A characterization of the finite soluble groups.

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1 Introduction.

Let G be a group and M a subgroup of G. Then we say that a subgroup A of G is a *supplement* to M in G, if $G = MA$. In 1965 O. H. Kegel [Ke] introduced the condition

(MC) Every maximal subgroup of G admits a supplement which is cyclic and of prime-power order.

and proved that every finite group G which satisfies condition (MC), is soluble [Ke, Proposition 1].

In the same paper Kegel conjectured that any finite group G whose maximal subgroups admit an abelian supplement is soluble.

Proposition 1 of [Ke] was generalized by R. M. Guralnik in a different way. He showed that if G is a finite group such that every maximal subgroup has prime power index, then $G/O_\infty(G) \cong 1$ or $L_2(7)$ ($O_\infty(G)$ is the maximal normal soluble subgroup of G) [Gu, Corollary 3].

In their paper [BR] J. C. Beidleman and D. J. S. Robinson considered finite groups G having the property that for every proper subgroup H of G there is a cyclic group $X = X(H)$ such that $XH = HX$. They proved that every group G enjoying this property is soluble [BR, Theorem A]. See also the related work by Z. Yaoqing [Ya] and by W. E. Deskins [De].

The proofs of Corollary 3 of [Gu] and of Theorem A of [BR] are based on the classification of finite simple groups.

Another characterization of soluble groups was obained by Y. Wang. He showed that a finite group G is soluble if and only if for every maximal subgroup H of G there exists a normal subgroup $N = N(H)$ of G such that $G = HN$ and $H \cap N \leq Core(H)$ [Wa, Theorem 3.1].

In this paper we will prove the conjecture of Kegel. Notice, that the proof of the conjecture implies Theorem A of [BR] as well as Theorem 3.1 of [Wa].

Let (MA) be the condition

(MA) Every maximal subgroup of G admits an abelian supplement.

We show

Theorem 1.1 *A finite group G which satisfies condition (MA) is soluble.*

Hence Theorem 1.1 proves the conjecture of Kegel. The theorem may be formulated in terms of permutation groups as follows.

Theorem 1.2 *Suppose that G is a finite group such that for any primitive action (G, Ω) of G, there exists an abelian subgroup of G which acts transitively on Ω. Then G is soluble.*

Notice, that a group G satisfying the condition that for any primitive permutation action (G, Ω) of G, there exists an abelian subgroup of G which acts transitively on Ω, is not necessarily soluble. For example let G be an extension of an elementary abelian group N by a non soluble group. Then G is not soluble, but N is a supplement to every maximal subgroup of G which does not contain N.

Notice also, that condition (MA) does not characterize the finite soluble groups as the following example shows.

Example. Let $G = SL_2(3) \cong Q_8 : \mathbf{Z}_3$. Let $Z = Z(O_2(G))$ and let T be a subgroup of order 3 of G. Then $M = ZT$ is maximal in G, but does not have an abelian supplement.

In order to characterize the soluble groups among the finite groups we introduce the condition.

(MC) Every maximal subgroup M of G admits a supplement whose commutator subgroup is contained in M.

Theorem 1.3 *A finite group G is soluble if and only if (MC) holds.*

Moreover we will determine the structure of the groups satisfying property (MN).

(MN) Every maximal subgroup M of G admits a nilpotent supplement.

We will show

Theorem 1.4 *Let G be a finite group which satisfies (MN). Then every composition factor of G is either soluble or isomorphic to $L_2(7)$. Moreover, $G/O_\infty(G)$ is isomorphic to 1 or $L_2(7)$.*

We also determine the groups with property

(MF) Every maximal subgroup of G admits a soluble supplement.

We proof:

Theorem 1.5 *The chief factors of a finite group G which satisfy (MF) are either p-groups or isomorphic to $L_2(q)$, $q \in \{5, 7, 9, 11\}$.*

But these groups still deserve a further analysis.

