Two side bounds on the inverses of diagonally dominant tridiagonal matrices

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Abstract

We establish upper and lower bounds for the entries of the inverses of diagonally dominant tridiagonal matrices. These bounds improve the bounds recently given by Shivakumar and Ji. Moreover, we show how to improve our bounds iteratively. For an $n \times n$ M–matrix this iterative refinement yields the exact inverse after $n-1$ steps.

1 Introduction

Tridiagonal matrices arise in many applications, e.g. the numerical solution of PDE's and cubic spline interpolation. Therefore this class of matrices is studied very well and a lot of theory is known about tridiagonal matrices and their inverses. One of the most beautiful result is due to Gantmacher and Krein [GK1], [GK2] who proved that the inverse of an irreducible symmetric tridiagonal matrix is given by two vectors. A similar result is also proved by Barrett in [Ba]. This result is generalized in several directions. For a review on symmetric tridiagonal matrices and their inverses see Meurant [M].

For many problems it is very helpful to have upper and lower bounds for the entries (or the absolute values of the entries) of the inverse of a matrix. For arbitrary diagonally dominant matrices Ostrowski [O2] already established such bounds. Later on Golub and Meurant gave in [GM] bounds for symmetric positive definite matrices established by using quadrature rules. Similar bounds were established by Robinson and Wathen in [RW]. Decay rates for the entries of inverses of certain banded matrices are established by Demko [D] and several other authors (see Meurant [M] and Nabben [N] and references therein). Of course these rates can also be used as approximations of entries of the inverse of a matrix.

For certain tridiagonal matrices arising in spline interpolation Kershaw derived in [K] upper and lower bounds. Recently, Shivakumar and Ji gave in [SJ] upper and lower bounds for diagonally dominant tridiagonal matrices.

Here, we establish new upper and lower bounds for the entries of the inverses of diagonally dominant tridiagonal matrices which improve the bounds given in [SJ], [K] and [O2]. Moreover, we show how these bounds can be improved iteratively. For an a $n \times n$ M–matrix we obtain after $n-1$ iterations sharp upper bounds for all entries of the inverse, i.e. by improving our bounds we have computed the inverse of the matrix. However, our examples show that one needs just a few iterations to get good bounds. For arbitrary diagonally dominant tridiagonal matrices A we obtain after $n-1$ steps of our iterative refinement the exact inverse of the comparison matrix of A. Moreover, we establish some further results of this iterative refinement for tridiagonal Toeplitz matrices.

This paper is organized as follows. In Section 2 we mention several known result for tridiagonal matrices which we use in the following section. Section 3 contains our main results while Section 4 gives a comparison of our bounds with known results as well as some numerical examples.

2 Preliminaries

We consider real tridiagonal matrices of the form

$$A = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & c_{n-2} & a_{n-1} & b_{n-1} \\ & & & c_{n-1} & a_n \end{bmatrix}, \quad c_i, b_i \neq 0. \tag{2.1}$$

Since A might be nonsymmetric we distinguish between diagonally dominance by rows, i.e.

$$|a_i| \geq |b_i| + |c_{i-1}| \quad \text{for all } i = 1, \dots, n, \tag{2.2}$$

where $c_0 = b_0 = c_n = b_n = 0$, and diagonally dominance by columns, i.e.

$$|a_i| \geq |b_{i-1}| + |c_i| \quad \text{for all } i = 1, \dots, n. \tag{2.3}$$

Furthermore, we sometimes assume that

$$|a_1| > |b_1| \quad \text{and} \quad |a_n| > |c_{n-1}|, \tag{2.4}$$

or

$$|a_1| > |c_1| \quad \text{and} \quad |a_n| > |b_{n-1}|. \tag{2.5}$$

It is well-known (see e.g. [I], [Hi], [N]) that the inverse of a tridiagonal matrix $A^{-1} = C = [c_{i,j}]$ is given by four vectors of real numbers $u = [u_i]$, $v = [v_i]$, $x = [x_i]$, $y = [y_i]$, where $u_i v_i = x_i y_i$ for all i, such that

$$c_{i,j} = \begin{cases} u_i v_j & i \leq j, \\ x_j y_i & i \geq j. \end{cases} \tag{2.6}$$

Using the Hadamard product (elementwise product) matrices of the form (2.6) can be written nicely as

$$C = \begin{bmatrix} u_1 & u_1 & \cdots & \cdots & u_1 \\ x_1 & u_2 & \cdots & \cdots & u_2 \\ x_1 & x_2 & u_3 & \cdots & u_3 \\ \vdots & \vdots & & \ddots & \vdots \\ x_1 & x_2 & \cdots & \cdots & u_n \end{bmatrix} \circ \begin{bmatrix} v_1 & v_2 & \cdots & \cdots & v_n \\ y_2 & v_2 & \cdots & \cdots & v_n \\ y_3 & y_3 & v_3 & \cdots & v_n \\ \vdots & \vdots & & \ddots & \vdots \\ y_n & y_n & \cdots & \cdots & v_n \end{bmatrix}. \tag{2.7}$$

