Graphs of some CAT(0) complexes

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Abstract. In this note, we characterize the graphs (1–skeletons) of some piecewise Euclidean simplicial and cubical complexes having nonpositive curvature in the sense of Gromov's CAT(0) inequality. Each such cell complex K is simply connected and obeys a certain flag condition. It turns out that if, in addition, all maximal cells are either regular Euclidean cubes or right Euclidean triangles glued in a special way, then the underlying graph $G(K)$ is either a median graph or a hereditary modular graph without two forbidden induced subgraphs. We also characterize the simplicial complexes arising from bridged graphs, a class of graphs whose metric enjoys one of the basic properties of CAT(0) spaces. Additionally, we show that the graphs of all these complexes and some more general classes of graphs have geodesic combings and bicombings verifying the 1– or 2–fellow traveller property.

1. Introduction

The idea of formulating nonpositive curvature for general geodesic metric spaces goes back to work of Alexandrov and Busemann [1, 2, 18]. Gromov [31] introduced the term CAT(0) for simply connected geodesic metric spaces in which the geodesic triangles are thinner than in the Euclidean plane. CAT(0) spaces have many nice equivalent definitions, reflecting different geometric features [6, 16, 17, 47]. A particularly important class of such spaces is formed by CAT(0) simplicial and cubical complexes [16]. Gromov [31] has shown that CAT(0) cubical complexes can be characterized in a purely combinatorial way; see [16, 17, 20, 21, 47] for related results and applications. Further properties of such complexes have been presented in the paper of Sageev [51] (he called them cubings). Niblo and Reeves [44] further investigate the properties of cubings and show how to find a bicombing of a Caley graph of their fundamental groups. On the other hand, Gersten and Short [29, 29] use the CAT(0) singular disk diagrams to provide an authomatic structure for fundamental groups of some two dimensional piecewise Euclidean complexes. Other examples of CAT(0) polygonal complexes were given in [16, 17, 31, 47].

In this article we show that certain CAT(0) simplicial or cubical cell complexes can be characterized by their underlying graphs. In particular, we show that the graphs of cubings are exactly the median graphs. They and the associated cubical complexes have many nice geometric, convexity and algebraic properties already known before [11, 35, 42, 43, 55]. This allows to simplify some results from [51]. Also we characterize the graphs of CAT(0) 2–complexes whose cells are right triangles as graphs in which all isometric cycles have length four and do not contain two forbidden subgraphs. Next we present a characterization of simplicial complexes arising from graphs in which all isometric cycles have length three (so–called bridged graphs). It is known that bridged graphs share one of the main properties of CAT(0) spaces, namely the convexity of the neighbourhoods of convex sets. We present examples of simplicial complexes with regular cells arising from bridged graphs which are not CAT(0). Nethertheless, in few important cases we obtain CAT(0) complexes. An important feature of bridged graphs is that they graph–metric admits a bicombing satisfying the k–fellow traveller condition for $k \leq 2$. This property (for different $k \leq 3$) is shared by all classes of graphs arising in this note, in particular, by folder graphs, median graphs (underlying graphs of cubings), and Helly graphs, a class of graphs containing the Caley graphs fundamental groups of cubings.

The paper is organized as follows. In Section 2 we will establish the notation and definitions used in the rest of the paper. In Section 3 we recall the basic characterizations of CAT(0) complexes; in Section 4 we introduce the classes of graphs and recall their main properties. In Section 5 we briefly disscuss the minimal disks diagrams which are used in the subsequent proofs. In Section 6 we establish the correspondence between the cubings and median complexes. Using this result we derive some properties of cubings. In Section 7 we focus our attention on CAT(0) 2–complexes whose cells are right triangles (we call them folder complexes). We characterize folder complexes and present their basic properties which can be derived from the specificity of their skeletons. In Section 8 we establish that the flag complexes whose graphs are bridged are exactly the simply connected simplicial complexes in which the links of vertices do not contain induced 4–cycles and 5–cycles. Finally, in Section 9 we present the results about combings and bicombings of the underlying graphs of complexes under consideration.

2. Preliminaries

This section summarizes some preliminaries from topology and metric spaces. Basic notions and results from topology can be found in any textbook on combinatorial topology (e.g. [52]).

2.1 Simplicial and cubical complexes

Let K be an abstract *simplicial complex,* i.e., a collection of sets (called *simplexes*) such that $\sigma \in K$ and $\sigma' \subseteq \sigma$ implies $\sigma' \in K$. A *cubical complex* is a set K of cubes of any dimensions which is closed under taking subcubes and nonempty intersections. We denote by $V(K)$ and $E(K)$ the *vertex set* and the *edge set* of K, namely, the set of all 0–simplexes and 1–simplexes of K. The pair $(V(K), E(K))$ is called the *(underlying) graph* or the 1*–skeleton* of K and is denoted by $G(K)$ or K^1 (the underlying graph of a cubical complex is defined similarly). More generally, for each $r \geq 0$ the r*–skeleton* of K is the simplicial complex $K^r = \{\sigma \in K : \#\sigma \leq r+1\}$. A *geometric realization* $|K|$ of K is a polyhedron obtained by the following construction. Let $\phi : V(K) \rightarrow \mathbb{R}^k$ be an injection such that for any two simplexes σ', σ'', $\text{conv}\phi(\sigma') \cap \text{conv}\phi(\sigma'') = \text{conv}\phi(\sigma' \cap \sigma'')$. Then $|K| = \cup\{\text{conv}\phi(\sigma) : \sigma \in K\}$.

The *link* of a simplex σ in K, denoted $\text{link}(\sigma, K)$ is the abstract simplicial complex consisting of all simplexes σ' such that $\sigma \cap \sigma' = \emptyset$ and $\sigma \cup \sigma' \in K$. The *star* $\text{star}(\sigma, K)$ of σ is the set of all simplexes containing σ. (The star of a cube in a cubical complex is defined analogously.) A simplicial complex K is a *flag complex* if any set of vertices is included in a member of K whenever each pair of its vertices is contained in a member of K. In the theory of hypergraphs this condition is called conformality. The link of any simplex in a flag complex is again a flag complex. A flag complex can be recovered by its underlying graph $G(K)$: the cliques of $G(K)$ are exactly the simplexes of K.

A topological space X is *1–connected* if every continuous mapping f of the 1–dimensional sphere S^1 into X can be extended to a continuous mapping of the disk D^2 with boundary S^1 into X. A simplicial complex is *simply connected* if its underlying space $|K|$ is 1–connected. It is well known that K is simply connected if and only if $|K^2|$ is 1–connected.

A *cycle* C is a sequence $(e_1, \ldots, e_m)$ of edges of the graph $G(K)$ such that the edges e_i and $e_{i+1(mod\ m)}$ are incident and $e_{i-1} \cap e_i \neq e_i \cap e_{i+1(mod\ m)}$ for $i = 1, \ldots, m$. If e_i and e_j are incident if and only if $j = i + 1(mod\ m)$ then C is called a *simple cycle* of the 1–skeleton of K. Equivalently, one can define a cycle as a sequence $(v_1, \ldots, v_m, v_1)$ of vertices such that the consecutive vertices are adjacent. If K is simply connected, then every cycle C of $G(K)$ is null–homologous, i.e., it can be written as a modulo 2 sum of 2–simplexes. Moreover, C can be transfered to a point by means of combinatorial deformations; for definitions see for example pp.163–167 of [52].

2.2. Piecewise Euclidean cell complexes

A *Euclidean cell* is a convex polytope in some Euclidean space. By a *piecewise Euclidean* (PE) *cell complex* we shall mean a space $\mathcal{X}$ formed by gluing together Euclidean cells via isometries of their faces, together with the decomposition of $\mathcal{X}$ into cells. Additionally we assume that the intersection of two cells either is empty or a single face of each of cells. If all cells of $\mathcal{X}$ are Euclidean simplexes (respectively, cubes), we will say that $\mathcal{X}$ is a *simplicial* (respectively, *cubical) cell complex*. With each simplicial or cubical complex $\mathcal{X}$ one can associate in a canonical way an abstract simplicial or cubical complex $K = K(\mathcal{X})$. If all cells of $\mathcal{X}$ are regular Euclidean simplexes or cubes (see Sections 7 and 8), then $\mathcal{X} = |K|$. A *piecewise spherical (PS) cell complex* is defined similarly using spherical cells. Such complexes arise naturally as links of cells and points (see [17, 20, 47] for more details).

Suppose that P is a Euclidean cell in $\mathbb{E}^n$ and $x \in P$. Then $\text{Link}(x, P)$ is the subset of the unit sphere S^{n-1} centered at x consisting of all unit vectors which originate in x and pointing into P. If P is a regular n–cube or a regular n–simpex and x is a vertex of P, then $\text{Link}(x, P)$ is a regular spherical simplex on n vertices. The distance between any two vertices of $\text{Link}(x, P)$ is $\pi/2$ in the first case and $\pi/3$ in the second case (recall that the spherical distance between two points u and v is the angle at x between the line segments joining x to u and v respectively). Now suppose that P is a cell in $\mathcal{X}$, $x \in P$, and $f : P \to \mathbb{E}^n$ is a homeomorphism onto a convex cell of $\mathbb{E}^n$. Put $\text{Link}(x, P)$=$\text{Link}(f(x), f(P))$, and $\text{Link}(x, \mathcal{X}) = \cup\ \text{Link}(x, P)$, where the union is taken over all cells P containing x. Then $\text{Link}(x, \mathcal{X})$ is a piecewise spherical cell complex, called the *link* of x in $\mathcal{X}$. If $\mathcal{X}$ is a simplicial PE complex and x is a vertex of K, then the abstract simplicial complex associated with $\text{Link}(x, \mathcal{X})$ coincides with $\text{link}(x, K(\mathcal{X}))$.

If P is a Euclidean cell, F is a k–face of P, then the set of unit vectors normal to the k–plane supported by F and pointing into P form $\text{Link}(F, P)$. If P is a cell of $\mathcal{X}$ and F is a face of P, then set $\text{Link}(F, P)$ =$\text{Link}(f(F), f(P))$ and $\text{Link}(F, \mathcal{X}) = \cup\text{Link}(F, \mathcal{X})$, where the union is taken over all cells containing F (if $F \subseteq Q \subseteq P$, then we identify $\text{Link}(F, Q)$ with the corresponding face of $\text{Link}(F, P)$). The PS complex $\text{Link}(F, \mathcal{X})$ is called the *link of F in $\mathcal{X}$*.

2.3. Metric spaces

Let (X, d) be a metric space. The closed *ball* of center x and radius $r \geq 0$ will be $B_r(x)$.

More generally, the closed *r–neighbourhood* of a set S is the set

$$B_r(S) = \{x \in V : d(x, S) \leq r\},$$

where $d(x,S) = \inf\{d(x,y) : y \in S\}$. A *path* is a map from a segment $[a,b]$ to X. A *geodesic* joining two points $x, y \in X$ is a map g from $[a,b]$ to X such that $g(a) = x, g(b) = y$ and $d(g(t), g(t')) = |t - t'|$ for all $t, t' \in [a,b]$ (in particular $d(x,y) = |a-b|$). A *geodesic segment* in X is the subset that is the image of a geodesic. A *geodesic metric space* is a metric space in which every pair of points can be joined by a geodesic segment. Geodesic metric spaces sometimes are called *convex:* for any points $x, y \in X$ and nonnegative real numbers $r(x)$ and $r(y)$ verifying $d(x,y) \leq r(x) + r(y)$, there is a point z such that $d(x,z) \leq r(x)$ and $d(y,z) \leq r(y)$. More generally, a metric space X is called *hyperconvex* if for any collection of closed balls in X, $B_{r_i}(x_i)$, $i \in I$, satisfying the condition that $d(x_i, x_j) \leq r_i + r_j$ for all $i, j \in I$, the intersection $\cap_{i \in I} B_{r_i}(x_i)$ is nonempty, i.e., the family of balls of X has the Helly property. The notion of hyperconvex spaces has been introduced by Aronszajn and Panitchpakdi [4] who proved that hyperconvex spaces are exactly the absolute retracts (alias injective spaces). To be more precise, here are the basic notions: a metric space (X, d) is *isometrically embedded* into a metric space (Y, d') if there is a map $h : X \to Y$ such that $d'(h(x), h(y)) = d(x,y)$ for all $x, y \in X$. In this case we say that X is a *subspace* of Y and that Y is an *extension* of X. Now, a *retraction* $h : Y \to X$ from a metric space (Y, d') to a subspace X is an idempotent ($h(x) = x$ for any $x \in X$) nonexpansive ($d'(h(x), h(y)) \leq d'(x,y)$ for any $x, y \in Y$) map; its image X is called a *retract* of Y. A metric space (X, d) is an *absolute retract* if X is a retract of every metric space in which X embeds isometrically.

The *interval* $I(x,y)$ between two points x and y is the set

$$I(x,y) = \{z \in X : d(x,y) = d(x,z) + d(z,y)\}.$$

A subset $Y \subseteq X$ is *convex* if Y includes every interval $I(x,y)$ with x, y in Y. A convex set Y with a convex complement $X - Y$ is called a *halfspace*. A subset S of a metric space is called *gated* [26] if for any point $x \notin Y$ there exists a (unique) point $x_Y \in Y$ (the *gate* for x in Y) such that $d(x,z) = d(x, x_Y) + d(x_Y, z)$ for all $z \in Y$. Evidently, every gated set is convex. One can easily show by induction that the family of gated sets has Helly property: if $S_1, \ldots, S_n$ is a collection of pairwise intersecting gated sets then $\cap_{i=1}^n S_i \neq \emptyset$. Pick a common vertex x_j of $S_1, \ldots, S_{j-1}, S_{j+1}, \ldots, S_n$, and let g be the gate of x_j in S_j. Since $g \in I(x_j, x_i)$ for all $x_i \neq x_j$, we conclude that $g \in \cap_{i=1}^n S_i$.

We conclude by recalling the following important notion. Fix a point b and consider a partial order $\leq_b$ on X defined as follows:

$$x \leq_b y \text{ iff } x \in I(y,b).$$

The relation $\leq_b$ is called the *base point order* at b [55].

3. CAT(0) CELL COMPLEXES

We will briefly review the definitions and some major results about the geometry of nonpositively curved cell complexes. This theory originates from the classical papers of Alexandrov, Busemann, Bruhat, Cartan, Hadamard, Tits, Toponogov and others; for the earlier developments see [1, 2, 18, 17]. In the most generality it has been outlined in the seminal paper of

Gromov [31]. The detailed proofs of the results from [31], further extensions and applications has been presented in a number of papers, among which we cite [6, 16, 20, 47]. For a survey in more depth and background, the reader should refer to the forthcoming book of Bridson and Haefliger [17], whose terminology we follow.

A *geodesic triangle* $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A *comparison triangle* for $\Delta(x_1, x_2, x_3)$ is a triangle $\Delta(x_1', x_2', x_3')$ in $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(x_i', x_j') = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. Gromov [31] defines a geodesic metric space X to be *nonpositively curved* if it locally satisfies CAT(0), the comparison axiom of Cartan–Alexandrov–Toponogov:

CAT(0): Let $\Delta(x_1, x_2, x_3)$ be a geodesic triangle in X, and let y be a point on the side of $\Delta(x_1, x_2, x_3)$ with vertices x_1 and x_2. Let y' denote a unique point on the line segment $[x_1', x_2']$ of the comparison triangle $\Delta(x_1', x_2', x_3')$, such that $d_{\mathbb{E}^2}(x_i', y') = d(x_i, y)$ for $i = 1, 2$. Then $d(x_3, y) \leq d_{\mathbb{E}^2}(x_3', y)$.

If X satisfies the CAT(0) globally, then X is called a *CAT(0) space.*

The *intrinsic pseudometric* of $\mathcal{X}$ is defined by assuming the distance between two points equal to the greatest lower bound on the length of piecewise linear paths joining them. The next theorem is a generalization of some results from [1, 18] for locally finite complexes.

THEOREM 3.1[16]. *If $\mathcal{X}$ is a PE complex with only finitely many isometry types of cells then $\mathcal{X}$ endowed with the intrinsic pseudometric is a complete geodesic metric space. If $\mathcal{X}$ is a CAT(0) space then every local geodesic is a geodesic.*

A cell complex $\mathcal{X}$ satisfies the *link condition* if for every $x \in \mathcal{X}$ any two points y, z in the PS complex $\mathrm{Link}(x, \mathcal{X})$ at distance less than π can be joined by a *unique geodesic.* The *systole* of $\mathcal{X}$ is defined to be the infimum of the length of an isometrically embedded circle. The following result collects some basic characterizations of CAT(0) cell complexes (the full list of criteria with detailed proofs is given in [16, 17]; see also [6, 47]). The most part of the following conditions characterize CAT(0) spaces as well.

THEOREM 3.2[31, 16]. *Let $\mathcal{X}$ be a simply connected PE complex with only finitely many isometry types of cells and endowed with the intrinsic metric. Then the following conditions are equivalent:*

- *(i) $\mathcal{X}$ satisfies CAT(0);*
- *(ii) any two points of $\mathcal{X}$ can be joined by a unique geodesic segment;*
- *(iii) the metric on $\mathcal{X}$ is convex, i.e., if α and β are geodesic segments in $\mathcal{X}$ then the function $f : [0, 1] \to \mathcal{X}$ given by $f(t) = d(\alpha(t), \beta(t))$ is convex;*
- *(iv) $\mathcal{X}$ satisfies the link condition;*
- *(v) for each cell C of $\mathcal{X}$ the systole of $Link(C, \mathcal{X})$ is at least 2π.*

If the dimensions of all cells of $\mathcal{X}$ are at most 2, then (i) is equivalent to

(i^+) for any vertex $x \in \mathcal{X}$ every simple cycle in Link$(x, \mathcal{X})$ has length at least 2π.

From (iii) one can immediately conclude that in a CAT(0) space the r–neighbourhoods $B_r(S)$ of convex sets S are convex. It also implies that a CAT(0) space does not contain isometrically embedded cycles; furthemore, in [17, 20] CAT(0) PE complexes were characterized as spaces of nonpositive curvature without isometrically embedded cycles. The convexity of balls yields that CAT(0) spaces are simply connected, even more, that they are contractible; see for example Chapter II of [17].

In the case of cube complexes, Gromov [31] presented a combinatorial characterization of CAT(0) condition (for a proof see [17, 47]).

PROPOSITION 3.3 [31]. *A cube complex $\mathcal{X}$ is CAT(0) if and only if $\mathcal{X}$ is simply connected and for any cube C the abstract simplicial complex of Link$(C, \mathcal{X})$ is a flag complex.*

This condition can be restated as follows: if three $(k+2)$–cubes of $\mathcal{X}$ share a common k–cube, and pairwise share common $(k+1)$–cubes then they are contained in a $(k+3)$–cube of $\mathcal{X}$. In [51, 44] the CAT(0) cube complexes were called *cubings*.

A rich class of CAT(0) 2–complexes is formed by PE complexes arising in small cancellation theory [40]. An (m, n)*–complex* is a 2–complex $\mathcal{X}$ in which each face has at least m sides and for any vertex x every simple cycle in Link$(x, \mathcal{X})$ has at least n edges. If $\mathcal{X}$ is a simply connected (m, n)–complex with $mn \geq 2(m+n)$ and each face of $\mathcal{X}$ is a regular Euclidean polygon, then from Theorem 3.2(i^+) follows that $\mathcal{X}$ is a CAT(0) space. In particular, (6,3), (4,4) and (3,6)–complexes are CAT(0).

Gersten and Short [29, 30] consider four types $A_1 \times A_1, A_2, B_2, G_2$ of 2–complexes of nonpositive curvature and show that the fundamental group of every such complex has an automatic structure. A complex of type $A_1 \times A_1$ is a piecewise Euclidean 2–complex in which each 2–cell is isometric to a square of side 1 in the Euclidean plane. A complex of type A_2 is a 2–complex in which the 2–cells are isometric to an equilateral triangle of side 1, i.e., they are the (3,6)–complexes defined above. The complexes of types B_2 and G_2 are defined similarly, but each 2–cell in a B_2 complex is isometric to an isosceles right triangle with short side of length one, and each 2–cell in a G_2 complex is isometric to a triangle with angles $\pi/6, \pi/3, \pi/2$ and with short side of length 1/2. The CAT(0) complexes of types $A_1 \times A_1$ and A_2 are 2–dimensional cases of median and bridged complexes defined below.