In the next section we prove that Theorem 1.5 holds for almost simple groups and that there is no almost simple group satisfying (MA) or (MC). These results are obtained by checking the list of maximal factorizations of the almost simple groups by M. Liebeck, C. Praeger and J. Saxl [LPS]. They determine for every almost simple group G the maximal subgroups A and B such that $soc(G)$ is neither a subgroup of A nor of B and such that $G = AB$. This paper of Liebeck, Praeger and Saxl [LPS] is based on the classification of the finite simple groups. Of course, it would be good, if one could prove the Lemmas 2.1 and 2.2 without quoting the classification. In the third section we prove Theorems 1.1, 1.3 and 1.5. The idea of the proofs of these theorems is to show first that the assertions hold if G is a wreath product $W = L \wr S_n$ and then to reduce the general case, that is G a group satisfying (MA) (resp. (MC) or (MF)) to the special situation that G is a wreath product. In the last section we prove Theorem 1.4.

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2 The almost simple groups satisfying (MF).

In this section P_i will always denote a parabolic subgroup of type i of an appropriate group of Lie type.

Lemma 2.1 *Let G be an almost simple group which satisfies (MF). Then $soc(G) \cong L_2(q),\ q \in \{5,7,9,11\}$. If $soc(G) \not\cong L_2(q),\ q \in \{5,7,9,11\}$, then there exists a maximal subgroup M of G with $G = M soc(G)$ which does not admit a soluble supplement.*

Proof. Our proof is quite similiar to the case by case analysis of Beidleman and Robinson [BR, p. 692ff] who show that every simple group has a maximal subgroup which does not admit a factorization satisfying certain conditions. Set $T = soc(G)$. In the following we will say that a proper subgroup M of G has a supplement if $G = MA$ for a subgroup A of G which does not contain T.

(1) The alternating groups.
Let $T = A_n,\ n \geq 5$.

Since $L_2(5) \cong A_5,\ L_2(9) \cong A_6$, we assume that $n \neq 5, 6$. If $n = 8$, then every maximal subgroup has a supplement but those to $M = N_G(N),\ N \cong E_{16}(S_3 \times S_3)$ are isomorphic to A_7. Thus $n \neq 8$. If $n \leq 12$, then according to the Atlas [CCNPW] there are maximal subgroups in G which have no supplement (clearly, under the assumption $n \neq 5, 6, 8$). Hence $n \geq 13$.

Set $\Omega = \{1, \ldots, n\}$. As $n \neq 6$, we have $G \leq Aut(\Omega) = S_n$. Let M be the stabilizer of a k-subset of Ω, $6 \leq k \leq n/2$. Then M is a maximal subgroup of G and A acts k-homogeneously on Ω by [LPS, Theorem D]. Since $k \geq 6$, the action of A is already k-transitive [Ka], so $A \cong A_n$ or S_n in contradiction with A being soluble.

(2) The projective groups.
Let $T = L_n(q),\ (n, q) \neq (2, 2), (2, 3)$.

Case $n = 2$. Assume $q \neq 4, 5, 7, 9, 11$ (recall $L_2(4) \cong L_2(5)$). Then according to [Kl] $M = N_G(N)$ is a maximal subgroup of G, where $N \cong D_{2(q-1)/d}$ with $d = (2, q-1)$. Therefore $q(q+1)/2$ divides the order of A which implies, as $q \neq 4$, that $A = soc(G)$; a contradiction.

Case $n \geq 4$, even. Since $L_4(2) \cong A_8$, we know $(n, q) \neq (4, 2)$. (Notice that in [LPS, Tables 3] a factorization for $L_4(2)$ is missing). Let M be the normalizer in G of the parabolic subgroup $P_{n/2}$ of $soc(G)$. Then M is a maximal subgroup of G and according to [LPS, Tables 1,3] M does not admit a supplement.

Case $n = 3$. Since $L_2(7) \cong L_3(2)$, we assume $q > 2$. Then $M = N_G(N)$, where $N = (\mathbf{Z}_{q-1})^2/\mathbf{Z}_d S_3$, $d = (3, q-1)$, is a maximal subgroup [Kl], which does not have a supplement [LPS, Tables 1,3].