The entries u_i, v_i, x_i and y_i of the vectors u, v, x, y which give A^{-1}, can be computed as follows (see e.g. [N])

$$\begin{aligned}
u_1 &= 1, \qquad u_2 = -\frac{a_1}{b_1}, \qquad u_i = -\frac{a_{i-1}u_{i-1} + c_{i-2}u_{i-2}}{b_{i-1}}, \\
x_1 &= 1, \qquad x_2 = -\frac{a_1}{c_1}, \qquad x_i = -\frac{a_{i-1}x_{i-1} + b_{i-2}x_{i-2}}{c_{i-1}}, \\
&\quad \text{for } i = 3, \dots, n \\
y_n &= \frac{1}{a_n x_n + b_{n-1}x_{n-1}}, \qquad y_{n-1} = -\frac{a_n}{c_{n-1}} y_n, \\
y_i &= -\frac{a_{i+1}y_{i+1} + b_{i+1}y_{i+2}}{c_i}, \quad \text{for } i = n-2, \dots, 1. \\
v_n &= \frac{1}{a_n u_n + c_{n-1}u_{n-1}}, \qquad v_{n-1} = -\frac{a_n}{b_{n-1}} v_n, \\
v_i &= -\frac{a_{i+1}v_{i+1} + c_{i+1}v_{i+2}}{b_i} \quad \text{for } i = n-2, \dots, 1.
\end{aligned}$$

For M–matrices of the form (2.1), i.e. $b_i < 0, c_i < 0$, which satisfy (2.2) and (2.4) it is shown in [N] that the sequence $\{u_i\}_{i=1}^n$ is strictly increasing while the sequence $\{y_i\}_{i=1}^n$ is strictly decreasing. For matrices which are diagonally dominant by columns and satisfies (2.5) the sequence $\{x_i\}_{i=1}^n$ is strictly increasing while the sequence $\{v_i\}_{i=1}^n$ is strictly decreasing.

Considering (2.7) we obtain that the entries of A^{-1} decay along a column away from the diagonal. Moreover with

$$\frac{1}{\rho_1} := \min_{i\geq 2} -\frac{a_i + c_{i-1}}{b_i} \qquad \frac{1}{\rho_2} := \min_{i\geq 2} -\frac{a_i + b_i}{c_{i-1}}$$
$$\frac{1}{\rho_3} := \min_{i\geq 2} -\frac{a_i + c_i}{b_{i-1}} \qquad \frac{1}{\rho_4} := \min_{i\geq 2} -\frac{a_i + b_{i-1}}{c_i}$$

it is shown in [N] for row diagonally dominant matrices which satisfies (2.4) that

$$c_{i,j} \leq \rho_1^{(j-i)} c_{j,j} \quad \text{for } i < j, \qquad c_{i,j} \leq \rho_2^{(i-j)} c_{j,j} \quad \text{for } i > j$$

where $C = [c_{i,j}] := A^{-1}$. Similarly, for column diagonally dominant matrices which satisfies (2.5) we obtain

$$c_{i,j} \leq \rho_3^{(j-i)} c_{i,i} \quad \text{for } i < j, \qquad c_{i,j} \leq \rho_4^{(i-j)} c_{i,i} \quad \text{for } i > j.$$

For symmetric tridiagonal matrices these results were established by Concus, Golub, and Meurant in [CGM].

In the next section we will specify this result and will give some upper and lower bounds for all entries of A^{-1}. These bounds improve the bounds given by Shivakumar and Ji in [SJ] and Kershaw in [K].

Ostrowski found in [O2] upper and lower bounds for the entries of the inverse of an arbitrary diagonally dominant matrix. This result is slightly generalized in [SWYM]. To state the result we first need some notations and definitions:

For $A = [a_{i,j}] \in \mathbb{R}^{n,n}$ and all $i \in N := \{1, \ldots, n\}$ define

$$\mu_i := \frac{1}{|a_{i,i}|} \sum_{j\neq i} |a_{i,j}|, \tag{2.8}$$

and let

$$J(A) := \{i \in N : \mu_i < 1\}.$$

Moreover A is called in [SWYM] weakly chained diagonally dominant (w.c.d.d.) if

$$\mu_i \leq 1 \quad \text{for all} \;\; i \in N \qquad \text{and if} \qquad J(A) \neq \emptyset$$

and if for all $i \in N, i \notin J(A)$ there exists indices $i_1, i_2, \ldots, i_k$ with $a_{i_r, i_{r+1}} \neq 0, 0 \leq r \leq k-1$ where $i_0 = i$ and $i_k \in J(A)$.

Bramble and Hubbard called w.c.d.d. matrices in [BH] matrices of positive type. The following lemma is then established in [SWYM]:

Lemma 2.1 *Let A be a w.c.d.d. matrix, then $A^{-1} = C = [c_{i,j}]$ exists and for $i \neq j$*

$$|c_{i,j}| \leq \mu_i |c_{i,i}| \leq |c_{i,i}|. \tag{2.9}$$

and for $i \in J(A)$

$$\frac{1}{|a_{i,i}|(1+\mu_i)} \leq |c_{i,i}| \leq \frac{1}{|a_{i,i}|(1-\mu_i)} \tag{2.10}$$

Lemma 2.1 was proved by Ostrowski in [O2] for strictly row diagonally dominant matrices.

