Connectedness of Opposite-Flag Geometries in Moufang Polygons

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Abstract

We show that the geometry of the elements opposite a certain flag in a Moufang polygon is always connected, up to some small cases. This completes the determination of all Moufang polygons for which this geometry is disconnected.

1 Introduction and statement of the Main Result

A *generalized n-gon*, $n \geq 2$, is a rank 2 geometry whose incidence graph has diameter n and girth $2n$, and each vertex has valency ≥ 3. If the latter condition is not satisfied , then we have a *weak generalized n-gon*. In this paper, we will always consider generalized n-gons with $n \geq 3$ (generalized 2-gons are trivial geometries). They are the irreducible spherical buildings of rank 2. A *generalized polygon* is a generalized n-gon, for some $n \geq 2$. We will view generalized polygons as geometries of rank 2 whose elements are points and lines. The *dual* is obtained by interchanging these names. A *flag* is an incident point-line pair and hence a chamber in the corresponding spherical rank 2 building. Generalized polygons were introduced by Tits [10] and are the basic rank 2 incidence geometries.

Let Γ be a generalized n-gon, $n \geq 3$. Given a fixed flag F in Γ, we define $\Gamma^{(F)}$ to be the set of all flags opposite F in Γ together with all points and lines occurring in these flags. So $\Gamma^{(F)}$ is a rank 2 subgeometry of Γ which we call

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opposite-flag geometry. The question arises what an opposite flag geometry looks like. In particular, is it connected? If Γ satisfies the Moufang condition, this plays an important role in the theory of twin buildings, see Condition (co) in MÜHLHERR & RONAN [6] and Condition $(*)$ in ABRAMENKO & MÜHLHERR [2]. In the present paper, we answer it for all such Moufang polygons. The result has already been used in the fundamental papers on twin buildings cited above, although there does not exist a complete proof in the literature. Let us briefly mention the history of the problem and its solution.

The question for arbitrary finite polygons was put to Andries Brouwer by the second author in connection with some characterizations of finite Moufang polygons. It was solved in BROUWER [3] by a matrix-technique. Around the same time, the first author determined the number of connected components of the opposite-flag geometries of all Moufang polygons (finite and infinite) using a group-theoretical argument (a sketch of this is contained in ABRAMENKO [1]). When the second author was confronted with this result, he managed to produce a geometric proof for the Moufang hexagons. In the present paper we present the original proof of the first author for Moufang octagons (relying on the classification of Moufang octagons, as given by TITS [15]), and we give two proofs for the case of Moufang hexagons: one entirely geometric and one group-theoretic, simplified by a geometric reduction. These proofs use the fact that infinite Moufang hexagons have at least 7, respectively 5 points on each line. This follows from the fact that appropriate Jordan algebras over finite fields are finite themselves. An alternative proof of the Main Result is given by Jacques Tits (lectures at Collège de France, 13 and 20 January 1998), based on Corollary 5 below. We comment on Tits' proof at the end of Subsection 3.1.

Our Main Result reads as follows.

Main Result. *Let Γ be a Moufang polygon, and let F be any flag of Γ. Then the geometry $\Gamma^{(F)}$ is connected, except in the following cases:*

(i) *Γ is the generalized quadrangle associated to the symplectic group $\mathbf{Sp}_4(2)$. In this case, $\Gamma^{(F)}$ has 2 connected components.*

(ii) *Γ is the generalized hexagon (or its dual) associated to the group $\mathbf{G}_2(2)$. In this case, $\Gamma^{(F)}$ has 4 connected components.*

(iii) *Γ is the generalized hexagon associated to the group $\mathbf{G}_2(3)$. In this case, $\Gamma^{(F)}$ has 3 connected components.*

(iv) Γ *is the generalized octagon (or its dual) associated to the group* ${}^2\mathbf{F}_4(2)$. *In this case,* $\Gamma^{(F)}$ *has 2 connected components.*

Note that TITS [16](16.7) mentions without proof a statement which treats the same question. Only, the Moufang octagons are not considered in his context, and the counterexample *(iii)* is overlooked.

By a result of WEISS [20] (see also TITS [12, 14]), Moufang n-gons, $n \geq 3$, only exist for $n = 3, 4, 6, 8$. There are examples in each of these cases, and in fact there is a complete classification, see TITS & WEISS [18].

