On the zeta function of quaternionic Shimura varieties

by

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Introduction

This paper is a supplement to the monograph [9]. We consider the Shimura variety Sh_D associated with the multiplicative group $D^\times$ of a quaternion division algebra D over a totally real number field F. It is defined over a number field $E(D) \subset \mathbb{C}$ and carries a continuous right action of the group $D^\times(\mathbb{A}_f)$ such that, for every open compact subgroup $C \subset D^\times(\mathbb{A}_f)$, $Sh_{D,C} = Sh_D/C$ is a projective variety over $E(D)$. The cohomology groups $H^i_C = \mathrm{H}^i(Sh_{D,C}(\mathbb{C}),\mathbb{C})$ form an inductive system with respect to C, and their limit H^i is a module over the Hecke algebra $\mathcal{H} = \mathcal{H}(D^\times(\mathbb{A}_f))$. There is a canonical decomposition

$$H^i_C = \bigoplus_{\pi\in\mathcal{A}^\infty} H^i(\pi)\otimes\pi_f^C\ , \quad H^i(\pi) = \mathrm{Hom}_{\mathcal{H}}(\pi_f\,,H^i)\,, \tag{0}$$

where $\mathcal{A}^\infty$ denotes the set of isomorphism classes of automorphic representations $\pi = \pi_f\otimes\pi_\infty$ of $D^\times$ such that π_∞ has trivial central and infinitesimal characters, and π_f^C is the subspace of invariants of C in π_f. The dimension of $H^i(\pi)$ can be computed using Lie algebra cohomology. One obtains

$$\dim H^i(\pi) = \begin{cases} \binom{d}{j} & \text{if } \dim\pi = 1 \text{ and } i = 2j \text{ for } j\in\{0,\dots,d\}\,;\\ 2^d & \text{if } \dim\pi = \infty \text{ and } i = d\,;\\ 0 & \text{otherwise}\,; \end{cases}$$

where d denotes the dimension of $Sh_{D,C}$. We assume that $d > 0$.

Let E be the completion of $E(D)$ at a prime $\wp$, and let $I_E \subset W_E \subset \mathrm{Gal}(\overline{E}|E)$ be the inertia group and the Weil group, respectively. We choose a rational prime $\ell\in\mathbb{Z}\smallsetminus\wp$ and an embedding $\mathbb{Q}_\ell\to\mathbb{C}$. Via the canonical isomorphisms

$$H^i_C \cong \mathrm{H}^i(Sh_{D,C}\otimes_{E(D)}\overline{E}\,,\mathbb{Q}_\ell)\otimes_{\mathbb{Q}_\ell}\mathbb{C}\,,$$

the continuous actions of $\mathrm{Gal}(\overline{E}|E)$ on the ℓ–adic cohomology groups induce algebraic representations of the Weil-Deligne group $W'_E = \mathbb{C}\rtimes W_E$ on H^i_C. Passing to the limit, we obtain an action of $W'_E\times\mathcal{H}$ on H^i and, for every $\pi\in\mathcal{A}^\infty$, an algebraic representation of W'_E on the 2^d–dimensional $\mathbb{C}$–vector space

$$H(\pi) = \begin{cases} \bigoplus_{i=0}^d H^{2i}(\pi) & \text{if } \dim\pi = 1\,;\\ H^d(\pi) & \text{if } \dim\pi = \infty\,. \end{cases}$$

With any algebraic representation τ of W'_E, one associates its L–function

$$L(\tau,s) = \det(\,1 - q^{-s}\tau(\Phi)|_{\tau^{\mathrm{nr}}}\,)^{-1}\,,$$

where q is the number of elements in the residue class field of E, $\Phi \in W_E$ is a geometric Frobenius element, and $\tau^{\mathrm{nr}} \subseteq \tau$ denotes the subspace of invariants of $\mathbb{C} \times I_E \subset W'_E$. The decomposition (0) of H^i_C yields the expression

$$Z_\wp(Sh_{D,C}, s) = \prod_{\pi \in \mathcal{A}^\infty} L(H(\pi), s)^{m(\pi,C)}$$

for the local factor of the zeta function at $\wp$, where

$$m(\pi, C) = m(\pi) \dim \pi_f^C \ , \quad m(\pi) = \begin{cases} 1 & \text{if } \dim \pi = 1\,; \\ (-1)^d & \text{if } \dim \pi = \infty\,. \end{cases}$$

On the other hand, one can attach to any $\pi \in \mathcal{A}^\infty$ a unique isomorphism class of automorphic representations π^* of $\mathrm{Gl}_2(F)$ (considered as an algebraic group over $\mathbb{Q}$) which has the same central character as π and satisfies the following condition. If π has dimension 1, then π^* has dimension 1 and is given by the same character of $F^\times(\mathbb{A})$. If π is infinite-dimensional, then π^* is cuspidal and, for every prime p such that $D \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{M}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$, the local factors π_p and π_p^* are isomorphic.

Let $\overline{E(D)} \subset \mathbb{C}$ be the algebraic closure of $E(D)$, and recall that there is a natural 2^d–dimensional representation ϱ° of the common L–group

$${}^L D^\times_{E(D)} = \widehat{D^\times} \rtimes \mathrm{Gal}(\overline{E(D)}\,|\,E(D))$$

of $D^\times$ and $\mathrm{Gl}_2(F)$ (cf. [9], §11). π and π^* as above give rise to unique $\widehat{D^\times}$–conjugacy classes of homomorphisms

$$\varphi_\wp(\pi) \text{ and } \varphi_\wp(\pi^*) \colon W'_E \longrightarrow {}^L D^\times_E = \widehat{D^\times} \rtimes \mathrm{Gal}(\overline{E}\,|\,E)\,.$$

If $\dim \pi = 1$, then $\mathbb{C} \subset \ker(\varphi_\wp(\pi^*))$ and $\varphi_\wp(\pi^*)|_{W_E} = \varphi_\wp(\pi)|_{W_E}$, and if $\dim \pi = \infty$, then $\varphi_\wp(\pi^*) = \varphi_\wp(\pi)$. We extend the natural embedding $E(D) \to E$ to $\overline{E(D)} \to \overline{E}$ and consider the local factors

$$L_\wp(\pi, \varrho^\circ, s) = L(\varrho^\circ \circ \varphi_\wp(\pi), s) \ , \quad L_\wp(\pi^*, \varrho^\circ, s) = L(\varrho^\circ \circ \varphi_\wp(\pi^*), s)$$

of the L–functions attached to π and π^*, respectively.

