \otimes V \otimes B$. The multiplication is defined on basis elements as follows:

$$(a \otimes b \otimes x) \cdot (c \otimes d \otimes y) := a \otimes d \otimes x\varphi(b, c)y.$$

We need an additional property, namely an involution on A: Assume, there is an involution i on B. Assume moreover, that φ satisfies $i(\varphi(v, w)) = \varphi(w, v)$. Then we can define an involution j on A by putting $j(a \otimes b \otimes x) = b \otimes a \otimes i(x)$.

This definition makes A into an associative R–algebra (possibly without unit), and j is an involutory anti–automorphism of A.

We call A an **inflation** of B along V.

If V has dimension one and the image of φ contains the unit element of B, then A is clearly isomorphic to B. Otherwise, A need not have a unit element, but it may contain idempotents.

Now we are going to produce iterated inflations.

Assume we are given an algebra B (maybe without unit) – which in practise will be an inflation of some other algebra – and an algebra C (with unit). We define an algebra structure on $A := B \oplus C$ which extends the given structures and which makes B into a two–sided ideal such that A/B becomes isomorphic to C. Multiplication is defined by fixing the eight summands of a multiplication map $(B \oplus C) \otimes (B \oplus C) \to (B \oplus C)$. In order to make B into an ideal we put the summands $B \otimes B \to C$, $C \otimes B \to C$ and $B \otimes C \to C$ all zero. The summands $C \otimes C \to C$ and $B \otimes B \to B$ are defined to be the given multiplications on C and B, respectively. Thus we have to choose three bilinear maps $\alpha : C \otimes C \to B$, $\beta : B \otimes C \to B$ and $\gamma : C \otimes B \to B$. Then multiplication in A is defined by $(b_1 + c_1)(b_2 + c_2) = b_1 b_2 + \beta(b_1, c_2) + \gamma(c_1, b_2) + \alpha(c_1, c_2) + c_1 c_2$.

This multiplication is associative if and only the following conditions are satisfied:

(I) β is a homomorphism of left B–modules,

(II) γ is a homomorphism of right B–modules,

(III) for all b in B and c_1, c_2 in C there is an equality: $\beta(\beta(b, c_1), c_2) = b\alpha(c_1, c_2) + \beta(b, c_1 c_2)$,

(IV) for all b in B and c_1, c_2 in C there is an equality: $\gamma(c_1, \gamma(c_2, b)) = \gamma(c_1 c_2, b) + \alpha(c_1, c_2)b$,

(V) for all c_1, c_2, c_3 in C there is an equality: $\alpha(c_1 c_2, c_3) + \beta(\alpha(c_1, c_2), c_3) = \alpha(c_1, c_2 c_3) + \gamma(c_1, \alpha(c_2, c_3))$,

(VI) for all b_1, b_2 in B and c in C there is an equality: $\beta(b_1, c)b_2 = b_1 \gamma(c, b_2)$,

(VII) for all c_1, c_2 in C and b in B there is an equality: $\beta(\gamma(c_1, b_1), c_2) = \gamma(c_1, \beta(b, c_2))$.

We need another condition in order to make sure that A has a unit element which is mapped to the unit element $1(C)$ of C by the quotient homomorphism.

(U) There exists an element b in B such that $b + 1(C)$ is a unit element in A.

Some other (obvious) conditions are needed in order to extend the given involutions on B and C to an involution on A. More conditions (again all of them obvious) make sure that all the inflation pieces are subquotients of ideals in the algebra which is the result of the iterated inflation.

Applying this procedure inductively to algebras $C, B_1, B_2, \ldots, B_n$ where each B_j itself is an inflation of an algebra B'_j we define an **iterated inflation** A of $C_1, B'_1, B'_2, \ldots, B'_n$.

In [16] (theorem 4.1) we have shown, that an algebra is cellular if and only if it can be written as an iterated inflation of copies of full matrix algebras. Moreover ([16], proposition 3.4) an iterated inflation of cellular algebras always is cellular again.

3.2 Inflated algebras

Let B be any algebra, $\varphi : V \otimes V \to B$ a bilinear form (for any finite dimensional k–vector space V) and $A = V \otimes V \otimes B$ the resulting inflation of B. We are going to discuss the ring structure of A.

Since B need not have a unit element, we have to be careful, when talking about its radical. By radical we always mean the Jacobson radical which is the intersection of the annihilators of the simple left modules and which contains any nilpotent ideal.

The following lemma is obvious.

Lemma 3.1 *(a) The vector space $I := V \otimes V \otimes rad(B)$ is a two–sided ideal of A which is contained in $rad(A)$. Moreover, I is nilpotent of nilpotence degree smaller than or equal to the nilpotence degree of B.*

(b) If B admits a decomposition $B = B_1 \oplus B_2$ into two-sided ideals, then A also admits a decomposition $A = (V \otimes V \otimes B_1) \oplus (V \otimes V \otimes B_2)$.

For all the questions we will be interested in later on, we may without loss of generality assume that B has the following properties:

(1) It is semisimple, since by corollary 8.4 in [16] the maximal semisimple quotient of a cellular algebra carries an induced cellular structure.