For $n = 3$, i.e., for projective planes, the Main Result is immediate (it is true for all projective planes, Moufang or not). Likewise, for generalized quadrangles, the result is true without the Moufang condition, see BROUWER [3] (cf. VAN MALDEGHEM [19](1.7.15)). Hence, in order to prove the Main Result, we may restrict our attention to Moufang hexagons and octagons.

Note that for $n = 6, 8$, the Moufang condition cannot be dispensed with as one can construct (by applying an appropriate modification of the "free construction" of generalized polygons given in TITS [13]) for arbitrary $n \geq 5$ infinite generalized n-gons with opposite-flag geometries having an infinite number of connected components, see ABRAMENKO [1], Chapter II, §2, Proposition 9.

Remark. The question whether $\Gamma^{(F)}$ is connected or not is a special case of a general problem which is treated in detail in ABRAMENKO [1], Chapter II. Namely, let Δ be an arbitrary thick spherical building of rank r and c a chamber of Δ. We denote by $\Delta^o(c)$ the subcomplex of Δ consisting of all simplices $y \in \Delta$ such that there exists a chamber x opposite c with $y \subseteq x$. The problem is to decide whether the geometric realization of $\Delta^o(c)$ is $(r-1)$-spherical, i.e. homotopy equivalent to a bouquet of $(r-1)$-spheres. The conjecture is that this is always true if Δ is a not "too small" Moufang building. The case $r = 2$ is discussed in the present paper. The conjecture is proved by ABRAMENKO [1] for "sufficiently big" (an explicit bound being given there) buildings of type A_r, B_r, C_r and D_r, provided that in the C_3 case the building is not associated to a non-embeddable polar space. So it remains to investigate the latter case and buildings of the exceptional types F_4, E_6, E_7 and E_8.

2 Moufang hexagons – a geometric approach

2.1 Proof of the Main Result for hexagons

In our geometric approach, our aim is to prove that all infinite Moufang hexagons have connected opposite-flag geometries, the finite case being treated by Brouwer [3].

We first recall some geometric definitions and facts concerning Moufang hexagons.

Let Γ be a generalized hexagon. For $i \in \{1,2,3,4,5,6\}$, let $\Gamma_i(x)$ be the set of all elements of Γ at distance i (measured in the incidence graph) from the element x (which is a point or a line). Also, elements at distance 6 from each other are called *opposite*. If two elements x, y are not opposite, then there exists a unique element incident with x and at minimal distance from y, and we denote that element by $\mathrm{proj}_x y$ (it is directly related to the usual projection mapping in buildings, see Tits [11], Subsection 3.19). If two elements x, y are opposite in Γ, then the set $\Gamma_i(x) \cap \Gamma_{6-i}(y)$, $i = 2, 3$, is non-empty (it has the same cardinality as $\Gamma_1(x)$ and $\Gamma_1(y)$) and is denoted for short by $x^y_{[i]}$. For $i = 2$, we sometimes write $x^y_{[2]} = x^y$, see e.g. Ronan [8]. The distance between two elements x and y is denoted by $\delta(x, y)$.

Let L be a line of Γ. Then we say that L is *distance-i-regular*, $i = 2, 3$, if for all lines $M, N \in \Gamma_6(L)$, the condition $|L^M_{[i]} \cap L^N_{[i]}| \geq 2$ implies $L^M_{[i]} = L^N_{[i]}$. Ronan [8](3.7),(5.9) showed that all lines of any Moufang hexagon are distance-3-regular and that, up to duality, all lines of any Moufang hexagon are distance-2-regular (but he used another terminology; we follow Van Maldeghem [19], Section 1.9). The set L^M is called a *trace*.

A *path* is a sequence of consecutively incident elements. *Confluent* lines are lines which are incident with a common point. The number of points on a line of any generalized polygon is a constant, which we call the *length* of any line.

Now let Γ be a Moufang hexagon. Without loss of generality (replacing Γ by its dual if necessary), we may assume that Γ has distance-2-regular and distance-3-regular lines. We show a lemma.