Now, suppose that D is a PE disk. For a vertex v of D, let $\alpha(v)$ denote the sum of the corner angles of the 2–cells of D with vertex v. If v is an interior vertex of D, define the curvature at v to be $\kappa(v) = 2\pi - \alpha(v)$. When v is a vertex in the boundary ∂D, define the turning angle at v to be $\tau(v) = \pi - \alpha(v)$. A vertex $v \in \partial D$ with $\tau(v) > 0$ is called a *corner* of D. The following Lyndon's Curvature Theorem [40] is a PE version of the Gauss–Bonnet Theorem:

$$\sum_{v \in D - \partial D} \kappa(v) + \sum_{v \in \partial D} \tau(v) = 2\pi.$$

If a PE disk D is CAT(0), then $\kappa(v) \leq 0$ for any interior vertex v. Consequently, D has at least two corners.

4. Weakly modular graphs

In this section we briefly introduce some classes of graphs and recall their basic properies (more detailed presentation of each class will be given in the next sections). All graphs occuring here are connected, and without loops or multiple edges. The *distance* $d_G(u,v)$ between two vertices u and v of a graph $G=(V,E)$ is the length of a shortest path between u and v. The set of all vertices w on shortest paths between u and v is the *interval* $I(u,v)$ between u and v. An induced subgraph H of a graph G is *isometric* if the distance between any pair of vertices in H is the same as that in G. By $G(S)$ we denote the subgraph induced by a set $S \subseteq V$. An *n–cycle* is a simple cycle with n edges.

A graph G is *weakly modular* [10] if its shortest–path metric d_G satisfies the following two conditions:

Triangle condition: for any three vertices u,v,w with

$$1 = d_G(v,w) < d_G(u,v) = d_G(u,w)$$

there exists a common neighbour x of v and w such that $d_G(u,x) = d_G(u,v) - 1$;

Quadrangle condition: for any four vertices u,v,w,z with

$$d_G(v,z) = d_G(w,z) = 1 \quad \text{and} \quad d_G(u,v) = d_G(u,w) = d_G(u,z) - 1,$$

there exists a common neighbour x of v and w such that $d_G(u,x) = d_G(u,v) - 1$.

We can define weakly modular graphs using the concept of metric triangle. Recall that three vertices u,v and w form a *metric triangle* uvw if the intervals $I(u,v), I(v,w)$ and $I(w,u)$ pairwise intersect only in the common end vertices. According to [22], G is weakly modular if and only if for every metric triangle uvw all vertices of the interval $I(v,w)$ are at the same distance $k = d_G(u,v)$ from u. The number k is called the *size* of the metric triangle uvw. A metric triangle uvw is a *pseudo–median* of the triplet x,y,z if the following metric equalities are satisfied:

$$d_G(x,y) = d_G(x,u) + d_G(u,v) + d_G(v,y),$$
$$d_G(y,z) = d_G(y,v) + d_G(v,w) + d_G(w,z),$$
$$d_G(z,x) = d_G(z,w) + d_G(w,u) + d_G(u,x).$$

A graph in which every metric triangle is degenerate, that is, has size 0 is called *modular* [9]. In other words, a graph is modular if $m(x,y,z) = I(x,y) \cap I(y,z) \cap I(z,x)$ is nonempty for every triplet x,y,z. In this case, each vertex from $m(x,y,z)$ is called a *median* of x,y,z. If the median is unique for all triplets, then such a graph is called *median.*

Lemma 4.1 [9, 42]. *A graph G is modular if and only if it is triangle–free and satisfies the quadrangle condition. A graph G is median if it is modular and does not contain $K_{2,3}$ as an induced subgraph.*

Median garphs and the related median structures have many nice characterizations and properties, investigated by several authors; [5, 7, 11, 15, 14, 35, 42, 43, 54, 55, 56] is a sample of papers on this subject. In next sections we will investigate median graphs in more details.

In order to check whether a given subgraph of a weakly modular graph G is convex or gated the following lemma is useful. It can be proved quite easily by induction.

LEMMA 4.2 [22]. *A connected subgraph H of a weakly modular graph G is convex if and only if it is 2–convex, i.e., $I(x,y) \subseteq H$ whenever $x,y \in H$ and $d_G(x,y) = 2$. Every convex set of a modular graph G is gated.*

A graph is called *hereditary weakly modular* if every isometric subgraph is weakly modular. In a similar way we can define *hereditary modular* graphs.

THEOREM 4.3 [9]. *A graph G is hereditary modular if and only if all isometric cycles of G have length four. Equivalently, G is hereditary modular if and only if it is modular and does not contain induced 6–cycles.*

This result has been extended to hereditary weakly modular graphs (by a *house* is meant the graph obtained by gluing a 3–cycle and a 4–cycle along a common edge):

THEOREM 4.4 [22]. *A graph G is hereditary weakly modular if and only if it does not contain induced houses and isometrically embedded n–cycles with $n \geq 5$.*

From the characterization of weakly modular graphs via metric triangles we conclude that the balls of weakly modular graphs are isometric subgraphs. Therefore, the balls of hereditary weakly modular graphs induce hereditary modular graphs. An important class of hereditary weakly modular graphs is formed by bridged graphs. A graph is called *bridged* if all isometric cycles of G have lengh three, i.e. every cycle of length larger that three has a bridge.

THEOREM 4.5 [33, 53]. *A graph G is bridged if and only if for any convex subset S of G and any integer $k \geq 0$ the k–neighbourhood $B_k(S)$ is convex.*

A cycle C of a graph G is *well–bridged* [33] if, for each vertex x of C, either the two neighbours u,v of x in C are adjacent, or there is a bridge from x to another vertex of C. In a bridged graph G all cycles C are well–bridged [33, 53]. Indeed, if the length of C is $2k$, y is the antipode of x in C and $d_G(x,y) = k$, then from the convexity of the ball $B_{k-1}(y)$ we infer that u,v are adjacent. Similarly, if C has length $2k+1$, y,z are the antipodes of x in C and $d_G(x,y) = d_G(x,z) = k$, then from the convexity of $B_{k-1}(\{y,z\})$ we obtain that u and v are adjacent. We continue with the characterization of graphs with convex balls.

THEOREM 4.6 [33, 53]. *In a graph G all balls $B_k(v)$ are convex if and only if every cycle in G of length other than 5 is well–bridged.*

The last class of weakly modular spaces is that of absolute retracts of reflexive graphs. A graph G is an *absolute retract* if G is a retract of any graph in which G is embedded as an isometric subgraph. A *Helly graph* is a graph in which every family of pairwise intersecting balls has a nonempty intersection. The following result is a discrete analogous of the characterization of absolute retracts due to Aronszajn and Panitchpakdi (for recursive characterizations of Helly graphs see [12]).

THEOREM 4.7 [36, 46, 50]. *For a graph G, the following conditions are equivalent:*

(i) G is an absolute retract;

(ii) G is a Helly graph;

(iii) G is a retract of a direct product of paths.

(Recall, the direct product of paths is a subspace of an l_∞–normed space: it has the integer points of some vector space as vertices and two vertices are adjacent if and only if their l_∞–distance is one.)

5. Minimal disks

Let D and K be two simplicial complexes. A map $f : V(D) \to V(K)$ is called *simplicial* if $f(\sigma) \in K$ for all $\sigma \in D$. Any simplicial map $f : V(D) \to V(K)$ induces a continuous map $f : |D| \to |K|$ by extending f affinely over the geometric simplexes $\mathrm{conv}(\sigma), \sigma \in D$. If D is a planar triangulation (i.e. the 1–skeleton of D is an embedded planar graph whose all interior 2–faces are triangles) and $C = f(\partial D)$, then (D, f) is called a *singular disk diagram* (or Van Kampen diagram) for C; (for more details see [40], Chapter V). According to Van Kampen's lemma ([40], pp.150–151), for every cycle of a simply connected simplicial complex one can construct a singular disk diagram. A singular disk diagram with no cut vertices (i.e., its 1–skeleton is 2–connected) is called a *disk diagram.* A *minimal (singular) disk* of C is a (singular) disk diagram D of C with a minimum number of 2–faces. This number is called the *(combinatorial) area* of C and is denoted Area(C).

The following result precises the structure of minimal disks of simple cycles in 1–skeletons of simply connected simplicial complexes.

Lemma 5.1.

(1) Let $C = (e_1, \ldots, e_m)$ be a simple cycle in the 1–skeleton of a simply connected simplicial complex K. There exists a disk diagram (D, f) such that f bijectively maps ∂D to C.

(2) If (D, f) is a minimal disk for a cycle C, then (a) the image of a 2–simplex of D under f is a 2–simplex, and (b) adjacent 2–simplexes of D have distinct images under f.

Proof. The first assertion can be easily proved by induction on the number of 2–simplexes involved in a (combinatorial) homotopy of C to a point. To prove the second assertion, let (D, f) be a minimal disk of C. Suppose that $f(x) = f(y)$ for two adjacent vertices x and y of D. First notice that each 3–cycle (x, y, z, x) bounds a face of D. Indeed, otherwise deleting the interior vertices, edges and simplexes in the region bounded by this cycle, we will obtain a new triagulation D' with $\partial D' = \partial D$. The restriction of f to D' is a simplicial map, because it maps the new simplex xyz to a vertex or to an edge of K, contrary to the choice of (D, f). Therefore x and y have only two common neighbours p and q, and $\tau = pxy$ and $\sigma = qxy$ are simplexes of D. Consider the simplicial complex D' obtained by identifying zx and zy for $z \in \{p, q\}$, by creating a vertex xy, and by ignoring the simplexes τ and σ. One can check that D' is a disk–triangulation and that the map $f' : V(D') \to V(K)$

defined by $f'(u) = f(u)$ if $u \notin \{x, y\}$ and $f'(xy) = f(x) = f(y)$ obeys the conditions, contradicting the minimality of (D, f).

Suppose that $f(\sigma) = f(\tau)$ where $\sigma = xyp, \tau = xyq$. Then $f(p) = f(q)$ by (a). Consider the triangulation D' obtained from D by deleting the edge xy and the triangles σ, τ, and by adding the edge pq and the 2–simplexes pxq and pyq. The map $f : V(D') \to V(K)$ is simplicial, because $V(D') = V(D)$ and f maps pxq and pyq to the edges $f(x)f(p)$ and $f(y)f(q)$ of K. Therefore, (D', f) is a minimal disk, contrary to (a). $\square$

5.2. Remark. Let (D, f) be a minimal disk of C, and let P be a path of D which intersect ∂D only in the end–vertices. These two vertices split ∂D in two paths P_1, P_2, while P divides D in two disk–triangulations D_1 and D_2 with $\partial D_1 = P_1 \cup P, \partial D_2 = P_2 \cup P$. Let $C_1 = f|_{\partial D_1}, C_2 = f|_{\partial D_2}$ (these cycles are not necessarily simple). Then $(D_1, f), (D_2, f)$ are minimal disks of C_1 and C_2, respectively. Conversely, if (D', f_1) and (D'', f_2) are minimal disks of C_1 and C_2, then gluing them together by identifying the paths $f_1^{-1}(f(P))$ and $f_2^{-1}(f(P))$ we will obtain a minimal disk of C.

6. Median complexes

In this section we will show that the cubings (alias CAT(0) cube complexes) coincide with the cubical cell complexes arising from median graphs. We will apply this to derive some results from [51].

6.1. Cubings as median complexes

Every median graph G gives rise to an abstract cubical complex $K(G)$ consisting of all cubes of G, i.e., subgraphs of G isomorphic to cubes of any dimensions. The geometric realization $|K(G)|$ is called a *median complex* (in [27] they are called Buneman complexes). Trivially, G is recovered from its complex $|K(G)|$ as the underlying graph. We shall prove the following result.

Theorem 6.1. *Cubings and median complexes are the same.*

Proof. Let G be a median graph. The proof that $K(G)$ is simply connected is straightforward (see also the proof of (iii)$\Rightarrow$(i) in Theorem 7.1 below). To prove that $|K(G)|$ is a cubing we have to show that if three $(n+2)$–cubes q_1, q_2, q_3 of $K(G)$ share a common n–cube q, and pairwise share common $(n+1)$–cubes q_{ij} then they are contained in a $(n+3)$–cube of $K(G)$. Choose an arbitrary vertex $x \in q$ and let x_{ij} denote its neighbour in the facet of q_{ij} disjoint from q. Further, denote by x_i the second common neighbour (it belongs to q_i) of the vertices x_{ij} and x_{ik}. The median of the triplet x_{ij}, x_{jk}, x_{ki} is a vertex m_x outside $q_1 \cup q_2 \cup q_3$, otherwise we will get an induced $K_{2,3}$, contrary to Lemma 4.1. For the same reason, $m_x \neq m_y$ if x and y are distinct vertices of q. Moreover, one can easily verify that two vertices x and y of q are adjacent if and only if the vertices m_x and m_y are adjacent. Hence the set $\{m_x : x \in q\}$ induces a n–cube q' which together with q_1, q_2 and q_3 yield the desired $(n+3)$–cube.

To prove the converse, we actually use a weaker flag condition. By a *3–wheel* in a cubical complex we will mean three 2–cubes which share a common vertex, and pairwise share common edges. Let K be a simply connected abstract cubical complex in which each 3–wheel is contained in a 3–cube of K. We claim that the 1–skeleton of K is a median graph.

First, triangulate $|K^2|$ by adding one diagonal in each 2–cube of K, thus obtaining a simply connected simplicial 2–complex. Let (D, f) be a minimal disk triangulation of a cycle C of $G(K)$ (we do not require that C is simple, although from Lemma 5.1 we know that for simple cycles such disks exist). Removing the edges of D which are mapped to diagonals of 2–cells, we obtain a disk–quadrangulation Q and a (dimension–preserving) cellular map $f : Q \to K^2$ such that $f|_{\partial Q} = C$ (we also call (Q, f) a minimal disk of C). Applying Lemma 5.1(b) to (D, f) we deduce that f maps incident faces of Q to distinct incident 2–cubes of K. In particular, every 3–wheel of Q is mapped to a 3–wheel of K. Finally, two faces of Q either are disjoint, or intersect in a single vertex or a single edge (in particular, the graph of Q does not contain $K_{2,3}$ as an induced subgraph).

Claim. *If a cycle C of $G(K)$ admits a disk diagram, then there exists a minimal disk (Q, f) of C with at least three corners.*

We argue by induction on the area of a minimal disk (Q, f) of C. Let $\partial Q = (e_1, e_2, \ldots, e_n)$, where $e_i = x_i x_{i+1(mod\ n)}$. First suppose that there exists an edge $e = x_i x_j$ (with $1 < i < j < n$ for simplicity) which splits ∂Q in two cycles $C_1 = (e, e_{j+1}, \ldots, e_{i-1})$ and $C_2 = (e_i, e_{i+1}, \ldots, e_j, e)$. Assume, in addition, that among minimal disks of C containing shortcuts the disk Q and the edge e of Q are selected to minimize $n - |i - j|$. By the induction assumption we can find two minimal disks (Q_1, f_1) and (Q_2, f_2) each containing at least three corners and such that $f_1|_{\partial Q_1} = C_1, f_2|_{\partial Q_2} = C_2$. Piece them together along the preimages of e in ∂Q_1 and ∂Q_2, thus obtaining a minimal disk (Q', f') for C. The corners of Q_1 and Q_2 are corners of Q', unless they coincide with x_i or x_j. If both x_i and x_j are corners of Q_1, then their neighbours x_{i-1} and x_{j+1} in ∂Q_1 are adjacent. From the choice of e we deduce that $x_{i-1}x_{j+1} \in \partial Q_1$. Then x_{i-1}, x_{j+1} are corners, showing that Q' has the desired property.

Further we may assume that x_i, x_j are adjacent in Q if and only if they are consecutive vertices of ∂Q. First we will show that Q can be choosen to have at least one corner. Indeed, suppose that e_1 belongs to the face $q = x_1 y_1 y_2 x_2$ of Q. If y_1 or y_2 belongs to ∂Q, then x_1 or x_2 is a corner. Otherwise, replace in ∂Q the edge e_1 by the path (x_1, y_1, y_2, x_2). By the induction assumption, the obtained cycle has a minimal disk Q'_0 with at least three corners. Let Q' be obtained by gluing q and Q'_0. If y_1 or y_2 is a corner of Q'_0, then we obtain two faces of Q' which intersect more than in one edge, which is impossible because Q' is a minimal disk. Thus at least one corner of Q'_0 is a corner of Q'.

So, let Q contain at least one corner, say x_1. Denote by $q_1 = x_1 x_2 y_1 x_n$ the unique face of Q containing x_1 and set $Q'_1 = q_1$. Let Γ_1 be a cycle obtained from ∂Q by replacing x_1 by y_1. By the induction assumption there exists a minimal disk Q_1 with $\partial Q_1 = \Gamma_1$ possessing at least three corners. If x_2 and x_n are not corners in Q_1, then gluing Q_1 and q_1 we obtain a minimal disk of C with the desired property. So, assume without loss of generality that x_2 is a corner of Q_1. Again, denote by $q_2 = x_2 x_3 y_2 y_1$ the unique face of Q_1 containing x_2, set $Q'_2 = Q'_1 \cup \{q_2\}$, and let Γ_2 be the cycle obtained from Γ_1 by replacing x_2 by y_2. Continuing this way, assume that we have defined the 2–chain Q'_i being the union of the faces $q_1, q_2, \ldots, q_i$, and the cycle Γ_i obtained from ∂Q by replacing the path $P_i = (x_n, x_1, x_2, \ldots, x_{i+1})$ by the path $P'_i = (x_n, y_1, y_2, \ldots, y_i, x_{i+1})$; see Figure 1 for an illustration. By the induction hypothesis, Γ_i has a minimal disk Q_i with at least three corners. Clearly $Q' = Q_i \cup Q'_i$ is a minimal disk of C. The vertex y_i is not a corner of Q_i, otherwise Q' would contain an induced $K_{2,3}$. If $y_i = x_{i+2}$, then x_{i+1} is a corner of Q' and we can find a required minimal disk applying the transform described below. So, let $y_i \neq x_{i+2}$. If x_{i+1} is a corner of Q_i and $x_{i+1} \neq x_n$, then define new $y_{i+1}, q_{i+1}, \Gamma_{i+1}, Q'_{i+1}$, and Q_{i+1}. Finally, assume that x_{i+1} is not a corner of Q_i. Since any corner of Q_i located on the path $\partial Q - P'_i$ is a corner of Q', it suffices to consider the case when all corners of Q_i except possibly one belong to P'_i. Pick a corner $y_j \in P'_i$. Let $q = y_{j-1}y_jy_{j+1}z$ be the unique face of Q_i containing y_j. The faces q, q_j, q_{j+1} form a 3–wheel W in Q' which will be mapped to a 3–wheel of K. By the flag condition, the image of W is contained in a 3–cube Z of K. Perform the following transform with Q' : remove the faces q, q_j, q_{j+1} and add a new vertex z' and new faces $q' = x_jx_{j+1}x_{j+2}z', q'_j = x_{j+2}z'zy_{j+1}, q'_{j+1} = y_{j-1}x_iz'z$. We will obtain a new minimal disk of C in which the vertex x_{j+1} becomes a corner. Indeed, one can map q', q'_j, q_{j+1} to the 2–faces of the cube Z opposite to q, q'_j, q'_{j+1}, respectively. Performing this operation step by step with all corners from P'_i, we obtain a minimal disk Q'' of C in which the corners of $Q_i \cap P'_i$ are replaced by corners of Q''. Clearly, Q'' obeys the desired property. The same reasoning applies when $x_{i+1} = x_n$; three arising configurations in this case are illustrated in Figure 1. This concludes the proof of the claim.