Case $n \geq 5$, odd. First let $(n, q) = (5, 2)$. Then every maximal subgroup has a supplement. But the maximal subgroup $N_G(N)$, $N \cong \mathbf{Z}_{31} : \mathbf{Z}_5$ has only subgroups isomorphic to $E_{16}L_4(2)$ as supplements. Therefore $(n, q) \neq (5, 2)$. We distinguish the two cases $G \leq P\Gamma L_n(q)$ and $G \not\leq P\Gamma L_n(q)$. Assume first $G \leq P\Gamma L_n(q)$. Then $M = N_G(P_2)$ is a maximal subgroup which does not have a supplement [LPS, Tables 1,3]. Hence $G \not\leq P\Gamma L_n(q)$. Then, as n is odd, $M = N_G(P_{(n-1)/2} \cap P_{(n+1)/2})$ is a maximal subgroup of G without a supplement [LPS, Tables 1,3].

(3) The symplectic groups.
Assume $T = PSp_{2n}(q),\ n \geq 2$ and $(n,q) \neq (2,2)$.

Case $n = 2$. If q is odd, then P_1 and P_2 are not isomorphic, as $O_p(P_1) \not\cong O_p(P_2)$, where p is the defining characteristic. Therefore $N_G(P_2)$ does not have a supplement according to [LPS, Tables 1,2,3]. Now let q be even. Since $Sp_4(2) \cong S_6$, we may assume $q = 2^r,\ r \geq 2$. If $q = 4$, then every maximal subgroup has a supplement. But $N_G(N),\ N \cong S_6$, has only groups isomorphic to $L_2(16)$ or $L_2(16)2$ as supplements. Assume $q = 16$ and set $M = N_G(N)$ with $N = Sp_4(4)$. Then M has a supplement A if and only if $G = Aut(T)$ [LPS, Table 1]. Then $A \cong L_2(2^8)$ [LPS, Table 1], so $q \neq 16$. Let e be a prime divisor of r. Then $N_G(N)$ with $N = PSp_4(2^{r/e})$ is a maximal subgroup of G which has no supplement [LPS, Tables 1,2,3].

Case $n > 2$. Then $M = N_G(P_2)$ is a maximal subgroup which does not have a supplement [LPS, Tables 1,3].

(4) The unitary groups.
Assume $T = U_n(q),\ n \geq 3$ and $(n,q) \neq (3,2)$.

Case n odd. In this case there are only exceptional factorizations occuring for the groups $U_3(3),\ U_3(5),\ U_3(8),\ U_9(2)$ [LPS, Table 3]. But then in the respective groups G for $N = 4S_4,\ M_{10}, \mathbf{Z}_3 \times L_2(8)$ the maximal subgroup $M = N_G(N)$ does not have a supplement in G according to [CCNPW]. If $soc(G) = U_9(2)$, then $M = N_G(P_2)$ does not have a supplement.

Case n even. We may assume $(n,q) \neq (4,2)$, since $U_4(2) \cong PSp_4(3)$. If $soc(G) = U_4(3)$, then $N_G(N)$ for $N \cong M_{10}$ does not have a supplement. In the remaining cases $M = N_G(P_1)$ does not have a supplement [LPS, Tables 1,3].

(5) The orthogonal groups $P\Omega_{2n+1}(q)$.
Assume $T = P\Omega_{2n+1}(q), n \geq 3, q$ odd. These groups are excluded, since $M = N_G(P_2)$ does not admit a supplement [LPS, Tables 1,2,3].

(6) The orthogonal groups $P\Omega^-_{2n}(q)$.
Assume $T = P\Omega_{2n}(q), n \geq 4$. Choose again $M = N_G(P_2)$ [LPS, Tables 1,3].

(7) The orthogonal groups $P\Omega^+_{2n}(q)$.
Assume $T = P\Omega_{2n}(q), n \geq 5$. Choose $M = N_G(P_2)$ [LPS, Tables 1,2,3].

(8) The orthogonal groups $P\Omega^+_8(q)$.
Assume $T = P\Omega^+_8(q)$. Then again $M = N_G(P_2)$ does not have a supplement [LPS, Table 4].

(9) The exceptional groups.
The only exceptional groups of Lie type admitting a factorization are those with $soc(G) \cong G_2(q)$ or $F_4(q)$. In the first case $M = N_G(P_1)$ does not have a supplement [LPS, Table 5]. Let $soc(G) = F_4(q)$. If G contains no graph automorphism then we set $N = P_1$ and else $N = P_2 \cap P_3$. Then $M = N_G(N)$ is a maximal subgroup which does not have a supplement [LPS, Table 5].