(2) It is indecomposable as a ring, since a block of a cellular algebra again is cellular.

Hence it is simple and thus (because it is cellular, see [15], proposition 3.4) a full matrix ring over k.

We decompose B as a left and as a right module into ${}_BB = \oplus L_m$ and $B_B = \oplus R_m$. Finally, we fix a basis $v_1, \ldots, v_l$ of V.

Lemma 3.2 *Let V be non-zero vector space. Fix a basis $v_1, \ldots, v_l$ of V and a decomposition of the left B–module B into simple submodules as follows: ${}_BB = \oplus_m L_m$.*

(a) For each m and n the vector space $V \otimes v_n \otimes L_m$ is a left A–module which we denote by $P(m, n)$.

(b) Fix m and n. Denote by $N(m,n)$ the subspace of $P(m,n)$ defined as follows: An element $\sum_{v\in V, l\in L_m} v\otimes v_n\otimes l$ is in $N(m,n)$ if and only if for all $w\in V$, the sum $\sum_{v,l}\varphi(w,v)l$ equals zero.

Then $N(m,n)$ is a submodule of $P(m,n)$ and every element of A acts as zero (from the left) on $N(m,n)$.

(c) If $\varphi\neq 0$, then the quotient module $P(m,n)/N(m,n)$ is simple. Denote it by $L(m,n)$.

(d) If $\varphi = 0$, then the multiplication in A is zero.

Proof. (a) By definition, a basis element $u\otimes v\otimes b$ of A acts on an element $w\otimes v_m\otimes x$ on the left by sending it to $u\otimes v_m\otimes b\varphi(v,w)x$ which is an element of $P(m,n)$. Thus $P(m,n)$ is a submodule of A. As a vector space, A is a direct sum of the subspaces $V\otimes v_m\otimes L_n$ (where m and n run), hence each of them is a module. This proves (a).

(b) Let $u\otimes w\otimes x$ be an element in A and multiply it with an element of $N(m,n)$. We get:

$$(u\otimes w\otimes x)\cdot(\sum_{v\in V, l\in L_m} v\otimes v_n\otimes l) = u\otimes v_n\otimes x(\sum_{v,l}\varphi(w,v)l) = 0$$

(c) If $\varphi\neq 0$, then $N(m,n)\neq P(m,n)$. In fact, if $P(m,n) = N(m,n)$, then for any $w,v\in V$ and $l\in L_m$ we have $\varphi(w,v)l = 0$. This implies that $\varphi(w,v)L_m = 0$. Since B is a simple artinian algebra, L_m is a faithful B–module. Thus $\varphi(w,v) = 0$ for all $w,v\in V$., which is a contradiction. Hence $N(m,n)$ is a proper subspace of $P(m,n)$.

Pick an element $z := \sum_{v\in V, x\in L_m} v\otimes v_n\otimes x$ which is in $P(m,n)$, but not in $N(m,n)$. We show that z generates all of $P(m,n)$ as a left A–module. Multiplying z with an element of A from the left we get the following expression

$$(u\otimes w\otimes x)\cdot(\sum_{v\in V, l\in L_m} v\otimes v_n\otimes l) = u\otimes v_n\otimes x(\sum_{v,l}\varphi(w,v)l),$$

in which u, w, and x can be choosen arbitrarily. Since z is not in $N(m,n)$ we can choose w in such a way that $\sum_{v,l}\varphi(w,v)l$ is not zero. But then we can let x vary in such a way, that the expression $x(\sum_{v,l}\varphi(w,v)l)$ runs through all of L_m (which is simple). Hence the claim follows.

This implies that any non–zero element in $P(m,n)/N(m,n)$ generates all of it, hence this is a simple A–module.

(d) This is trivial by the definition of A. ■

We note that similar statements hold true for right modules. In order to formulate them appropriately for later use, one has to observe that the involution i on A restricts to a k–isomorphism $V \to V$ which identifies the first and the second copy of V in $V \otimes V \otimes B = A$. This provides V with another basis $i(v_1), i(v_2), \ldots, i(v_l)$.

We note that both the description of the maximal semisimple quotient of A immediately carries over to the general situation where B is not assumed to be semisimple.

We have to determine the radical of the algebra A. The previous lemma gives us an obvious candidate, since it provides us with simple modules of the form $P(m,n)/N(m,n)$. We shall show that there really is a simple quotient algebra having as simple left modules precisely the $L(m,n)$.

Lemma 3.3 *Let $\hat{N}_L := \sum_{m,n} N(m,n)$ and $\hat{N}_R$ the analogous sum of right modules. Let $\hat{N}$ be the sum (of vector spaces)*

$$\hat{N} := \hat{N}_L + \hat{N}_L \cdot A + \hat{N}_R + A \cdot \hat{N}_R,$$

where $\hat{N}_L \cdot A$ is the k–space generated by products of the form $n \cdot a$ with $a \in A$ and $n \in \hat{N}_L$. Then $\hat{N}$ is a two-sided ideal of A, it is nilpotent of degree three, i.e. $\hat{N}^3 = 0$, and $A/\hat{N}$ is a simple algebra which is Morita equivalent to $End_k(L(m,n))$ for any m,n. In particular, $\hat{N}$ equals the Jacobson radical of A.