Conjecture 1 *For every* $\pi \in \mathcal{A}^\infty$, $L(H(\pi), s) = L_\wp(\pi^*, \varrho^\circ, s - \frac{d}{2})$.

If $C \subset D^\times(\mathbb{A}_f)$ is an open compact subgroup such that conjecture 1 holds for all $\pi \in \mathcal{A}^\infty$ with $\pi_f^C \neq 0$, then

$$Z_\wp(Sh_{D,C}, s) = \prod_{\pi \in \mathcal{A}^\infty} L_\wp(\pi^*, \varrho^\circ, s - \tfrac{d}{2})^{m(\pi,C)}\,.$$

There is a related problem which we shall discuss now. For every algebraic representation τ of W'_E, let $\mathrm{gr}(\tau)$ denote the corresponding representation on the graded module associated with the monodromy filtration of τ, and define the semi-simple L–function of τ by

$$L^{\mathrm{ss}}(\tau, s) = L(\mathrm{gr}(\tau), s)\,.$$

For every $\pi \in \mathcal{A}^\infty$, we consider the local factor

$$L^{\mathrm{ss}}_\wp(\pi, \varrho^\circ, s) = L^{\mathrm{ss}}(\varrho^\circ \circ \varphi_\wp(\pi), s)$$

of the semi-simple L–function attached to π (and π^*). Then

$$L_\wp(\pi^*, \varrho^\circ, s) = \begin{cases} L^{\mathrm{ss}}_\wp(\pi, \varrho^\circ, s) & \text{if } \dim \pi = 1\,; \\ L_\wp(\pi, \varrho^\circ, s) & \text{if } \dim \pi = \infty\,. \end{cases}$$

Conjecture 2 *For every* $\pi \in \mathcal{A}^\infty$, $L^{\mathrm{ss}}(H(\pi), s) = L^{\mathrm{ss}}_\wp(\pi, \varrho^\circ, s - \frac{d}{2})$.

In the present paper, we assume that $\wp$ divides a rational prime p which is unramified in F. Then $E|\mathbb{Q}_p$ is unramified, too. Let

$$\mathcal{A}^{\infty,p} = \{\, \pi \in \mathcal{A}^\infty \mid \pi_p \text{ unramified} \,\}\,.$$

We prove that Conjecture 2 holds for all $\pi \in \mathcal{A}^{\infty,p}$. This is a stronger version of Theorem 11.6 in [9]. In fact, let $C \subset D^\times(\mathbb{A}_f)$ be a sufficiently small open compact subgroup containing a maximal compact subgroup of $D^\times(\mathbb{Q}_p)$, and consider the local factor

$$Z^{\mathrm{ss}}_\wp(Sh_{D,C}, s) = \prod_{i=0}^{2d} L^{\mathrm{ss}}(H^i_C, s)^{(-1)^i}$$

of the semi-simple zeta function. Since all $\pi \in \mathcal{A}^\infty$ with $m(\pi, C) \neq 0$ are contained in $\mathcal{A}^{\infty,p}$, the decomposition (0) of H^i_C and the present result imply

$$Z^{\mathrm{ss}}_\wp(Sh_{D,C}, s) = \prod_{\pi \in \mathcal{A}^\infty} L^{\mathrm{ss}}_\wp(\pi, \varrho^\circ, s - \tfrac{d}{2})^{m(\pi,C)}\,.$$

In §2, we discuss the modifications of the proof in [9] that are necessary to yield the stronger result. In particular, we have to replace the Lefschetz fixed point formula by a more general trace formula. For this we use a variant of a theorem of Pink [6] which is explained in §1.

In §3, we show further that Conjecture 1 holds for all $\pi \in \mathcal{A}^{\infty,p}$ if the monodromy filtration on H^i_C is pure of weight i for every $i \in \{0, \ldots, 2d\}$ and every sufficiently small open compact subgroup $C \subset D^\times(\mathbb{A}_f)$ containing a maximal compact subgroup of $D^\times(\mathbb{Q}_p)$. For this we follow the sketch of a proof of Theorem 4.4 in [7].

§1. A trace formula for unramified correspondences

Let κ be a finite field, X and B two schemes of finite type over κ, and

$$b = (b_1, b_2) \colon B \longrightarrow X \times_\kappa X$$

a morphism of κ–schemes such that b_1 factors through the Frobenius endomorphism of X over κ and b_2 is unramified. Let $B^\Delta \to B$ denote the pull-back of the diagonal morphism of X over κ with respect to b.