3 Upper and Lower Bounds

First define for $i = 1, \ldots, n$

$$\tau_i = \frac{|b_i|}{|a_i| - |c_{i-1}|}, \qquad \omega_i = \frac{|c_{i-1}|}{|a_i| - |b_i|}, \tag{3.1}$$

$$\alpha_i = \frac{|c_i|}{|a_i| - |b_{i-1}|}, \qquad \beta_i = \frac{|b_{i-1}|}{|a_i| - |c_i|}, \tag{3.2}$$

where $c_0 = b_0 = 0 = c_n = b_n$. If the tridiagonal matrix A is diagonally dominant by rows, i.e. $|a_i| \geq |b_i| + |c_{i-1}|$ for all i we have $\tau_i \leq 1$ and $\omega_i \leq 1$. If A is diagonally dominant by columns we obtain $\alpha_i \leq 1$ and $\beta_i \leq 1$.

Theorem 3.1 *Let A be a nonsingular tridiagonal matrix of the form* (2.1)*. Let $A^{-1} = C = [c_{i,j}]$. If A is diagonally dominant by rows, then*

$$\begin{aligned} |c_{i,j}| &\leq |c_{j,j}| \prod_{k=i}^{j-1} \tau_k \quad \textit{for } i < j, \\ |c_{i,j}| &\leq |c_{j,j}| \prod_{k=j+1}^{i} \omega_k \quad \textit{for } i > j. \end{aligned}$$

If A is diagonally dominant by columns, then

$$\begin{aligned} |c_{i,j}| &\leq |c_{i,i}| \prod_{k=i}^{j-1} \beta_k \quad \textit{for } i < j, \\ |c_{i,j}| &\leq |c_{i,i}| \prod_{k=j+1}^{i} \alpha_k \quad \textit{for } i > j. \end{aligned}$$

Proof. Since $u_2 = -\frac{a_1}{b_1}$ it is clear that $|u_1| \geq |u_2|$. Thus

$$\begin{aligned} |u_i| &= \left|\frac{a_{i-1}u_{i-1} + c_{i-2}u_{i-2}}{b_{i-1}}\right| \geq \frac{|a_{i-1}u_{i-1}| - |c_{i-2}u_{i-2}|}{|b_{i-1}|} \\ &= \frac{|a_{i-1}||u_{i-1}| - |c_{i-2}||u_{i-2}|}{|b_{i-1}|} \\ &\geq \frac{|a_{i-1}| - |c_{i-2}|}{|b_{i-1}|}|u_{i-1}| = \frac{1}{\tau_{i-1}}|u_{i-1}|. \end{aligned}$$

Therefore, $|u_i| \geq |u_{i-1}|$ or more preciously $\tau_{i-1}|u_i| \geq |u_{i-1}|$. Hence for $i < j$

$$|c_{i,j}| = |u_i||v_j| \leq \tau_i|u_{i+1}||v_j| \leq \prod_{k=i}^{j-1} \tau_k|u_j||v_j| = \prod_{k=i}^{j-1} \tau_k|c_{j,j}|.$$

Similarly we obtain $|y_{n-1}| \geq |y_n|$ and

$$|y_i| \geq \frac{|a_{i+1}| - |b_{i+1}|}{|c_i|}|y_{i+1}| = \frac{1}{\omega_{i+1}}|y_{i+1}|.$$

Thus, $|y_i| \geq |y_{i+1}|$ or $\omega_{i+1}|y_i| \geq |y_{i+1}|$. Hence, for $i > j$ we obtain

$$|c_{i,j}| = |x_i||y_i| \leq \omega_i|x_j||y_{i-1}| \leq \prod_{k=j+1}^{i} \omega_k|x_j||y_j| = \prod_{k=j+1}^{i} \omega_k|c_{j,j}|.$$

Considering A^T gives the bounds for column diagonally dominant matrices.

$\square$

Next we give upper and lower bounds for the diagonal entries.

Theorem 3.2 *Let A be a nonsingular tridiagonal matrix of the form* (2.1)*. Let $A^{-1} = C = [c_{i,j}]$. If A is diagonally dominant by rows then*

$$\frac{1}{|a_i| + \tau_{i-1}|c_{i-1}| + \omega_{i+1}|b_i|} \leq |c_{i,i}| \leq \frac{1}{|a_i| - \tau_{i-1}|c_{i-1}| - \omega_{i+1}|b_i|}$$

provided that the denominator of the upper bound of is not zero. If A is diagonally dominant by columns then

$$\frac{1}{|a_i| + \alpha_{i-1}|b_{i-1}| + \beta_{i+1}|c_i|} \leq |c_{i,i}| \leq \frac{1}{|a_i| - \alpha_{i-1}|b_{i-1}| - \beta_{i+1}|c_i|}$$

provided that the denominator of the upper bound is not zero.

Proof. First we consider row diagonally dominant matrices. Since $AA^{-1} = I$ we have

$$c_{i-1}c_{i-1,i} + a_i c_{i,i} + b_i c_{i+1,i} = 1.$$

Thus

$$|c_{i-1}||c_{i-1,i}|| + |b_i||c_{i+1,i}| \geq |1 - a_i c_{i,i}|.$$

Using Theorem 3.1 we obtain

$$|1 - a_i c_{i,i}| \leq (\tau_{i-1}|c_{i-1}| + \omega_{i+1}|b_i|)|c_{i,i}|.$$

Hence we get the desired bounds. Similarly we get (3.3). □

In the following we establish some lower bounds for the entries of the inverse.

