High dimensional knots corresponding to the Fractional Fibonacci groups *

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Abstract

We prove that the natural HNN-extensions of the Fractional Fibonacci groups are fundamental groups of high-dimensional knot complements. We give also some characterization and interpretation of these knots. In particular we shall show that some of them are 2-knots.

1 Introduction

The *fractional Fibonacci groups* $F^{k/l}(2,m)$ were introduced in [6] for integers k, l, m such that $m \geq 3$ and $(k,l) = 1$:

$$F^{k/l}(2,m) \;=\; \langle a_1, a_2, \ldots, a_m \mid a_i^l \, a_{i+1}^k \;=\; a_{i+2}^l, \quad i = 1, \ldots, m \rangle \qquad (1)$$

with all indices reduced module m. In the case $k = l = 1$ we get the well-known Fibonacci groups $F(2,m)$ introduced by J. Conway in 1965. The topological and geometric properties of three-dimensional closed orientable manifolds uniformized by groups $F(2,2n)$ and by groups $F^{k/l}(2,2n)$ were

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studied in [2] and [6], respectively. Almost all of these groups (and manifolds) are hyperbolic. Concerning the case m odd, Helling has pointed out to us that the groups $F(2, m)$ have torsion, and it was shown in [7] that these groups cannot be fundamental groups of hyperbolic 3-manifolds or hyperbolic 3-orbifolds.

The investigation of HNN-extensions of the Fibonacci groups as high-dimensional knot groups was started by J. Hillman in [3], where he considered the natural extension of the group $F(2, 6)$ that is the fundamental group of the Euclidean Hantzsche-Wendt manifold. This studying was continued in [11], where it was proven that the same HNN extension of the Fibonacci groups $F(2, m)$ also arise as high-dimensional knot groups. In the present paper we shall demonstrate that natural HNN-extensions of the Fractional Fibonacci groups $F^{k/l}(2, m)$ are high-dimensional knot groups if and only if $k = 1$.

We recall that a group G is said to be an *n-dimensional knot group*, for $n \geq 1$, if $G = \pi_1(S^{n+2} \setminus K(S^n))$ for some n-dimensional knot $K : S^n \to S^{n+2}$. Kervaire [10, section 11D] obtained the following algebraic characterization of such groups: *A group G is a 3-knot group (and so, n-knot group for any $n \geq 3$) if and only if it is finitely presentable, $H_1(G, \mathbb{Z}) = \mathbb{Z}$, $H_2(G, \mathbb{Z}) = 0$ and G has weight 1 (that is G is the normal closure of some single element).* In dimensions $n = 1, 2$ the above conditions are necessary but no longer sufficient. Our studying of the group extensions of $F^{k/l}(2, m)$ will be based on this criterion and on Zeeman's twist spun construction of knots [12].

2 The natural HNN-extension

The group $F^{k/l}(2, m)$ with the presentation (1) is an example of a cyclically presented group in the sense of [5]. We recall that a group G is *cyclically presented* if for some n and w it has a presentation

$$G = \langle x_1, x_2, \ldots, x_m \mid w, \eta(w), \ldots \eta^{m-1}(w) \rangle, \tag{2}$$

where $\eta : \mathbb{F}_m \to \mathbb{F}_m$ is the automorphism of the free group $\mathbb{F}_m = \langle x_1, \ldots, x_m \rangle$ of rank m given by $\eta(x_i) = x_{i+1}, i = 1, \ldots, m$, and $w \in \mathbb{F}_m$ is a cyclically reduced word. The fractional Fibonacci groups with the presentation (1) arise in this construction for $w = x_1^l x_2^k x_3^{-l}$.