Lemma 1.1 *B^Δ is finite over κ.*

Proof. The assumptions on b imply that the natural map $B^\Delta \to X$ is unramified and factors through the Frobenius endomorphism of B^Δ over κ. Hence the latter is unramified, too. Since B^Δ is of finite type over κ, this implies the claim. ∎

Let ℓ be a prime which is different from the characteristic of κ and Λ a commutative ring which is finite over $\mathbb{Z}_\ell$ or $\mathbb{Q}_\ell$. We fix an algebraic closure $\bar\kappa|\kappa$ and use a bar to denote base change from κ to $\bar\kappa$. We consider an object M of the category $D^b_{ctf}(\overline{X}_{et}, \Lambda)$ (cf. [4], 1.1.2–3) and assume that X and B are proper over κ and that there is a sequence $(\, f_i \colon W_i \to X \mid i = 0, \ldots, m\,)$ of morphisms of κ–schemes such that f_0 is surjective and, for every $i \in \{0, \ldots, m\}$, the following conditions are satisfied:

W_i is a proper smooth κ–scheme;

if Y_i denotes the scheme theoretic image of f_i and $Y_{m+1} = \varnothing$, then $Y_{i+1} \subset Y_i$, f_i is an isomorphism over $U_i = Y_i \smallsetminus Y_{i+1}$, and $\overline{f_i^{-1}(Y_{i+1})} \subset \overline{W}_i$ is a union of smooth divisors with, at most, normal crossings;

$b_2^{-1}(Y_i) \subseteq b_1^{-1}(Y_i)$;

for every $j \in \mathbb{Z}$, $\mathcal{H}^j(M)|_{\overline{U}_i}$ is constant.

For any cohomological correspondence

$$u \colon \bar b_{2*} \bar b_1^* M \longrightarrow M$$

with support in $\bar b$, let u_* denote the resulting endomorphism of $\mathrm{R}\Gamma(\overline{X}, M)$, and for every geometric point $\beta \in B^\Delta(\bar\kappa)$, let u_β denote the endomorphism of $M_{b_1(\beta)}$ induced by u.

Theorem 1.2 *If b, M, and u are as above, then*

$$\mathrm{Tr}\,(u_*) = \sum_{\beta \in B^{\Delta}(\bar{\kappa})} \mathrm{Tr}\,(u_\beta)\,.$$

Proof. The assertion follows by a slight modification of the proof of Theorem 7.2.2 in [6]. Our correspondences $\bar{b}$ and u play the roles of $b^{(1)}$ and $u^{(1)}$, Lemma 1.1 replaces Lemma 7.1.2 (loc. cit.), and the morphisms f_i considered above provide the data of type 7.2.1 (loc. cit.). Since our b_2 is assumed to be unramified, formula (c) on page 516 (loc. cit.) holds with $d = 1$. Hence the assertion of Proposition 7.3.4 (loc. cit.) is valid for $n = 1$. Moreover, since X and B are proper and all geometric data are defined over κ, we do not need the considerations on page 518 (loc. cit.). The application of Pink's argument with these modifications shows that, in our situation, the bound m_0 in Theorem 7.2.2 (loc. cit.) may be replaced by 1. ∎

§2. Semi-simple L–functions

We consider the situation described in the introduction. In particular, E denotes the completion of the field of definition $E(D)$ of the Shimura variety Sh_D at a prime $\wp$. We assume that $\wp$ divides a rational prime p which is unramified in the center F of D, and that the set $\mathcal{Z}$ of infinite places of F at which D is split is not empty. $d = \#\mathcal{Z}$ is the dimension of Sh_D. The fixed embedding $\overline{E(D)} \to \overline{E}$ induces a map from the set of infinite places of F onto the set of primes of F over p. Let d' be the number of elements in $\mathcal{Z}$ which are mapped to a prime at which D is ramified. We fix an algebraic closure $\bar{\kappa}$ of the residue class field κ of E, and a maximal compact subgroup $C_p \subset D^\times(\mathbb{Q}_p)$. Let

$$Sh_D^p = Sh_D / C_p \otimes_{E(D)} E\,.$$

We will say that an open subgroup $C \subset D^\times(\mathbb{A}_f^p)$ is sufficiently small if there is an O_F–order $O_D \subset D$ and an integer $N > 2$ such that $C_p \times C \subseteq (1 + NO_D \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}})^\times$.

In part I of [9] we constructed a flat O_E–scheme $\widetilde{Sh}_D^p$ together with a continuous action of $D^\times(\mathbb{A}_f^p)$ and an equivariant isomorphism

$$\widetilde{Sh}_D^p \otimes_{O_E} E \cong Sh_D^p$$

such that, for every sufficiently small open subgroup $C \subset D^\times(\mathbb{A}_f^p)$,

$$\widetilde{Sh}_{D,C}^p = \widetilde{Sh}_D^p / C$$

is a normal projective O_E–scheme. The most important properties of these schemes are summarized in Theorem 6.6 (loc. cit.). For the application in the present paper, we need the following supplement to this theorem.

Proposition 2.1 (i) *If $C' \subseteq C \subset D^\times(\mathbb{A}_f^p)$ are sufficiently small open subgroups, then the canonical morphism*

$$\widetilde{Sh}^p_{D,C'} \longrightarrow \widetilde{Sh}^p_{D,C}$$

is étale.

(ii) *For every integer $i \in \{0, \ldots, d'\}$, there are a κ–scheme W_i together with a continuous action of $D^\times(\mathbb{A}_f^p)$ and an equivariant morphism*

$$f_i \colon W_i \longrightarrow \widetilde{Sh}^p_D \otimes_{O_E} \kappa$$

such that f_0 is surjective and, for the scheme theoretic image Y_i of f_i (and $Y_{d'+1} = \varnothing$), the following conditions are satisfied: Y_{i+1} is contained in Y_i, f_i is an isomorphism over $U_i = Y_i \smallsetminus Y_{i+1}$, and, for every sufficiently small open subgroup $C \subset D^\times(\mathbb{A}_f^p)$, W_i/C is a smooth projective κ–scheme of pure dimension $d - i$,

$$\overline{f_i^{-1}(Y_{i+1})}/C \subset \overline{W}_i/C$$

is a union of smooth divisors with, at most, normal crossings, and the restrictions of the sheaves of vanishing cycles to $\overline{U}_i/C \subseteq \widetilde{Sh}^p_{D,C} \otimes_{O_E} \overline{\kappa}$ are constant.