Lemma 1 *Let L, M be two opposite lines in Γ and let p be any point of Γ. Suppose that $|\Gamma_3(L) \cap \Gamma_3(M) \cap \Gamma_4(p)| \geq 3$. Then there is a unique point x of $\Gamma_3(L) \cap \Gamma_3(M)$ collinear with p, and all other points of $\Gamma_3(L) \cap \Gamma_3(M)$ belong to $\Gamma_4(p)$.*

Proof. Let $x_1, x_2, x_3 \in \Gamma_3(L) \cap \Gamma_3(M) \cap \Gamma_4(p)$. Note that x_1, x_2, x_3 are mutually opposite points. The path (p, L_i, p_i, M_i, x_i), $i = 1, 2, 3$, defines the elements L_i, p_i and M_i. Let L' be the unique element of $\Gamma_3(x_2) \cap \Gamma_2(M_1)$. If $L' \neq M$, then by the distance-3-regularity we have $x_1, x_2, x_3 \in \Gamma_3(L') \cap \Gamma_3(M)$, hence replacing L' by L, we may assume without loss of generality that M_1 is confluent with L. If $L' = M$, then by interchanging the names of L and M, we also obtain that M_1 meets L. Let $\{q_1\} = \Gamma_1(L) \cap \Gamma_1(M_1)$, and let (x_2, M_2', q_2, L) be a path connecting x_2 with L. Note that we may assume that $L_1 \neq L$, otherwise the assertion follows. So it is clear that $q_1 \neq p_1$, since otherwise all points of L_1 different from p_1 are opposite x_2. Consequently we have the ordinary hexagon $(p, L_1, p_1, M_1, q_1, L, q_2, M_2', x_2, M_2, p_2, L_2, p)$, showing that L_2 is opposite L. We can also see from this that $\{M_1, M_2'\} \subseteq L^M \cap L^{L_2}$. Hence, by the distance-2-regularity of lines, every element of L^M is at distance 4 from L_2. Consider the line $M_3' := \mathrm{proj}_{x_3}(L)$. By the foregoing, M_3' is at distance 4 from L_2, Since p is clearly opposite $\mathrm{proj}_L(x_3)$ (indeed, there is a path $(p, L_1, p_1, M_1, q_1, L, \mathrm{proj}_L(x_3)))$, the unique line N meeting both L_2 and M_3' is not incident with p. But if it were not incident with x_3 either, then $\delta(x_3, p) = 6$, a contradiction. Hence $L_2 \in \Gamma_3(x_2) \cap \Gamma_3(x_3)$, and hence by the distance-3-regularity, $L_2 \in \Gamma_3(x)$, for all $x \in \Gamma_3(L) \cap \Gamma_3(M)$. So we may now take $L_2 = M$, without loss of generality and the assertion follows easily. $\square$

We can now show:

Proposition 2 *Let Γ be a Moufang hexagon with distance-2-regular lines of length ≥ 7, and let F be some flag of Γ. Then the opposite-flag geometry $\Gamma^{(F)}$ is connected.*

Proof. We put $F = \{p, L\}$, with p a point of Γ and L a line. Let M and N be two distinct lines opposite L.

We note that $|\Gamma_1(x)| \geq 7$, for all points x of Γ by VAN MALDEGHEM [19](1.9.5)).

First we assume that $M_p := \mathrm{proj}_p M \neq \mathrm{proj}_p N =: N_p$, and that M and N are opposite. Let S be some indexing set with the same cardinality as $\Gamma_1(M)$ (and hence $\ell := |S|$ is the length of any line of Γ). Then we can put $\Gamma_1(M) = \{x_i(M) : i \in S\}$. Let $(x_i(M), L_i(M, N), p_i(\{M, N\}), L_i(N, M), x_i(N), N)$ be a path connecting $x_i(M)$ and N, with $i \in S$. If, for some $i \in S$, $\delta(p_i(\{M, N\}), p) = 2$, then clearly $M_p = \mathrm{proj}_p p_i(\{M, N\}) = N_p$, contradicting our hypothsis. Hence, by the previous lemma, there are at least $\ell - 2 \geq 5$ elements i of S such that $p_i(\{M, N\})$ is opposite p. At least

$\ell - 4 \geq 3$ amongst these are such that both $x_i(M)$ and $x_i(N)$ are opposite p. We gather such i in $S' \subseteq S$. By the distance-2-regularity of lines, we have either that for at most one element $j \in S$, the line $L_j(N, M)$ is not opposite L, or that for all elements $j \in S$, the line $L_j(N, M)$ is not opposite L. Similarly for the $L_j(M, N)$'s. Without loss of generality, we may assume that no line $L_j(N, M)$ is opposite L, for all $j \in S$ (because in the other case, there exist at least $|S'| - 2 \geq \ell - 6 \geq 1$ elements $i \in S'$ such that the path $(M, x_i(M), L_i(M, N), p_i(\{M, N\}), L_i(N, M), x_i(N), N)$ is contained in $\Gamma^{(F)}$). Now fix $i \in S'$. Let N', $N' \neq N$, be any line through $x_i(N)$, opposite both L and M (N' exists since $\Gamma_1(x_i(N))| \geq 7$). We let N' play the same role as N and we use similar notation (note that we do not require $\mathrm{proj}_p N' \neq M_p$).