Obviously, η induces an automorphism $\Phi : G \to G$ given by $\Phi(x_i) = x_{i+1}, i = 1, \ldots, m$. Let us define a *natural HNN-extension* $\mathcal{G}$ of a cyclically presented group G:

$$\mathcal{G} = \{G, t \mid tgt^{-1} = \Phi(g), \quad g \in G\}. \tag{3}$$

Thus for the group $F^{k/l}(2,m)$ with the presentation (1) we consider the automorphism $\Phi : F^{k/l}(2,m) \to F^{k/l}(2,m)$ given by $\Phi(a_i) = a_{i+1}, i = 1, \ldots, m$, and the natural HNN-extension

$$\mathcal{F}_m^{k/l} = \{F^{k/l}(2,m), t \mid tft^{-1} = \Phi(f), \quad f \in F^{k/l}(2,m)\} \tag{4}$$

which is defined for the same k, l, m as the group $F^{k/l}(2,m)$ did. We have

Theorem. *The group $\mathcal{F}_m^{k/l}$ is a 3-knot group if and only if $k = 1$.*

Proof: We shall check the conditions of Kervaire's criterion.

(1) First we prove that Φ is a *meridional* automorphism if and only if $k = 1$; in other words that the normal closure $A^{k/l}(2,m)$ in $F^{k/l}(2,m)$ of $\{f^{-1}\Phi(f) | f \in F^{k/l}(2,m)\}$ is $F^{k/l}(2,m)$ only for $k = 1$ (see [3, page 123]). In fact, there is an obvious epimorphism

$$h : F^{k/l}(2,m) \to \mathbb{Z}_k = \langle \gamma \mid \gamma^k = 1 \rangle$$

given by $h(a_i) = \gamma$ for all i. This epimorphism induces an epimorphism of the abelianization:

$$h^{ab} : F^{k/l}(2,m)^{ab} \to \mathbb{Z}_k.$$

Thus $H_1(\mathcal{F}_m^{k/l}) = \mathbb{Z}$ if and only if $k = 1$. So, from now we will assume that $k = 1$ in our considerations.

(2) We shall demonstrate that $\mathcal{F}_m^{1/l}$ is the normal closure of the element $b = t^{-1}a_1$. In fact,

$$t^{-1}a_2 = t^{-1}(ta_1t^{-1}) = a_1t^{-1} = t(t^{-1}a_1)t^{-1} = tbt^{-1},$$

and for $i = 1, \ldots, m$ we get

$$t^{-1}a_i = t^{-1}(t^{i-1}a_1t^{-(i-1)}) = t^{i-2}a_1t^{-(i-1)} = t^{i-1}(t^{-1}a_1)t^{-(i-1)} = t^{i-1}bt^{-(i-1)}$$

Let $\mathcal{B}$ be the normal closure of b in $\mathcal{F}_m^{1/l}$. Consider the canonical projection

$$\rho : \mathcal{F}_m^{1/l} \to \mathcal{F}_m^{1/l}/\mathcal{B},$$

and denote $\rho(a) = \bar{a}$ for $a \in \mathcal{F}_m^{1/l}$. Then $\bar{a_1} = \ldots = \bar{a_m} = \bar{t}$. For any i we get $\bar{a_i}^l \bar{a_{i+1}} = \bar{a_{i+2}}^l$, hence $\bar{t} = e$ is the neutral element. Therefore $\bar{a_1} = \ldots = \bar{a_m} = \bar{t} = e$ and $\mathcal{F}_m^{1/l} = \mathcal{B}$. Thus $\mathcal{F}_m^{1/l}$ has weight 1.
(3) From the short exact sequence of groups

$$1 \to F^{k/l}(2,m) \to \mathcal{F}_m^{k/l} \to \mathbb{Z} \to 1$$

we have the Hochschild–Serre spectral sequence $E^2_{p,q} = H_p(\mathbb{Z}, H_q(F^{k/l}(2,m)))$, [1, p. 171]. So, it is enough to prove that