The proof of this lemma will be postponed to the end of the next section. Let us first state a few corollaries of the lemma.

Corollary 3.4 *If $\hat{N}$ is not equal to A, then $A/\hat{N}$ is Morita equivalent to B.*

Corollary 3.5 *The algebra A has a unit element if and only if $N(m,n) = 0$ for all m,n, or equivalently, for some m and n there holds $N(m,n) = 0$.*

Proof. If A has a unit, say e, then e must act non–trivially on $\hat{N}$, which implies that $N(m,n)$ equals zero for all m,n. Conversely, if $N(m,n)$ is zero, then A is a simple artinian ring, hence has a unit element. ■

We have now precise knowledge on the simple A–modules. To be consistent with our use of (Jacobson) radical, a module M has to be called simple if $AM \neq 0$ and there are no non–trivial submodules. Hence we have the following description of simple modules.

If $\hat{N} = 0$, then A has precisely one isomorphism class of simple modules, represented by any indecomposable direct summand of A. Moreover, A acts non–triviallly on any non–zero A–module.

If $0 \neq \hat{N} \neq A$, then A has precisely one isomorphism class of simple modules, again represented by any indecomposable direct summand of $A/\hat{N}$. But there is also the module k with trivial A–action.

If $A = \hat{N}$, then A has no simple module, but there is again the module k with trivial A–action.

Suppose B is semisimple. We call the bilinear form φ **singular** if $N_\varphi(L, n) \neq 0$ for some simple B-module L and n, where $N_\varphi(L, n)$ stands for the subset of $V \otimes v_n \otimes L$ consisting of all elements $\sum_{v \in V, l \in L} v \otimes v_n \otimes l$ with $\sum_{v,l} \varphi(w, v)l = 0$ for all $w \in V$.

If B is arbitrary, we call φ **singular** if $\bar{\varphi} : V \otimes V \longrightarrow B \longrightarrow B/rad(B)$ is singular.

4 Matrix presentation of inflated algebras

In this section we use matrices to present and study the inflated algebras (see Brown's generalised matrix algebras [3]). We assume that V is a non-zero vector space with a basis $\{v_1, \cdots, v_m\}$ and that B is an arbitrary k–algebra with identity. Again we denote by A the inflated algebra (B, V, φ). Let $M_m(B)$ denote the full matrix algebra over B.

Clearly, the bilinear form φ can be described by an $m \times m$ matrix Φ over B, that is, $\Phi = (\varphi(i, j))$, where $\varphi(i, j) = \varphi(v_i, v_j)$ for all $1 \leq i, j \leq m$. Now we use this matrix to define a new multiplication $\circ$ on $M_m(B)$ in the following way:

$$X \circ Y := X\Phi Y \quad \text{for all} \quad X, Y \in M_m(B).$$

This multiplication is associative and hence $(M_m(B), \circ)$ is an associative algebra. (The addition of two matrices is the usual matrix addition). We denote this algebra by $(M_m(B), \Phi)$.

The following lemma shows that the inflated algebra A can be described by this new algebra.

Lemma 4.1 *The inflated algebra $A(B, V, \varphi)$ is isomorphic to $(M_m(B), \Phi)$ as k–algebra.*

Proof. Since each element of $V \otimes V \otimes B$ can be uniquely written in the form $\sum_{i,j}^m v_i \otimes v_j \otimes b_{ij}$, this gives us a matrix $(b_{ij}) \in M_m(B)$. Thus we may define a linear map (depending on the choice of basis in V)

$$\mu : A = V \otimes V \otimes B \longrightarrow (M_m(B), \Phi), \quad \sum_{i,j} v_i \otimes v_j \otimes b_{ij} \mapsto (b_{ij}).$$

It is easy to check that this is an algebra homomorphism. Comparing the dimensions of the two spaces, we see easily that μ is an isomorphism. ∎

From this matrix presentation of inflated algebras, we have the following fact. Note that in an artinian algebra left and right invertible is the same, hence a left unit is a right unit and so on.

Proposition 4.2 *The following are equivalent for the inflated algebra A:*

(1) The algebra A has a unit element.

(2) The matrix Φ is invertible, that is, there is a matrix $X \in M_m(B)$ such that $\Phi X = I_m$, where I_m denotes the identity matrix in $M_m(B)$.

(3) The matrix $\bar{\Phi} := (\overline{\varphi(i,j)}) \in M_m(\bar{B})$ is invertible, where $\bar{B}$ stands for $B/rad(B)$ and $\bar{b}$ stands for the residue class of an element $b \in B$.

Proof. The equivalence of (1) and (2) follows from Lemma 4.1. It is obvious that (3) is implied by (2).