Theorem 3.3 *Let A be a nonsingular tridiagonal matrix of the form* (2.1)*. Let $A^{-1} = C = [c_{i,j}]$. If A is diagonally dominant by rows then*

$$\begin{array}{rcll} \displaystyle\prod_{k=i+1}^{j} \frac{|c_{k-1}|}{|a_k| + |b_k|}|c_{i,i}| & \leq & |c_{i,j}| & \text{for } i < j \\ \displaystyle\prod_{k=j}^{i-1} \frac{|b_k|}{|a_k| + |c_{k-1}|}|c_{i,i}| & \leq & |c_{i,j}| & \text{for } i > j \end{array}$$

Proof. Let $A^{r,s}$ $s \geq r$ be the square submatrix of order $s - r + 1$ of A whose diagonal entries are $a_{r,r}, a_{r+1,r+1}, \ldots, a_{s,s}$. It is shown in [SJ] that

$$\frac{|c_{i,j}|}{|c_{i,i}|} = \prod_{k=i+1}^{j} |c_{k-1}| \prod_{p=i+1}^{j} \frac{|\det A^{(p+1,n)}|}{|\det A^{(p,n)}|},$$

where $\det A^{(n+1,n)} = 1$. Next we consider τ_1 and μ_1 as defined in (3.1) and (2.8). For the matrix $A^{(p,n)}$ we have $\tau_1 = \mu_1$. Thus we can use Lemma 2.1, (2.10), for $A^{(p,n)}$ and get for $i < j$

$$\prod_{k=i+1}^{j} |c_{k-1}| \prod_{p=i+1}^{j} \frac{1}{|a_p| + |b_p|} \leq \frac{|c_{i,j}|}{|c_{i,i}|}$$

Similarly we get the desired result for $i > j$. $\square$

Note that the Theorems 3.1-3.3 depend on the values τ_i and ω_i. Moreover, the bounds established in these Theorems are related to each other, i.e. Theorem 3.1 gives bounds for the off-diagonal entries related to the diagonal entries. This information is used then to obtain bounds for the diagonal entries. Thus, if we have some other estimates for the relation of $c_{i,i}$ with $c_{i,i+1}$ and $c_{i+1,i}$ we also get some other bounds for the diagonal entries.

In the following we show how the upper bounds for the off-diagonal entries and diagonal entries can be improved iteratively.

We have shown inductively in the proof of Theorem 3.1 that $\tau_{i-1}|u_i| \geq |u_{i-1}|$. But for the induction step we have only used the fact $|u_i| \geq |u_{i-1}|$. Hence we would get

$$\begin{aligned} |u_i| &= \left|\frac{a_{i-1}u_{i-1} + c_{i-2}u_{i-2}}{b_{i-1}}\right| \geq \frac{|a_{i-1}u_{i-1}| - |c_{i-2}u_{i-2}|}{|b_{i-1}|} \\ &\geq \frac{|a_{i-1}||u_{i-1}| - |c_{i-2}||u_{i-2}|}{|b_{i-1}|} \geq \frac{|a_{i-1}| - \tau_{i-2}|c_{i-2}|}{|b_{i-1}|}|u_{i-1}|. \end{aligned}$$

Thus, if we define

$$\tau_{i-1,2} := \frac{|b_{i-1}|}{|a_{i-1}| - \tau_{i-2}|c_{i-2}|},$$

we obtain $\tau_{i-1,2}|u_i| \geq |u_{i-1}|$. Of course this procedure can be repeated. Hence, if we define

$$\begin{aligned} \tau_{i,1} &= \tau_i \\ \tau_{i,j} &= \begin{cases} \tau_{i,j-1} & \text{for } \; i < j \\ \frac{|b_i|}{|a_i| - \tau_{i-1,j-1}|c_{i-1}|} & \text{else,} \end{cases} \end{aligned}$$

$$\begin{aligned} \omega_{i,1} &= \omega_i \\ \omega_{i,j} &= \begin{cases} \omega_{i,j-1} & \text{for } \; n-j+1 < i \\ \frac{|c_{i-1}|}{|a_i| - \omega_{i+1,j-1}|b_i|} & \text{else,} \end{cases} \end{aligned}$$

$$
\begin{aligned}
\alpha_{i,1} &= \alpha_i \\
\alpha_{i,j} &= \begin{cases} \alpha_{i,j-1} & \text{for } \; i < j \\ \frac{|c_i|}{|a_i| - \alpha_{i-1,j-1}|b_{i-1}|} & \text{else,} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\beta_{i,1} &= \beta_i \\
\beta_{i,j} &= \begin{cases} \beta_{i,j-1} & \text{for } \; n-j+1 < i \\ \frac{|b_{i-1}|}{|a_i| - \beta_{i+1,j-1}|c_i|} & \text{else,} \end{cases}
\end{aligned}
$$

we get

Theorem 3.4 *Let A be a nonsingular tridiagonal matrix of the form* (2.1)*. Let $A^{-1} = C = [c_{i,j}]$. If A is diagonally dominant by rows then for each $t = 1, \ldots, n-1$ we have*

$$
\begin{aligned}
|c_{i,j}| &\leq |c_{j,j}| \prod_{k=i}^{j-1} \tau_{k,t} \quad \textit{for } i < j \\
|c_{i,j}| &\leq |c_{j,j}| \prod_{k=j+1}^{i} \omega_{k,t} \quad \textit{for } i > j.
\end{aligned}
$$