Proof. It is clear from the construction of Sh^p_D in [9] (§6) that (i) follows from the analogous statement concerning the schemes $\widetilde{Sh}_{G,C}$ constructed in [9] (§2). Let $R \to R_0$ be a surjective homomorphism of strictly henselian local O_E–algebras. By Proposition 2.14 (loc. cit.), points of $\widetilde{Sh}_{G,C}$ with values in these rings are given by the functor $\underline{\mathcal{M}}_{G,C}$ introduced in Definition 2.12 (loc. cit.). Let $C' \subseteq C \subset G(\mathbb{A}_f)$ be subgroups which satisfy condition (2.11) (loc. cit.). We have to show that the natural map

$$\underline{\mathcal{M}}_{G,C'}(R) \longrightarrow \underline{\mathcal{M}}_{G,C}(R) \underset{\underline{\mathcal{M}}_{G,C}(R_0)}{\times} \underline{\mathcal{M}}_{G,C'}(R_0)$$

is bijective. Let $\mathfrak{a}_0 = (A_0, \iota_0, \Lambda_0, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(R_0)$ be the image of any given point $\mathfrak{a} = (A, \iota, \Lambda, \overline{\eta}) \in \underline{\mathcal{M}}_{G,C}(R)$, and consider a point $\mathfrak{a}_0' \in \underline{\mathcal{M}}_{G,C'}(R_0)$ lifting $\mathfrak{a}_0$. It is represented by $(A_0, \iota_0, \Lambda_0, \overline{\eta}')$ with $\overline{\eta}' \subseteq \overline{\eta}$, and $\mathfrak{a}' = (A, \iota, \Lambda, \overline{\eta}')$ is an element of $\underline{\mathcal{M}}_{G,C'}(R)$ which is mapped to the pair $(\mathfrak{a}, \mathfrak{a}_0')$. Any $\mathfrak{a}'' \in \underline{\mathcal{M}}_{G,C'}(R)$ which has this property is represented by $(A, \iota, \Lambda, \overline{\eta}'')$ for some $\overline{\eta}'' \subseteq \overline{\eta}$, and there is an $\alpha_0 \in \operatorname{Aut} \mathfrak{a}_0$ such that $\widehat{\mathrm{V}}^p(\alpha_0) \circ \overline{\eta}' = \overline{\eta}''$. Lemma 2.13 (loc. cit.) implies that the natural map

$$\operatorname{Aut} \mathfrak{a} \longrightarrow \operatorname{Aut} \mathfrak{a}_0$$

is an isomorphism. Hence α_0 can be lifted to an isomorphism $\alpha \colon \mathfrak{a}' \to \mathfrak{a}''$. This completes the proof of (i).

For the proof of (ii), recall the notion of an admissible set introduced in [9] (p. 46). By Theorem 6.6 (loc. cit.), every such set Z gives rise to a $D^\times(\mathbb{A}_f^p)$–invariant closed subscheme

$$Sh^p_{D,Z} \subseteq \widetilde{Sh}^p_D \otimes_{O_E} \overline{\kappa}$$

such that, for every sufficiently small open subgroup $C \subset D^\times(\mathbb{A}_f^p)$, $Sh^p_{D,Z}/C$ is a smooth projective $\overline{\kappa}$–scheme of pure dimension $d(Z)$ with $d-d' \leqslant d(Z) \leqslant d$. These subschemes are permuted by the action of the Frobenius automorphism over κ. Hence, for every $i \in \{0,\dots,d'\}$ there is a κ–scheme W_i together with a continuous action of $D^\times(\mathbb{A}_f^p)$ and an equivariant isomorphism

$$\overline{W}_i \cong \coprod_Z Sh^p_{D,Z}\,,$$

where Z runs through the set of all admissible sets with $d(Z) = d-i$. Moreover, there are natural equivariant morphisms $f_i\colon W_i \to \widetilde{Sh}^p_D \otimes_{O_E} \kappa$. That these morphisms satisfy the conditions in (ii) is a consequence of Theorem 6.6 (loc. cit.). In particular, the fact that the sheaves of vanishing cycles are constant on $\overline{U}_i/C$ (which has already been used in the proof of assertion (v) of that theorem) is obtained using the results of [8] (§3). ∎

For every sufficiently small open subgroup $C \subset D^\times(\mathbb{A}_f^p)$, we consider the spectral sequence

$$E_2^{i,k} = \mathrm{H}^i(\widetilde{Sh}^p_{D,C} \otimes_{O_E} \overline{\kappa}\,, \mathrm{R}^k\Psi_C) \Longrightarrow \mathrm{H}^{i+k}(Sh_{D,C_p\times C} \otimes_{E(D)} \overline{E}\,, \mathbb{Q}_\ell)\,,$$

where $\mathrm{R}^k\Psi_C$ denotes the k–th $\mathbb{Q}_\ell$–sheaf of vanishing cycles on the geometric special fibre of $\widetilde{Sh}^p_{D,C}$. These spectral sequences are $\mathrm{Gal}(\overline{E}|E)$–equivariant and form an inductive system (with respect to C) which is compatible with the action of $D^\times(\mathbb{A}_f^p)$. In particular, we obtain a $W_E' \times \mathcal{H}_C^p$–equivariant filtration $F^\bullet$ on $H^i_{C_p\times C}$, where $\mathcal{H}_C^p = \mathcal{H}(D^\times(\mathbb{A}_f^p)/\!/C)$. The inertia group I_E acts trivially on the sheaves of vanishing cycles (cf. [9], Theorem 6.6). Hence I_E acts trivially on $\mathrm{gr}^{F^\bullet}(H^i_{C_p\times C})$. The image of this module in the Grothendieck group of finite dimensional $\mathbb{C}$–vector spaces with an action of $W_E' \times \mathcal{H}_C^p$ does not change if $F^\bullet$ is replaced by the monodromy filtration (cf. [9], Lemma 8.2 and its proof). For every $f \in \mathcal{H}_C^p$ and every $j > 0$, let

$$\mathrm{Tr}^{\mathrm{ss}}_C(f,j) = \sum_{i=0}^{2d} (-1)^i \,\mathrm{Tr}\left(\Phi^j \times f \;\middle|\; \mathrm{gr}(H^i_{C_p\times C})\right).$$