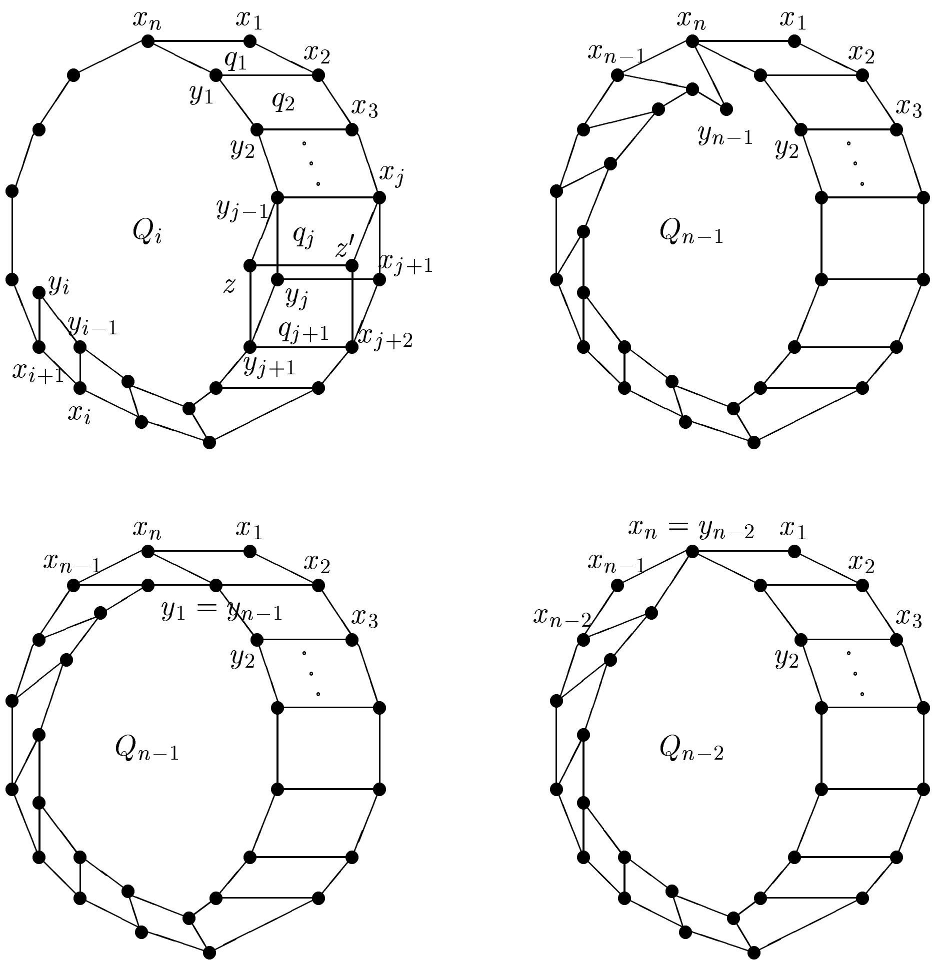

FIGURE 1.

From this claim we conclude that the graph $G(K)$ is triangle–free and every its 4–cycle spans a 2–face of K. Therefore $G(K)$ does not contain induced $K_{2,3}$. To establish that $G(K)$ is median, in view of Lemma 4.1 it suffices to show that if u, v, w, z are distinct vertices and $v, w \in I(u, z)$ are neighbours of z, then there exists a common neighbour x of v and w such that $d_{G(K)}(u, x) = d_{G(K)}(u, v) - 1$. We proceed by induction on $k = d_{G(K)}(u, v)$, departing from the trivial case $k = 1$. We may assume that $I(u, v) \cap I(u, w) = \{u\}$, otherwise we replace u by a closest to v vertex from this intersection and apply the induction hypothesis. Let P' and P'' be shortest paths connecting the pairs u, v and u, w, respectively. Let C be the (simple) cycle composed of P', P'' and the edges vz, zw. Suppose that among all shortest paths between u, v and u, w the paths P', P'' are selected to form a cycle C with minimal Area(C). Suppose, additionally, that the quadrangle condition holds for all quadruples of vertices defining a cycle C' with Area(C') <Area(C). Choose a minimal disk Q of C with at least three corners. For simplicity, we will use the same labels for the vertices of Q and their images in K. If at least one corner of Q is distinct from v, w, then necessarily we will find a corner p distinct from u, v, w, z. Let $p \in P'$, and by a, b denote the neighbours of p in P'. Consider the face $pacb$ of Q containing p. If we replace in P' the vertex p by c, we obtain a shortest path between u and w. The area of a minimal disk of the cycle formed by this path, P'', and the edges wz, zv is smaller than Area(C), contrary to the choice of P'.

Now, assume that Q has exactly three corners u, v, w. Let y be the neighbour of v in P' and $vyv'z$ be the unique face of Q containing v. Clearly $y \in I(v', u)$. Consider a cycle C' formed by the path P'', the path $(w, z, v'y)$ and the part of P' comprised between y and u. Since Area(C') <Area(C) and $v', w \in I(z, u)$, by the choice of u, v, w, z, we can find a vertex y' adjacent to v', w at distance $k-1$ to u. Since $y, y' \in I(v', u)$, by the induction assumption there is a common neighbour u' of y, y' at distance $k-2$ to u. The 2–cells $zvyv', zv'y'w$ and $v'yu'y'$ form a 3–wheel W of K. By the flag condition, W is contained in a 3–cube Z of K. Let x be the unique vertex in $Z - W$. Then x is adjacent to u', v, and w, therefore $d_{G(K)}(x, u) = k - 1$. From this we infer that x is a median of the triplet u, v, w, whence $G(K)$ is a median graph. Since all 4–cycles of G span 2–faces of K, an easy induction on the dimension together with the flag condition imply that all graphic cubes of $G(K)$ induce cells of K. This concludes the proof. □

In view of Theorem 6.1, the following result is obvious.

COROLLARY 6.2. *$\mathcal{X}$ is a CAT(0) complex of type $A_1 \times A_1$ if and only if its 1–skeleton G is a cube–free median graph and $\mathcal{X} = |K(G)|$.*

From the claim of Theorem 6.1 we obtain that if we fix in the 1–skeleton of a cubing two shortest paths P', P'' with common end–vertices u, v, then there exists a sequence of shortest paths P_i from u to v with $P_1 = P', P_n = P''$, such that the simmetric difference $P_i \Delta P_{i+1}$ is a boundary of some 2–cube. This result has been established in [51] and is a rather pleasant way to deform one shortest path to another by means of special elementary homotopies. Notice that the same holds true for shortest paths in the leg graphs of F–complexes investigated in the next section. Actually, it holds for all graphs verifying the quadrangle condition.

6.2. Properties of median complexes

Recently, Sageev [51] and Niblo and Reeves [44] investigated different aspects of groups acting on cubings. For these purposes, the authors of both papers thoroughly explore various geometrical and structural properties of cubings. Theorem 6.1 essentially facilitate this, moreover, most of such properties have been already known for median graphs; [11, 14, 15, 27, 42, 43, 54, 55] are just few sources. On the other hand, the CAT(0) property shade a new light on median complexes. Below, we present a brief account of properties of median graphs and median complexes, attempting a unifying approach in view of Theorem 6.1.

Let G be a underlying graph of a cubing $\mathcal{X}$. For an edge uv of G let

$$G(u,v) = \{x \in V(G) : d_G(u,x) < d_G(v,x)\}, G(v,u) = \{x \in V(G) : d_G(v,x) < d_G(u,x)\}$$

(for brevity, we use this notation also for the subgraphs induced by these sets).

LEMMA 6.3. *$G(u,v), G(v,u)$ are complementary halfspaces of G.*

PROOF. Since G is bipartite, $G(v,u) \cup G(u,v) = V(G)$. If $x \in G(u,v)$, then $I(x,u) \subseteq G(u,v)$, whence $G(u,v)$ is connected and we can apply Lemma 4.2. Pick two vertices $x, y \in G(u,v)$ at distance two and a vertex z adjacent to x and y. Suppose by way of contradiction that $z \in G(v,u)$. Then $d_G(x,u) = d_G(z,v) = d_G(y,u)$. Let $m = m(x,y,u)$. Since $m \in I(x,u) \subset G(u,v)$, we conclude that $m \neq z$. Since m and z are adjacent to x and y and are equidistant to v, we deduce that z is also a median of the triplet x, y, v, which is impossible. □

Following Djoković [25], define for edges uv and $u'v'$ of G,

$$uv \; \Theta \; u'v' \Leftrightarrow u' \in G(u,v) \text{ and } v' \in G(v,u).$$

Lemma 6.3 implies that $uv \; \Theta \; xy$ if and only if $G(u,v) = G(u',v')$ and $G(v,u) = G(v',u')$, i.e., the relation Θ is transitive (and hence an equivalence relation on $E(G)$). We may compare Θ to the following relation Ψ. For two edges e and e' define $e \Psi e'$ if they either are equal or there exists edges $e_0, e_1, \dots, e_n$ such that $e = e_0, e' = e_n$ and $e_i, e_{i+1} (i = 0, \dots, n-1)$ constitute opposite edges on some 4–cycle of G. If uv and $u'v'$ are opposite edges of a 2–cube, then $u' \in G(u,v), v' \in G(v,u)$, hence $uv \; \Theta \; u'v'$. A trivial induction yields $\Psi \subseteq \Theta$. To show that Ψ includes Θ, proceed by induction on the distance k between two distinct edges uv and $u'v'$ in the relation Θ. Select an arbitrary neighbour y of u' on a path joining the pair v, v'. Let x be the median of the triplet u, u', y. Since $d_G(u,x) = d_G(v,y) < k$, x is distinct from v'. Convexity of $G(u,v)$ yields $x \in G(u,v)$, thus $uv \; \Theta \; xy$. By the induction assumption, $uv \; \Psi \; xy$. Since Ψ is an equivalence relation and $xy \; \Psi \; u'v'$, we infer that $uv \; \Psi \; u'v'$. This yields $\Theta \subseteq \Psi$, and in conjunction with $\Psi \subseteq \Theta$ establishes the following result.

LEMMA 6.4. $\Theta = \Psi$.

According to the theorem of Djoković [25] transitivity of Θ ensures isometric embeddability into a hypercube (a *hypercube* $H(\Lambda)$ has all finite subsets of some set Λ as vertices, and the symmetric difference $\#A \Delta B$ as the distance between A, B). Let $E_i, i \in \Lambda$ be the equivalence classes of $\Theta = \Psi$ and let G'_i, G''_i be the complementary halfspaces defined by E_i. Define the following embedding ϕ of G into the power set of Λ. Pick a base point b and set $\phi(b) = \emptyset$. For a vertex x, $\phi(x)$ consists of all $i \in \Lambda$ such that b and x lie in different halfspaces defined by E_i. Clearly, each $\phi(x)$ is finite. For arbitrary vertices x, y we have $d_G(x,y) = \#\phi(x) \Delta \phi(y)$. Indeed, the convexity of halfspaces implies that if $x \in G'_i$ and $y \in G''_i$ then every shortest path connecting x and y contains exactly one edge of E_i. Conversely, if an edge on a shortest path between x and y belongs to E_i, then x and y are located in different halfspaces defined by E_i. Then apply this to shortest paths between x, y and b which pass via $m(x, y, b)$.

REMARK 6.5. Proposition 3.4 and the previous result remain valid if one replace the cubical cells by boxes. Then all edges from one equivalence class E_i of Θ have the same length μ_i. We will call such box complexes also cubings.

For a class E_i of equivalent edges assume that $b \in G'_i$. Let F'_i be the subgraph induced by all vertices of G'_i which have a neighbour in G''_i. Analogously define F''_i. Set $F_i = F'_i \cup F''_i$. Let H'_i, H''_i be the cubical cell complexes induced by F'_i, F''_i. Obviously, H'_i and H''_i are isomorphic. Moreover, given $0 < \mu < 1$, one can define a cubical complex $H_i(\mu)$ isomorphic to H'_i as follows: on each edge $x'x'' \in E_i$ pick the point x_μ verifying $d(x_\mu, x') = \mu, d(x_\mu, x'') = 1 - \mu$. Define a graph $F_i(\mu)$ with this set of points as vertices: two vertices $x_\mu \in x'x''$ and $y_\mu \in y'y''$ are adjacent is and only if $x'x''$ and $y'y''$ are opposite edges of some 2–cube of G. Then $H_i(\mu)$ is the cubical complex of $F_i(\mu)$. Following Sageev [51], we call $H_i(\mu)$ *geometric hyperplanes.* An important result in [51] is to show that the geometric hyperplanes of a cubing partition the complex into two connected components. The second important moment is to establish that a hyperplane in a cubing is a cubing and that any finite collection of pairwise intersecting hyperplanes has a point in common. The first assertion is immediate from previous properties, because $\mathcal{X}$ is the union of $|G'_i| \cup (|H'_i| \times [0, \mu])$ and $([\mu, 1] \times |H''_i|) \cup |G''_i|$ glued together along $H_i(\mu)$. In particular, $H_i(\mu)$ separates any pair of vertices from different halfspaces G'_i and G''_i.

PROPOSITION 6.6[51]. *Every geometric hyperplane in a cubing is a cubing.*

PROOF. First we prove that F'_i, F''_i and F_i are convex subgraph of G. From Lemma 6.4 we deduce that F'_i is connected, therefore it suffices to show 2–convexity of F'_i. Pick $x', y' \in F'_i$ at distance 2, their neighbours x'', y'' in F''_i, and a common neighbour z. Since G''_i is convex and G is bipartite, we infer that $d_G(x'', y'') = 2$. Hence the median v of x'', y'', z is a vertex of G''_i adjacent to each vertex from the triplet, whence $z \in F'_i$. Thus F'_i, F''_i and their union are convex. Since convex subgraphs of a median graph are median, by Theorem 6.1 and the isomorphism of H'_i and $H_i(\mu)$ we conclude that $H_i(\mu)$ is a cubing. $\square$

As to the Helly property, let $H_{i_j}(\mu_j)$ $(j = 1, \ldots, m)$ be a collection of pairwise intersecting geometric hyperplanes. First notice that $H_i(\mu), H_k(\lambda)$ intersect if and only if the combinatorial hyperplanes F_i and F_k intersect in G. Therefore $F_{i_1}, \ldots, F_{i_m}$ is a collection of pairwise intersecting convex sets of G. Being gated, by the Helly property for gated sets, they share a common vertex x'. Clearly, if $x' \in F'_{i_j}$ then its neigbour $x'' \in F''_{i_j}$ is also a vertex from the intersection. Thus in the intersection we can find a box $|C|$ sharing a facet C_{i_j} with every F'_{i_j}. Take a point $p \in |C|$ whose distance to C_{i_j} is μ_j. Evidently, $p \in \cap_{j=1}^{m} H_{i_j}(\mu_j)$, thus establishing the Helly property.

We continue with an useful property established by different authors. asserting that finite median graphs have thin halfspaces.

LEMMA 6.7 *If G'_i is a minimal proper halfspace of a finite median graph G, then $G'_i = F'_i$, i.e. any vertex of G'_i is adjacent to a vertex of G''_i.*

Suppose not; then we can find a vertex $v \in G'_i - F'_i$ adjacent to a vertex $u' \in F'_i$. Let u'' be the neighbour of u' in F''_i. Suppose that the edge vu' belongs to the equivalence class E_j. Let G'_j be the halfspace containing v. By the choice of G'_i, there is a vertex $x \in G'_j \cap G''_i$. From the convexity of these halfspaces we deduce that the median of x, u'', v is a common neigbour of u'', v and belongs to G''_i. This shows that v is a vertex of F'_i, contrary to our assumption.

Given a geometric hyperplane $H_i(\mu)$, one can refine the cellular structure of $\mathcal{X}$ by subdividing the cells $|C|$ intersected by $H_i(\mu)$ and adding the intersections $|C| \cap H_i(\mu)$ as new cells. The underlying graph of the new box complex is obtained from G by taking on each edge of E_i the corresponding vertex of $F_i(\mu)$ and adding the edges of the graph $F_i(\mu)$. Since the new cubical complex has $\mathcal{X}$ as the geometric realization, from Theorem 6.1 (see Remark 6.5) we conclude that the new graph is again median. More generally, we can refine $\mathcal{X}$ by consequently cutting it with a finite number of geometric hyperplanes, arriving at a new CAT(0) box complex whose 1–skeleton is a median graph.

Every point $p \in \mathcal{X}$ belongs to the relative interior of a unique cell $|C|$. The geometric hyperplanes $H_{i_1}(\mu_{i_1}), \ldots, H_{i_n}(\mu_{i_n})$ passing through p are all parallel to the facets of $|C|$. Define $\mathbf{p} = (p_1, p_2, \ldots,)$, where $p_i = \mu_{i_j}$ if $i = i_j$ and $p_i = 0$ otherwise. If p is a vertex, then clearly $\mathbf{p}$ is the characteristic vector of the set $\phi(p)$. If we endow each cell with the l_1–distance and extend it to the intrinsic distance ρ_1 on $\mathcal{X}$, then one can show that for any points $x, y \in \mathcal{X}$ we have

$$\rho_1(x,y) = ||\mathbf{x} - \mathbf{y}||_1 = \sum_{i \in \Lambda} |x_i - y_i|.$$

(The sum is well–defined, because each of the vectors $\mathbf{x}$, $\mathbf{y}$ has only a finite number of nonzero coordinates.) $(\mathcal{X}, \rho_1)$ is a median space, i.e., for any three points $x, y, z \in \mathcal{X}$ there exists a unique point $m = m(x,y,z)$ such that m belongs to all intervals between x, y, z; for details about this and previous result see [55]. In view of previous results, one can sketch the following proof of this fact: refine $\mathcal{X}$ by cutting it with all geometric hyperplanes passing through at least one of the points x, y, z. We will obtain a cubing whose 1–skeleton is a (weighted) median graph isometrically embedded in $(\mathcal{X}, \rho)$. Since x, y, z are vertices of this graph, their median will be the median in $(\mathcal{X}, \rho)$ (its unicity easily follows).

THEOREM 6.8[55]. *If $\mathcal{X}$ is a cubing, then $(\mathcal{X}, \rho_1)$ is a median space, which embeds isometrically into a space of L_1–type.*

Median spaces are particular instances of a remarkable algebraic structure. A *median operator* on a set X is a function $m : X^3 \to X$ satisfying the following conditions:

(absorption law) $m(a,a,b) = a$;
(symmetry law) if σ is any permutation of a, b, c, then

$$m(\sigma(a), \sigma(b), \sigma(c)) = m(a,b,c);$$

(transitive law) $m(m(a,b,c),d,c) = m(a, m(b,c,d), c)$.

The resulting pair (X, m) is called a *median algebra*. Median algebras and related convexity and metric median structures have a rich theory. Their study goes back to Birkhoff, Kiss and Sholander; for an extensive survey, see [11, 55]. A more general notion of median algebra is given in [35]. We only mention that there is a bijection between finite (discrete) median algebras and median graphs; with a discrete median algebra (X, m) one can associate a median graph by taking X as the vertex–set and all pairs xy, such that $m(x,y,z)=x$ or y for all $z\in X$, as edges.

Mai and Tang [41] established that any finite collapsible simplicial complex X is injectively metrizable. To prove this, they showed that X can be subdivided to a collapsible cubical complex K, which is an injective space with respect to the intrinsic l_∞–metric. Recall, this metric ρ_∞ is defined in the following way: given two points $x,y\in|K|$ so that if x and y are in a common cell, say in a k–cell, then $\rho_\infty(x,y)=\max_i|x_i-y_i|$, where $x=(x_1,\ldots,x_k), y=(y_1,\ldots,y_k)$; otherwise the distance is the length of a shortest path joining them. Recently, van de Vel [56] observed that the cubical complexes occuring in [41] are actually the median complexes, leading to the following result.

THEOREM 6.9 [41, 55]. *If $\mathcal{X}$ is a locally finite cubing, then $(\mathcal{X},\rho_\infty)$ is an injective space.*

A discrete version of the result of Mai and Tang is given by Bandelt and van de Vel [15]. It turns out that there is a parallel between their result and the approach of Niblo and Reeves [44] to bicombing the fundamental group of a cubing. Given the median graph G of a cubing $\mathcal{X}$, let G^Δ be the graph having the same vertex set as G, where two vertices are adjacent if and only if they belong to a common cube of G. Halfspaces $G_{i_1},G_{i_2},\ldots,G_{i_q}$ of G form a chain if

$$G_{i_1}\subset G_{i_2}\subset\ldots\subset G_{i_q}.$$

Equivalently, for any $0<\mu<1$, the geometric hyperplanes $H_{i_1}(\mu),\ldots,H_{i_q}(\mu)$ are pairwise disjoint.