Now assume that $\bar{\Phi}$ is invertible, that is, there is a matrix $X = (x_{ij}) \in M_m(B)$ such that $\bar{\Phi}\bar{X} = I_m \in M_m(\bar{B})$, where $\bar{X} = (\bar{x}_{ij}) \in M_m(\bar{B})$. Since $rad(M_m(B)) = M_m(rad(B))$ and $M_m(B/rad(B)) = M_m(B)/rad(M_m(B))$, there is a matrix $Z \in M_m(rad(B))$ such that $\Phi X - I_m = Z$. Since Z is a nilpotent matrix over B, the matrix $I_m - Z$ is invertible in $M_m(B)$. This shows that the matrix ΦX is invertible and therefore Φ is invertible. ∎

Corollary 4.3 *If Φ is invertible in $M_m(B)$, then for any B-module M we have $N_\varphi(M, n) = 0$.*

Proof. As remarked above, the fact that $M_m(B)$ is an artinian algebra implies, that always $X\Phi = I_m$ if $\Phi X = I_m$ holds true.

Suppose $\sum_i^t u_i \otimes v_n \otimes l_i \in N(M, n)$, where we may assume that $l_1, \cdots, l_t$ are k–linear independent. By definition, we have for any v_j that $\sum_i \varphi(v_j, u_i) l_i = 0$. Now let $u_j = \sum_k \alpha_{jk} v_k$. Then $\sum_{k=1}^m \sum_{i=1}^t \alpha_{ik} \varphi(v_j, v_k) l_i = 0$. If $X = (x_{ij})$, then multiplying by $x_{s,j}$ and summing up over j, we have

$$0 = \sum_{i,k} \alpha_{ik} \sum_j x_{sj} \varphi(v_j, v_k) l_i = \sum_{i,k} \alpha_{ik} \delta_{sk} l_i = \sum_i \alpha_{is} l_i.$$

since $l_1, \cdots, l_t$ are linear independent, $\alpha_{is} = 0$ for all i and s. Hence $u_i = 0$ for all i. ∎

Now let us give the proof of the lemma 3.3.

Proof of 3.3. We keep all notation introduced before.

Assume now that B is a matrix algebra $M_s(k)$ over k. The algebra A acts trivially from the left on $\hat{N}_L$ and from the right on $\hat{N}_R$. It follows that $\hat{N}$ is a two-sided ideal. Products of elements in $\hat{N}$ are elements of $\hat{N}_L \cdot \hat{N}_R + \hat{N}_L \cdot A \cdot \hat{N}_R$. But this space is annihilated by all of A both from the left and from the right. Hence triple products of elements in $\hat{N}$ are zero. Thus $\hat{N}$ is contained in the Jacobson radical of A.

Define $E\binom{ij}{\lambda\mu} = v_i \otimes v_j \otimes E_{\lambda\mu}$, where $E_{\lambda\mu}$ is the matrix units in $M_s(k)$. Then the elements $E\binom{ij}{\lambda\mu} = v_i \otimes v_j \otimes E_{\lambda\mu}, 1 \leq i, j \leq l, 1 \leq \lambda, \mu \leq s$, form a basis of A, and the multiplication in A with respect to this basis reads as follows:

$$E\binom{ij}{\lambda\mu} E\binom{mn}{\gamma\delta} = \varphi\binom{jm}{\mu\gamma} E\binom{in}{\lambda\delta}$$

where $\varphi\binom{ij}{\lambda\mu}$ is defined by

$$\varphi(v_i, v_j) = \sum_{\lambda,\mu} \varphi\binom{ij}{\lambda\mu} E_{\lambda\mu}.$$

Define $\Phi = (\varphi\binom{ij}{\lambda\mu})$ to be the matrix with $(\binom{i}{\lambda}, \binom{j}{\mu})$-th entry $\varphi\binom{ij}{\lambda\mu}$, then by a similar argument as in [3] we will prove that the radical of A has the k-dimension $l^2 s^2 - r^2$, where r is the rank of the matrix Φ.

We fix a left ideal L_{μ_0} in B and its basis $\{E_{\lambda\mu_0}\}$. Let $\sigma : V \otimes L_{\mu_0} \longrightarrow V \otimes L_{\mu_0}$ be the linear map defined by $v_i \otimes E_{\lambda\mu_0} \mapsto \sum_{j,\mu} \varphi\binom{ji}{\mu\lambda} v_j \otimes E_{\mu\mu_0}$. Clearly, the linear map has Φ as its matrix. The kernel of σ consists precisely of the elements of the form $\sum_{i,\lambda} \alpha_{i\lambda} v_i \otimes E_{\lambda\mu_0}$ with $\sum_{i,\lambda} \varphi(\binom{ji}{\mu\lambda})\alpha_{i\lambda} = 0$ for all j and μ. An equivalent condition is that $\sum_{i,\lambda} \alpha_{i\lambda} v_i \otimes v_n \otimes E_{\lambda\mu_0}$ lies in $N(\mu_0, n)$. Now we identify A with $(V \otimes L_{\mu_0}) \otimes (V \otimes L_{\mu_0})$ via the k-linear map $v_i \otimes v_j \otimes E_{\lambda\mu} \mapsto (v_i \otimes E_{\lambda\mu_0}) \otimes (v_j \otimes E_{\mu\mu_0})$. Thus, as in [8], we prove that the dimension of $A/\hat{N}$ is r^2. (Note that the dimension of the image of σ is the rank of the matrix Φ.) Hence the dimension of $\hat{N}$ is equal to the dimension of the radical A. It follows from $\hat{N} \subset \mathrm{rad}(A)$ that the radical of A coincides with $\hat{N}$. Now the rest of the lemma 3.3 becomes transparent. ■