This generalizes the semi-simple traces considered in [9] (§8). Recall the definition of the functions $f_p^{(j)} \in \mathcal{H}(D^\times(\mathbb{Q}_p)/\!/C_p)$ and $f_\infty \in C_c^\infty(D^\times(\mathbb{R})/F^\times(\mathbb{R}))$ in [9] (§10). Then $f^{(j)} = f_\infty \cdot f \cdot f_p^{(j)} \in C_c^\infty(D^\times(\mathbb{A})/F^\times(\mathbb{R}))$ acts on the space $\mathcal{A}$ of automorphic forms on $D^\times F^\times(\mathbb{R})\backslash D^\times(\mathbb{A})$. We will modify the arguments in [9] (§7–10) to obtain the following generalization of Proposition 10.9 (loc. cit.).

Proposition 2.2 *For every $f \in \mathcal{H}_C^p$ and every $j > 0$, $\mathrm{Tr}_C^{\mathrm{ss}}(f,j) = \mathrm{Tr}(f^{(j)}|\mathcal{A})$.*

Proof. Using the spectral sequence of vanishing cycles we obtain

$$\mathrm{Tr}_C^{\mathrm{ss}}(f,j) = \sum_{i,k=0}^{\infty} (-1)^{i+k}\, \mathrm{Tr}\left(\Phi^j \times f \;\middle|\; \mathrm{H}^i(\widetilde{Sh}^p_{D,C} \otimes_{O_E} \overline{\kappa}, \mathrm{R}^k\Psi_C) \otimes_{\mathbb{Q}_\ell} \mathbb{C}\right).$$

We normalize the Haar measure on $D^\times(\mathbb{A}_f^p)$ such that $\mathrm{vol}(C) = 1$, and denote the Frobenius endomorphism of $\widetilde{Sh}^p_{D,C} \otimes_{O_E} \kappa$ over κ by Frob. For every $\xi \in D^\times(\mathbb{A}_f^p)$, we consider the geometric correspondence

$$b_\xi^{(j)} = (\mathrm{Frob}^j \circ b_{\xi,1}\,,\, b_{\xi,2})\colon \widetilde{Sh}^p_{D,C^\xi} \otimes_{O_E} \kappa \longrightarrow \widetilde{Sh}^p_{D,C} \otimes_{O_E} \kappa \;\times\; \widetilde{Sh}^p_{D,C} \otimes_{O_E} \kappa\,,$$

where $C^\xi = \xi C \xi^{-1} \cap C$, $b_{\xi,1}$ denotes the morphism induced by the action of ξ, and $b_{\xi,2}$ is the natural projection. Let $u_\xi^{(j)}$ be the cohomological correspondence with support in $\overline{b_\xi^{(j)}}$ on $\mathrm{R}\Psi_C \in D^b_{ctf}\big((\widetilde{Sh}^p_{D,C} \otimes_{O_E} \overline{\kappa})_{et}, \mathbb{Q}_\ell\big)$ which is induced by the canonical isomorphisms

$$(\mathrm{Frob}^j)^*\mathrm{R}\Psi_C \cong \mathrm{R}\Psi_C\,, \quad \bar{b}^*_{\xi,1}\mathrm{R}\Psi_C \cong \mathrm{R}\Psi_{C^\xi} \cong \bar{b}^*_{\xi,2}\mathrm{R}\Psi_C\,,$$

and the trace map

$$(\bar{b}_{\xi,2})_*\bar{b}^*_{\xi,2}\mathrm{R}\Psi_C \longrightarrow \mathrm{R}\Psi_C\,,$$

and let $f_\xi \in \mathcal{H}_C^p$ denote the characteristic function of the double coset $C\xi C$. The actions of $u_\xi^{(j)}$ and $\Phi^j \times f_\xi$ on $\mathrm{H}^i(\widetilde{Sh}^p_{D,C} \otimes_{O_E} \overline{\kappa}, \mathrm{R}^k\Psi_C) \otimes_{\mathbb{Q}_\ell} \mathbb{C}$ coincide. Moreover, by Proposition 2.1, $b_\xi^{(j)}$ and $\mathrm{R}\Psi_C$ satisfy the assumptions on b and M in §1. Hence we may apply Theorem 1.2 and obtain

$$\mathrm{Tr}_C^{\mathrm{ss}}(f_\xi,j) = \sum_{k=0}^{\infty} (-1)^k \sum_\beta \mathrm{Tr}\left((u_\xi^{(j)})_\beta \;\middle|\; (\mathrm{R}^k\Psi_C)_{\bar{b}_{\xi,2}(\beta)}\right), \tag{2.3}$$

where the second sum is taken over the set

$$\{\, \beta \in \widetilde{Sh}^p_{D,C^\xi}(\overline{\kappa}) \mid \mathrm{Frob}^j(\bar{b}_{\xi,1}(\beta)) = \bar{b}_{\xi,2}(\beta) \,\}\,.$$

As in [9] (7.3) this set may be identified with

$$\coprod_{\mathfrak{t}\in\mathfrak{T}_D} \coprod_{h\in\mathfrak{R}(\mathfrak{t})} \Big(H_h(\mathbb{Q})\backslash\big(Y_p(h) \times Y^p(h,\xi)\big)\Big)\,,$$

where $\mathfrak{T}_D$ is defined on page 34 (loc. cit.), $\mathfrak{R}(\mathfrak{t})$, H_h and $Y_p(h)$ are explained on page 60 (loc. cit.), and

$$Y^p(h,\xi) = \{\, d \in D^\times(\mathbb{A}_f^p) \mid d^{-1}hd \in \xi C \,\}/C^\xi\,.$$