PROPOSITION 6.10 [15]. *If G is a median graph, then G^Δ is a Helly graph. The distance $d_{G^\Delta}(x,y)$ is the largest possible length of a chain of nontrivial halfspaces separating x and y.*

Each path $P_{(v_0,v_n)}=(v_0,v_1,\ldots,v_{n-1},v_n)$ of G^Δ corresponds to a chain of cubes [15]: this is a sequence of cubes $C_0,C_1,\ldots,C_n$, such that C_i is the cube of minimal dimension containing v_i and v_{i+1}. Call a chain of cubes verifying $C_i\cap C_{i+1}=\{v_{i+1}\}$ for all $i<n$ a *cube–path*. A cube–path is called a *normal cube–path* [44] if C_{i+1} intersects $\mathrm{star}(C_i,\mathcal{X})$ in v_{i+1} (recall, $\mathrm{star}(C_i,\mathcal{X})$ consists of all cubes containing C_i). The following result is a nice observation of Niblo and Reeves.

PROPOSITION 6.11 [44]. *Given two vertices v,w of a median graph G, there is a unique normal cube–path from v to w and a unique normal cube path from w to v. Both paths are shortest paths in the graph G^Δ.*

Using Proposition 6.11 and Lemma 4.2 one can readily check that G^Δ is a Helly graph. Indeed, as in the proof of Mai and Tang or that of Bandelt and van de Vel, it suffices to show that every ball $B_r(b)$ of G^Δ is a convex set of G (recall, the convex sets of a median graph are gated). By Lemma 4.2, it is enough to show that if $x,y\in B_r(b), d_G(x,y)=2$, then any their common neighbour z in G belongs to $B_r(b)$. Consider the last cubes C' and C'' in the normal cube–paths $P_{(b,x)}$ and $P_{(b,y)}$ connecting b with x and y, respectively. The star of at least one cube, say of C', must contain the vertex z, otherwise, adding to $P_{(b,x)}$ and $P_{(b,y)}$ the edges xz and yz, respectively, we will get two distinct normal–cube paths between b and z. Therefore $d_{G^\Delta}(b,z)\le d_{G^\Delta}(b,x)$, whence $z\in B_r(b)$.

In some sense, Proposition 6.1 expresses the fact that given a base–point b of a median graph G, the pair $(V(G), \leq_b)$ is a median semilattice; e.g. [55]: each order interval $[b, u]$ is distributive, and a nonempty finite set of vertices have an upper bound provided each pair of its vertices have an upper bound. To construct the normal cube–path $P_{(v,w)}$ from v to w, consider the set F of all neighbours of v in the interval $I(v, w)$. Take the upper bound u of F. Then $I(v, u)$ induces a cube C. Define $P_{(v,w)}$ be the union of C and $P_{(u,w)}$. From the choice of u we conclude that any cube containing C intersects the first cube of the path $P_{(u,w)}$ in u, and by the induction assumption $P_{(v,w)}$ is indeed a normal cube–path.

Remark 6.12. In fact, it is possible to deduce Theorem 6.9 from Proposition 6.10, and Proposition 6.11 from Lemma 6.7, but we will not present them here.

We conclude the list of properties of median structures with the following important characterization of median graphs.

Theorem 6.12 [7]. *Median graphs are precisely the retracts of hypercubes.*

Bandelt and van de Vel [13] have proven that every edge–preserving map $f : G \to G$ of a finite median graph has an invariant cube, i.e. a cube of G which is mapped isomorphically onto itself by f. On the other hand, Kirk [39], generalizing a result of Penot [48], proved that every nonexpansive map of a convex bounded weakly countably compact metric space in which all balls are convex has a fixed point. Van de Vel (p.504 of [55]) noticed that "the similarities between..." two results "...are too detailed to be pure coincidence". It seems that CAT(0) property of median complexes gives a partial explanation to this fact. Namely, assume additionaly that $f : G \to G$ is a cell–to–cell map. Then f induces a continuous map $f : |K(G)| \to |K(G)|$ by extending f affinely over the geometric cubes. As a result, we obtain a nonexpansive map of a CAT(0) space $|K(G)|$. From Theorem 3.2(iii) and the result of Kirk, f has a fixed point p. Consider the smallest cube C of G such that p belongs to the relative interior of $|C|$. Clearly, C is an invariant cube of the initial map f. This remark can be extended to obtain fixed simplexes of edge–preserving maps of underlying graphs of CAT(0) simplicial complexes, but does extend to arbitrary edge–preserving maps of CAT(0) polysimplicial complexes (cell complexes whose cells are products of simplexes).

7. Simplicial complexes with right triangular cells

In this section we assume that $\mathcal{X}$ is a PE 2–complex with only finitely many isometry types of cells whose 2–cells are right triangles and the attaching map verifies the following conditions:

(a) if the 2–cells σ', σ'' share a common edge e, then e is either the hypotenuse of both σ', σ'' or a leg of both σ', σ'';

(b) if the 2–cells σ', σ'' are glued along the hypotenuse, then $\sigma' \cup \sigma''$ is isometric to a Euclidean rectangle.

(c) each hypotenuse belongs to at least two triangles.

Simple arguments show that in a cell complex obeying (a) and (b) all maximal cells are triangles, thus the assumption that $\mathcal{X}$ is two–dimensional does not restrict the generality. Further, a 2–complex verifying (a) and (b) can be complemented by adding "pendant triangles" to fulfills (c) as well. Let $K = K(\mathcal{X})$ denote the abstract simplicial complex associated with $\mathcal{X}$. A *leg graph* $LG(K)$ is obtained by deleting the hypotenuse–edges from the 1–skeleton $G(K)$ of K. Clearly, if a hypotenuse h is shared by n triangles, then the subgraph of $LG(K)$ induced by $\mathrm{star}(h, K)$ is the complete bipartite graph $K_{2,n}$ (further called a *biclique).* Following [37], we call the union of all triangles of $\mathcal{X}$ sharing the hypotenuse h a *folder* and denote it by F_h. An arbitrary bipartite graph G gives rise to a cell complex $\mathcal{X}(G)$ which fulfills the requirements (a), (b) and (c). For this, in each biclique $K_{2,n}$ of G peak two vertices x, y which are adjacent to all other vertices $z \in K_{2,n}$ (x, y are uniquely defined unless $n = 2$). Define the right triangle xyz with the right angle at z, and set $\mathcal{X}(G)$ be the union of all such triangles. The obtained complex $\mathcal{X}(G)$ verifies (a) and (c). If we require that the opposite edges in every 4–cycle of G have equal lengths, then $\mathcal{X}(G)$ obeys the condition (b) as well. Trivially, G is recovered from its complex $\mathcal{X}(G)$ as the leg graph. In general, the structure of $\mathcal{X}(G)$ can be quite complicated: consider, for example, the complexes generated by the graphs $K_{3,3}$ and $K^-_{3,3}$ from Figure 2.

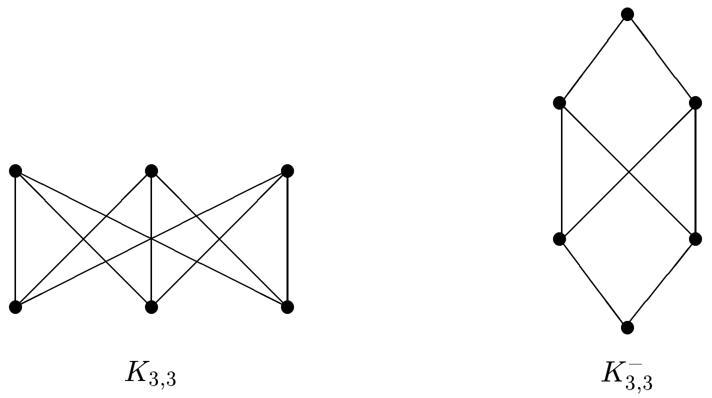

FIGURE 2.

7.1. Folder–complexes

We say that a complex $\mathcal{X}$ obeying (a) and (b) is a *folder complex (F–complex)* if the intersection of any two folders does not contain incident legs and $\mathcal{X}$ does not contain any three folders $F_{h_1}, F_{h_2}, F_{h_3}$ and three distinct legs e_1, e_2, e_3 sharing a common vertex such that e_i belongs to F_{h_j} exactly when $i \neq j$. (This is a "folder" version of the flag condition for cubical complexes).

THEOREM 7.1. *For a simply connected cell 2–complex $\mathcal{X}$ verifying the conditions (a),(b) and (c) the following conditions are equivalent:*

(i) $\mathcal{X}$ satisfies CAT(0);

(ii) $\mathcal{X}$ is an F–complex;

(iii) $\mathcal{X} = \mathcal{X}(G)$ for a hereditary modular graph G without $K_{3,3}$ and $K^-_{3,3}$ as induced subgraphs.

If $\mathcal{X}$ is simply connected and satisfies (a) and (b) then (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii).

PROOF. (i)$\Rightarrow$ (ii): Suppose by way of contradiction that $\mathcal{X}$ contains three folders $F_{h_1}, F_{h_2}, F_{h_3}$ and three distinct legs xy_1, xy_2, xy_2 such that xy_i belongs to F_{h_j} exactly when $i \neq j$. The hypotenuse h_j is either y_iy_k $(i,k \neq j)$ or xx_j. In the first case, y_iy_k belongs to the PS complex $\text{Link}(x,\mathcal{X})$ and its spherical length is $\pi/2$. In the second case, y_ix_j and y_kx_j belong to $\text{Link}(x,\mathcal{X})$, and by condition (b) the spherical length of the path $y_ix_jy_k$ again is $\pi/2$. Concatenating such hypotenuses and paths, we obtain a simple cycle of $\text{Link}(x,\mathcal{X})$ of length $2\pi/3$, contrary to Theorem 3.2.

Now, suppose that the folders $F_{h'}$ and $F_{h''}$ share two incident legs xy and xz. At least one of h' or h'', say h', is distinct from yz. Then $h' = xu$, the union of the 2–cells xuy and xuz being a rectangle. If $h'' = yz$, then $\text{Link}(x,\mathcal{X})$ contains a simple cycle (z,y,u,z) of length π. Otherwise, if $h'' = xv$, the simple cycle (z,v,y,u,z) of $\text{Link}(x,\mathcal{X})$ has spherical length π, in contradiction to Theorem 3.2.

(ii)$\Rightarrow$ (iii): Let C be a simple cycle in the graph $LG(\mathcal{X})$. Consider a minimal disk (D,f) of C. Delete from D all edges whose images under f are hypotenuses of $\mathcal{X}$. Since the map f is dimension–preserving, the resulting planar graph Q is a quadrangulation, i.e., all (interior) faces of Q are quadrangles.

CLAIM. *The degree of every interior vertex x of Q is at least 4.*

Suppose not. First, assume that x has exactly two neighbours y and z, and let $yxzu$ and $yxzv$ be the faces of Q incident to x. Necessarily, x is adjacent in D to at least one of the vertices u,v. We may thus assume that xu is an edge of D. If x is adjacent to v too, by Lemma 5.1 the triangles xuy, xuz, xvy, xvz will be mapped to four distinct triangles from the folders $F_{f(xu)}$ and $F_{f(xv)}$. Since the legs $f(xy), f(xz)$ are incident and belong to both folders, we obtain a contradiction with the assumption that $\mathcal{X}$ is a folder complex. The same contradiction arises in the case when y and z are adjacent in D and $f(v) \neq f(u)$. So, let $f(v) = f(u)$. Then $F_{f(yz)}$ and $F_{f(xu)}$ share the 4–cycle $(f(y), f(x), f(z), f(u) = f(v), f(y))$, a contradiction.

Now, suppose that x is adjacent in Q to three vertices y,z,w. Let $q_1 = x_1yxz, q_2 = x_2zxw, q_3 = x_3wxy$ be the faces of Q incident to x, each of them being a union of two triangles of D; see Figure 3 in which the edges mapped to hypotenuses are heavy. Let e_i be the common edge of two triangles constituting q_i. If e_1, e_2, e_3 have distinct images in K, we obtain three folders $F_{f(e_1)}, F_{f(e_2)}, F_{f(e_3)}$ and three edges $f(xy), f(xz), f(xw)$ which contradicts the hypothesis that $\mathcal{X}$ is a folder complex. From this and Lemma 5.1 we immediately conclude that x cannot be adjacent to all three vertices x_1, x_2, x_3, and that the vertices y,z,w are not pairwise adjacent, thus settling the first two cases from Figure 3. An easy analysis of the remaining cases show that every simplicial map f with $f(e_i) = f(e_j)$ violate one of the conditions of Lemma 5.1. This proves the claim.

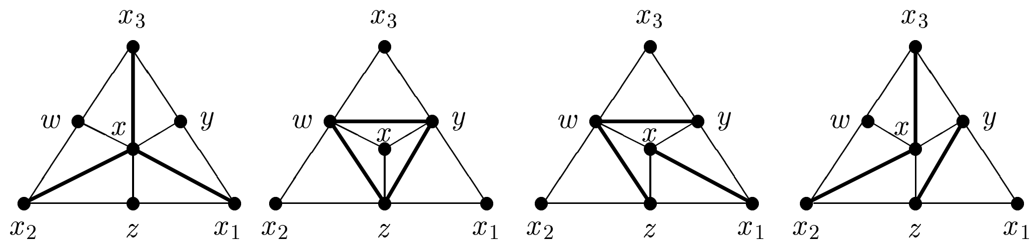

FIGURE 3.

From this claim and Gauss–Bonnet formula one can easily show that ∂Q contains at least four corners (in this case they are vertices of degree two of Q). Indeed, Q is a CAT(0) (4,4)–complex, thus $\kappa(v) \leq 0$ for all interior vertices v. Since the turning angles are nonpositive or $\pi/2$, necessarily Q contains at least four corners. From this we immediately obtain that $LG(K)$ is triangle–free. Now, suppose that C is a 4–cycle of $LG(K)$. Then Q consists of a single quadrangle, i.e., C is the boundary of a union of two triangles of $\mathcal{X}$ sharing a common hypotenuse, whence $LG(K)$ is bipartite. As $\mathcal{X}$ is a folder complex, a simple analysis in conjunction with this imply that $LG(K)$ does not contain $K_{3,3}$ and $K_{3,3}^-$ as induced subgraphs. To prove that $LG(K)$ is hereditary modular, in view of Theorem 4.3 we must show that $LG(K)$ does not contain isometric cycles of length at least 6. Let C be a simple cycle with at least 6 edges, and let (Q,f) be a minimal disk quadrangulation of C. From Theorems 6.1 and 4.3 we have that Q is a cube–free median graph, whence a hereditary modular graph. Hence ∂Q is not an isometric cycle of Q. Since $f: Q \to LG(K)$ is nonexpansive, the image of ∂Q is a non–isometric cycle of $LG(K)$, yielding that $LG(K)$ is hereditary modular.

(iii)⇒(i): Let G be a hereditary modular graph which does not contain $K_{3,3}$ and $K_{3,3}^-$ as induced subgraphs and denote $\mathcal{X} = \mathcal{X}(G)$. Each 4–cycle is the boundary of two triangles of $\mathcal{X}$. To show that $\mathcal{X}$ is simply connected, it suffices to prove that every simple cycle C of G can be (combinatorially) deformed to an arbitrary vertex $x \in C$. We apply the induction on $\sum_{w \in C} d_G(x,w)$. Walking around C starting from x, we will arrive at a vertex $y \in C$ such that its neighbours u,v in C obey $d_G(x,u) = d_G(x,v) < d_G(x,y)$. By modularity condition we can find a common neighbour y' of u and v one step closer to x. Let C' be the cycle obtained from C by replacing y by y' (and eventually removing one of the vertices u or v if $y' \in C$). Clearly, for C' we can apply the induction hypothesis, thus C' can be deformed to x. Since C can be transfered to C' using two elementary deformations in $\mathcal{X}$, we are done.

To establish that $\mathcal{X}$ is nonnositively curved, by Theorem 3.2 we must show that for a vertex x the length of every simple cycle in the PS complex $\mathrm{Link}(x,\mathcal{X})$ is at least 2π. To prove this, first we present few simple properties of $\mathcal{X}$. One can readily check that the maximal complete bipartite subgraphs of G are bicliques. Therefore every 4–cycle of G belongs to a unique folder of $\mathcal{X}$. Two folders either are disjoint or intersect in a vertex or in an edge, otherwise we will get a forbidden subgraph $K_{3,3}$ or $K_{3,3}^-$. This implies that $\mathrm{Link}(x,\mathcal{X})$ does not contain 4–cycles of G. Indeed, if we could find such a cycle (y,z,u,v,y), then x must be adjacent in G to some opposite vertices of this cycle, say to y and u. Hence xz and xv are hypotenuses, yielding two folders F_{xz} and F_{xv} with a forbidden intersection.

Consider a simple cycle C in $\mathrm{Link}(x,\mathcal{X})$. The legs of C combine in pairs l', l'' to form paths whose spherical length is $\pi/2$. Indeed, pick a leg $l' = yz$, and suppose that xy is the hypotenuse of the triangle xyz. If u is the neighbour of y in C, then xuy is a 2-cell, and therefore we can set $l'' = yu$. On the other hand, any hypotenuse $h = uv$ of C has length $\pi/2$. In addition, we can find a vertex $x_h \notin C$ such that (x, u, x_h, v, x) is a 4–cycle of G. If we take another hypotenuse h' of C, then the vertices x_h and $x_{h'}$ are distinct. Moreover, x_h cannot be adjacent in G to any end–vertex of h', otherwise G would contain a 3–cycle.

Now, if C contains at least 4 hypotenuses and/or $\pi/2$–pairs of legs, then the length of C is at least 2π and we are done. The subsequent case analysis will show that the converse cannot happen. First suppose that C has only three edges, say $C = (y, z, u, y)$. Since G is triangle–free, at least one edge of C, say $h = yu$, is a hypotenuse. If yz and zu are legs, then xz or xu, say the first one, is a hypotenuse, providing two folders F_h and F_{xz} with a forbidden intersection. Otherwise, if $h' = yz$ and $h'' = zu$ are hypotenuses, then $(u, x_h, y, x_{h'}, z, x_{h''}, u)$ is an induced 6–cycle of G, contrary to the fact that G is hereditary modular. The same contradiction arises when C has 6 edges and is a cycle of G. Then C is not induced, and this would imply that the link of x contains a 4–cycle of G, which we have proven to be impossible. If C has two hypotenuses $h' = yz$, $h'' = yv$ and two legs zu, uv, then in G we obtain an induced 6–cycle $(y, x_{h'}, z, u, v, x_{h''}, y)$. Indeed, as we have shown before $x_{h'}v, x_{h''}z \notin E(G)$. On the other hand, $uy \in E(G)$ would imply that $u \in F_{h'} \cap F_{h''}$, which is impossible because these folders intersect already along the edge xy. Finally, assume C consists of a hypotenuse yz and two pairs of legs zu, uv and vw, wy. Consider the 6–cycle (y, x_h, z, u, v, w, y) of G, which cannot be induced. However, if G contains anyone of the edges x_hv, yu or zw, then any two of the folders F_{yz}, F_{xu}, F_{xw} intersect more than in one edge. This proves that every simple cycle C in the link of x has length at least 2π, concluding the proof that $\mathcal{X}$ is a CAT(0) space. □

Motivated by Theorem 7.1, we call a hereditary modular graph which does not contain $K_{3,3}$ and $K^-_{3,3}$ as induced subgraphs an *F–graph.*

Obviously, a B_2 complex verifies the conditions (a) and (b), therefore, the CAT(0) complexes of this type can be characterized via Theorem 7.1. Namely, given an F–graph G, one can define a unique B_2 complex $\mathcal{X}(G)$ by setting all edges of G to have length one.

To present the next example, let $\mathcal{K}$ be a simply connected (4,4)–complex. Let $\mathcal{X}$ be the subdivision of $\mathcal{K}$ obtained by adding a new vertex at the barycenter of each 2–face, and then joining it in the face with the vertices on the boundary. Then $\mathcal{X}$ is a simplicial complex. Model each simplex of $\mathcal{X}$ as an isosceles triangle with angles $\pi/2, \pi/4, \pi/4$, the edge of $\mathcal{K}$ of length 1 and two new edges of length $1/\sqrt{2}$. Clearly, $\mathcal{X}$ is a B_2 complex, and that $G(\mathcal{K})$ coincides with the graph $HG(\mathcal{X}) = G(\mathcal{X}) - LG(\mathcal{X})$. The converse is not true. One can construct examples of F–complexes $\mathcal{X}$ in which the graph $HG(\mathcal{X})$ has a prescribed number of connected components, whose interplay can be quite complicated.