If A is diagonally dominant by columns then for each $t = 1, \ldots, n-1$ we have

$$
\begin{aligned}
|c_{i,j}| &\leq |c_{i,i}| \prod_{k=i}^{j-1} \beta_{k,t} \quad \textit{for } i < j \\
|c_{i,j}| &\leq |c_{i,i}| \prod_{k=j+1}^{i} \alpha_{k,t} \quad \textit{for } i > j.
\end{aligned}
$$

Proof. The proof follows immediately from the proof of Theorem 3.1 and the remarks before Theorem 3.4. □

Moreover, we obtain for the diagonal entries

Theorem 3.5 *Let A be a nonsingular tridiagonal matrix of the form* (2.1)*. Let $A^{-1} = C = [c_{i,j}]$. If A is diagonally dominant by rows then for $t = 1, \ldots, n-1$*

$$
\frac{1}{|a_i| + \tau_{i-1,t}|c_{i-1}| + \omega_{i+1}|b_i|} \leq |c_{i,i}| \leq \frac{1}{|a_i| - \tau_{i-1,t}|c_{i-1}| - \omega_{i+1,t}|b_i|}
$$

provided that the denominator of the upper bound of is not zero. If A is diagonally dominant by columns then for $t = 1, \ldots, n-1$

$$\frac{1}{|a_i| + \alpha_{i-1,t}|b_{i-1}| + \beta_{i+1}|c_i|} \leq |c_{i,i}| \leq \frac{1}{|a_i| - \alpha_{i-1,t}|b_{i-1}| - \beta_{i+1,t}|c_i|}$$

provided that the denominator of the upper bound of is not zero.

Proof. The proof follows the proof of Theorem 3.2 but uses Theorem 3.4 instead of Theorem 3.1. □

Note that for row diagonally dominant tridiagonal matrices which satisfies (2.4) we have $\tau_{i,j} < 1$. Thus the sequence $\{\tau_{i,j}\}_{j=1}^{n}$ and the sequence $\{\omega_{i,j}\}_{j=1}^{n}$ are decreasing, i.e.

$$\tau_{i,j} \leq \tau_{i,j-1} \quad \text{and} \quad \omega_{i,j} \leq \omega_{i,j-1} \tag{3.3}$$

Hence the bounds $\tau_{i,j}$ and $\omega_{i,j}$ become sharper in each step. The same holds for the $\alpha_{i,j}$ and $\beta_{i,j}$ for column diagonally dominant matrices. Moreover, in the M–matrix case the $\tau_{i-1,j}$ gives the exact ratio between u_i and u_{i-1} for all $i \leq j$. Thus, if we compute the $\tau_{i,j}$ and $\omega_{i,j}$ for $j = 1, \ldots, n-1$, Theorem 3.4 with $t = n-1$ gives exact bounds for the off-diagonal entries. If we then use Theorem 3.5 we get the inverse of A.

Hence, this gives another way to compute the inverse of a tridiagonal matrix rather then using the recurrence formulas for the u_i, v_i, x_i and y_i theirselves. The advantage of this new procedure is that the $\tau_{i,j}$ and $\omega_{i,j}$ remain bounded. In contrast the sequences u_i, v_i, x_i and y_i can have a huge dynamic range. Thus, computing the inverse by using the recurrence formulas often yields to overflow and underflow problems, see Higham ([Hi], p. 303).

For arbitrary diagonally dominant matrices the bounds $\tau_{i,j}$ and $\omega_{i,j}$ become also better in each step. But they do not yield to the exact inverse of the matrix. However, Theorem 3.4 and Theorem 3.5 with $t = n-1$ give the exact inverse of the comparison matrix $\mathcal{M}(A) = [m_{i,j}]$ of A, which is given by

$$\begin{aligned} m_{i,j} &= -|a_{i,j}| \quad \text{for } i \neq j \\ m_{i,i} &= |a_{i,i}|. \end{aligned}$$

Then the well-known result by Ostrowski [O1] says that

$$|A^{-1}| \leq (\mathcal{M}(A))^{-1}. \tag{3.4}$$

Thus our bounds cannot improve (3.4). But using (3.4) one still needs bounds for the entries of the comparison matrix of A which is an M–matrix.