$$E^2_{2,0} = H_2(\mathbb{Z}, H_0(F^{k/l}(2,m))) = 0,$$

$$E^2_{1,1} = H_1(\mathbb{Z}, H_1(F^{k/l}(2,m))) = H_1(F^{k/l}(2,m))_{\mathbb{Z}} = 0,$$

and

$$E^2_{0,2} = H_0(\mathbb{Z}, H_2(F(2,n))) = 0.$$

The first equality is obvious. For the proof of the next we shall use the Hopf's formula to get a 5-term exact sequence [1, p. 47]

$$H_2(\mathcal{F}_m^{k/l}) \to H_2(\mathbb{Z}) \to H_1(F^{k/l}(2,m))_{\mathbb{Z}} \to H_1(\mathcal{F}_m^{k/l}) \to H_1(\mathbb{Z}) \to 0,$$

where the $\mathbb{Z}$-action on $H_1(F^{k/l}(2,m))$ is induced by the conjugation action of $\mathcal{F}_m^{k/l}$ on $F^{k/l}(2,m)$. We have $H_2(\mathbb{Z}) = 0$, $H_1(\mathbb{Z}) = \mathbb{Z}$. Suppose $k = 1$, then by (1), $H_1(\mathcal{F}_m^{1/l}) = \mathbb{Z}$. Thus

$$0 \to H_1(F^{1/l}(2,m))_{\mathbb{Z}} \to \mathbb{Z} \to \mathbb{Z} \to 0,$$

and

$$H_1(F^{1/l}(2,m))_{\mathbb{Z}} = 0.$$

For the abelianization $F^{k/l}(2,m)^{ab}$ of the group $F^{k/l}(2,m)$ the following property holds.

Lemma. *For any k, l, m the group $F^{k/l}(2,m)^{ab}$ is a finite group of order*

$$|F^{k/l}(2,m)^{ab}| = d_m^{k/l} = \begin{cases} c_m^{k/l} + l^2\, c_m^{k/l}, & m \text{ - odd}, \\ c_m^{k/l} + l^2\, c_{m-2}^{k/l} - 2l^m, & m \text{ - even}, \end{cases} \tag{5}$$

where

$$c_m^{k/l} = k\, c_{m-1}^{k/l} + l^2\, c_{m-2}^{k/l}, \tag{6}$$

with $c_0^{k/l} = 1$ *and* $c_1^{k/l} = k$.

Proof The lemma is obvious by direct computation of the determinant of the exponential sum matrix [8].

$\square$

In particular, from (5) and (6) we get

$$d_5^{k/l} \;=\; k^5 \;+\; 5k^3l^2 \;+\; 5kl^4,$$

and

$$d_6^{k/l} \;=\; k^6 \;+\; 6k^4l^2 \;+\; 9k^2l^4.$$

Turn back to the proof of the theorem. We remark that $H_2(F^{k/l}(2,m)) = 0$. In fact, from Hopf's formula [1, p. 46] the number of generators of the group $H_2(F^{k/l}(2,m))$ is equal to $r - g + w$. Here g is the number of generators, r is the number of relations of the group $F^{k/l}(2,m)$, and $w = rank(H_1(F^{k/l}(2,m)))$. In our case $g = r = m$, and by the lemma above, $w = 0$. So, $H_2(F^{k/l}(2,m)) = 0$ and $E^2_{0,2} = 0$. Summing up we have shown that $H_2(\mathcal{F}_m^{1/l}) = 0$.

$\square$

3 The fibred 2-knots

As we remarked in the introduction the Kervaire conditions of the high dimensional knot groups are also necessary when $n = 1$ or 2, but are then no longer sufficient. But it is proven in [4, p. 34] that if a group G is a torsion free 3-knot group such that G' is the fundamental group of a closed orientable 3-manifold M whose factors are Haken, hyperbolic or Seifert fibred, then G is the group of a fibred 2-knot with closed fibre M.