Suppose that $L_j(N', M)$ is not opposite L, for all $j \in S$. Then we consider any line M' through $x_i(M)$ opposite both L and N, $M' \neq M$. Note that M' exists and is automatically opposite N'. Due to the distance-2-regularity of N and N', we have $N^{M'} \cap N^L = N'^{M'} \cap N'^L = \{L_i(N, M)\}$ (because the confluent lines M and M' define different traces $N^M \neq N^{M'}$ and $N'^M \neq N'^{M'}$). For a similar reason at most one of the traces M'^N or $M'^{N'}$ coincides with M'^L. Noticing that $p_i(\{M, N\}) \in \Gamma_3(M) \cap \Gamma_3(N) \cap \Gamma_3(M') \cap \Gamma_3(N')$ implies that at most two elements of $\Gamma_3(M') \cap \Gamma_3(N)$ (respectively $\Gamma_3(M') \cap \Gamma_3(N')$) are not opposite p, we now see that M' and either N or N' belong to the same connected component of $\Gamma^{(F)}$. Hence, since both $x_i(M)$ and $x_i(N)$ are opposite p, the lines M and N are in the same connected component of $\Gamma^{(F)}$.

So we may assume that $L_j(N', M)$ is opposite L, for all $j \in S \setminus \{i\}$, and for all lines $N' \in \Gamma_6(L) \cap \Gamma_6(M) \cap \Gamma_1(x_i(N))$ with $N' \neq N$. For at most one such N' we have $M^{N'} = M^L$. So there exists at least one choice for N' such that $|M^{N'} \cap M^L| \leq 1$. As in the previous paragraph, it follows that N' (and hence N) and M are in the same connected component of $\Gamma^{(F)}$.

If M and N are not opposite, then we claim that there is always a line N' opposite both M and L in the same connected component of $\Gamma^{(F)}$ as N. Moreover, N' can be chosen such that $\mathrm{proj}_p N' \neq M_p$. Indeed, consider any point $x \in \Gamma_1(N) \cap \Gamma_6(p)$, $x \neq \mathrm{proj}_N M$. Any line $N_1 \in \Gamma_1(x) \cap \Gamma_6(L) \cap \Gamma_6(M_p)$, $N_1 \neq N$, satisfies $\delta(M, N_1) = \delta(M, N) + 2$ and $\mathrm{proj}_p N_1 \neq M_p$. There are at least $|\Gamma_1(x)| - 3$ choices for N_1. If $\delta(M, N) = 4$, then we can take $N_1 = N'$, if $\delta(M, N) = 2$, then repeating the argument with M and N_1 proves the claim.

Finally, if $M_p = N_p$, then we may pick any line $P \in \Gamma_6(L)$ with $\mathrm{proj}_p P \neq M_p$ (this is always possible). By the previous part, both M and N belong to the same connected component of $\Gamma^{(F)}$ as P. $\square$

Remark. The Main Result for generalized hexagons now follows from the previous proposition, from Brouwer [3], and from the fact that no infinite Moufang hexagon has lines with finite length. Indeed, this follows from Tits' unpublished classification of Moufang hexagons (see Tits & Weiss [18]). However, we do not need the full strength of this classification here but only the following ingredients.

First of all, the root groups of a Moufang hexagon Γ constitute a root datum of type G_2. This is shown in Tits [17]. Secondly, the root groups corresponding to the long roots of a root datum of type G_2 can be coordinatized by the additive group of a (commutative) field $\mathbb{K}$, and those corresponding to the short roots by the additive group of a Jordan division algebra J over $\mathbb{K}$ (cf. Faulkner [5], Theorem 3.55). Recall that J is in particular a $\mathbb{K}$-vector space endowed with a cubic form $N : J \to \mathbb{K}$ such that $N(a) \neq 0$, for all $a \in J \setminus \{0\}$. In order to show that no infinite Moufang hexagon Γ has lines of finite length, it suffices to verify that J has to be finite dimensional if $\mathbb{K}$ is finite. However, this follows directly from the theorem of Chevalley-Warning which implies that every cubic form on a vector space of dimension at least 4 over a finite field necessarily has a nontrivial zero.