Let us note here that lemma 4.1 shows that an inflation is a generalized matrix algebra in the sense of Brown [3]. In his language, an equivalent version of the above proof runs as follows: The isomorphism class of the algebra A is not changed if we replace the 'multiplication matrix' Φ by a matrix $S\Phi T$ (with S and T being invertible). Hence one can replace Φ by its normal form, with some 1's on the diagonal and 0's elsewhere, say $\Phi = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$ with 1 denoting an identity matrix of size $r \times r$ and 0 denoting matrices of appropriate sizes. Then clearly $\hat{N}_L$ consists of matrices of the form $\begin{pmatrix} 0 & 0 \\ \star & \star \end{pmatrix}$ (with $\star$ denoting arbitrary entries), and $\hat{N}_R$ consists of matrices of the form $\begin{pmatrix} 0 & \star \\ 0 & \star \end{pmatrix}$. The two-sided ideal $\hat{N}$ generated by them according to the multplication rule in A consists of matrices of the form $\begin{pmatrix} 0 & \star \\ \star & \star \end{pmatrix}$. Hence the quotient algebra $A/\hat{N}$ has a basis represented by matrices of the form $\begin{pmatrix} \star & 0 \\ 0 & 0 \end{pmatrix}$, which implies the assertion of the lemma.

Under an additional assumption, which is easily seen to be satisfied by Brauer algebras, one can avoid some of the above computations by directly writing down a set of primitive idempotents in A. Assume that for the given basis $v_1, \dots, v_l$ all elements $\varphi(v_j, v_j)$ are non-zero scalar multiples of the identity of B. Then elements of the form

$$\frac{1}{\varphi(v_j, v_j)} v_j \otimes v_j \otimes e$$

(for e a primitive idempotent in B) are primitive idempotents.

5 Brauer algebras are iterated inflations

In this section we prove the main technical result of this paper: Brauer algebras are iterated inflations of group algebras of symmetric groups. We observed this fact already in [16] – without giving details of the proof – as a consequence of the rather complicated computations in [7]. The proof we give here is much more elementary and it works for related 'diagram algebras' as well. (We have checked this for Temperley–Lieb algebras and for Jones' annular algebras.)

As an immediate consequence we get that Brauer algebras are cellular and we also obtain a description of the simple modules of the Brauer algebra over a field.

We fix a commutative noetherian domain k, a natural number r, a parameter $\delta \in k$, and the corresponding Brauer algebra $A := B_k(r, \delta)$. Note that k need not be a field, so we cover in particular the case of δ being an indeterminate over a given field. It is well–known how to define an involution on A:

Lemma 5.1 *Sending a diagram d to a digram $i(d)$ which has the same edges, but top and bottom vertices are interchanged (that is, the vertex l of d is the vertex $r + l$ of $i(d)$ and the vertex $r + l$ of d is the vertex l of $i(d)$ for all l with $1 \leq l \leq r$), defines an involution $i : A \to A$.*

The first step in exhibiting the structure of an iterated inflation consists in defining an appropriate filtration which also is well–known:

Lemma 5.2 *Assign to a diagram d the number $t(d)$ of 'through strings', that is, edges which connect a top vertex with a bottom vertex. Let J_l be the k–space generated by all diagrams with $t(d) \leq l$. Then J_l is a twosided ideal of A. This defines a filtration of A by twosided ideals: $0 \subset J_a \subset J_{a+2} \subset \ldots \subset J_{r-2} \subset J_r$ (which starts with either $a = 0$ or $a = 1$ depending on whether r is even or odd).*

Proof. If a diagram d has $t(d)$ through strings, then it has $r - 2t(d)$ edges which connect top vertices with top vertices and it also has $r - 2t(d)$ edges which only involve bottom vertices. Multiplication of two diagrams d_1 and d_2 yields a scalar multiple of a diagram d_3 with $t(d_3) \leq min\{t(d_1), t(d_2)\}$, since those edges of d_1 involving only top vertices are not changed by the multiplication, and the same holds for the 'bottom' edges of d_2. ■

In the following we use the convention $k\Sigma_0 = k$ in order to simplify notation.