(10) The Sporadic groups.
The sporadic groups for which there is a maximal factorization are

$$M_{11},\ M_{22},\ M_{23},\ M_{24},\ J_2,\ HS,\ He,\ Ru,\ Suz,\ Fi_{22},\ Co_1,$$

by [LPS, Table 6]. Each of these groups has a subgroup which does not have a supplement [CCNPW]. □

Remark 1 *If G is an almost simple group such that every maximal subgroup of G admits a supplement which does not contain $soc(G)$, then $soc(G)$ is isomorphic to $L_2(q)$, $q \in \{5,7,9,11\}$; $L_4(2)$, $L_5(2)$ or $Sp_4(4)$.*

Lemma 2.2 *Let G be an almost simple group with $soc(G) \ncong L_2(7)$. Then there exists a maximal subgroup M in G such that*

(i) $G = MT$, where $T = soc(G)$;

(ii) M does not have a nilpotent supplement;

(iii) M does not have a supplement A with $A' \leq M$.

Proof. In the proof of Lemma 2.1 for every almost simple group G with $T \ncong L_2(q)$, $q \in \{5,7,9,11\}$ a maximal subgroup M is given which does not admit a soluble supplement. A direct check shows that these maximal subgroups also does not admit a supplement A with $A' \leq M$.

If $T \cong L_2(5)$, $L_2(9)$, $L_2(11)$, then $M = N_G(N)$ for $N \cong S_3$, S_4, D_{12} are, respectively, maximal subgroups of G whose supplements A are, respectively, isomorphic to D_{10}; D_{10} or A_5; $Frob(11:5)$. These supplements are neither nilpotent nor $A' \leq M$. This proves the Lemma. □

Lemma 2.3 *Let $G \cong L_2(7)$. Then there exists a maximal subgroup M in G such that*

(i) $G = MT$, where $T = soc(G)$;

(ii) M does not have an abelian supplement;

(iii) M does not have an supplement A with $A' \leq M$.

Proof. Set $M = N_G(N)$ with $N \cong Frob(7:3)$. Then (i) is satisfied and any supplement to M in G contains at least a subgroup isomorphic to D_8. Thus also (ii) holds. Since N is a maximal subgroup in $soc(G)$, (iii) is true as well. □

Remark 2 *In fact, we can show more than stated in Lemmas 2.2 and 2.3. Using the same proofs we could replace (iii) by*

(iii') M does not have an supplement A such that $N \leq M$ for some normal subgroup $1 \neq N$ of A.

3 Proofs of Theorems 1.1, 1.3 and 1.5.

Notation. For L a group, the *direct power* L^n will be thought of as the group of all functions

$$\Sigma \to L,\ \Sigma = \{1, ..., n\},$$

multiplication being defined pointwise, this means

$$i(a_1a_2) = (ia_1)(ia_2) \text{ for all } i \in \Sigma \text{ and all } a_1, a_2 \in L^n.$$

By a *wreath product* $L \wr S_n$ we mean the usual semidirect product of L^n with S_n: if $s \in S_n$ and $a \in L^n$, then $ia^s = i^{s^{-1}}a$ for all $i \in \Sigma$. Further for $U \leq L$ we denote by U_i the subgroup

$$U_i = \{a \in U^n \mid ja = 1 \text{ for } j \neq i\}$$

of L^n. Then in particular, $L^n = L_1 \times ... \times L_n$. Let W be a wreath product $W = L \wr S_n$. Then we denote by π_i the projection of

$$N_W(L_i) = L_i \times (L_1 \times \ldots \times L_{i-1} \times L_{i+1} \times \ldots \times L_n)S_{n-1}$$

onto L_i and for G a subgroup of W we denote by $G_{(i)}$ the images of i under the set of functions $\pi_i(N_G(L_i))$, that is $G_{(i)} = (i)\pi_i(N_G(L_i))$. So $G_{(i)}$ is a subgroup of L.

In the following we shall use a lemma of L. G. Kovács on subgroups of wreath products. To formulate the lemma we introduce further notation. For a wreath product $W = L \wr S_n$ we call the subgroup S_n the *top group* of W.