2.2 A reduction lemma for Moufang hexagons

In order to simplify a little bit the group theoretic approach, we now show a geometric reduction result. Let Γ be any Moufang hexagon. Up to duality, we may assume that all lines are distance-2-regular. We will use the following property, proved by Ronan [8](8.20) in an entirely geometric way. Every pair $\{p, p'\}$ of opposite points in Γ and every triple of distinct lines $L, L', L'' \in \Gamma_1(p)$ are contained in a unique subhexagon Γ' with the property that $\Gamma'_1(M) = \Gamma_1(M)$, for all lines M of Γ' (the latter property of Γ' is sometimes abbreviated by saying that the subhexagon Γ' is *full* in Γ), and such that Γ' is dual to the usual hexagon associated to a group of type G_2 (the latter being called a *split Cayley hexagon* in Van Maldeghem [19], Subsection 2.4.9). Its isomorphism class does not depend on the points p, p' and the lines L, L', L''.

Lemma 3 *With the above notation, let F be any flag of Γ, and let Γ' be ismorphic to any full dual split Cayley subhexagon of Γ. Let F' be some flag in Γ'. Then $\Gamma^{(F)}$ is connected whenever $\Gamma'^{(F')}$ is.*

Proof. Suppose that $\Gamma'^{(F')}$ is connected. We put $F = \{p, L\}$, p some point of Γ and L some line. Also, let M and N be two lines opposite L. Let x be any point on N opposite p and put $N' = \mathrm{proj}_x \mathrm{proj}_p M$. Then N' is opposite L and is contained in the same connected component of $\Gamma^{(F)}$ as N.

Now let $y = \mathrm{proj}_L x$ and $z = \mathrm{proj}_M p$. Consider a full dual split Cayley subhexagon Γ'' of Γ containing y, z and the three lines L, $\mathrm{proj}_y M$ and $\mathrm{proj}_y N'$. Put $\Gamma_2(x) \cap \Gamma_2(y) = \{u\}$ and $\Gamma_2(p) \cap \Gamma_3(N') = \{v\}$. Since u is incident with $\mathrm{proj}_y N'$, the point u belongs to Γ''. Since L belongs to Γ'', also the line $\mathrm{proj}_p z$ belongs to Γ''. Since v is incident with $\mathrm{proj}_p z$, it also belongs to Γ''. Let M' be any line of Γ'' at distance 3 from both v and u and distinct from L. By assumption, M and M' are contained in the same connected component of $\Gamma^{(F)}$.

Left to show is that M' and N' are contained in the same connected component of $\Gamma^{(F)}$. This follows immediately from the fact that M' and N' are contained in a full dual split Cayley subhexagon Γ''' containing the points v and u and the lines $\mathrm{proj}_u L$ (and hence Γ''' contains L), $\mathrm{proj}_u M'$ and $\mathrm{proj}_u N'$. $\square$

3 The group theoretic approach

3.1 A general lemma

The following discussion is based on ideas developed in Tits [16], Section 16 (see also Abramenko [1], Chapter II, §2). We have to introduce some notation. Let Δ be a spherical Moufang building of rank r (see Ronan [9], Chapter 6 for the basic properties of Moufang buildings). Fix an apartment Σ of Δ and a chamber $c \in \Sigma$. Denote by Φ the set of all roots (half apartments) of Σ, and set $\Phi^+ := \{\alpha \in \Phi : c \in \alpha\}$. For any $\alpha \in \Phi$, U_α will be the root group associated to α. Let $c_1, \ldots, c_r$ be the chambers of Σ which are adjacent but not equal to c. For any i, $1 \leq i \leq r$, we denote by α_i the unique element of Φ^+ not containing c_i. We set

$$U := \langle U_\alpha : \alpha \in \Phi^+ \rangle \text{ and } U' := \langle U_{\alpha_i} : 1 \leq i \leq r \rangle,$$

which are subgroups of $\mathrm{Aut}(\Delta)$. Finally, we define (slightly modifying the notation of the remark just preceding Section 2)

$$\Delta^o(c) := \{x \in \Delta : x \text{ and } c \text{ are opposite in } \Delta\}.$$

A connected component of $\Delta^o(c)$ is by definition a maximal subset M of $\Delta^o(c)$ such that any two chambers of M can be connected by a gallery in M.