From Theorem 3.4 we can easily obtain decay rates of the entries of the inverse. If we define for $t = 1, \dots, n-1$

$$\rho_{1,t} = \max \tau_{i,t}, \quad \rho_{2,t} = \max \omega_{i,t}, \quad \rho_{3,t} = \max \alpha_{i,t}, \quad \rho_{3,t} = \max \beta_{i,t},$$

we get

Corollary 3.6 *Let A be a nonsingular tridiagonal matrix of the form* (2.1)*. Let $A^{-1} = C = [c_{i,j}]$. If A is diagonally dominant by rows then for each $t = 1, \dots, n-1$ we have*

$$\begin{aligned} c_{i,j} &\leq \rho_{1,t}^{(j-i)} c_{j,j} \quad \textit{for } i < j \\ c_{i,j} &\leq \rho_{2,t}^{(i-j)} c_{j,j} \quad \textit{for } i > j. \end{aligned}$$

If A is diagonally dominant by columns and satisfies (2.5) *then for each $t = 1, \dots, n-1$ we have*

$$\begin{aligned} c_{i,j} &\leq \rho_{3,t}^{(j-i)} c_{i,i} \quad \textit{for } i < j \\ c_{i,j} &\leq \rho_{4,t}^{(i-j)} c_{i,i} \quad \textit{for } i > j. \end{aligned}$$

In the following we consider tridiagonal Toeplitz matrices, i.e. matrices of the form

$$A = \begin{bmatrix} a & b & & & \\ c & a & b & & \\ & \ddots & \ddots & \ddots & \\ & & c & a & b \\ & & & c & a \end{bmatrix}, \quad c, b \neq 0 \tag{3.5}$$

with

$$|a| \geq |b| + |c|. \tag{3.6}$$

Obviously, we have for these matrices

$$\begin{aligned} \tau_{i,j} = \tau_{i+1,j} = \beta_{i,j} = \beta_{i+1,j} \quad &\text{for all } j \\ \omega_{i,j} = \omega_{i+1,j} = \alpha_{i,j} = \alpha_{i+1,j} \quad &\text{for all } j. \end{aligned}$$

Hence we will only consider the τ_i and ω_i in the following.

For Toeplitz matrices one can calculate the $\tau_{i,j}$ and $\omega_{i,j}$ for j directly as well the limits of the sequences $\{\tau_{i,j}\}_{j=1}^{\infty}$ and $\{\omega_{i,j}\}_{j=1}^{\infty}$.

Lemma 3.7 *Let $a, b, c \in \mathbb{R}^+$ and let*

$$x_1 = \frac{b}{a}, \quad x_{i+1} = \frac{b}{a - x_i c} \qquad i = 1, 2 \dots \tag{3.7}$$

Then

$$x_i = \frac{r_+^i - r_-^i}{r_+^{i+1} - r_-^{i+1}},$$

where

$$r_\pm = \frac{a}{2b} \pm \sqrt{\frac{a^2}{4b^2} - \frac{c}{b}}.$$

Proof. Similarly to the proof of Lemma 2.5 of [M] we transform (3.7) into a linear difference equation. Let

$$x_i = \frac{\beta_{i-1}}{\beta_i}.$$

Then we obtain the recurrence relation

$$\beta_{i+1} - \frac{a}{b}\beta_i + \frac{c}{b}\beta_{i-1} = 0 \quad \text{with} \quad \beta_0 = b \;\; \beta_1 = a.$$

The solution of this difference equations is given by (see [He]):

$$\beta_i = c_1 r_+^{i+1} + c_2 r_-^{i+1}.$$

Using the initial conditions we obtain $c_1 + c_2 = 0$. Therefore

$$\beta_i = c_1 (r_+^{i+1} - r_-^{i+1}).$$

Thus,

$$x_i = \frac{r_+^i - r_-^i}{r_+^{i+1} - r_-^{i+1}}.$$

□

Lemma 3.8 *Let $a, b, c \in \mathbb{R}^+$ with $a \geq b + c$. Let*

$$x_1 = \frac{b}{a}, \quad x_{i+1} = \frac{b}{a - x_i c} \qquad i = 1, 2 \ldots$$

Then

$$\lim x_i = \tilde{x}_-,$$

where

$$\tilde{x}_- = \frac{a}{2c} - \sqrt{\frac{a^2}{4c^2} - \frac{b}{c}}$$

Proof. Since $x_1 < 1$ we easily get $x_{i+1} < x_i$ for $i \geq 2$. The solutions of the fixedpoint equation

$$x = \frac{b}{a - xc}$$

are

$$\tilde{x}_\pm = \frac{a}{2c} \pm \sqrt{\frac{a^2}{4c^2} - \frac{b}{c}}.$$

Obviously, $\tilde{x}_+ > 1$ and $\tilde{x}_- < 1$. Moreover, we have

$$\tilde{x}_- = \frac{b}{a - \tilde{x}_- c} \leq \frac{b}{a - c} = x_0.$$

Similarly, we show inductively that $\tilde{x}_- \leq x_i$ for all i. Thus the sequence of the x_i is bounded. Hence, the limit exists and is $\tilde{x}_-$. □

We have seen for arbitrary diagonally dominant tridiagonal matrices that

$$\tau_{i,j} \geq \tau_{i,j+1} \qquad \omega_{i,j} \geq \omega_{i,j+1}.$$

For Toeplitz matrices we show a relation between these values for i and $i + 1$.