Lemma 3.1 *[Ko, Lemma (2.2)] Let G be a subgroup of the wreath product $W = L \wr S_n$ such that G acts transitively on $\{L_1, \ldots, L_n\}$. Then there is some $g \in L_2 \times \ldots \times L_n$ such that G^g is a subgroup of $V = G_{(1)} \wr S_n$, where $V \leq W$ has the same top group as W.*

Proposition 3.2 *Let $G = W = L \wr S_n$, L an almost simple group. Let K be a self normalizing subgroup of L with $L = KT$, where $T = soc(L)$. Set $M = N_G(K^n) = K \wr S_n$. If M has a soluble (resp. nilpotent, resp. abelian) supplement in G, then K has a soluble (resp. nilpotent, resp. abelian) supplement in L. Furthermore, if M has a supplement A in G with $A' \leq M$, then K has a supplement B in L with $B' \leq K$.*

Proof. Assume that M has a supplement A which is soluble (resp. nilpotent, resp. abelian, resp. such that $A' \leq M$), but that K does not have a supplement B in L with the respective property.

Notice, if A is soluble (resp. nilpotent, resp. abelian), then also $N_A(T_1)$ and $A_{(1)} = (1)\pi_1(N_A(T_1))$. Now suppose $A' \leq M$. As

$$N_A(T_1)/(A' \cap N_A(T_1)) \cong N_A(T_1)A'/A'$$

is abelian, also $A_{(1)}/(A')_{(1)}$ is abelian which implies $(A_{(1)})' \leq (A')_{(1)}$. Thus $(A_{(1)})' \leq (1)\pi_1(N_{A'}(T_1)) \leq (1)\pi_1(N_M(T_1)) = K$. This shows that $A' \leq M$ implies $(A_{(1)})' \leq K$.

We shall show that $L = KA_{(1)}$, which yields by the paragraph above a contradiction to our assumption that K does not have a supplement B in L with the respective property.

We claim that A acts fixed point freely on $\Omega = \{T_1, \ldots, T_n\}$:
Assume that B normalizes T_1. Let $t \in L_1$. Then there are $m_i \in K_i$, $a_i \in L_i$, $x, y \in S_n$ such that $(a_1 \cdots a_n)y \in B$ and such that

$$t = (m_1 \cdots m_n)x(a_1 \cdots a_n)y.$$

Thus $t = (m_1 a_{1^x} \cdots m_n a_{n^x})xy$. So $x = y^{-1}$ fixes T_1, which implies $t = m_1 a_1$. This shows that $L = KA_{(1)}$ in contradiction with our assumption.

We may assume that A acts transitively on Ω:
Let Ω_1 be an orbit of B on Ω. Assume without loss of generality that $\Omega_1 = \{T_1, \ldots T_k\}$. Let C be the projection of A on $L \wr S_k$ Then we see in the same way as in the previous paragraph that $L \wr S_k = (K \wr S_k)C$. Thus we may assume $n = k$.

The final contradiction:
According to Lemma 3.1 there is an $g \in L_1 \times \cdots \times L_n$ such that A^g is a subgroup of

$A_{(1)} \wr S_n$ with $A^g_{(1)} = A_{(1)}$. Since $G = MA$ implies $G = MA^g$, we may assume that A is a subgroup of $A_{(1)} \wr S_n$. As A is transitive on Ω, clearly, $A_{(1)} = A_{(i)}$ for $i = 2, \dots n$. Therefore, we obtain again $L = KA_{(1)}$ in contradiction to our assumption. This proves the proposition. $\square$

Now we are prepared to prove Theorems 1.1, 1.3 and 1.5.

Proof of Theorem 1.5. Let G be a counterexample of minimal order to Theorem 1.5.

According to Lemma 2.1 and Proposition 3.2 G is not a wreath product $L \wr S_n$ for some almost simple group L. We will derive a contradiction to our assumption by showing that the minimal counterexample G is in fact such a wreath product.

(1) Let N be a normal subgroup of G. Then G/N has the property (MF).