Lemma 4 *The index $[U : U']$ is equal to the number of connected components of $\Delta^o(c)$. In particular, $\Delta^o(c)$ is (gallery-)connected if and only if $U = U'$.*

Proof. Let $c^o, c_1^o, \dots, c_r^o$ be the chambers of Σ which are opposite $c, c_1, \dots, c_r$, respectively, and let $M = M(c, c^o)$ be the connected component of $\Delta^o(c)$ containing c^o. Since U acts simple-transitively on $\Delta^o(c)$ (cf. Ronan [9], Theorem 6.15) and hence transitively on the set of its connected components, it suffices to show that $\mathrm{Stab}_U(M) = U'$.

First we show that $\mathrm{Stab}_U(M) \leq U'$. Assume that $u \in U$ stabilizes M. Then there is a gallery $\gamma = (c^o = x_0, x_1, \dots, x_\ell = u(c^o))$ in $\Delta^o(c)$ connecting c^o and $u(c^o)$. We prove $u \in U'$ by induction on the length ℓ of γ. If $\ell = 0$, then $u(c^o) = c^o$ and hence $u = 1$, again by Ronan [9], Theorem 6.15. For $\ell > 0$, there is an i such that the panels $x_1 \cap c^o$ and $c_i^o \cap c^o$ coincide. Since $c_i^o \in \alpha_i$ and $x_1, c^o \notin \alpha_i$, there exists a $u_i \in U_{\alpha_i}$ satisfying $u_i(x_1) = c^o$. Applying the induction hypothesis to the gallery $(c^o = u_i(x_1), u_i(x_2), \dots, u_i(x_\ell) = u_i u(c^o))$, we obtain $u_i u \in U'$ and hence $u \in U'$.

Now we show that $U' \leq \mathrm{Stab}_U(M)$. Clearly, it suffices to verify that $U_{\alpha_j} \subseteq \mathrm{Stab}_U(M)$, for all $j \in \{1, \dots, r\}$. Let $u \in U_{\alpha_j}$ be arbitrary. Since $u(c^o)$ contains the panel $c^o \cap c_j^o$, c^o and $u(c^o)$ are adjacent chambers. Hence $u(c^o) \in M$ and $u(M) = M$. $\square$

Now let Γ be a Moufang polygon and let F be a flag in Γ. Then Γ can be considered as a spherical Moufang building of rank $r = 2$, F as a chamber of Γ and $\Gamma^o(F)$ as the set of chambers of $\Gamma^{(F)}$. Choosing an apartment Σ of Γ which contains F, setting $c = F$ and defining U, U' as above, we obtain the following specialization of Lemma 4.

Corollary 5 *The number of connected components (in the usual graph theoretic sense) of $\Gamma^{(F)}$ is equal to $[U : U']$.*

Hence the problem is reduced to prove that in the generic case $U' = U$. So one must calculate U'. In fact, we will only need to perform an explicit calculation for split Cayley hexagons and for Moufang octagons. A way to avoid explicit calculations with commutation relations (as below) was given by Tits (unpublished) in his course at Collège de France (January 1998). His idea is the following. By considering the action of a (suitable) torus T, one turns the group

$$U'' := \langle U_\alpha : \alpha \in \Phi^+ \text{ and } \alpha \text{ contains every chamber adjacent to } c \rangle$$

into a group *with operators*. This allows one to show that, in the generic case, every subgroup (with operators) of U'' is a product of subgroups of the U_α, with $\alpha \in \Phi^+$ containing every chamber adjacent to c. It is then easy to deduce $U'' \leq U'$ and hence $U' = U$. The statement concerning U'' is achieved by showing that the U_α, α as above, have no isomorphic sub-quotients with respect to the operators. This method needs an explicit calculation in T, and it also relies on the classification of Moufang polygons. On the one hand, it is somewhat more involved than the method below, because it needs some additional lemmas. On the other hand, the calculations to perform afterwards are shorter.

3.2 Split Cayley hexagons

For a given split Cayley hexagon Γ, there exists a commutative field $\mathbb{K}$ such that Γ is the building associated to the (adjoint) Chevalley group $\mathbf{G}_2(\mathbb{K}) =: G$. In this special situation, we reinterpret the notions introduced in the previous subsection as follows:

The root groups U_α are 1-dimensional unipotent subgroups of G, and U is the unipotent radical of a Borel subgroup of G. We identify Φ with the elements of a root system of type G_2. Let $\{\alpha_1, \alpha_2\}$ be the base of Φ corresponding to Φ^+. We assume that α_1 is short, α_2 is long and set

$$\begin{array}{lll} U_1 := U_{\alpha_1}, & U_2 := U_{3\alpha_1+\alpha_2}, & U_3 := U_{2\alpha_1+\alpha_2}, \\ U_4 := U_{3\alpha_1+2\alpha_2}, & U_5 := U_{\alpha_1+\alpha_2}, & U_6 := U_{\alpha_2}, \end{array}$$

which implies $U = \langle U_i : 1 \leq i \leq 6\rangle$ and $U' = \langle U_1 \cup U_6\rangle$. It is well known (cf. for example Carter [4], Section 5.2) that there exist parametrizations (which are isomorphisms) $u_i : (\mathbb{K}, +) \to U_i$ $(1 \leq i \leq 6)$ such that the following commutation relations hold (here a, b are elements of $\mathbb{K}$ and $[x, y] := xyx^{-1}y^{-1}$):

(1) $[u_1(a), u_6(b)] = u_2(a^3b)u_3(a^2b)u_4(-a^3b^2)u_5(ab)$;

(2) $[u_1(a), u_5(b)] = u_2(3a^2b)u_3(2ab)u_4(-3ab^2)$;

(3) $[u_1(a), u_3(b)] = u_2(3ab)$; (3′) $[u_3(a), u_5(b)] = u_4(3ab)$;

(4) $[u_2(a), u_6(b)] = u_4(-ab)$;

(5) $[U_1, U_2] = [U_1, U_4] = [U_2, U_3] = [U_2, U_4] = [U_2, U_5] = [U_3, U_4] = [U_3, U_6] = [U_4, U_5] = [U_4, U_6] = [U_5, U_6] = \{1\}$.

Lemma 6 *If $\mathbb{K}$ has at least 4 elements, then $U' = U$.*

Proof. From (1), (4) and (5), we deduce for all $b \in \mathbb{K}$

$$[[u_1(1), u_6(1)], u_6(b)] = [u_2(1)u_3(1)u_4(-1)u_5(1), u_6(b)] = u_4(-b),$$

hence

(6) $U_4 \leq U'$.

Now (1), (5) and (6) imply $u_2(a^3b)u_3(a^2b)u_5(ab) \in U'$, for all $a, b \in \mathbb{K}$. Replacing a by 1 and b by $-ab$, we also have that $u_2(-ab)u_3(-ab)u_5(-ab) \in U'$. In view of (3′), (5) and (6), this implies

(7) $u_2((a^3 - a)b)u_3((a^2 - a)b) \in U'$, for all $a, b \in \mathbb{K}$.

Suppose now $\mathrm{char}\mathbb{K} \neq 3$. Then we choose a such that $a \neq a^2$ (possible, since $\mathbb{K} \neq \mathbf{GF}(2)$) and deduce from (3), (5) and (7) that for all $c \in \mathbb{K}$

$$[u_1(c), u_2(a^3 - a)u_3(a^2 - a)] = [u_1(c), u_3(a^2 - a)] = u_2(3(a^2 - a)c) \in U'.$$

This implies successively $U_2 \leq U'$, $U_3 \leq U'$ (by (7)), $U_5 \leq U'$ (by (1)) and hence $U' = U$.

So we may assume that $\mathrm{char}\mathbb{K} = 3$. We can use (1), (2), (3) and (5) to show $U_3 \leq U'$:

$$\begin{aligned}
[u_1(a), [u_1(1), u_6(1)]] &= [u_1(a), u_2(1)u_3(1)u_4(-1)u_5(1)] \\
&= [u_1(a), u_5(1)] \\
&= u_3(2a) \in U', \text{ for all } a \in \mathbb{K}.
\end{aligned}$$

Now (7) shows (since $\mathbb{K} \neq \mathbf{GF}(3)$) that $U_2 \leq U'$. Hence again $U_5 \leq U'$ by (1), and $U' = U$ follows. □

In view of Corollary 5, the above lemma immediately implies:

Corollary 7 *If in the split Cayley hexagon Γ the lines have length ≥ 5, then $\Gamma^{(F)}$ is connected.*

Remarks. For $\mathbb{K} = \mathbf{GF}(2)$, one immediately verifies that $U' = \{\prod_{i=1}^{6} u_i(a_i) : a_2 = a_3 = a_5\}$, hence $[U : U'] = 4$, and for $\mathbb{K} = \mathbf{GF}(3)$, it is easily seen that $U' = \{\prod_{i=1}^{6} u_i(a_i) : a_2 = a_5\}$, hence $[U : U'] = 3$.