7.2. Properties of F–complexes and F–graphs

Since every two folders intersect in an edge or a vertex, with some abuse of terminology, we can assume that $\mathcal{X}$ is a cell complex, whose 2–cells are the folders and the 1–cells are the legs of triangles. Let G be an F–graph endowed with the standard graph–metric d_G. Due to the bijection between folders and bicliques, in some places below, by a folder we actually mean the biclique of G induced by a folder of $\mathcal{X}(G)$. The first two assertion can be easily derived from the definitions and Lemma 4.2.

LEMMA 7.2 *Every folder is a gated subgraph of G.*

LEMMA 7.3 *Each ball $B_k(x)$ induces an isometric subgraph of G.*

LEMMA 7.4 *If $d_G(b,v) = k+1$, then either v has a unique neighbour v^b in $B_k(b)$ or there is a unique vertex v^b at distance $k-1$ to b which is adjacent to all neighbours of v in $B_k(b)$.*

PROOF. Pick two neighbours $x,y \in B_k(b)$ of v, and let $v' \in m(x,y,b)$. Assume that v'' is another median of this triplet. Both v' and v'' are common neighbours of x,y at distance $k-1$ to b. Let $z \in m(v',v'',b)$. Then the vertices v,x,y,v',v'',z induce the forbidden $K_{3,3}^-$. Hence the median v' is unique. Let $z \in B_k(b)$ be another neighbour of v. We assert that z is adjacent to v'. Indeed, suppose not, and let z', z'' be the medians of the triplets x,z,b and y,z,b, respectively. Then we have constructed an induced 6–cycle (x,v',y,z'',z,z',x), a contradiction. □

This property implies that in $B_{k+1}(b)$ the vertex v belongs either to a unique edge vv^b or to a unique folder F. From Lemma 7.2 we infer that v^b is the gate of b in F. Therefore, a total order $\prec_b$ refining $\leq_b$ has the property that in the subgraph induced by $\{z : z \prec_b v\}$ the vertex v belongs to a unique cell F_{vv^b} spanned by the pair vv^b (we will call this cell *pendant*, specifying its type, if necessary). In particular, this implies that finite F–complexes are collapsible.

A *folder–path* between two vertices v', v, is a sequence of folders and/or edges $F_0, \ldots, F_n$, such that $v' \in F_1, v \in F_n$ and $F_i \cap F_{i+1} = \{v_i\}$ for $i = 0, \ldots, n-1$. By induction on $n = d_G(b,v)$ we will define a special folder–path $P_{(v,b)}$ between b and each vertex v. Namely, take the pendant cell F_{vv^b} spanned by v and v^b and set $P_{(v,b)} = \{F_{vv^b}\} \cup P_{(v^b,b)}$. One can be readily checked that if $F_0, F_1, \ldots, F_n$ are the cells of $P_{(v,b)}$, then any maximal cell containing F_i intersects F_{i+1} in exactly one vertex for all $i < n$. Due to the similarity with normal cube–paths, we will call the folder–paths obeying this condition a *normal folder–path.* As in the case of cube–paths, every folder–path can be identified with a sequence $(v_0, v_1, \ldots, v_n)$ of vertices, such that each pair of consecutive vertices spans an edge or a folder. We can extend such a path to a (geometric) path in the complex $\mathcal{X}(G)$. If we take the union of all paths $P_{(v,b)}$ pointing at b, we will obtain a tree T_b (a dendron in $\mathcal{X}(G)$). One can construct examples of F–complexes in which certain normal folder–paths are not paths of the underlying graph. However, as in the case of cubings, folder–paths correspond to paths in an appropriate graph G^Δ associated with an F–graph G. Namely, let G^Δ be the graph having the same vertex set as G, where two vertices v, w are adjacent iff they belong to a common folder or edge of $\mathcal{X}$.

PROPOSITION 7.5.

(1) $P_{(v,b)}$ is the unique normal folder–path between v and b. All normal folder–paths are shortest paths in G^Δ.

(2) G^Δ is a Helly graph.

PROOF. (1) By induction on $d_{G^\Delta}(b,v)$ we will prove that $P_{(v,b)}$ is a shortest path in G^Δ. In G^Δ pick a neighbour x on a shortest path between v and b. By the induction assumption, $P_{(x,b)}$ is a shortest path. If $d_G(x,b) > d_G(v,b)$, Lemma 7.4 would imply that x, v, and x^b lie in a common folder, thus v and x^b are adjacent in G^Δ, contrary to the choice of x. Hence $d_G(x,b) \leq d_G(v,b)$. If $x = v^b$, we are done. If x and v are adjacent in G, then v^b

is adjacent to both x and x^b. Consequently, v^b is adjacent to both v and x^b in $G^\triangle$, whence $d_{G^\triangle}(v^b,b) < d_{G^\triangle}(v,b)$. Finally, assume that $d_G(x,b) = d_G(v,b)$. The median m of the triplet v,x,b is adjacent to v^b and x^b, therefore m, v^b, x^b belong to a common folder. We obtain $d_{G^\triangle}(v^b,b) < d_{G^\triangle}(v,b)$, again establishing that $P_{(v,b)}$ is a shortest path. As a consequence one obtains that $u \leq_b v$ in G implies $d_{G^\triangle}(b,u) \leq d_{G^\triangle}(b,v)$.

Suppose now that there exists another normal folder–path $P'_{(v,b)}$, and let x be the neighbour of v in it. We may assume that $P_{(x,b)}$ is the unique normal folder–path connecting x and b. This implies $d_G(x,b) \leq d_G(v,b)$. If v and x are adjacent in G, then the folders spanned by vv^b and xx^b intersect in an edge xv^b, contrary to the fact that $F_{vv^b} \in$star$(vx, \mathcal{X})$. Finally, if $d_G(v,b) = d_G(x,b)$, we obtain the same contradiction: the cell spanned by vx intersects F_{xx^b} in the edge xm, where m denotes the median of v,x,b.

Let $B^\triangle_r(b)$ be a ball in $G^\triangle$ with centre x and radius r. Using Lemma 4.2 and the induction on r we will show that $B^\triangle_r(b)$ is a convex set in G. Pick $y,z \in B^\triangle_r(b)$ with $d_G(y,z) = 2$ and let x be their common neighbour in G. If $d_G(b,x) < \max\{d_G(b,y), d_G(b,z)\}$ then $d_{G^\triangle}(b,x) \leq \max\{d_{G^\triangle}(b,y), d_{G^\triangle}(b,z)\} \leq r$, and we are done. So assume that $k = d_G(b,y) = d_G(b,z) < d_G(b,x)$. If x^b is the unique neighbour in $B_{k-1}(b)$ of y or of z, we infer that $x \in B^\triangle_r(b)$. Hence in $B_{k-1}(b)$ we can find the neighbours u and w of y and z, respectively. If $y^b = z^b$ then $(x,y,u,y^b = z^b, z, x)$ is an induced 6–cycle of G. So, $y^b \neq z^b$. By the induction hypothesis $d_{G^\triangle}(b,x_b) \leq \max\{d_{G^\triangle}(b,y^b), d_{G^\triangle}(b,z^b)\} \leq r-1$, leading us to the conclusion that $x \in B^\triangle_r(b)$. Therefore $B^\triangle_r(b)$ is a convex set of G, and by Lemma 4.2 it is gated in G. From the Helly property for gated sets we infer that $G^\triangle$ is a Helly graph. □

The following property, being of independent interest, has been presented in [37] for minimizable graphs.

Lemma 7.6. *Let F be a folder of G and let $x,y \notin F$ be two vertices of G whose gates g_x, g_y in F are distinct and nonadjacent. Then $d_G(x,y) = d_G(x,g_x) + d_G(g_y,y) + 2$, i.e., g_x and g_y lie on a common shortest path between x and y.*

Proof. We argue by induction on $k = d_G(x,g_x) + d_G(y,g_y)$. Suppose by way of contradiction that $d_G(x,y) < k+2$. First, let $k = 2$, i.e., x, g_x and y, g_y are adjacent. Then $d_G(x,y) = 2$. Take a common neighbour z of x and y, and two common neighbours $u,v \in F$ of g_x and g_y. To avoid an induced 6–cycle, z must be adjacent to both u and v. But then x,z,y,u,v,g_x,g_y induce the forbidden $K^-_{3,3}$. So, let $k \geq 3$, say $d_G(x,g_x) \geq 2$. In $I(x,g_x)$ pick a neighbour w of x, and a neighbour $p \neq x$ of w. Clearly g_x is also the gate of w and p. By the induction hypothesis $d_G(w,y) = k+1$. Since $x,p \in I(w,y)$, we will find a common neighbour w' of p and x one step closer to y. As $w' \in I(x,g_x)$, we arrive at a contradiction to the induction assumption. □

We continue by presenting a special subdivision of $\mathcal{X} = \mathcal{X}(G)$. Let E_i $i \in \Lambda$ be the classes of the equivalence relation Ψ defined in the previous section. Clearly, all edges from a folder $K_{2,n}, n \geq 3$, belong to one equivalence class. If G is an F–graph then all edges in E_i have the same length λ_i. An *ij–rectangle* is a rectangle of $\mathcal{X}$ with two opposite edges in E_i and two other opposite edges in E_j. In this case define $\lambda_{ij} = \lambda_i/\lambda_j$.

Let $0 < \lambda < \lambda_i$. Consider an ij–rectangle $R = xuvy$ with $xy, uv \in E_i$ and $xu, vy \in E_j$. Define the points $p(x), p(y) \in [x,y]$, $p(u), p(v) \in [u,v]$, $q(x), q(u) \in [x,u]$, $q(y), q(v) \in [y,v]$, such that

$$|x - p(x)| = |y - p(y)| = |u - p(u)| = |v - p(v)| = \lambda$$

and

$$|x - q(x)| = |y - q(y)| = |u - q(v)| = |v - q(v)| = l \cdot \lambda_{ji}.$$

Subdivide R into 9 rectangles by means of the segments $[p(x), p(u)], [p(y), p(v)], [q(x), q(y)]$ and $[q(u), q(v)]$. In $\mathcal{X}(G)$ R is a union of two right triangles with a common hypotenuse h. Each of these triangles will be subdivided into 3 rectangles and 3 right triangles whose hypotenuses constitute h. Refine this subdivision further by taking in each of 3 rectangles the diagonal parallel to h. If we provide this operation with all ij–rectangles, we obtain a 2–complex $\mathcal{X}'$ whose 2–cells are right triangles. Clearly $\mathcal{X}'$ obeys the conditions (a),(b) and (c). Since $\mathcal{X}$ and $\mathcal{X}'$ have the same underlying space, $\mathcal{X}'$ is CAT(0). By Theorem 7.1, the leg graph of $\mathcal{X}'$ is an F–graph. More generally, given a finite subset of equivalence classes $\Lambda' \subseteq \Lambda$, we can apply the same procedure step by step for all $i \in \Lambda'$, yielding an F–complex and an F–graph as the leg graph. We summarize this in the following property.

LEMMA 7.7. *Performing the subdivision operation with respect to a finite subset of equivalence classes of an F–graph we obtain again an F–graph.*

Recently, Karzanov [37] presented a characterization of minimizable graphs; we refrain from giving definitions here, refering to [37, 38] for precise formulations and interesting details. It turns out that minimizible graphs form a proper class of F–graphs: these are hereditary modular graphs whose edges can be oriented so that in every ij–rectangle each pair of opposite edges have the same orientation. The main (and rather difficult) step in the proof of the result from [37] is to associate with each minimizable graph G the B_2 complex $\mathcal{X}(G)$ endowed with the intrinsic l_1–metric and to prove that every tight extension of the graph–metric d_G is a subspace of this metric space. Below we will show that for an F–complex $\mathcal{X}$ the intrinsic l_1–metric is hyperconvex. In view of lemma of Aronszajn and Panitchpakdi [4] characterizing hyperconvex spaces as injective spaces, the result of [37] will follow. In [34] Isbell introduced collapsible cubical 2–complexes and proved that their rectilinear metric is injective. One can show that such complexes are exactly the CAT(0) $A_1 \times A_1$ complexes (alias cube–free median complexes).

Let $\mathcal{X}$ be a 2–complex obeying the requirements (a),(b),(c). Metrize $\mathcal{X}$ so that each solid ij–rectangle is a copy of the rectangle $[0, \lambda_i] \times [0, \lambda_j]$ in the l_1–plane; define the distance between two points not in a common rectangle as the length of a shortest piecewise–linear path joining them. The length of such a path P is computed in the following way: suppose that there is a subdivision $t_0 < \ldots < t_k$ such that $P(t_i, t_{i+1})$ is contained in some cell. Then sum up the lengths of $P(t_i, t_{i+1})$, the length inside a cell being measured with respect to the l_1–metric. We denote the obtained metric space by $(\mathcal{X}, \rho_1)$. One can see that the restriction of ρ_1 on $LG(K)$ coincides with the weghted shortest–path metric. If $\mathcal{X}$ is a B_2 complex, then $LG(K)$ endowed with the standard graph–metric $d_{LG(K)}$ is isometrically embedded in $(\mathcal{X}, \rho)$. Since the F–graphs are hereditary modular, from [8, 10] we deduce that in F–complexes $\mathcal{X}$ every shortest path of $(LG(K), \rho)$ is a shortest path of $(LG(K), d_{LG(K)})$ and vice versa. Now we formulate the second characterization of F–complexes.

THEOREM 7.8. *A locally–finite 2–complex $\mathcal{X}$ verifying (a),(b), and (c) is an F–complex if and only if $(\mathcal{X}, \rho_1)$ is hyperconvex.*

PROOF. Assume that the metric ρ_1 is injective (thus $\mathcal{X}$ is simply connected), however $\mathcal{X}$ is not an F–complex. First, let $\mathcal{X}$ contain three folders $F_{h_1}, F_{h_2}, F_{h_3}$ and three distinct legs e_1, e_2, e_3 sharing a common vertex such that e_i belongs to F_{h_j} exactly when $i \neq j$. Denote by ν_1, ν_2, ν_3 the lengths of e_1, e_2, e_3, respectively. In F_{h_i} pick a vertex v_i which does belong to other two folders. Set $r_i = \max\{\nu_j, \nu_k\}$, where $i \neq j, k$. One can verify that the balls $B_{r_1}(v_1), B_{r_2}(v_2), B_{r_3}(v_3)$ pairwise intersect, but their common intersection is empty. Second, suppose that two folders F_{h_1} and F_{h_2} have two legs xy and xz in common. Let $v \in F_{h_1}$ and $w \in F_{h_2}$ be some common neighbours of y and z in $LG(K)$. Let μ_1, μ_2 be the lengths of xy, xz, where $\mu_1 \leq \mu_2$. If $v \neq w$, consider the collection of balls with x, y, z, v, w as centers and the following radii: $r(x) = r(y) = r(z) = \mu_2$ and $r(v) = r(w) = \mu_1$. The balls pairwise intersect, however their intersection is empty, because $B_{\mu_2}(x), B_{\mu_1}(v), B_{\mu_1}(w)$ intersect in at most two points z, y, but $z \notin B_{\mu_2}(y), y \notin B_{\mu_2}(z)$. So assume that $v = w$, i.e., we have two solid rectangles R_1, R_2 with the common boundary $xyvz$. Take the balls $B_{\mu_2}(x), B_{\mu_2}(y), B_{\mu_1}(z), B_{\mu_1}(v)$. They intersect in two interior points $p_1 \in R_1, p_2 \in R_2$. Add the balls centered at p_1 and p_2 each having radius $\mu_1/2$. We will get a collection of six pairwise intersecting balls with an empty intersection, a contradiction.

As to the converse, since the balls are compact, it suffices to verify the Helly property only for finite collections of balls. It will be enough to check this only for F–complexes $\mathcal{X}$ with a finite number of cells. Indeed, any finite collection of balls is contained in the subcomplex spanned by a sufficiently large ball of the leg graph. Since $\mathcal{X}$ is locally finite, this ball is finite and by Lemma 7.3 it spans an F–complex. Second, it suffices to verify the Helly property for pairwise intersecting balls whose centers are all located at vertices of $\mathcal{X}$. Indeed, we can subdivide sufficiently many times $\mathcal{X}$ as described before, arriving at a new complex with the same underlying space as $\mathcal{X}$, but in which all ball–centers are vertices. By Lemma 7.7 the resulting complex will be an F–complex as well.

So assume that $\mathcal{X}$ is a finite F–complex, and let $\mathcal{B} = \{B_{r_i}(x_i) : i = 1, \ldots, n\}$ be a collection of pairwise intersecting balls whose centers are vertices of $\mathcal{X}$. Suppose that the result holds for any collection of less than n balls (evidently it holds if $\mathcal{X}$ is a rectangle). To prove that $\cap_{i=1}^{n} B_{r_i}(x_i) \neq \emptyset$ we proceed by induction on the number of vertices of $\mathcal{X}$. We will show how to replace in $\mathcal{B}$ certain balls by balls which are their proper subsets, to obtain a new collection $\mathcal{B}'$ of pairwise intersecting balls in a proper subcomplex of $\mathcal{X}$.

By Lemma 7.4 we can find a vertex v which belongs to a unique edge or to a pendant folder F (as v one can take any vertex at maximum distance from some base point b). Let z be the gate in F of b. If no ball is centered at v, then delete v and the cells incident to v, thus obtaining an F–complex $\mathcal{X}'$ with less vertices. So we may assume that $v = x_1$.

First suppose that F has more than 4 vertices and v is not incident to the hypotenuse of F (notice that this also covers the case $\mathcal{X} = F$). From the choice of v we deduce that v and all non–neighbours of v in F except z have degree 2 in $LG(\mathcal{X})$. Delete from $\mathcal{B}$ the balls around such vertices except the vertex with the smallest radius. The reduced family of pairwise intersecting disks has a point p in common. Clearly, p belongs to all deleted balls, and by the induction assumption we are done. So, further we may assume that $x_1 z$ is the hypotenuse of F. If F is a rectangle, then its edges have two lengths λ and μ (which eventually may coincide), otherwise all edges have the same length $\lambda = \mu$.

For a ball-center x_i ($i > 1$) by g_i denote its gate in F. Let $r_i' = r_i - \rho_1(x_i, g_i)$. If z is the gate of all x_i, then replace $B_{r_1}(x_1)$ by $B_{r_1 - \lambda - \mu}(z)$ if $r_i' < 0$ for some x_i, otherwise replace every ball $B_{r_i}(x_i)$ by $B_{r_i'}(z)$. In both cases we obtain a collection $\mathcal{B}'$ of pairwise intersecting balls in a smaller F–complex (the same procedure applies when x_1 is pendant and z is its unique neighbour). Therefore, at least one such gate is different from z. Consider two vertices

x_i and x_j whose gates g_i and g_j are distinct and both different from z. By Lemma 7.6 and since the shortest paths in $(LG(\mathcal{X}), \rho_1)$ and $(LG(\mathcal{X}), d_{LG(\mathcal{X})})$ are the same, we conclude that

$$r_i + r_j \geq \rho_1(x_i, x_j) = \rho_1(x_i, g_i) + \lambda + \mu + \rho_1(g_j, x_j), \qquad (*)$$

i.e., $r_i' + r_j' \geq \lambda + \mu$.

Suppose $r_1 \geq \lambda = \rho_1(x_1, g_i)$. Then replace $B_{r_1}(x_1)$ by $B_{r_1'}(g_i)$, where $r_1' = r_1 - \lambda$. Note that $r_j + r_1' \geq \rho_1(x_j, g_i)$ holds for all x_j whose gates in F coincide with z or g_i. If this inequality is true when g_i and g_j are not adjacent, by the induction hypothesis the balls intersect. Since $B_{r_1'}(g_i) \subset B_{r_1}(x_1)$, we are done. Hence, further we may assume that there exists at least one x_j such that $r_1' + r_j < \lambda + \mu + \rho_1(x_j, g_j)$, i.e.,

$$r_1' + r_j' < \lambda + \mu. \qquad (**)$$

Next we will show that actually all r_i' are positive. This is evident for centers whose gates coincide with z, because, otherwise $r_1 \geq \lambda + \mu$. Replacing $B_{r_1}(x_1)$ by $B_{r_1 - \rho_1(x_1, g_i)}(g_i)$, where g_i is the gate closest to x_1, we obtain the collection $\mathcal{B}'$ with the desired property. Now let $r_i \leq \rho_1(x_i, g_i)$ for a vertex x_i with $g_i \neq z$. From $(*)$ we have $r_j \geq \rho_1(g_i, x_j)$ for any x_j with $y_j \neq y_i, z$. Since $r_1 \geq \lambda = \rho_1(x_1, g_i)$, we obtain a contradiction with $(**)$.