Lemma 3.9 *Let A be as in* (3.5) *and let $\tau_{i,j}$ and $\omega_{i,j}$ as in* (3.1). *Then*

$$\tau_{i,j} \leq \tau_{i+1,j}, \qquad \omega_{i,j} \leq \omega_{i+1,j}.$$

Proof. For the Toeplitz matrix A we have

$$\tau_{i,1} = \tau_{i+1,1} = \frac{|b|}{|a| - |c|}, \qquad \omega_{i,1} = \omega_{i+1,1} = \frac{|c|}{|a| - |b|}.$$

Since $\tau_{i,j} \le 1$ we get

$$\tau_{i,j} = \frac{|b|}{|a| - \tau_{i-1,j-1}|c|} \le \frac{|b|}{|a| - \tau_{i,j-1}|c|} = \tau_{i+1,j}.$$

Similarly, we get the desired result for $\omega_{i,j}$. □

Summarizing we obtain for diagonally dominant tridiagonal Toeplitz matrices

Theorem 3.10 *Let A be as in (3.5) a nonsingular diagonally dominant tridiagonal Toeplitz matrix. Then*

$$\tau_{i,j} = \frac{r_+^j - r_-^j}{r_+^{j+1} - r_-^{j+1}} \quad \textit{for } i \ge j, \quad \omega_{i,j} = \frac{s_+^j - s_-^j}{s_+^{j+1} - s_-^{j+1}} \quad \textit{for } i \le j$$

where

$$r_\pm = \frac{a}{2b} \pm \sqrt{\frac{a^2}{4b^2} - \frac{c}{b}}, \qquad s_\pm = \frac{a}{2c} \pm \sqrt{\frac{a^2}{4c^2} - \frac{b}{c}}.$$

Moreover,

$$\lim_{j\to\infty} \tau_{i,j} = s_-, \qquad \lim_{j\to\infty} \omega_{i,j} = r_-$$

The decay rate can be given as

$$\begin{aligned} |c_{i,j}| &\le \tau_{n-1,n-1}^{j-i} |c_{j,j}| \quad \textit{for } i \le j, \\ |c_{i,j}| &\le \omega_{n-1,n-1}^{i-j} |c_{j,j}| \quad \textit{for } i \ge j. \end{aligned} \tag{3.8}$$

or

$$\begin{aligned} |c_{i,j}| &\le x_-^{j-i} |c_{j,j}| \quad \textit{for } i \le j, \\ |c_{i,j}| &\le y_-^{i-j} |c_{j,j}| \quad \textit{for } i \ge j. \end{aligned} \tag{3.9}$$

Proof. Theorem 3.10 follows immediately from the Lemmata above. □

Note that the decay rates in (3.8) are sharp at least for some entries. Moreover, the decay rates in (3.9) are approximatively best possible.

4 Comparisons and Examples

In this section we compare our bounds theoretically with some other well-known bounds. Moreover, we give some numerical examples.

For tridiagonal matrices A of the form (2.1) the values μ_i defined in (2.8) are just

$$\mu_i = \frac{|c_{i-1}| + |b_i|}{|a_i|}.$$

Since

$$\tau_i = \frac{|b_i|}{|a_i| - |c_{i-1}|}, \qquad \omega_i = \frac{|c_{i-1}|}{|a_i| - |b_i|},$$

we obtain for diagonally dominant matrices

$$\tau_i \leq \mu_i, \qquad \omega_i \leq \mu_i.$$

Thus Theorem 3.1 improves the upper bound given by Ostrowski in [O2], i.e. (2.9) for strictly diagonally dominant matrices.

Shivakumar and Ji established in [SJ] (Theorem 2.3) the following Theorem

Theorem 4.1 *Let A be a nonsingular row diagonally dominant tridiagonal matrix. Let $C = [c_{i,j}] = A^{-1}$. Then*

$$\begin{aligned} \frac{\prod_{k=i+1}^{j} |c_{k-1}|}{\prod_{k=i}^{j} |a_k|(1+\mu_k)} \leq |c_{i,j}| \leq \frac{\prod_{k=i+1}^{j} \mu_k}{|a_i| - |c_{i-1}|\mu_{i-1} - |b_i|\mu_{i+1}}, \quad i \leq j, \\ \frac{\prod_{k=j}^{i-1} |b_k|}{\prod_{k=j}^{i} |a_k|(1+\mu_k)} \leq |c_{i,j}| \leq \frac{\prod_{k=j}^{i-1} \mu_k}{|a_i| - |c_{i-1}|\mu_{i-1} - |b_i|\mu_{i+1}}, \quad i \geq j. \end{aligned}$$

Since $\tau_i \leq \mu_i$ and $\omega_i \leq \mu_i$ we immediately obtain that the upper bounds given in Theorem 3.1 or Theorem 3.4 with $t = 1$ combined with Theorem 3.2 improve the upper bounds due to Shivakumar and Ji. Note that we get even better bounds with $t > 1$.

For comparing the lower bounds we have

$$\frac{1}{|a_k|(1+\mu_k)} = \frac{1}{|a_k| + |c_{k-1}| + |b_k|} \leq \frac{1}{|a_k| + |b_k|},$$

and

$$\frac{1}{|a_i|+|b_i|+|c_{i-1}|} \leq \frac{1}{|a_i|+\omega_{i+1}|b_i|+\tau_{i-1}|c_{i-1}|}.$$

Hence we have improved the bounds due to Shivakumar and Ji.