Here we have used the obvious equality $F^\times \cap (C_p \times C^\xi) = F^\times \cap (C_p \times C)$. The sets $Y_p(h)$ and the traces occurring in (2.3) can be computed as in [9] (§7) using Theorem 6.6 (loc. cit.). Let $r = j \cdot [E : \mathbb{Q}_p]$, and denote the quotient field of the Witt ring $\mathrm{W}(\mathbb{F}_{p^r})$ by $\mathcal{K}_r$. Let $\Phi_j \colon D^\times(\mathcal{K}_r) \to \mathbb{Q}$ be the map introduced in Proposition 7.7 (loc. cit.) and f'_ξ the characteristic function of ξC. We obtain the following variant of (8.6) (loc. cit.):

$$\mathrm{Tr}^{\mathrm{ss}}_C(f_\xi, j) = \sum_{\mathfrak{t} \in \mathfrak{T}_D} \sum_{h \in \mathfrak{R}'(\mathfrak{t})} \sum_{(d_1, d_2)} \Phi_j(d_1^{-1} \delta \sigma(d_1)) f'_\xi(d_2^{-1} h d_2)\,, \tag{2.4}$$

where the third sum is extended over

$$(d_1, d_2) \in H_h(\mathbb{Q}) \Big\backslash \Big(D^\times(\mathcal{K}_r) / \big(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}_{p^r})\big)^\times \times D^\times(\mathbb{A}_f^p)/C^\xi \Big)\,,$$

$\mathfrak{R}'(\mathfrak{t}) \subset \mathfrak{R}(\mathfrak{t})$ is the subset introduced on page 62 (loc. cit.), and $\delta \in D^\times(\mathcal{K}_r)$ is chosen as in Proposition 7.7 (loc. cit.).

Now we choose a set $\{c_1, \ldots, c_s\} \subset C$ of representatives for the factor space C/C^ξ. Then $\{c_1\xi, \ldots, c_s\xi\}$ is a set of representatives for $C\xi C/C$. This implies that, for any $d \in D^\times(\mathbb{A}_f^p)$, $f_\xi(d^{-1}hd) = 1$ if and only if there is an $i \in \{1, \ldots, s\}$ (which is uniquely determined) such that $f'_\xi((dc_i)^{-1}h(dc_i)) = 1$. Hence, we may replace f'_ξ and C^ξ in (2.4) and the range for the third sum, respectively, by f_ξ and C. Since $\mathcal{H}_C^p$ is generated as a $\mathbb{C}$–vector space by the set of functions $\{\, f_\xi \mid \xi \in D^\times(\mathbb{A}_f^p)\,\}$, we obtain, for every $f \in \mathcal{H}_C^p$,

$$\mathrm{Tr}^{\mathrm{ss}}_C(f, j) = \sum_{\mathfrak{t} \in \mathfrak{T}_D} \sum_{h \in \mathfrak{R}'(\mathfrak{t})} \sum_{(d_1, d_2)} \Phi_j(d_1^{-1} \delta \sigma(d_1)) f(d_2^{-1} h d_2)\,,$$

where the third sum is extended over

$$(d_1, d_2) \in H_h(\mathbb{Q}) \Big\backslash \Big(D^\times(\mathcal{K}_r) / \big(O_D \otimes_{\mathbb{Z}} \mathrm{W}(\mathbb{F}_{p^r})\big)^\times \times D^\times(\mathbb{A}_f^p)/C \Big)\,.$$

This generalizes (8.6) (loc. cit.), and we may proceed as in §8–10 (loc. cit.) to prove the assertion of the proposition. ∎

For the final step in the present section, we need the following lemma. It is, probably, well known. We include a proof since we could not find an appropriate reference. We will use the following notation. For any finite set S of rational primes, let $\mathbb{A}_f^S = \mathbb{Q} \otimes_{\mathbb{Z}} \prod_{\ell \in T} \mathbb{Z}_\ell$, where T denotes the complement of S in the set of all finite places of $\mathbb{Q}$, and for every automorphic representation π of $D^\times$, let π^S denote the representation of $D^\times(\mathbb{A}_f^S)$ which is obtained as the restricted tensor product

$$\pi^S = \bigotimes_{\ell \in T} \pi_\ell\,,$$

where π_ℓ denotes the local factor of π at ℓ.

Lemma 2.5 *Let $\pi_0, \ldots, \pi_n$ be automorphic representations of $D^\times$ such that, for $i \in \{1, \ldots, n\}$, $\pi_i \ncong \pi_0$, let S be a finite set of rational primes, let $C \subset D^\times(\mathbb{A}_f^S)$ be an open compact subgroup, and let $f_0 \in \mathcal{H}_C^S = \mathcal{H}(D^\times(\mathbb{A}_f^S)//C)$ be any function. Then there is a function $f \in \mathcal{H}_C^S$ such that*

$$\pi_i^S(f) = \begin{cases} \pi_i^S(f_0) & \text{if } \ i = 0\,; \\ 0 & \text{if } \ i \in \{1, \ldots, n\}\,. \end{cases}$$