So $r_i' > 0$ for all $i > 1$. Replace every $B_{r_i}(x_i)$ by $B_{r_i'}(g_i)$. Together with $B_{r_1}(x_1)$ these balls constitute $\mathcal{B}'$. Notice that $r_1 + r_k' \geq \lambda + \mu$ for all x_k with $g_k = z$. On the other hand, by $(*)$ $r_i' + r_j' \geq \lambda + \mu$ for all x_i, x_j whose gates are distinct and both different from z. If $r_i' + r_k' \geq \rho_1(g_i, g_k)$ for all $g_i \neq g_k = z$, then in F we will find a common point of balls from $\mathcal{B}'$, which belongs to all balls from $\mathcal{B}$ as well. Hence, $r_i' + r_k' < \rho_1(g_i, g_k) = \mu$ for some g_i, g_k. From two previous inequalities we infer that

$$r_1 + r_k' + r_i' + r_j' \geq 2\lambda + 2\mu > 2\lambda + \mu + r_i' + r_k',$$

i.e., $r_1 - \lambda + r_j' > \lambda + \mu$, contrary to $(**)$. This concludes the proof. □

An alternative metrization of an F–complex $\mathcal{X}$ is obtained by extending the l_∞–metric from the cells. Each ij–rectangle is a copy of the rectangle $[0, \lambda_i] \times [0, \lambda_j]$ in the l_∞–plane. As in the previous section, denote the resulting intrinsic metric on $\mathcal{X}$ by ρ_∞. The (weighted) Helly graph G^Δ associated to the 1–skeleton G of $\mathcal{X}$ is isometrically embedded in $(\mathcal{X}, \rho_\infty)$. This suggests that $(\mathcal{X}, \rho_\infty)$ is injective. We only sketch the proof of this property in the case when all cells are isosceles right triangles. It suffices to show that any ball $B_r(x)$ of $(\mathcal{X}, \rho_\infty)$ is a convex set of $(\mathcal{X}, \rho_1)$ (the second space being modular, all its compact convex sets are gated). It is sufficient to show this for finite F–complexes and balls with rational radii. Since regular subdivision of all cells with some given mesh yields again an F–complex, it suffices to consider a ball $B_r(x)$ whose center is a vertex and whose radius is an integer. Then one can easily see that $B_r(x)$ is a union of cells of $\mathcal{X}$. Moreover, from the proof of Proposition 7.5 we infer that if two nonadjacent vertices of some folder F belong to $B_r(x)$, then the whole folder lie in $B_r(x)$. By Lemma 4.2 the subgraph of G induced by $B_r(x)$ is convex in G. This implies that $B_r(x)$ is convex in $(\mathcal{X}, \rho_1)$. Therefore, we have proven the following property of flag complexes.

PROPOSITION 7.9. *If $\mathcal{X}$ is a locally finite F–complex, then $(\mathcal{X}, \rho_\infty)$ is an injective space.*

8. Bridged complexes

Every bridged graph G gives rise to an abstract simplicial complex $K(G)$ whose simplexes are the complete subgraphs of G. Trivially, $K(G)$ is a flag complex. The geometric realization $|K(G)|$ is called a *bridged complex*. In this section we present a characterization of flag complexes whose underlying graphs are bridged.

THEOREM 8.1. *For a simplicial flag complex K the following conditions are equivalent:*

(i) K is simply connected and the link of every vertex v does not contain induced 4–cycles and 5–cycles;

(ii) $G(K)$ is weakly modular and does not contain induced 4–cycles and 5–cycles;

(iii) $G(K)$ is bridged.

PROOF. (i)$\Rightarrow$(ii): We actually need a weaker version of the flag condition: if three 2–simplexes share a common vertex, and pairwise share common edges then they are contained in a 3–simplex of K. Consider a simple cycle C in the graph $G(K)$ and let (D, f) be a minimal disk of C.

CLAIM 1. *The degree of every interior vertex x of D is at least 6.*

Let $x_1, \ldots, x_n$ be the neighbours of x, where $\sigma_i = xx_ix_{i+1(mod\ n)}$ $(i = 1, \ldots, n)$ are the faces incident to x. Trivially, $n \geq 3$. Suppose by way of contradiction that $n \leq 5$. From Lemma 5.1 we conclude that $f(\sigma_1), \ldots, f(\sigma_n)$ are distinct 2–simplexes of K. If $n = 3$ then $f(\sigma_1), f(\sigma_2), f(\sigma_3)$ intersect in $f(x)$ and pairwise share an edge of K. Therefore they are contained in a 3–simplex of K. This implies that $\delta = f(x_1)f(x_2)f(x_3)$ is a 2–face of K. Let D' be a disk triangulation obtained from D by deleting the vertex x and the triangles $\sigma_1, \sigma_2, \sigma_3$, and adding the 2–simplex $x_1x_2x_3$. The map $f : V(D') \to V(K)$ is simplicial, because it maps $x_1x_2x_3$ to δ. Therefore (D', f) is a disk diagram for C, contrary to the choice of D. Now, let x has four neighbours. The cycle $(x_1, x_2, x_3, x_4, x_1)$ is sent to a 4–cycle in $\text{link}(f(x), K)$, in which two opposite vertices, say $f(x_1)$ and $f(x_3)$, are adjacent. Consequently, $\delta' = f(x_1)f(x_3)f(x_2)$ and $\delta'' = f(x_1)f(x_3)f(x_4)$ are 2–faces of K. Let D' be a disk triangulation obtained from D by deleting the vertex x and the triangles $\sigma_i (i = 1, \ldots, 4)$, and adding the 2–simplexes $\sigma' = x_1x_3x_2$ and $\sigma'' = x_1x_3x_4$. The map f remains simplicial, since it sends σ', σ'' to δ', δ'', respectively, contrary to the choice of D. Finally, suppose that $n = 5$. Consider the 5–cycle of $\text{link}(f(x), K))$ induced by the vertices $f(x_1), f(x_2), f(x_3), f(x_4), f(x_5)$. By the initial hypothesis concerning K, we deduce that there is a vertex, say $f(x_1)$, which is adjacent to all other vertices of this cycle. This implies that $f(x_1)f(x_3)f(x_2); f(x_1)f(x_4)f(x_3)$ and $f(x_1)f(x_5)f(x_4)$ are 2–faces of K. Let D' be obtained from D by deleting the vertex x and the faces $\sigma_i (i = 1, \ldots, 5)$, and adding three new faces $x_1x_3x_2, x_1x_4x_3$, and $x_1x_5x_4$. As in preceding cases, we can conclude that (D', f) is a disk diagram for C, yielding a contradiction to the minimality of C.

From this claim we obtain that any minimal disk D is a (3,6)–complex, thus a CAT(0) space. The turning angle $\tau(v)$ at any corner v of D is $2\pi/3$ or $\pi/3$. The corners of first type are the boundary vertices of degree two. The corners of second type are boundary vertices of degree three. In the first case two neigbours of v are adjacent. In the second case v and its neigbours in ∂D are adjacent to the third neighbour of v. From Gauss–Bonnet formula we infer that D contains at least three corners, and if D has exactly three corners then all they are of first type. Together with the claim this ensures that all 3–cycles of $G(K)$ bounds 2–simplexes of K and that $G(K)$ does not contain induced 4–cycles and 5–cycles.

Next we show that $G(K)$ is weakly modular. To verify the triangle condition, pick three vertices u, v, w with $1 = d_{G(K)}(v, w) < d_{G(K)}(u, v) = d_{G(K)}(u, w) = k$. We claim that if $I(u, v) \cap I(u, w) = \{u\}$, then $k = 1$. Suppose not. Pick two shortest paths P' and P'' joining the pairs u, v and u, w, respectively, such that the cycle C composed of P', P'' and the edge vw has minimum area. Choose a minimal disk D of C, which necessarily has a corner x distinct from u, v, w. Let $x \in P'$. Notice that x is a corner of second type, otherwise its neighbours y, z in P' are adjacent, contrary to the assumption that P' is a shortest path. Let p be the vertex of D adjacent to x, y, z. If we replace in P' the vertex x by p, we will obtain a new shortest path between u and v. Together with P'' and the edge vw this path forms a cycle C' whose area is strictly smaller than $\mathrm{Area}(C)$, contrary to the choice of C. As to the quadrangle condition, suppose by way of contradiction that we can find distinct vertices u, v, w, z such that $v, w \in I(u, z)$ are neighbours of z and $I(u, v) \cap I(u, w) = \{u\}$, however u is not adjacent to v and w. Again, select two shortest paths P' and P'' between u, v and u, w, respectively, so that the cycle C composed of P', P'' and the edges vz and zw has minimum area. Choose a minimal disk D of C. From the initial hypothesis concerning the vertices u, v, w, z we deduce that D has at most one corner of first type located at u. Hence D contains at least four corners. If one corner x is distinct from u, v, w, z, then proceeding in the same way as before, we will obtain a contradiction with the choice of P', P''. Therefore u, v, w, z are the only corners of D, where $\tau(u) = 2\pi/3, \tau(v) = \tau(w) = \tau(z) = \pi/3$. Since the sum of turning angles at corners is smaller than 2π, we arrive at a contradiction with Gauss–Bonet formula. The proof of (ii)$\Rightarrow$(iii)$\Rightarrow$(i) follows from the following result.

Claim 2. *A graph G is bridged if and only if it is weakly modular and does not contain 4–cycles and 5–cycles as induced subgraphs.*

If G is bridged, then trivially G does not contain induced 4–cycles and 5–cycles. Weak modularity of G easily follows from the fact that the cycles of G are well–bridged. Conversely, let G be a weakly modular graph without induced 4–cycles and 5–cycles. To show that G is bridged, by Theorem 4.5 and since $B_{k+r}(S) = B_k(B_r(S))$, it suffices to show that if S is convex then the neighbourhood $B_1(S)$ of S is convex, too. By Lemma 4.2 it suffices to check 2–convexity only. Pick two vertices $x, y \in B_1(S)$ at distance two. We assume without loss of generality that $x, y \notin S$, otherwise immediately we obtain $I(x, y) \subseteq B_1(S)$. Let x' and y' be neighbours of x and y in S minimizing $d_G(x', y')$. Suppose by way of contradiction that there is a common neighbour z of x and y outside $B_1(S)$. Since S is convex $d_G(x', y') \leq 3$. If x' and y' were adjacent, then the vertices x', x, z, y, y' would induce a forbidden cycle. If $d_G(x', y') = 3$, then necessarily $d_G(x, y') = 3$. By the triangle condition there exists a common neighbour u of x and x' at distance 2 to y'. Since S is convex, $u \in S$, contrary to the choice of x'. Finally, assume $d_G(x', y') = 2$, and let z' be adjacent to x' and y'. If $d_G(z, z') = 3$, by the quadrangle condition we will find a common neigbour v of x', y', z. Since $v \in I(x', y') \subseteq S$, we arrive at a contradiction to the choice of z. Otherwise, if $d_G(z', z) = 2$, by the triangle condition we can find a common neighbour s of z, x', y' and a common neighbour t of z, z', y'. In order to avoid forbidden 4–cycles and 5–cycles, the vertices s, t must be adjacent to x, x', y, y'. This implies $s, t \in I(x', y') \subseteq S$, contrary to the choice of z. This concludes the proof. □

COROLLARY 8.2. *$\mathcal{X}$ is a CAT(0) complex of type A_2 if and only if it is a bridged 2–complex. In particular, any triangle building is bridged.*

Another example of a CAT(0) complex whose 1–skeleton is bridged is provided by any tiling of the plane into acute angled triangles.

Although bridged graphs are characterized by a discrete variant of a property describing CAT(0) spaces (see the remarks after Theorem 3.2), not all bridged complexes are CAT(0). We present two such examples. First, take a 6–wheel consisting of a 6–cycle $(x, u_1, u_2, y, u_3, u_4, x)$ and a central vertex z, and replace the vertices u_1, u_2, u_3, u_4 by n–cliques C_1, C_2, C_3, C_4. Make x adjacent to all vertices of C_1 and C_4, y to all vertices of C_2 and C_3, and z to the vertices of all four cliques. Finally, assume that $C_1 \cup C_2$ and $C_3 \cup C_4$ form $(n+1)$–cliques. We obtain a new bridged graph G_n. One can imagine G_n as glued from two identical halves along the path (x, z, y). G_1 coincides with the initial 6–wheel. In this case (x, z, y) is the unique geodesic in $|G_1|$ connecting the vertices x and y. This is not longer true in $|G_n|$ for sufficiently large n. Indeed, let b_i be the barycenter of the simplex $|C_i \cup \{z\}|$, $i = 1, \ldots, 4$. Then $d(x, b_1) = d(x, b_4) = d(y, b_2) = d(y, b_3) = \sqrt{\frac{n+1}{2n}}$ and $d(b_1, b_2) = d(b_3, b_4) = \frac{\sqrt{n-1}}{n}$. The paths (x, b_1, b_2, y) and (x, b_4, b_3, y) have length $\sqrt{\frac{2(n+1)}{n}} + \frac{\sqrt{n-1}}{n}$, which is smaller than 2. This shows that the path (x, z, y) is not a geodesic. Due to the symmetry between the halves, we conclude that x and y can be connected with more than one shortest path, showing that $|G_n|$ is not CAT(0).

Even not all bridged 3–complexes are CAT(0). For this, let Γ be the bridged graph with vertices $z, x_1, x_2, x_3, u_1, u_2, v_1, v_2, w_1, w_2$, where z is adjacent to all vertices, u_1, u_2 are adjacent to x_1, x_2; v_1, v_2 are adjacent to x_2, x_3, and w_1, w_2 are adjacent to x_3, x_1. In the link of z consider the geodesic segments joining each pair of vertices x_1, x_2, x_3. For example, the geodesic segment between x_1 and x_2 consists of two edges sharing the midpoint of the segment u_1u_2. Their union is an isometric cycle of $\mathrm{Link}(z, |\Gamma|)$ of length $5\pi/2$, thus $|\Gamma|$ is not CAT(0).

Bridson [16] established that all simplicial 3–complexes whose cells are regular and the links of all vertices are PE (3,6)–disks are CAT(0). Clearly, such complexes are bridged. Following the lines of the proof from [16], one can show that all bridged 3–complexes in which the links of vertices are bridged are CAT(0). Bridged complexes K in which the links of all simplexes are bridged form a very special class of simplicial complexes. As we will show below their skeletons do not contain induced cycles of length larger than three (such graphs are the well known *chordal graphs*).

COROLLARY 8.3. *The links of all simplexes of a simply connected flag complex K are bridged if and only if the 1–skeleton $G(K)$ is a chordal graph.*

PROOF. Let σ be a maximal simplex whose link contains an induced cycle C of length larger than three. Since $\mathrm{link}(\sigma, K)$ is bridged, one can construct a minimal disk D for C. We may suppose that all cycles of $\mathrm{link}(\sigma, K)$ whose minimal disks have smaller area are not induced. From the proof of Theorem 8.1 we know that D is a (3,6)–disk, thus it has a corner x. Since C is induced, the turning angle at x is $\pi/3$, i.e., it belongs to exactly two faces xyz_1

and xyz_2 of D. The cycle C' obtained from C by replacing x by y has smaller area, thus it is not induced. Therefore y is adjacent to some vertex $v \neq x$ of C. The edge yv splits C' in two cycles of smaller area. Continuing this way, we obtain that y is adjacent to all vertices of C. This means that C is a cycle in the link of the simplex $\sigma \cup \{y\}$, contrary to the choice of σ. The converse is immediate, because the skeletons of links of all simplexes will be chordal graphs. □

It is well known (cf. [32]) that every finite chordal graph G has a simplicial vertex, i.e. a vertex x such that its neighbours form a complete subgraph. If G is the graph of a complex K, this means that x belongs to a unique maximal simplex σ. Consequently, $|K|$ can be obtained by gluing $|\sigma|$ and $|K'|$, where K' is the subcomplex of K spanned by $\sigma' = \sigma - \{x\}$ and the maximal simplexes distinct from σ. Since the gluing is performed along a convex set $|\sigma'|$, from the result of Rechetniak (cf. Theorem 4.5 of [17]) we obtain that $|K|$ is CAT(0) if and only if $|\sigma|$ and $|K'|$ are CAT(0). In particular, we have the following observation (to pass from finite to arbitrary complexes we use Theorem 3.2).

Corollary 8.4. *If the graph of a flag complex K is chordal, then $|K|$ is CAT(0).*

9. Combings and bicombings of some graphs: a clin d'oeil

Both notions of combing and bicombing have been introduced in [28] for standard word metric on the Caley graph of a group in order to define an automatic or a biautomatic structure on the group. A (geodesic) combing of a group is a selection for each element g of one shortest path beginning at the identity and ending at g. A reasonable requirement to a combing is that the paths to neighbouring points are uniformly close. A bicombing is a choice of shortest paths between each pair of points which, in some sense, imitates the convexity of the distance function between geodesics in a CAT(0) space. The study of infinite groups by means of the geometry of their Caley graphs has proved to be a powerful technique for understanding large classes of groups; [19, 28, 29, 30, 44, 49] are only few references.

In this section we will investigate the concepts of combing and bicombing in the framework of graphs. We will show that for all classes of graphs introduced in the previous sections there exists combings and bicombings having tight fellow traveller property. Their particular instances are Caley graphs of groups acting on some CAT(0) complexes, for example, A_2 and B_2 complexes and cubings.

During this section we assume that the graphs are locally finite (i.e. every vertex has a finite number of neighbours). Let $[0,n]^*$ denote the set of integer points from the segment $[0,n]$. Given a path of length n of a graph G, we can parametrize it and denote $p : [0,n]^* \to G$. It is often convenient to extend p over a larger interval, $p : [0,m]^* \to G$, for some $m > n$. We do this by setting $p(t) = p(n)$ for all $t \geq n$. (In particular, when comparing two such paths, we may assume that they are defined on the same interval).

Suppose that $p, q : [0,n]^* \to G$ are two paths in G, and k a positive integer. We say that p and q are *k–fellow travellers* if $d_G(p(t), q(t)) \leq k$ for all integers t with $0 \leq t \leq n$. Following [29], we say that two paths $p, q : [0,n]^* \to G$ *lie in a k–Hausdorff neighboudhood* of each other if every vertex of p is at distance at most k from q and vice versa, i.e., for every $t \in [0,n]$ there exist $s', s'' \in [0,n]$ such that $d_G(p(t), q(s')) \leq k$ and $d_G(p(s''), q(t)) \leq k$.

Let b be a base point of G. A *(geodesic) k–combing* is a choice of a shortest path $P_{(b,x)}$ between b and each vertex x, such that $P_{(b,v)}$ and $P_{(b,w)}$ are k–fellow travellers for any adjacent vertices v and w. One can imagine the union of combing paths as a spanning tree T_b of G rooted at b and preserving the distances from b to all vertices. The neighbour of v in the unique path of T_b connecting v with the root will be called the *father* of v.

A *(geodesic) k–bicombing* is a choice of a shortest path $P_{(v,v')}$ between each pair of vertices v, v', such that $P_{(v,v')}$ and $P_{(w,w')}$ are k–fellow travellers for any vertices v, v', w, w' such that $d_G(v,w) \leq 1$ and $d_G(v',w') \leq 1$. A particularly strong form of bicombing is the case in which the paths $P_{(v,v')}$ and $P_{(v',v)}$ coincide. In this case we say that the bicombing is *symmetric.*

9.1. Combings

Given a base point b, a natural way to comb a graph G is to find a total order $\prec_b$ refining the base point order $\leq_b$. Let α be a strictly isotone map from $V(G)$ to $\mathbb{N}$, i.e., $\alpha(v) < \alpha(w)$ if and only if $v \prec_b w$. Denote by G_n the subgraph of G induced by vertices v with $\alpha(v) \leq n$. The *father* of a vertex $v \neq b$ is the neighbour v_b of v with the least value $\alpha(v_b)$. Clearly, $d_G(b, v_b) < d_G(b,v)$. Define the combing path $P_{(b,v)}$ be the union of the path $P_{(b,v_b)}$ and the edge vv_b.