We recall [6] that a fractional Fibonacci group $F^{1/l}(2,2n)$, $l \geq 1$, $n \geq 3$, is the fundamental group of a closed orientable 3-manifold $\mathcal{M}_n^{1/l}$, which can be obtained as n-fold cyclic covering of the two-bridge knot $(2l + \frac{1}{2l})$. The manifolds $\mathcal{M}_n^{1/l}$ are hyperbolic with one exceptional case $\mathcal{M}_3^1$ which is the Euclidean Hantzsche-Wendt manifold which is Seifert fibred. Thus we have the following corollary of the above theorem.

Proposition 1. *For* $l \geq 1$ *and* $n \geq 3$ *the group* $\mathcal{F}_{2n}^{1/l}$ *is a fibred 2-knot group with closed fibre* $\mathcal{M}_n^{1/l}$.

Recall that by the definition (4) the automorphism Φ corresponding to the group $\mathcal{F}_{2n}^{1/l}$ is of order $2n$.

We will show that there are other HNN-extensions of the group $F^{1/l}(2,2n)$. Indeed, the n-fold cyclic covering $\mathcal{M}_n^{1/l}$ of the 2-bridge knot $(2l+\frac{1}{2l})$ is induced by the automorphism $\Psi : F^{1/l}(2,2n) \to F^{1/l}(2,2n)$ given by $\Psi(a_i) = a_{i+2}, i = 1,\ldots,2n$.

Let us define an HNN-extension

$$\mathcal{G}_n^{l,p} = \{F^{1/l}(2,2n), t \mid tft^{-1} = \Psi^p(f), \quad f \in F^{1/l}(2,2n)\}. \tag{7}$$

By the general construction of a twist spun knot ([9], [10, section 3L]) we get that for $0 < p < n$, $(p,n) = 1$, the group $\mathcal{G}_n^{l,p}$ is the fundamental group of a p-twist spun of a $(2l+\frac{1}{2l})$-knot, which can be described as a fibred knot with fibre $punc(\mathcal{M}_n^{1/l})$. We denote this 2-knot by $\mathcal{K}_n^{l,p}$. Here $punc(\mathcal{M}_n^{1/l})$ denotes a manifold $\mathcal{M}_n^{1/l}$ minus an open ball around the branch set. As a corollary of this considerations, we get

Proposition 2. *For $n \geq 3$, $l \geq 2$, $0 < p < n$, $(p,n) = 1$ the group $\mathcal{G}_n^{l,p}$ is a fibred 2-knot group with fiber $punc(\mathcal{M}_n^{1/l})$.*

For integers p, q satisfying $0 < p, q < n$, $(p,n) = (q,n) = 1$ we can distinguish knots $\mathcal{K}_n^{l,p}$ and $\mathcal{K}_n^{l,q}$ in virtue of [9]. The groups $\mathcal{G}_n^{l,p}$ and $\mathcal{G}_n^{l,q}$ are isomorphic if and only if Ψ^p is conjugate to $\Psi^{\pm q}$ in $Out(F^{1/l}(2,2n))$. The knot exteriors are homeomorphic only if Ψ^p is conjugate to $\Psi^{\pm q}$ in $Aut(F^{1/l}(2,2n))$.

We remark that the extension (7) also can be regarded as a natural HNN-extension of a cyclically presented group. Indeed, if we suppose $y_i = a_{2i}$, for $i = 1,\ldots,n$, then $a_{2i+1} = y_i^{-l} y_{i+2}^l$. Therefor we get the following cyclic presentation for the fractional Fibonacci group

$$F^{1/l}(2,2n) = \langle y_1,\ldots,y_n \mid (y_i^{-l} y_{i+1}^l)^l y_{i+1} = (y_{i+1}^{-l} y_{i+2}^l)^l, i = 1,\ldots,n\rangle, \tag{8}$$

and the automorphism $\Psi(y_i) = y_{i+1}$. Thus the group $\mathcal{G}_n^{l,1}$ is the natural HNN-extension of the cyclically presented group (8).

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