Lemma 5.3 *Fix an index l and let B be the k–algebra (possibly without unit) $B := J_l/J_{l-2}$. Then B is isomorphic (as a k–algebra) to an inflation $V_l \otimes V_l \otimes k\Sigma_l$ of the group algebra of the symmetric group Σ_l along a free k–module V_l. The k–rank of V_l equals the number of possibilities to draw $(r-l)/2$ edges between $r-l$ out of r vertices. The bilinear form φ will be defined in the proof.*

We remark that the mentioned number of configurations of $f := (r-l)/2$ edges in the top row of a diagram equals

$$\frac{r!}{2^f l! f!},$$

hence the dimension of B equals

$$\frac{(r!)^2}{4^f l! (f!)^2}.$$

Proof. Denote the vector space $V_l \otimes V_l \otimes k\Sigma_l$ by C. We define a k–isomorphism $\psi : B \to C$ as follows: B has a k–basis consisting of diagrams d with $t(d) = l$, that is, there are $(r-l)/2$ edges connecting top vertices with top vertices, another $(r-l)/2$ edges connecting bottom vertices with bottom vertices and another l through strings. Fix a diagram d. Denote by e the configuration formed by the edges relating top vertices only and by f the configuration formed by the edges relating bottom vertices only. Renumber the top vertices of the through strings as $1, \ldots, l$ (from left to right) and their bottom vertices also (from left to right) as $1, \ldots, l$. Then the through strings define a permutation σ of the set of l letters by sending the numbers assigned to the top vertex of a string to the numbers of its bottom vertex.

Define $\psi : B \to C$ to send d to

$$\psi(d) := e \otimes f \otimes \sigma.$$

This clearly defines an isomorphism of k–modules.

In order to give C a multiplicative structure we have to define φ. This will be done using the isomorphism ψ, thus verifying at once that ψ is a ring isomorphism. Let $x := e \otimes f \otimes \sigma$ and $y := g \otimes h \otimes \tau$ be two basis elements of C. The product xy has to be of the form $e \otimes h \otimes \sigma\varphi(f,g)\tau$. We have to distinguish two cases.

First case: Each edge in f has at least one common vertex with an edge in g (after identifying as usual the vertices in f with those in g and thus forming a graph Γ consisting of these vertices together with all edges of f and of g). Then we define $\varphi(f,g) := \delta^m \cdot \sigma$ where the data m and σ are defined as follows: The group element σ is the permutation defined by the through strings of the product $\psi^{-1}(f \otimes f \otimes Id) \cdot \psi^{-1}(g \otimes g \otimes Id)$. The number m counts the connected components of the graph Γ, which is defined as follows: Denote the bottom vertices of e by $r+1, r+2, \ldots, 2r$ and the bottom vertices of f by $1, 2, \ldots, r$ of f. Then identify $r+j$ with j for each $j : 1 \leq j \leq r$. The set of edges of Γ is the union of the set of edges in e and the set of edges in f.

The product xy defined in this way obviously coincides with $\psi(\psi^{-1}(x)\psi^{-1}(y))$.

Second case: There is at least one edge γ in f which does not share a vertex with any edge in g. Then we define $\varphi(f,g) := 0$. We have to show that this fits together with multiplication in B. In fact, forming the product of the two diagrams representing $\psi^{-1}(x)$ and $\psi^{-1}(y)$ inside A gives a result which has a strictly smaller number of through strings, since the edge γ yields an additional bottom edge in the product. Hence, inside B the product is zero.

This defines φ and shows that ψ is an isomorphism of k–algebras. ■

It is easy to check that ψ 'preserves' the involution i:

Lemma 5.4 *Under ψ the involution $i : B \to B$ corresponds to the standard involution on C which sends $e \otimes f \otimes \sigma$ to $f \otimes e \otimes \sigma^{-1}$.*

This finishes our consideration of the candidate layers of the iterated inflation. It remains to make sure, that the layers fit together (which is more than just having a filtration by two-sided ideals). For simplicity we call all the isomorphisms identifying a layer with some inflation by the same letter ψ.

Lemma 5.5 *Let $d_1 \in J_m \backslash J_{m-2}$ and $d_2 \in J_n \backslash J_{n-2}$ be two diagrams in A and let their respective ψ–images be $a \otimes b \otimes \sigma$ and $e \otimes f \otimes \tau$. Assume that $m \geq n$. Then the product $d_1 d_2$ either is an element of J_{n-2} or it is an element of $J_n \backslash J_{n-2}$ and in the latter case it corresponds under ψ to a scalar multiple of an element $c \otimes e \otimes \mu\tau$ where c is an element in V_n and μ is an element in $k\Sigma_n$.*

A similar assertion is valid in the case of $m \leq n$. The proof is similar to that of lemma 5.3.

The outcome of this lemma is that the bilinear maps α, β, γ can now be defined which are needed for getting an iterated inflation. The precise definitions are very similar to that of ψ given in lemma 5.3 and thus we omit them.

Altogether we have proved the following theorem.

Theorem 5.6 *The algebra $A = B_k(r, \delta)$ is an iterated inflation of group algebras of symmetric groups.*

More precisely: as a free k–module, A is equal to

$$k\Sigma_r \oplus (V_{r-2} \otimes V_{r-2} \otimes k\Sigma_{r-2}) \oplus (V_{r-4} \otimes V_{r-4} \otimes k\Sigma_{r-4}) \oplus \dots,$$

and the iterated inflation starts with $k\Sigma_r$, inflates it along $V_{r-2} \otimes V_{r-2} \otimes k\Sigma_{r-2}$ and so on, ending with an inflation of $k = k\Sigma_1$ or $k = k\Sigma_0$ as bottom layer (depending on whether r is odd or even).