Proof. If π is an automorphic representation of $D^\times$ such that π^S does not have a non-zero fixed vector for C, then $\pi^S(f) = 0$ for all $f \in \mathcal{H}_C^S$. Hence we may assume that $\pi_0^S, \ldots, \pi_n^S$ admit non-zero fixed vectors for C. We choose a finite set S' of rational primes such that $S \subseteq S'$ and

$$C = C' \times C_0\,, \quad C' = \prod_{\ell \in T'} C_\ell\,,$$

where T' denotes the complement of S' in the set of all finite places of $\mathbb{Q}$, for every $\ell \in T'$, $C_\ell \subset D^\times(\mathbb{Q}_\ell)$ is a hyperspecial subgroup, and

$$C_0 \subset G_0 = \prod_{\ell \in S' \smallsetminus S} D^\times(\mathbb{Q}_\ell)$$

is an open compact subgroup. The function f_0 can be expressed in the form

$$f_0 = \sum_{j=1}^{m} f_{0,j} \cdot g_j\,,$$

where $f_{0,j} \in \mathcal{H}_{C'}^{S'}$ and $g_j \in \mathcal{H}(G_0//C_0)$. If $f_j \in \mathcal{H}_{C'}^{S'}$ satisfies the assertion of the lemma with S, C, and f_0 replaced by S', C', and $f_{0,j}$, respectively, then

$$f = \sum_{j=1}^{m} f_j \cdot g_j\,,$$

solves our problem. Hence we may assume that C is a product of hyperspecial subgroups. This implies that $\mathcal{H}_C^S$ is commutative and that, for $i \in \{0, \ldots, n\}$, the set of C–invariant elements in π_i^S is an $\mathcal{H}_C^S$–module which has dimension 1 over $\mathbb{C}$. Let $\mathfrak{m}_i \subset \mathcal{H}_C^S$ be its annihilator. Since $D^\times$ satisfies strong multiplicity one, we have $\pi_i^S \ncong \pi_0^S$ for $i \neq 0$. Since a spherical representation τ of $D^\times(\mathbb{Q}_\ell)$ is uniquely determined by the homomorphism $\omega\colon \mathcal{H}(D^\times(\mathbb{Q}_\ell)//C_\ell) \to \mathbb{C}$ such that, for every $h \in \mathcal{H}(D^\times(\mathbb{Q}_\ell)//C_\ell)$ and every C_ℓ–invariant vector v of τ, $\tau(h)(v) = \omega(h) \cdot v$ (cf. [2], §IV), this implies $\mathfrak{m}_i \neq \mathfrak{m}_0$ for $i \neq 0$. Hence the assertion of the lemma follows from the Chinese remainder theorem. ∎

We are now ready for the main result of the present section.

Theorem 2.6 *For every* $\pi \in \mathcal{A}^{\infty,p}$, $L^{\mathrm{ss}}(H(\pi), s) = L^{\mathrm{ss}}_{\wp}(\pi, \varrho^\circ, s - \frac{d}{2})$.

Proof. Let $S = \{p\}$, and fix a sufficiently small open subgroup $C \subset D^\times(\mathbb{A}_f^p)$ such that π^S admits a non-zero C–fixed vector. If we apply the preceding lemma to the (finite) set Π_C of all elements of $\mathcal{A}^{\infty,p}$ for which there is a non-zero C–fixed vector, then we find a function $f \in \mathcal{H}_C^p$ such that

$$\mathrm{Tr}\left(\pi^S(f)\right) = \mathrm{Tr}\left(\pi_\infty(f_\infty)\right) = m(\pi)$$

and, for all $\widetilde{\pi} \in \Pi_C \smallsetminus \{\pi\}$, $\widetilde{\pi}^S(f) = 0$. This implies that, for every $j > 0$,

$$\mathrm{Tr}^{\mathrm{ss}}_C(f, j) = \mathrm{Tr}^{\mathrm{ss}}(H(\pi), j)\ , \quad \mathrm{Tr}(f^{(j)} | \mathcal{A}) = \mathrm{Tr}\left(\pi_p(f_p^{(j)})\right),$$

where, for every algebraic representation τ of W'_E,

$$\mathrm{Tr}(\tau, j) = \mathrm{Tr}(\tau(\Phi^j)_{|\tau^{\mathrm{nr}}})\ , \quad \mathrm{Tr}^{\mathrm{ss}}(\tau, j) = \mathrm{Tr}(\mathrm{gr}(\tau), j)\,.$$

For the first of these equations, we have used the decomposition (0) of $H^i_{C_p \times C}$, and for the second, the fact that π occurs in $\mathcal{A}$ with multiplicity 1. Together with Proposition 2.2 above and [9] (Lemma 11.4) they imply

$$\mathrm{Tr}^{\mathrm{ss}}(H(\pi), j) = q^{jd/2}\, \mathrm{Tr}^{\mathrm{ss}}_{\wp}(\pi, \varrho^\circ, j)$$

for all $j > 0$, where

$$\mathrm{Tr}^{\mathrm{ss}}_{\wp}(\pi, \varrho^\circ, j) = \mathrm{Tr}^{\mathrm{ss}}(\varrho^\circ \circ \varphi_{\wp}(\pi), j)\,.$$

The assertion of the theorem follows from these equations using the power series expansions

$$\log L^{\mathrm{ss}}(H(\pi), s) = \sum_{j=1}^{\infty} \mathrm{Tr}^{\mathrm{ss}}(H(\pi), j) \frac{q^{-js}}{j}$$

$$\log L^{\mathrm{ss}}_{\wp}(\pi, \varrho^\circ, s) = \sum_{j=1}^{\infty} \mathrm{Tr}^{\mathrm{ss}}_{\wp}(\pi, \varrho^\circ, j) \frac{q^{-js}}{j}\,. \qquad \blacksquare$$

§3. L–functions and purity

We will say that an automorphic representation $\pi \in \mathcal{A}^\infty$ satisfies the purity condition (with respect to the fixed embedding $\mathbb{Q}_\ell \to \mathbb{C}$) if, for every $i \in \{0, \ldots, 2d\}$, the monodromy filtration on $H^i(\pi)$ is pure of weight i. It follows from the decomposition (0) that this holds if there is an open compact subgroup $C \subset D^\times(\mathbb{A}_f)$ such that $\pi_f^C \neq 0$ and the monodromy filtration on H^i_C is pure of weight i. This is an instance of Deligne's conjecture on the purity of the monodromy filtration.