(2) Let N be a minimal normal subgroup of G. Then N is the direct product of isomorphic simple groups $T_1, \dots, T_n$, where $T_i \not\cong L_2(q),\ q \in \{5, 7, 9, 11\}$ and $C_G(N) = 1$: As $N \neq 1$, we have $|G/N| < |G|$. Since G/N satisfies (MF) by (1), G/N has as composition factors p-groups or groups isomorphic to $L_2(q),\ q \in \{5, 7, 9, 11\}$ by induction. Therefore N is not abelian and the first part of (2) holds. Thus $C_G(N) \cap N = 1$ and as G is a minimal counterexample we obtain with (1) $C_G(N) = 1$.

¿From (2) follows that $G \leq Aut(T_1) \wr S_n$.

(3) Since G acts transitively on $\Omega = \{T_1, \dots, T_n\}$ by Lemma 3.1 G is conjugate in $Aut(T_2) \times \dots \times Aut(T_n)$ to a subgroup H of $G_{(1)} \wr S_n$ such that $H_{(1)} = G_{(1)}$. Then H acts transitively on Ω and therefore $H_{(i)} = H_{(1)} = G_{(1)}$ for $i = 1, \dots, n$. Since G is a counterexample, H is as well and we may (and do) assume $G = H$. Set $W = G_{(1)} \wr S_n$.

(4) By Lemma 2.1 there is a maximal subgroup $M_{(1)}$ of $G_{(1)}$ such that

(i) $G_{(1)} = M_{(1)}T_{(1)}$ and

(iii) there does not exist a soluble subgroup $A_{(1)}$ of $G_{(1)}$ with $G_{(1)} = M_{(1)}A_{(1)}$.

Set $M = (M_{(1)})^n$.

(5) $K = N_W(M) \cap G$ is maximal in G:
As $N \leq G$ and as $M_{(1)}T_{(1)} = G_{(1)}$, we have $G = KN$. Let $K < U \leq G$ and let $u \in U \setminus K$. Then $u \notin N_W(M)$. Assume $(M \cap N)^u = M \cap N$. Since M is the normalizer of $N \cap M$ in $G_1 \times \dots \times G_n$ and since u normalizes $G_1 \times \dots \times G_n$, we obtain in contradiction with the choice of u that u normalizes M. Therefore $(M \cap N)^u \neq M \cap N$, so, $\pi_i(M \cap N)^u) \neq \pi_i(M \cap N)$ for some $i \in \{1, \dots, n\}$. As $M_{(i)} \cap N_{(i)} = (M \cap N)_{(i)}$, it follows $((M \cap N)^u)_{(i)} \not\leq M_{(i)}$. As $K_{(i)} = M_{(i)}$, we obtain that $U_{(i)} = G_{(i)}$. Since K is transitive on Ω, in fact $U_{(k)} = G_{(k)}$ for $k = 1, \dots, n$. ¿From the fact that $M \cap N \leq K$ follows $N \leq U$. So, as $G = KN$, we obtain $U = G$, the assertion.

By condition (MF) there is a soluble subgroup A of G such that $G = KA$. Notice that $W = (M_{(1)} \wr S_n)G$. Therefore

(6) $W = (M_{(1)} \wr S_n)G = (M_{(1)} \wr S_n)KA = N_W(M)A$.

Now we obtain a contradiction with Proposition 3.2. This proves the theorem. $\square$

Proposition 3.3 *If G satisfies (MA) or (MC), then G is soluble.*

Proof.

Let G be a counterexample of minimal order. We shall almost copy the proof of Theorem 1.5. Statement (1) clearly also holds here. Let N be a minimal normal subgroup of G. Since G is soluble if and only if G/N and N are soluble, N is the direct product of isomorphic non-abelian simple groups $T_1, \dots, T_n$. As in (2) of the proof of Theorem 1.5 we obtain again $C_G(N) = 1$. The next paragraph in the proof of Theorem 1.5 is concerned with subgroups of wreath products, so here also (3) of the proof of Theorem 1.5 holds. (4) has to be replaced by

(4') By Lemmas 2.2 and 2.3 there exists a maximal subgroup $M_{(1)}$ of $G_{(1)}$ such that

(i) $G_{(1)} = M_{(1)}T_{(1)}$;

(ii) $M_{(1)}$ does not have an abelian supplement;

(iii) $M_{(1)}$ does not have a supplement $A_{(1)}$ with $(A_{(1)})' \leq M_{(1)}$.