Perhaps the simplest algorithmic way to define $\prec_b$ is provided by the well known *Breadth–First Search (BFS)* procedure. In the Breadth–First Search the vertices of G are numbered in increasing order. We number with 1 the vertex b put it on an initially empty queue of vertices. We repeatedly remove the vertex v at the head of the queue and consequently number and place onto the queue all still unnumbered neighbours of v. BFS constructs a spanning tree T_b of G with the vertex b as a root. A vertex x is the father of any of its neighbours in G included in the queue when x is removed (a detailed description of the BFS procedure is presented, for example, in the book of Golumbic [32]). In [24] we have investigated the properties of BFS orderings of weakly modular graphs. In particular, we have shown that for all house–free weakly modular graphs this ordering is distance preserving, i.e. each G_n is an isometric subgraph of G. In [23] we established that any ordering of a bridged graph produced by BFS is a domination ordering. We continue this line of research here by showing that BFS can be used for combing hereditary weakly modular graphs and graphs with convex balls (for an illustration see the first graph from Figure 4). A part of next result is implicit in [24].

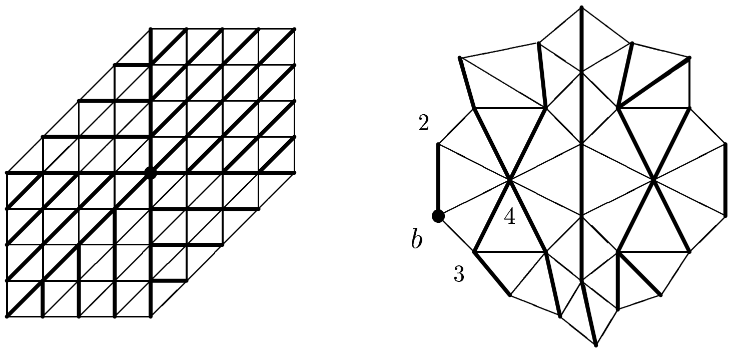

FIGURE 4. BFS–combing and a partial bicombing of bridged graphs.

PROPOSITION 9.1. *Let G be a hereditary weakly modular graph and b a base point. Then the paths $\{P_{(b,v)} : v \in V(G)\}$ constructed by BFS form a geodesic 2–combing of G. If G is bridged, then this combing satisfies the 1–fellow traveller property.*

PROOF. Let v and w be two adjacent vertices of G with $v \prec_b w$, and let v_b and w_b be their fathers. First assume that $d_G(v,b) = d_G(w,b) = k$. To prove that the paths $P_{(b,v)}$ and $P_{(b,w)}$ are 2–fellow travellers it suffices to establish that v_b and w_b are adjacent or coincide and that w_b and v are adjacent. We proceed by induction on k. If $k = 1$, then $v_b = b = w_b$ and we are done. So, let $k > 1$. Suppose, for a contradiction, that $d_G(v_b, w_b) > 1$. Since $v \prec_b w$, according to BFS $v_b \prec_b w_b$. Thus v_b and w must be non–adjacent.

First let $d(v_b, w_b) = 3$. Since $d_G(v_b, b) = d_G(w_b, b)$, each pseudo–median of b, v_b, w_b has size 3 or size 1. In the first case it consists of v_b, w_b and a vertex x at distance 3 from v_b and w_b. From the properties of weakly modular graphs presented in Section 4 we deduce that $d_G(v,x) = d_G(w,x) = 3$. Since $d_G(x,b) = k-4$, we get a contadiction with $d_G(b,v) = d_G(b,w) = k$. So, assume that the pseudo–median y,z,x of v_b, w_b, b has size 1. Then $d_G(y,b) = d_G(z,b) = k-2$ and $d_G(x,b) = k-3$. Since $v, y \in I(v_b, w_b)$, by the quadrangle condition there is a common neighbour s of v, y and w_b. As G is house–free, s and z must be adjacent. But then the vertices v_b, v, y, s, z induce a house, because $d_G(v_b, w_b) = 3$ and $d_G(y,b) = d_G(z,b) = k-3$.

Now, assume $d_G(v_b, w_b) = 2$. By the triangle condition applied to the vertices v_b, w_b, w, we can find their common neighbour z. In order to avoid an induced house, the vertices z and v must be adjacent. According to BFS $\alpha(z) > \alpha(w_b)$. We assert that $d_G(z,b) = k-1$. Indeed, otherwise $d_G(z,b) = k$ and $v_b, w_b \in I(z,b)$. By the quadrangle condition there is a common neighbour r of v_b and w_b at distance $k-2$ from b. Then we get two houses, induced by the vertices v, w, v_b, w_b, z, r. So, let $d_G(z,b) = k-1$. Consider the fathers x, z_b, y of the vertices v_b, z, w_b, respectively. By the induction assumption $d_G(x, z_b) \le 1$ and $d_G(z_b, y) \le 1$. In addition, z_b must be adjacent to both v_b and w_b, because $\alpha(v_b) < \alpha(z) > \alpha(w_b)$. Then, however, the vertices z_b, v_b, v, w, w_b induce a 5–cycle, which is impossible. So, $d_G(v_b, w_b) \le 1$.

It remains to prove that if $v_b \ne w_b$, then w_b and v are adjacent. Suppose the contrary. Since v_b and w_b are at distance $k-1$ to b, by the triangle condition there is a common neighbour b' of v_b and w_b at distance $k-2$ to b. As a result we will get a house, induced by v, w, v_b, w_b, b'.

To complete the proof, consider the second case $k = d_G(b,v) < d_G(b,w)$. By induction on k we will show that v_b and w_b are adjacent. We may suppose that w_b is not adjacent to v, otherwise from the preceding case we are done. Let y be the father of w_b. Necessarily y is not adjacent to v, otherwise we are in contradiction with BFS. By the quadrangle condition we can find a common neighbour z of v and w_b at distance $k-1$ to b. Since $\alpha(v_b) < \alpha(z) > \alpha(w_b)$, by the induction assumption, v_b and y must be adjacent to the father z_b of z. The 6–cycle $(v, w, w_b, y, z_b, v_b, v)$ must contain at least one diagonal, otherwise by Lemma 10 of [24] there is a vertex adjacent to all vertices of this cycle, contrary to the fact that $d_G(v, z_b) = 3$. But z_b is not adjacent to any of v, w_b, w, because $d_G(z_b, b) = k-2$. Before we noticed that v is not adjacent to w_b, y. Therefore v_b and y are adjacent. To avoid an induced 5–cycle, the vertices v_b and w_b must be adjacent, as desired.

If G is bridged and $k = d_G(b,v) < d_G(b,w)$, then necessarily w_b is adjacent to v, reducing the second case to the first one, thus establishing the 1–fellow traveller property. □

The tth *level* of an interval $I(b,v)$ consists of all vertices of $I(b,v)$ at distance t to b. For simplicity, let $v_t = P_{(b,v)}(t), w_t = P_{(b,v)}(t)$. The following simple property of BFS–orderings is essential.

REMARK 9.2.

(1) v_t is the vertex of the tth level of $I(b,v)$ with the minimum value of α;

(2) If $d_G(b,v) = d_G(b,w) = n$ and $t \in [0,n]^*$, then $v \prec_b w$ if and only if $v_t \prec_b w_t$.

PROPOSITION 9.3. *Let G be a graph with convex balls and b a base point. Then the paths $\{P_{(b,v)} : v \in V(G)\}$ constructed by BFS form a geodesic 2–combing of G.*

PROOF. Pick two adjacent vertices v and w of G such that with $v \prec_b w$, $d_G(b,v) = n$ and $v_b \neq w_b$. If $d_G(b,v) < d_G(b,w)$, from the convexity of $B_n(b)$ we infer that v and w_b are adjacent. Thus assume that $d_G(b,w) = n$. Let b' be the lowest common ancestor of v and w in the tree T_b, i.e. the furthest from b common vertex of the combing paths $P_{(b,v)}$ and $P_{(b,w)}$. Consider the cycle C formed by the edge vw and the subpaths of $P_{(b,v)}$ and $P_{(b,w)}$ between b' and v and w, respectively. If C is a 3– or a 5–cycle, then clearly $d_G(v_t, w_t) \leq 2$ for any nonnegative integer $t \leq n$. Otherwise, since C is well–bridged either v and v_b are adjacent to w_b, or there is a bridge from w to a vertex $x \neq b'$ on the shortest path of C between v and b'. Since $d_G(v,b) = d_G(w,b)$, we obtain that $d_G(v,x) = d_G(w,x)$, whence $x \in I(b,w)$. Let y be the vertex on the shortest path of C between w and b' with $d_G(y,b') = d_G(x,b')$. Since x and y have the same distance to b, Remark 9.2(1) implies $y \prec_b x$. But then from Remark 9.2(2) we conclude that $w \prec_b v$, which is impossible. □

To find a combing of a Helly graph G we define the rooted tree T_b as follows. Given a vertex v at distance n to b, there exists a neighbour v_b of v at distance $n-1$ to b and which is adjacent to all neighbours x of v with $d_G(b,x) \leq n$ (in fact, this property characterizes the Helly graphs [12]). Indeed, take the collection of pairwise intersecting balls consisting of $B_{n-1}(b), B_1(v)$ and $B_1(x)$ for all neighbours x of v with $d_G(b,x) \leq n$. Then we can select any point from their intersection as the father v_b of v. Pick two adjacent vertices v, w. If $d_G(b,v) = d_G(b,w)$, then either their fathers coincide or v, w, v_b, w_b are pairwise adjacent. Otherwise, if $d_G(b,w) > d_G(b,v)$ and $w_b \neq v$, necessarily w_b is adjacent to v and v_b, thus establishing the following observation.

PROPOSITION 9.4. *Every Helly graph G has a geodesic 1–combing.*

We continue with combings of hypercubes and median graphs. Notice that one cannot depart from an arbitrary BFS ordering, even if we want to comb the 3–cube. Let G be a median graph with a base point b isometrically embedded into the hypercube $H(\Lambda)$ as was presented in Section 6. To comb $H(\Lambda)$, we define $\prec_b$ be the lexicographic order on the finite subsets of the totally ordered set Λ ($\prec_b$ can be obtained by a stonger version of BFS, namely, by the lexicographic breadth–first search). The father of a vertex A in the combing tree T_b is the vertex $A - \{\lambda\}$, where λ is the largest element in A. If B is a neighbour of A, say $B = A - \{\mu\}$, then $A - \{\lambda, \mu\}$ is the father of B, showing that T_b provides a geodesic 2–combing of $H(\Lambda)$.

Since G is a retract of $H(\Lambda)$, G has a geodesic 2–combing as well. In fact, we can provide an explicit 2–combing of G. For this, let $\Lambda_n = \{i \in \Lambda : i \leq n\}$ and $G(n)$ be the subgraph induced by the vertices of G contained in Λ_n. Clearly $G(1) \subseteq G(2) \subseteq \ldots$ and $G = \cup_{n\in\Lambda} G(n)$. Every $G(n)$, being the intersection of G with the hypercube $H(\Lambda_n)$, is a convex subgraph of G. The equivalence class E_i splits G in two halfspaces G'_{n+1} and G''_{n+1}, where $G(n)$ is contained in one of them, say $G(n) \subseteq G'_{n+1}$. One can easily show that the complement H'' of $G(n)$ in $G(n+1)$ consists of all vertices v'' of H''_{n+1} whose neighbours v' in H'_{n+1} belong to $G(n)$. (In other words, H'' is a thin halfspace of $G(n+1)$.) Let H' denote the subgraph of $G(n)$ induced by the vertices v'. Notice that both H', H'' are convex subgraphs of the graphs $G(n+1)$ and $G(n)$ (see subsection 6.2). The vertex v' will be called the father of v''.

Assume that we have defined the combing tree $T_b^+(n)$ in the graph $G(n)$ and we wish to extend it to the tree $T_b^+(n+1)$ of $G(n+1)$. For this, add to $T_b^+(n)$ all edges of the type $v'v''$, thus v' becomes the unique neighbour of v'' in $T_b^+(n+1)$. If $v'', w'' \in H''$ are adjacent, then by the convexity of $G(n)$ we infer that their fathers are adjacent as well. This shows that the resulting tree $T_b^+ = \cup_{n\in\Lambda} T_b^+(n)$ provides a geodesic 2–combing for G.

There is yet another way to grow a combing tree T_b^- of G. Namely, given the current tree $T_b^-(n)$, we intersect it with H' and consider the connected components of this intersection. In each such component C we pick the vertex x_C closest to the root b. (In fact, one can view C as a tree rooted at x_C). In H'' take the copy of each C, add it to $T_b^-(n+1)$, previously connecting it with C by the edge between x_C and its neighbour in H''. Set $T_b^- = \cap_{n\geq 1} T_b^-(n)$. Before establishing the 2–fellow property we pause to emphasize two properties of the algorithm. First, notice that there are no 4–cycles of G in which exactly two incident edges belong to T_b^-. Second, if a 4–cycle C share with T_b^- three edges, then the fourth edge belongs to an equivalence class with a smaller number than that of the equivalence class which contains the two opposite edges from $C \cap T_b^-$.

Let v, w be adjacent vertices of $G(n+1)$. Consider the paths $P_{(b,v)}$ and $P_{(b,w)}$ of T_b^- from b to v and w. To check that $P_{(b,v)}$ and $P_{(b,w)}$ are 2–fellow travellers, by induction on n we will prove the following stronger condition, which we call the *"1.5–fellow traveller property":*

each vertex of $P_{(b,v)}$ is either adjacent to a vertex of $P_{(b,w)}$ or its neighbours in $P_{(b,v)}$ are adjacent to a common vertex in $P_{(b,w)}$ and vice versa.

The result easily follows if $v \in H'$ and $w \in H''$. So suppose that $v, w \in H''$, say $v = v'', w = w'$. If v' and w' belong to a common connected component of $T_b^-(n) \cap H'$, or they belong to different components C', C'', but neither v' nor w' are the roots, the result easily follows by the induction hypothesis and the algorithm. Finally, suppose that $v' \in C'$ is the root of C', while the root of the connected component C'' containing w' is a vertex $x' \neq w'$. According to the used procedure, $P_{(b,v)}$ consists of $P_{(b,v')}$ and the edge $v'v$. The path $P_{(b,w'')}$ consists of $P_{(b,x')}$, the edge $x'x''$, and the image in H'' of the path $P_{(x',w')}$. Notice that $P_{(b,w')}$ consists of $P_{(b,x')}$ and $P_{(x',w')}$. Let z' be the neighbour of w' in $P_{(x',w')}$ and y, u be the last two vertices in $P_{(b,v')}$. By the induction assumption, either u is adjacent to z' or w' and y are adjacent. In the first case $u \in I(v', z') \subseteq H'$, because H' is convex. Then $u \in C'$, contrary to the assumption that v' is the closest to b vertex of C'. Otherwise, if w' is adjacent to y, then we will get a 4–cycle (y, u, v', w', y) which contradicts the properties established before.

For an illustration of both methods, see Figure 5. In this figure, the median graph is isometrically embedded in $\mathbb{R}^3$ endowed with the l_1–metric. The equivalence classes are performed in the order indicated by the labels of edges in the tree.

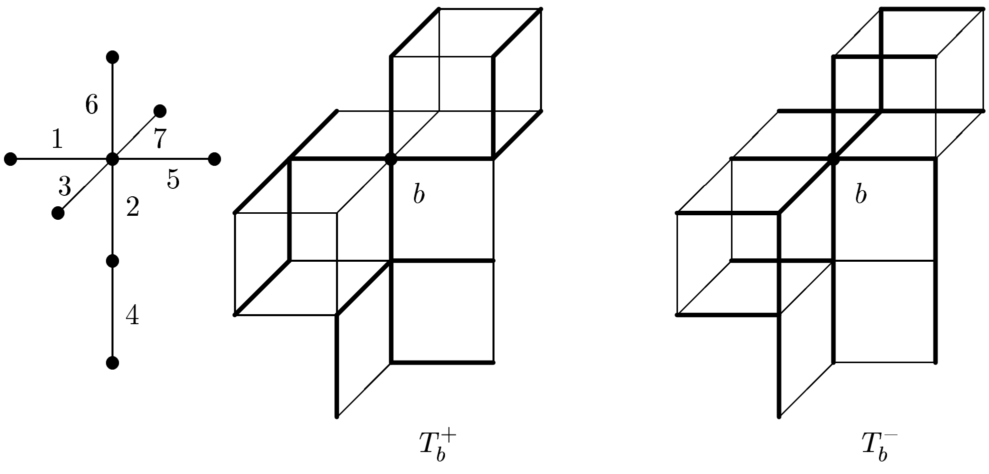

FIGURE 5. Combing trees T_b^+ and T_b^- of a median graph.

Slightly modifying the first procedure, one can obtain a geodesic 1–combing of the associated to G Helly graph G^Δ. For this, notice that a cube C of G intersects every $G(n)$ in a subcube. Therefore, if C intersects both $G(n)$ and $H'' = G(n+1) - G(n)$, then it shares with each of them a facet. Let $T_b^\Delta = \cup_{n\in\Lambda} T_b^\Delta(n)$; we extend the combing tree $T_b^\Delta(n)$ in the following way. For a new vertex v'' we consider the neighbour x of v' in the current tree $T_b^\Delta(n)$. If xv'' spans a cube, then add the edge $v''x$ to $T_b^\Delta(n+1)$. Otherwise add the edge $v'v''$ to $T_b^\Delta(n+1)$. By induction on n we will show that the path $P_{(b,v'')}$ of $T_b^\Delta(n+1)$ connecting b and v'' is a normal cube–path in $G(n+1)$ (and therefore in G). Let C, C' be the cubes spanned by first two edges of $P_{(b,v'')}$. We must show that $C\cap$ star$(C', \mathcal{X})$ consists of one vertex, namely the neighbour of v'' in $T_b^\Delta(n+1)$. If v'' and v' have a common neighbour x in $T_b^\Delta(n+1)$, then a cube Q from star$(C', \mathcal{X})$ intersects the facet of C from H'' if and only if it intersects the complementary facet spanned by the edge $v'x$, contrary to the induction assumption. Now, let v' be the neighbour of v'' in $T_b^\Delta(n+1)$. Then C consists only of v', v''. If $v'' \in Q$, then v'' and the neighbour x of v' both belong to Q, implying that actually x must be the predecessor of v''. The 1–fellow traveler property easily follows by induction. Since every normal cube path is a geodesic of G^Δ, we recover the same combing as that of Niblo and Reeves [44]. Concluding, we have proven the following result.

PROPOSITION 9.5. *If G is a locally finite median graph then G and G^Δ have geodesic 2–combings and 1–combings defined by the trees T_b^+, T_b^- and T_b^Δ.*

9.2. Bicombings

Finding a bicombing of a graph seems to be more difficult than finding a combing. For example, Gersten and Short [29, 30] use the special structure of minimal disk diagrams, Noskov [45] presented a bicombing of triangle buildings based on geometric properties of the tesselation of the plane into equilateral triangles. We continue with algorithms for bicombing bridged graphs and F–graphs. Using some results from [24] we can extend them to all hereditary weakly modular graphs and graphs with convex disks, but the correction proofs become more technical and will be presented elsewhere. Notice that one cannot extend these results to all graphs in which the isometric cycles have length three or four. For this take the square grid $n \times n$, delete each of four corners and make its neighbours adjacent. We will obtain an octagon with four sides of length 1 and four sides of length $n-1$ subdivided into n^2-4 squares and 4 triangles. The resulting graph Γ does not contain other isometric cycles. Let vw and $v'w'$ be the opposite short sides of the octagon. Then $d_\Gamma(v,v') = d_\Gamma(w,w') = 2n-3$. The pairs v,v' and w,w' are connected by unique shortest paths which are not k–fellow travellers for any $k < n$. One can obtain the same effect if one triangulate the previous graph Γ by adding in each square the diagonal parallel to vw. In the resulting graph all isometric cycles have length three, except two cycles which have length five.