As immediate consequences we get several results from the literature:

Corollary 5.7 *(Graham and Lehrer [7]) The Brauer algebra $B_k(r, \delta)$ over any commutative noetherian domain k is cellular.*

Corollary 5.8 *(Graham and Lehrer [7]) Let k be a field. If δ is not zero, then the simple representations of the Brauer algebra $B_k(r, \delta)$ are indexed by pairs $(l, L(\lambda))$ where l runs through the non–negative integers $r, r-2, \dots$ and $L(\lambda)$ runs through the isomorphism classes of simple modules over $k\Sigma_l$.*

In the case of $\delta = 0$, the above assertion is also valid except that the case $l = 0$ (which occurs only for r even) does not contribute a simple module.

The exception in the case of r even and $\delta = 0$ comes from the fact that precisely then there is a piece in the iterated inflation which has $\varphi(-, -)$ equal to zero.

Corollary 5.9 *(a) The radical of a Brauer algebra $B_k(r,\delta)$ is nilpotent of degree less than or equal to $\prod_l 3^{nil(k\Sigma_{r-2l})}$, the product of powers of 3 to the nilpotence degrees of the radicals of the group algebras occuring in the inflation.*

(b) (Hanlon and Wales [10], corollary 7.1.5) If k has characteristic zero or bigger than r, then the radical of $B_k(r,\delta)$ is nilpotent of degree less than or equal to $3^{[r/2]+1}$, where $[r/2]$ is the integer part of $r/2$.

Proof. An inflation of a semisimple algebra has radical of nilpotence degree up to three. Thus part (a) follows by writing a group algebra of a symmetric group as an iterated inflation. Under the assumption of (b), the occuring group algebras all are semisimple. ■

6 Block decomposition

Throughout this section k is assumed to be a field. We are going to determine the blocks of the Brauer algebra in case δ is not singular in the following sense.

Recall that a bilinear form φ defining an inflation is called **singular** if $N_{\bar{\varphi}}(L,n) := \{\sum_{v\in V, l\in L} v\otimes v_n \otimes l \mid \sum_{v\in V,l\in L} \bar{\varphi}(w,v)l = 0\}$ does not vanish for some simple B–module L and some n.

We have seen that for non–singular φ, the algebra $(V\otimes V\otimes B)/(V\otimes V\otimes rad(B))$ has a unit element (by lemma 3.5). (We note that this statement becomes wrong if one replaces the field k by a noetherian domain, say a polynomial ring in one variable.) Now by proposition 4.2, the algebra $V\otimes V\otimes B$ has a unit element.

The parameter δ is called **singular** if for some l the associated bilinear form φ is singular. (This is well-defined since the singularity of φ depends upon $\bar{\varphi}$ and the latter depends only on δ.) As we have seen in section four, the singularity of δ is decided by the vanishing of at least one of finitely many polynomials, hence there are only finitely many singular values of δ.

Based on some partial results as well as on extensive computations Hanlon and Wales [8] formulated a precise conjecture (in the case of characteristic zero) for which values δ should be singular, namely only integers of 'small' absolute value. This conjecture has been proved by Wenzl [18]. Thus in characteristic zero, most integer values of δ are not singular whereas in prime characteristic it may happen (depending on the choice of r) that all 'integer' values are singular.

We will show that in the case of δ not singular the Brauer algebra $A = B_k(r,\delta)$ is a direct sum of the group algebras of symmetric groups occuring in the above picture of iterated inflation. Thus the blocks of A are known since the blocks of group algebras of symmetric groups are known by Nakayama's rule.

Lemma 6.1 *Let A be any ring and J a twosided ideal in A. Then the inclusion $J \subset A$ induces a ring decomposition $A = J \oplus A/J$ if and only if J (seen as a subalgebra of A) has a unit element.*

Proof. If A has such a block decomposition, then J clearly has a unit element. Conversely, assume J has a unit element, say e. Then J equals eJe which is contained in eAe which is contained in the twosided ideal J containing e. Thus J equals eAe. Moreover, $(Id_A - e)Ae$ is contained in $(Id_A - e)J$ which equals $(Id_A - e)eJ$, hence it is zero. Similarly, $eA(Id_A - e)$ is zero. Thus A is a direct sum $eAe \oplus (Id_A - e)A(Id_A - e)$ and the first summand is J, thus the second one is isomorphic to A/J. ∎

This lemma will be used to find the block decomposition of the Brauer algebra. Alternatively, one may use the following statement, which is in terms of cellular algebras and which contains some information of independent interest.

Lemma 6.2 *Let A be a cellular algebra (over a field) with cell chain $0 \subset J_1 \subset \ldots \subset J_n = A$.*

(a) If L is a simple A–module which occurs as a composition factor of a standard module $\Delta(l)$ for some l, then there is some $m \geq l$ such that J_m/J_{m-1} maps onto L. In other words: composition factors of standard modules are associated with 'higher' layers in the cell filtration.

(b) If some ideal J_l in the cell chain has a unit element, then A has a ring decomposition $A = J_l \oplus A/J_l$. If l equals 1, then in addition J_1 is a full matrix ring over k.