As in the proof of Theorem 1.5 we set $M = (M_{(1)})^n$. Then $K = N_W(M) \cap G$ is maximal in G. Therefore K has a supplement A which satisfies $A' \leq K$. Moreover, we obtain as in the proof of Theorem 1.5 that $W = (M_{(1)} \wr S_n)G = (M_{(1)} \wr S_n)KA = N_W(M)A$, which yields a contradiction with Proposition 3.2. $\square$

Notice that Proposition 3.3 proves Theorem 1.1.

Lemma 3.4 *Let G be a soluble group. Then G possesses properties (MN) and (MC).*

Proof. Let M be a maximal subgroup of G. It is well known that a maximal subgroup of a soluble group G has prime power index in G. Therefore, G is the product of M and of an appropriate Sylow p-subgroup of G. Hence G satisfies property (MN).

Moreover, it is well known that the chief factors of G are abelian p groups. Let $G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_r = 1$ be a chief series of G. Then there is some $i \in \{1, \ldots, r\}$ such that $G = G_{i-1}M$ and $G_i \leq M$. Since G_{i-1}/G_i is abelian G also enjoys property (MC). $\square$

Now Proposition 3.3 and Lemma 3.4 prove Theorem 1.3.

4 Proof of Theorem 1.4.

Suppose that G satisfies property (MN). To show Theorem 1.4 we shall apply the ideas of the proofs of Theorem 1.5 and Proposition 3.3 as well as of the proof of Corollary 3 by Guralnick [Gu].

By the same arguments as used in the proof of Proposition 3.3, we obtain, if G satisfies (MN), then the composition factors of G are either cyclic of prime order or isomorphic to $L_2(7)$. Now we want to prove that $G/O_\infty(G)$ is trivial or isomorphic to $L_2(7)$. Since $G/O_\infty(G)$ still satisfies (MN), we may assume (and do) that $O_\infty(G) = 1$.

Let N be a minimal normal subgroup of G. Then N is the direct product of groups $T_1, \ldots, T_n$ isomorphic to $L_2(7)$. By the same argument as above suppose $C_G(N) = 1$. Thus G is a subgroup of $W = Aut(T_1) \wr S_n$. Again by [Ko, Lemma (2.2)] we may (and do) assume that $G \leq G_{(1)} \wr S_n$.

First we claim that $G_{(1)}$ is isomorphic to $L_2(7)$. Assume $G_{(1)} \cong PGL_2(7)$. Let $T \in Syl_2(N)$ and let $K = N_G(T)$. As in (5) of the proof of Theorem 1.5 we see that K is a maximal subgroup of G. Therefore, we obtain as in (6) that $W = N_W(T)A$, where A is a nilpotent subgroup of G with $G = KA$. Now Proposition 3.2 yields that $K_{(1)}$ has a nilpotent supplement in $Aut(T_1)_{(1)}$, which is false as $K_{(1)}$ is isomorphic to D_{16}. Thus $G_{(1)} \cong L_2(7)$, as claimed.

Now we want to show $n = 1$. Since $G_{(1)} \cong L_2(7)$, we have $G \leq T_1 \wr S_n$, so $G = T_1 \wr X$ with X a transitive subgroup of S_n. Therefore, the diagonal

$$M = \{n \in N \mid in = 1n \ \text{ for } i = 2, \ldots, n\}$$

is a maximal subgroup of G. Hence, there is a nilpotent subgroup A of G with $MA = G$. Here

$$| A : A \cap M | = | L_2(7) |^{n-1} .$$

Thus

$$2^{3(n-1)} 3^{n-1} 7^{n-1}$$

divides the order of A. Since $| S_n |_p \leq p^{(n-1)/(p-1)}$, we obtain that

$$2^{2(n-1)} 3^{(n-1)/2} 7^{(5n-5)/6}$$

divides $| A \cap N |$. As $A_{(i)}$ is nilpotent for $i = 1, \ldots, n$, for any i $A_{(i)}$ is a p-group for some $p \in \{2, 3, 7\}$. This implies $n = 1$. Using the same argumentation we obtain that $C_G(N) = 1$. This proves Theorem 1.4.

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