We will analyse few simple ways of generating a geodesic bicombing in a graph. First, we can fix a base point b and an ordering $\prec_b$ refining $\leq_b$ (in our cases this will be an BFS ordering). Suppose that every G_k is an isometric subgraph of G (this is the case of graphs occuring in this section; see [24]). Assume that we have defined the bicombing paths between each pair of vertices of G_{k-1} and we wish to extend this bicombing further. Let $\alpha(v) = k$. For any vertex v' with $\alpha(v') < k$, let $P_{(v',v)}$ be the union of the path $P_{(v',v^*)}$ and the edge v^*v, where v^* is the neigbour of v in the interval $I(v,v')$ with the smallest $\alpha(v^*)$. Clearly, if the father v_b of v belongs to $I(v',v)$ then v^* coincides with v_b. If the neighbour x of v_b in $P_{(v',v_b)}$ is adjacent to v, then $v^* = x$. The resulting path $P_{(v',v)}$ is unimodal with respect to the function α. Namely, when moving from v to v' along $P_{(v',v)}$, α decreases until we arrive at the vertex where it achieves the minimum value and then it strictly increases until we reach v'. As a result, we obtain a symmetric geodesic bicombing. (In Figure 4 we present certain paths in a bicombing of a bridged graph obtained by this procedure. We do not list the complete BFS–ordering, only the labels of the neighbours of the root–vertex b.)

If we let $P'_{(v',v)}$ be the union of the path $P'_{(v',v_b)}$ and the edge v_bv, we will obtain a symmetric bicombing, but the bicombing paths are not necessarily geodesics. If v_b *dominates* the vertex v in G_k (this means that every neigbour of v in G_k is a neighbour of v_b, too, then one can easily shows by induction that $\{P'_{(v',v)} : v',v \in V(G)\}$ constitutes a bicombing which enjoys the 1–fellow travellers property. Such graphs are known in the literature under the name *dismantlable graphs.* It is known and it can be easily shown that Helly graphs are dismantlable [12]. That bridged graphs are dismantlable have been established in [3]. In [23] is shown that this can be done by BFS. In this case a vertex v with $\alpha(v) = k$ is dominated in G_k by its father in the BFS–tree.

PROPOSITION 9.6. *Let G be a locally finite bridged graph. Then the symmetric geodesic bicombing $\{P_{(v',v)} : v',v \in V(G)\}$ satisfies the 2–fellow traveller property.*

PROOF. Let $\alpha(v) = k$. By induction on n we will show that if $d_G(v,w) \leq 1, d_G(v',w') \leq 1$ and all v',w',w,v' belong to G_k, then $P_{(v',v)} = (v' = v_0, v_1, \ldots, v_{n-1}, v_n = v)$ and $P_{(w,w')} = (w' = w_0, w_1 \ldots, w_{m-1}, w_m = w)$ lie in a 1–Hausdorff neighbourhood of each other.

If v is adjacent to some vertex w_i or w is adjacent to a vertex v_i, then we may apply the induction hypothesis. The same applies if the vertices v_{n-1} and w_{m-1} are adjacent. Thus we may assume the converse. Let v_b be the father of v and denote by x the neighbour of v_b in the path $P_{(v',v_b)}$. Since v_b dominates v, v_b is adjacent to w and v_{n-1} (from our assumption, clearly $v_b \neq v_{n-1}$).

Case 1: $x = v_{n-1}$.

By the induction assumption, the paths $P_{(w',w)}$ and $P_{(v',v_b)}$ lie in a 1–Hausdorff neighbourhood of each other. Then we obtain the desired property, unless some vertex w_i is adjacent in $P_{(v',v_b)}$ only to v_b. Clearly $i \geq m-2$. To avoid induced 4–cycles, v_b and w_{m-1} must be adjacent. If w_{m-1} is adjacent to some v_j, then $j \geq n-3$. To avoid induced 5–cycles or 4–cycles, the vertices w_{m-1} and v_{n-1} must be adjacent, which is impossible. Therefore $i = n-1$. By the induction assumption, v_{n-1} is adjacent to a vertex w_l. Since $P_{(w,w')}$ is a shortest path, necessarily $l = m-2$ or $m-3$. If v_{n-1} is adjacent to w_{m-3} but not to w_{m-2}, then the vertices $v_b, v_{n-1}, w_{m-3}, w_{m-2}, w_m$ induce a 5–cycle, or v_b and w_{m-3} are adjacent, and we get a 4–cycle formed by $v_b, v_{n-1}, w_{m-3}, w_{m-2}$. Thus v_{n-1} and w_{m-2} are adjacent. But then we have obtained an induced 5–cycle $(w, v, v_{n-1}, w_{m-2}, w_{m-1}, w)$. This setlles the case $x = v_{n-1}$; for an illustration see Figure 6.

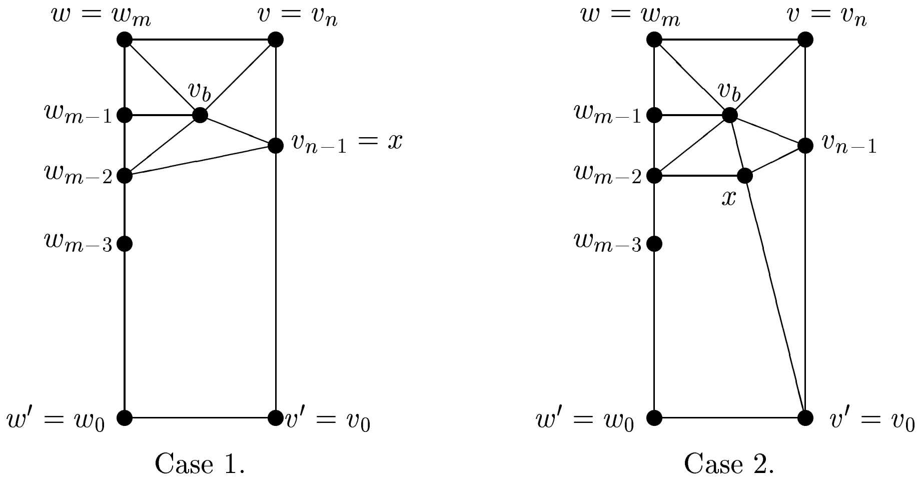

FIGURE 6.

Case 2: $x \neq v_{n-1}$.

This implies that $x, v_{n-1} \in I(v_b, v')$, therefore x and v_{n-1} are adjacent. On the other hand, x is not adjacent to v. Applying the induction assumption to the paths $P_{(w',w)}$ and $P_{(v',v_b)}$, we conclude that x is adjacent to w or to a vertex $w_i, i \geq m-3$. If w and x are adjacent, then the vertices w, v, v_{n-1}, x form an induced 4–cycle. A forbidden 5–cycle $(w, v, v_{n-1}, x, w_{m-1}, w)$ occurs if x and w_{m-1} are adjacent. If x is adjacent to w_{m-3} then $w_{m-2}, x \in I(w_{m-3}, w)$, therefore x and w_{m-2} must be adjacent. Hence $i = m-2$. To avoid forbidden cycles induced by $w, v_b, x, w_{m-2}, w_{m-1}$, the vertex v_b must be adjacent to both w_{m-1}, w_{m-2}; see Figure 6. This implies $v_b \in I(w, w')$, whence $P = (v_b, w_{m-2}, w_{m-3}, \ldots, w_1, w')$ is a shortest path. On the other hand, we know that $Q = (v_b, v_{n-1}, v_{n-2}, \ldots, v_1, v')$ is also a shortest path. Consider the cycle formed by the paths P, Q and the edge $v'w'$. It cannot be well–bridged, because the neighbours v_{n-1} and w_{m-2} of v_b in this cycle are not adjacent, thus settling Case 2.

This concludes the proof that $P_{(v',v)}$ and $P_{(w',w)}$ lie in a 1–Hausdorff neighbourhood of each other. To prove that these paths are 2–fellow travellers we must show that a vertex v_i is adjacent to at least one of the vertices w_{i-1}, w_i, w_{i+1}. Suppose that v_i is adjacent to a vertex w_j. Since $P_{(v',v)}$ and $P_{(w',w)}$ are shortest paths, we have $|i-j| \leq 2$. If $j = i+2$, then

$v_i, w_{i+1} \in I(w_{i+2}, w')$, thus v_i and w_{i+1} are adjacent, as desired. Otherwise, if $j = i-2$, then $d_G(w_{i-2}, w) \geq d_G(v_i, v)$ and $d_G(w_{i-2}, v) > d_G(v_i, v)$. Consider the cycle C formed by the edges vw, v_iw_{i-2} and the portions of the paths $P_{(v',v)}$ and $P_{(w',w)}$ comprised between v, v_i and w, w_{i-2}, respectively. Since C is well–bridged, from the previous inequalities we deduce that the neigbours v_i and w_{i-1} of w_{i-2} in C are adjacent, concluding the proof. □

REMARK 9.7. From the proof one can conclude that if $|d_G(v, v') - d_G(w, w')| \leq 1$, then the paths $P_{(v',v)}, P_{(w',w)}$ are 1–fellow travellers (this is also true for bicombings of F–graphs). We do not have an example of a bridged graph without a geodesic 1–bicombing. However, one can construct bridged graphs and BFS–bicombings which do not satisfy the 1–fellow traveller property.

To find a bicombing of an F–graph G we use a slightly modified version of the previous procedure. It concerns the case when v belongs to a pendant folder F of G_k, where $\alpha(v) = k$. For a vertex $v' \prec_b v$, let v^* be the neighbour of v in $I(v, v')$ with the value of α. If $v^b \notin I(v, v')$, then let $P_{(v',v)}$ be the union of the path $P_{(v',v^*)}$ and the edge v^*v, otherwise, if $v^b \in I(v, v')$, then the combing path $P_{(v',v)}$ is the union of the paths $P_{(v',v^b)}$ and (v^b, v^*, v). Before presenting its analysis, we formulate the following property of BFS–orderings of G.

LEMMA 9.8. *Let G be a locally finite F–graph. If x, y are two opposite vertices of a folder F and $x \prec_b y$, then the father of y in the BFS ordering belongs to F.*

PROOF. Suppose $y_b \notin F$. Then $x_b \notin F$, otherwise $y_b = x_b$ by BFS. Pick a vertex $z \in F$ adjacent to both x and y, such that $z \prec_b y$. By Proposition 9.1 its father z_b is adjacent to y. If z_b is distinct from x, then by Proposition 9.1 z_b is adjacent to x_b or z is adjacent to the father of x_b. In the first case we obtain a forbidden configuration of three folders. In the second case we obtain $z \prec_b y_b$, which is impossible. So suppose that $z_b = x$. By Proposition 9.1 either x and y_b are adjacent, implying $y_b \in F$, or z is adjacent to the father s of y_b. Then s must be adjacent to x_b, yielding an induced 6–cycle. □

PROPOSITION 9.9. *Let G be a locally finite F–graph. Then the symmetric geodesic bicombing $\{P_{(v',v)} : v', v \in V(G)\}$ satisfies the 2–fellow traveller property.*

PROOF. Let $\alpha(v) = k$, $d_G(v, w) \leq 1, d_G(v', w') \leq 1$, and suppose that v', w', w, v' belong to G_k. Denote $P_{(v',v)} = (v' = v_0, v_1, \ldots, v_{n-1}, v_n = v)$ and $P_{(w',w)} = (w' = w_0, w_1, \ldots, w_{m-1}, w_m = w)$. By induction on $n+m$ we will show that every v_i either is adjacent to a vertex of $P_{(w',w)}$ or both its neighbours v_{i-1}, v_{i+1} have a common neighour in $P_{(w',w)}$ and vice versa. From this we immediately obtain the 2–fellow traveller property. We know that in G_k the vertex v either is pendant or belongs to a pendant folder F. The either case is trivial, because then $v_{n-1} = w$. Thus let v, w belong to a common folder F. Without loss of generality we may assume that $v', w' \notin F$ and $v \neq w$. Recall that by $v^b \in F$ we denote the vertex of F opposite to v. From Lemma 9.8 we conclude that v^b is the father of at least one of the vertices v_{n-1} and w. If $v^b = v_{n-2}$ then $\alpha(w) > \alpha(v_{n-1})$ and v^b is the father of w. Then either $w_{m-1} = v^b$ or $d_G(v^b, v') = d_G(w, w')$. In both cases we can apply the induction hypothesis. The same holds when $v^b = w_{m-1}$. So assume $v^b \neq v_{n-2}, w_{m-1}$. By Lemma 7.6 we conclude that if v' and w' have distinct gates in F, then v^b is the gate of at least one of them. From the algorithm and our agreement we deduce that v^b is the gate of w' and the father of v_{n-1}. Then $w_{m-1} \prec_b v^b$ and $v^b \notin I(v_{n-1}, v')$. Since G is bipartite, this implies that $d_G(v_{n-1}, v') = d_G(v^b, w')$, and therefore $d_G(v, v') = d_G(w, w')$. Let u be the neighbour of v^b in the combing path $P_{(w',v^b)}$. By the induction assumption we infer that v_{n-2} and u are adjacent. If u is adjacent to w_{m-1}, we obtain three folders with a forbidden intesection. Thus v^b is adjacent to w_{m-2}. Since $v^b \in I(w, w')$, from the choice of w_{m-1} we deduce that $w_{m-1} \prec_b v^b$. By Lemma 9.8 w_{m-2} must be the father of v^b. Since $w_{m-2} \in I(v^b, w')$ we obtain a contradiction with the choice of the vertex u. Finally, assume that v' and w' have the same gate in F different from v^b. Evidently this gate is the vertex v_{n-1}. Then $v, v^b, w_{m-1} \in I(w, w')$, and by Lemma 7.4 v_{n-1} and w_{m-1} are adjacent. Since $\alpha(w_{m-1})$ and $\alpha(v^b)$ are smaller that $\alpha(w)$, by the algorithm we have $w_{m-2} = w^b$. To avoid forbidden subgraphs induced by the vertices $v, v_{n-1}, w, v^b, w_{m-1}, w^b$, necessarily $w^b = v_{n-1}$. □

Although there exist geodesic 1–bicombings of G^{Δ} derived by a procedure similar to those for bridged graphs or F–graphs, we present the analysis of the bicombing generated by normal folder–paths.

Proposition 9.10. *Let G be a locally finite F–graph. Then the bicombing of G^{Δ} generated by normal folder–paths satisfies the 1–fellow traveller property.*

Proof. As before, suppose that $d_{G^{\Delta}}(v, w) \leq 1, d_{G^{\Delta}}(v', w') \leq 1$ and let $P_{(v,v')} = (v = v_0, v_1, \ldots, v_n = v'), P_{(w,w')} = (w = w_0, w_1, \ldots, w_m = w')$ be the normal folder–paths connecting v, v' and w, w', respectively. Let F' and F'' be the cells (folders or edges) spanned by v, w and v', w', respectively. It is sufficient to show that v_1 and v_2 are adjacent in G^{Δ}, and the general result follows by induction on $n + m$ and the fact that a subpath of a normal folder–path is itself normal. This can be checked directly, using the properties of F–graphs. We will outline a shorter proof, using the result from [44] in Proposition 9.12 below.

Every folder–path lifts to a path of G obtained by replacing each pair $v_i v_{i+1}$ spanning a folder F_i by a path consisting of v_i, v_{i+1} and their common neighbour in F_i. If we will perform this operation with all folders in $P_{(v,v')}$ and $P_{(w,w')}$, we will obtain two genuine paths P', P'' of G. These paths together with some paths connecting v, w and v', w' in F', F'' form a cycle C of G. Assume that C is selected so that the area of a minimal singular disk quadrangulation (Q, f) of C is as small as possible. Pick a path of length two $(v_i, a, v_{i+1}) \subset \partial Q$. Its image is contained in the folder F_i. Add a new face q_i to Q by taking a vertex a' adjacent to v_i and v_{i+1}. Performing this operation with all folders from both normal folder–paths and with F', F'' as well, we will surround Q by a ring of new quadrangles, thus obtaining a new quadrangulation Q' (some care is needed while performing this operation for F' and F''). Since two consecutive folders from one normal folder–path intersect exactly in one vertex, this implies that f can be extended to a cellular map from Q' to G (namely, it will map q_i to a 4–cycle of the folder F_i). From Theorem 6.1 and the claim in the proof of Theorem 7.1 we infer that Q' is a cube–free median graph. The vertices v and v' are connected in Q' by a normal cube–path consisting of new faces and boundary edges mapped to the folders and edges of $P_{(v,v')}$ (a normal cube–path for w, w' is defined similarly). By Proposition 9.12, the vertices v_1 and w_1 belong to a common face of Q', therefore they belong to a common folder in G. □

Geodesic bicombings of median graphs G and associated Helly graphs G^{Δ} can be found by modifying the combing procedure from subsection 9.2. Assume that we have defined the combing paths between any two vertices of the graph $G(n)$ and we wish to extend this bicombing to the graph $G(n+1)$. We denote the union of the combing paths of $G(n)$ pointing from a vertex u by $T_u(n)$. To extend this tree, if $u \in G(n)$, then we let a vertex $v'' \in H''$ be adjacent in $T_u(n+1)$ with its father v'. If $u \in H''$, say $u = u''$, then define $T_{u''}(n+1)$ be the image in H'' of the tree $T_{u'}(n) \cap H'$ (that this indeed is a tree follows from the convexity of H'). Put $T_u = \cup_{n \geq 1} T_u(n)$. Clearly, the obtained bicombing consists of shortest paths only. By induction on n easily follows that if $v, w \in T_u$ are adjacent in G, then the paths of T_u between u and v, w are "1.5–fellow travellers". Using this fact, a simple case analysis and the induction on n, one can prove the following result.

PROPOSITION 9.11. *If G is a locally finite median graph, then the geodesic bicombing defined by the collection of trees $T_v, v \in V(G)$ satisfies the 3–fellow traveller property.*

We continue with the following result of Niblo and Reeves.

PROPOSITION 9.12 [44]. *If G is a locally finite median graph, then the bicombing of G^{Δ} generated by normal cube–paths satisfies the 1–fellow traveller property.*

Using the preprocessing of a median graph G defined before, we will outline an algorithmic way to find the normal cube–paths between two vertices v, w of G. Suppose that this task is accomplished for $G(n)$. If both vertices are in H'', say they are v'', w'', then define the paths $P_{(v'',w'')}$ and $P_{(w'',v'')}$ as the images in H'' of the paths $P_{(v',w')}$ and $P_{(w',v')}$, respectively. Now, assume that one vertex v is in $G(n)$ and another $w = w''$ is in H''. Let x be the neighbour of w' in $P_{(v,w')}$. Again, if the pair xw'' spans a cube, then set $P_{(v,w'')} = P_{(v,x)} \cup \{xw''\}$, otherwise let $P_{(v,w'')} = P_{(v,w')} \cup \{w'w''\}$. The proof that the paths constructed in this way are normal is the same as in subsection 9.1. To find the path $P_{(w'',v)}$ we use the fact that H' and H'' are convex and gated. If $v \in H'$ and y is the neighbour of w' in $P_{(w',v)}$, then one can show that y and w'' span a cube. Then $P_{(w'',v)}$ consists of the pair $w''y$ and the path $P_{(y,v)}$. Otherwise, if $v \notin H'$, let $v_{H'}$ be its gate in H'. Consider the neighbour z of w'' in the path $P_{(w'',v_{H'})}$. Then set $P_{(w'',v)}$ be the union of $w''z$ and the path $P_{(z,v)}$.

Proposition 9.12 can be used to establish that Helly graphs have a geodesic 1–bicombing. Indeed, the basic example of a cubing is the Euclidean space tiled into regular cubes. The underlying graph is the rectilinear grid Γ, which can be viewed as a Cartesian product of paths. The associated Helly graph Γ^{Δ} is the direct product of paths. By Theorem 4.7, Helly graphs are exactly the retracts of such products (maybe with an infinite number of factors). Therefore, at least for Helly graphs G isometrically embeddable into the direct product of a finite number of paths, the following holds.

COROLLARY 9.13. *Helly graphs have geodesic 1–bicombings.*

We were not successful in developing an algorithm for finding such a bicombing, and leave this as a question.

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