Theorem 3.1 *If $\pi \in \mathcal{A}^{\infty,p}$ satisfies the purity condition, then*

$$L(H(\pi), s) = L_{\wp}(\pi^*, \varrho^\circ, s - \tfrac{d}{2})\,.$$

Proof. Let τ denote the Φ–semi-simplification of $H(\pi) \otimes |\;|^{-d/2}$ (cf. [3], §8). The representations $\sigma = \varrho^\circ \circ \varphi_{\wp}(\pi)$ and $\sigma^* = \varrho^\circ \circ \varphi_{\wp}(\pi^*)$ are Φ–semi-simple, since $\varphi_{\wp}(\pi)$ and $\varphi_{\wp}(\pi^*)$ are admissible (cf. [1], §8), and Theorem 2.6 implies

$$\mathrm{gr}(\tau) \cong \mathrm{gr}(\sigma) \tag{3.2}$$

since I_E acts trivially on both sides. For τ, this is obvious from the considerations preceding Proposition 2.2, and for σ, this has been observed in [9] (Remark 11.1). Using the purity condition, we will deduce an isomorphism $\tau \cong \sigma^*$ which implies the assertion of the theorem. We have to consider two cases.

Assume first that π is infinite-dimensional. As noted in the introduction, this implies that σ^* and σ are isomorphic. Moreover, since $H(\pi) = H^d(\pi)$, the purity condition implies that, for all $j \in \mathbb{Z}$, $\mathrm{gr}_j(\tau)$ is pure of weight j. If D splits at all places above p, the monodromy filtrations on both sides of (3.2) are trivial by [9] (Remark 11.7). Therefore, we assume that D is ramified at some finite place of F. In this case, we can apply a result of Ohta concerning the Ramanujan-Petersson conjecture (cf. [5], Corollary 3.2.3 with $w = k_1 = \cdots = k_g = 0$). It shows that the local factors of π at all places of F dividing p where D splits are tempered. The local factors at the remaining places dividing p are unitary characters. Using [9] (Remark 11.5), we conclude that, for all $j \in \mathbb{Z}$, $\mathrm{gr}_j(\sigma)$ is pure of weight j. Therefore, the isomorphism (3.2) induces, for every $j \in \mathbb{Z}$, an isomorphism

$$\mathrm{gr}_j(\tau) \cong \mathrm{gr}_j(\sigma)\,.$$

According to [3] (Proposition 8.9) this implies that τ and σ are isomorphic.

Now we assume that $\dim \pi = 1$. In this case, σ^* and $\mathrm{gr}(\sigma)$ are isomorphic, and we have a natural decomposition

$$\tau = \bigoplus_{i=0}^{d} \tau_i \quad \text{with} \quad \dim \tau_i = \tbinom{d}{i}\,.$$

The purity condition implies that, for every $j \in \mathbb{Z}$ and every $i \in \{0, \ldots, d\}$, $\mathrm{gr}_j(\tau_i)$ is pure of weight $2i - d + j$. We will, in fact, show that $\mathrm{gr}_j(\tau_i) = 0$ if $j \neq 0$, i.e. $\mathrm{gr}(\tau) = \tau$.

Recall that in [9] (proof of Lemma 11.4) we have constructed a basis $\{\, y_J \mid J \subseteq \mathcal{Z}\,\}$ of $\mathrm{gr}(\sigma)$ and that, since π is a unitary character, for some $\lambda \in \mathbb{C}^\times$ with $|\lambda| = 1$ and all $J \subseteq \mathcal{Z}$,

$$\Phi y_J = q^{(d-2\#J)/2} \lambda y_{\Phi J}\,.$$

Therefore, for all $k \in \{0,\ldots,d\}$, the subspace σ_k generated by all y_J with $\#J = k$ is pure of weight $d-2k$. Using (3.2), we obtain an isomorphism

$$\sigma_k \cong \bigoplus_{i=0}^{d} \mathrm{gr}_{2(d-k-i)}(\tau_i)\,, \tag{3.3}$$

and, for every $i \in \{0,\ldots,d\}$ and $j \in \mathbb{Z} \smallsetminus \{\,2l \mid l \in \mathbb{Z},\ 0 \leqslant i+l \leqslant d\,\}$, $\mathrm{gr}_j(\tau_i) = 0$. Let $k \in \{0,\ldots,d\}$ be minimal with the property

$$\mathrm{gr}_{2l}(\tau_i) = 0 \quad \text{for all } i \in \{0,\ldots,d\},\ l \in \{1,\ldots,d\} \text{ with } i+l > k\,.$$

In particular, $\tau_i = \mathrm{gr}_0(\tau_i)$ for all $i \geqslant k$ since, for every j, $\dim \mathrm{gr}_{-j}(\tau_i) = \dim \mathrm{gr}_j(\tau_i)$. Using the isomorphism (3.3), we obtain

$$\sum_{i=0}^{k} \dim \mathrm{gr}_{2(k-i)}(\tau_i) \leqslant \dim \sigma_{d-k} = \tbinom{d}{k} = \dim \tau_k = \dim \mathrm{gr}_0(\tau_k)\,.$$

By the minimality of k, this is only possible if $k = 0$, and the theorem is proved. ∎

Errata in [9]

p. 2_{14-13}: Replace "a projective system" by "an inductive system";

p. 84_{14}: Insert "either $\chi_{i,1} = |\ |^{1/2}\chi_i$, $\chi_{i,2} = |\ |^{-1/2}\chi_i$ for an unramified character χ_i of $\mathbb{Q}_p^\times$ such that π_i is the 1–dimensional representation $\chi_i \circ \mathrm{Nm}_{D_{\mathfrak{p}_i}|\mathbb{Q}_p}$, or" after the word "and";

p. 86^3: Replace "$\mathrm{gr}^{F^\bullet} V$" by "$(\mathrm{gr}^{F^\bullet} V)^{I'}$";

p. 86^{9+10}: Replace "$\mathrm{gr}^{F^\bullet} V^\circ$" by "$(\mathrm{gr}^{F^\bullet} V^\circ)^{I'}$";

p. 93^6: Delete a spurious "$=$".

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