Kershaw gives in [K] upper and lower bounds for the entries of the inverse of matrices A arising in spline approximation, i.e.

$$A = \begin{bmatrix} \lambda_1 & 1-\alpha_1 & & & \\ \alpha_2 & \lambda_2 & 1-\alpha_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \alpha_{n-1} & \lambda_{n-1} & 1-\alpha_{n-1} \\ & & & \alpha_n & \lambda_n \end{bmatrix},$$

where $0 < \alpha_i < 1$ and $\lambda_i > 1$ for all i. It was shown in [SJ] that the upper bounds in Theorem 4.1 improve the bounds given by Kershaw. Hence our upper bounds also improve Kershaw's bounds.

In [RW] Robinson and Wathen establish bounds using some information on the spectrum of the matrix. An example in [RW] shows that in general neither the bounds due to Robinson and Wathen improve Kershaw's bounds nor the other way. Thus our bounds are not worse then the bounds in [RW].

In the following we consider some examples for different matrices A and compare the entries of A^{-1} with our bounds. Let $UB = [ub_{i,j}]$ denote the matrix whose entries are the upper bounds for the entries of $C = [c_{i,j}] = A^{-1}$. Similarly, we define $LB = [lb_{i,j}]$ containing the lower bounds. Then define $e_u := \max_{i,j}\{ub_{i,j} - c_{i,j}\}$ and $e_l := \max_{i,j}\{c_{i,j} - lb_{i,j}\}$.

Example 1. Let $A = toplitz(-1, 4, -1) \in \mathbb{R}^{100\times 100}$, i.e. A is of the form (3.5) with $a = 4, b = c = -1$. The following table gives the maximal error for the upper bounds for different numbers t of iterations as in Theorem 3.4 and Theorem 3.5.

t	1	2	3	4	5
e_u	0.0230	0.0016	$0.0283 * 10^{-3}$	$0.7668 * 10^{-5}$	$0.5613 * 10^{-6}$

We can see that we get good results with just a few number of iterations. The maximal error of the lower bounds is $e_l = 0.0614$.

Example 2. Let $A = toplitz(-0.9, 5, -1.1) \in \mathbb{R}^{100\times 100}$. For this matrix Eijkhout amd Polman gives in [EP] decay rates which are the same as ours.
We obtain

t	1	2	3	4
e_u	0.0104	$4.4358 * 10^{-4}$	$1.91060 * 10^{-5}$	$8.2320 * 10^{-7}$

Thus also for this nonsymmetric matrix just a few number of iterations gives good upper bounds. The error of the lower bounds is $e_l = 0.0325$

Example 3. Here we consider a random tridiagonal nonsymmetric M-matrix $A = [a_{i,j}] \in \mathbb{R}^{100\times 100}$ which is diagonally dominant by rows. A is constructed as

$$a_{i,i+1} = -rand \qquad a_{i+1,i} = -rand, \qquad a_{i,i} = -a_{i,i-1} - a_{i,i+1} + rand,$$

where $rand$ is the random function of Matlab which generates numbers in the interval $(0, 1)$. Thus, the diagonal dominance of this matrix is not very large. We get

t	1	2	3	4	5	6
e_u	5.2309	1.8224	0.2380	0.0257	0.0044	$5.7763 * 10^{-4}$

As expected the error in each iteration is larger than the error in the previous Examples. But again, for a small number of iterations we get good bounds. The error of the lower bounds is $e_l = 2.0385$.

Example 4. Again we consider a random matrix 100×100 which is constructed similarly to the one of Example 3. However the signs of the off-diagonal entries are also random. i.e. we do not have an M–matrix anymore. Moreover we have

$$a_{i,i} = -a_{i,i-1} - a_{i,i+1} + 2.$$

For the error $UB - |A^{-1}|$ we get

t	1	2	3	4	5
$max_{i,j}\{ub_{i,j} - \lvert c_{i,j}\rvert\}$	0.0690	0.0629	0.0627	0.0627	0.0627

Thus we observe an almost stagnation of the error rounded to four digits. But the maximal error between $|A^{-1}|$ and the comparison matrix $\mathcal{M}(A)$ is just $d = 0.0627$. Thus we have

t	1	2	3
$max_{i,j}\{ub_{i,j} - \lvert c_{i,j}\rvert - d\}$	0.006314	$1.4669 * 10^{-4}$	$2.2602 * 10^{-6}$

The error of the lower bounds is $max_{i,j}\{|c_{i,j}| - lb_{i,j}\} = 0.1227$.

5 Conclusion

We have established upper and lower bounds for the entries of the inverses of diagonally dominant tridiagonal matrices. These bounds improve several known bounds. Moreover, our upper bounds can be improved iteratively. For $n \times n$ M–matrices this iterative refinement yields the inverse after $n-1$ steps. However, our numerical examples show that one needs just a few iterations to get good bounds. For arbitrary diagonal dominant tridiagonal matrices we obtain after $n-1$ iterations the inverse of the comparison matrix. We have also shown how our bounds can be computed easily for tridiagonal Toeplitz matrices.

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