Proof. We first recall from [15] that a cell ideal J in an algebra A over a field is either nilpotent (in which case its square is already zero) or generated (as a twosided ideal) by a primitive idempotent, say f, and in the latter case J is a heredity ideal, that is, it has the form $Af \otimes_k fA$ and the endomorphism ring fAf equals k.

For the proof of (a) we use induction on n. If n equals 1, then A is simple (see [15], proposition 3.4), hence the assertion is obvious. Assume now that n is bigger than 1 and pick any l. If l is bigger than 1, then both L and $\Delta(l)$ are modules over the cellular algebra A/J_1, thus the assertion follows by induction. Hence we can assume $l = 1$. If J_1 is nilpotent with square zero, then again both L and $\Delta(l)$ are modules over the cellular quotient algebra A/J_1 and again the assertion follows by induction. In the other case, J_1 is a heredity ideal of the form $Af \otimes fA$. In particular, the endomorphism ring fAf of $\Delta(1) \simeq Af$ is isomorphic to k. Thus either L is the unique top composition factor of Af – and then it is a quotient of J_1 itself – or it is a composition factor of $rad(Af)$ and hence a module over A/J_1 and again we can apply induction. This proves (a).

If the ideal J_l has a unit element, say e, then $\Delta(l)$ equals $e\Delta(l)$, hence it cannot have any composition factor L on which A/J_l acts non–trivially. By part (a) also A/J_l does not have any composition factor L which occurs as a quotient of any of the ideals $J_1, \ldots, J_l$. Hence A can be decomposed into $J_l \oplus A/J_l$.

In the case $l = 1$ we have moreover, that $\Delta(1)$ cannot have any other composition factors than just its top factor. Thus J_1 is a simple algebra. Since it is cellular, it must be a full matrix ring over k by [15]. ■

Theorem 6.3 *Let k be any field, r any integer, and choose a parameter $\delta \in k$ which is not singular for the Brauer algebra $B_k(r,\delta)$. Let Λ be the set of all nonnegative integers smaller than or equal to r and congruent to r modulo two.*

Then the Brauer algebra $B_k(r,\delta)$ decomposes into a direct sum

$$B_k(r,\delta) \simeq \bigoplus_{l \in \Lambda} V_l \otimes V_l \otimes k\Sigma_l$$

of the layers occuring when writing $B_k(r,\delta)$ as iterated inflation as in theorem 5.6.

Moreover, $B_k(r,\delta)$ is Morita equivalent to

$$\bigoplus_{l \in \Lambda} k\Sigma_l$$

hence the block decomposition of $B_k(r,\delta)$ is given by applying Nakayama's rule for block decomposition of group algebras of symmetric groups.

For a precise statement of Nakayama's rule we refer to [11, 12].

Proposition 4.2 implies that in the singular case the statement of the theorem is not true.

Proof. The previous section tells us how to write $B_k(r,\delta)$ as an iterated inflation the layers of which are inflations of group algebras of symmetric groups. Since δ is not singular, each of the layers has a unit element (by 4.2 and 3.5), hence applying 6.1 inductively we can split off layers as direct summands. ∎

We noticed already that the theorem is not true if one replaces k by a polynomial ring in one variable or if k is a field, but δ is singular. The method of proof can however be used to split off some layers in the singular case as well. In case of characteristic zero, Wenzl [18] (theorem 3.4 and corollary 3.5) obtained a block decomposition with rather different methods which seem to be restricted to this case (or rather to the case of a field k over which all group algebras of symmetric groups occuring in the iterated inflation are semisimple). At present we have to leave open the following problem:

Problem: Let k be a field of prime characteristic or a polynomial ring (of any characteristic). Determine the blocks of a Brauer algebra with singular parameter δ.

Finally, we list a few corollaries to the theorem.

Corollary 6.4 *(Brown [4], Wenzl [18] theorem 3.2(a)) If the characteristic of k is zero or bigger than r and if δ is not singular, then the Brauer algebra $B_k(r,\delta)$ is split semisimple.*

We note that under the assumptions of this corollary one may deduce branching rules and character formulae for the Brauer algebra. We do not formulate these results since they are (properly) contained in results of Wenzl [18] and Ram [17].

Corollary 6.5 *Let k be any field, r any integer and δ not singular. Then a block of the Brauer algebra $B_k(r,\delta)$ has finite representation type, that is, there are up to isomorphism only finitely many indecomposable modules, if and only if this block is up to Morita equivalence a Brauer tree algebra. In this case, the Brauer tree is a line with no exceptional vertex.*

Proof. We refer to [1] for the definition and a discussion of Brauer tree algebras and their ring structure (which is completely known). There it is proven, that a block of a group algebra of a finite group (over a splitting field) has finite representation type if and only if it is a Brauer tree algebra. This applies to our situation since any field is a splitting field for the symmetric group. In [15] it has been shown that a Brauer tree of a cellular algebra must be a line. In [14] it is moreover shown that in the case of an integral cellular algebra (which the Brauer algebra is) no exceptional vertex can occur. ■

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