It is possible to show by similar but more involved calculations that $U' = U$ for all Moufang hexagons different from the split Cayley hexagons over $\mathbf{GF}(2)$ and $\mathbf{GF}(3)$. The general proof is based on the coordinatization of Moufang hexagons by Jordan algebras and on the commutator formulae given in FAULKNER [5], Theorem 3.55.

Putting together Corollary 7 and Lemma 3, the Main Result for hexagons is proved for all Moufang hexagons with distance-2-regular lines of length ≥ 5. To finish the proof of the Main Result in the case of hexagons, we can refer to BROUWER [3] if we would know that Moufang hexagons with distance-2-regular lines of length 3 or 4 are finite. If Γ is a Moufang hexagon with distance-2-regular lines of length 3, then the end of the proof of Theorem 6.3.2 in VAN MALDEGHEM [19] shows in an elementary way that Γ is finite (see also RONAN [7]). So only in the case of Moufang hexagons with distance-2-regular lines of length 4, we have to appeal here to the Jordan algebra argument, as in the remark at the end of 2.1, to conclude that Γ is finite.

3.3 Moufang octagons

If Γ is a Moufang octagon, it is hard to construct connecting paths in $\Gamma^{(F)}$ analogously to our proceeding in Section 2. However, the group theoretic approach works here as well, yielding $U' = U$ with only one exception. Our discussion will be based on TITS' classification of Moufang octagons developed in [15], and we shall also use the notations introduced there.

Given a Moufang octagon Γ, there exist a field $\mathbb{K}$ of characteristic 2 and an endomorphism σ of $\mathbb{K}$ satisfying $\sigma^2(a) = a^2$, for all $a \in \mathbb{K}$, such that Γ is (isomorphic to) the building associated to the group ${}^2\mathbf{F}_4(\mathbb{K}, \sigma) =: G$. We can identify the root groups U_α ($\alpha \in \Phi$) with the subgroups U_i ($1 \leq i \leq 16$) of G introduced in TITS [15], Section 1, and U with $\langle U_i : 1 \leq i \leq 8\rangle$, as well as U' with $\langle U_1 \cup U_8\rangle$. Recall that there are parametrizations $x_{2j+1} : \mathbb{K} \to U_{2j+1}$ and $x_{(2j)} : \mathbb{K}^2 \to U_{2j}$ satisfying $x_{2j+1}(a)x_{2j+1}(b) = x_{2j+1}(a + b)$ and

$$x_{(2j)}(a, b)x_{(2j)}(\tilde{a}, \tilde{b}) = x_{(2j)}(a + \tilde{a}, b + \tilde{b} + \sigma(a)\tilde{a}), \text{ for all } a, b, \tilde{a}, \tilde{b} \in \mathbb{K}.$$

Furthermore, $x_{2j}(a) = x_{(2j)}(a, 0)$, $x'_{2j}(b) = x_{(2j)}(0, b)$ and $U'_{2j} = \{x'_{2j}(b) : b \in \mathbb{K}\}$. As TITS [15], we shall often abbreviate $x_i(a)$ by a_i and $x'_{2i}(b)$ by $b_{2i'}$. If there is no danger of ambiguity, we shall also use the notation $i := 1_i = x_i(1)$ and $2i' := 1_{2i'} = x'_{2i}(1)$.

Note that $U(\mathbf{GF}(2), \mathrm{id})$ can always be considered as a subgroup of U. It is therefore useful to determine $U'(\mathbf{GF}(2), \mathrm{id})$ at first.

Lemma 8 *For* $\mathbb{K} = \mathbf{GF}(2)$ *and* $\sigma = \mathrm{id}$ *the subgroup* U' *of* U *coincides with*

$$\widetilde{U} := \{(a_1)_1(a_2)_2(b_2)_{2'} \dots (a_8)_8(b_8)_{8'} : a_i, b_{2j} \in \mathbf{GF}(2); a_2 + a_4 + a_6 = 0\}.$$

Proof. First we observe that $\widetilde{U}$ is in fact a subgroup of U (and hence contains U'). This follows mainly from an inspection of the formulae (1) up to (15) given in TITS [15], Subsection 1.7, for commutators $[u_i, u_j]$ with $u_i \in U_i$, $u_j \in U_j$ and $1 \le i < j \le 8$. One notes that $[u_i, u_j]$ can always be expressed as a product with factors from $U_{i+1}, U_{i+2}, \dots, U_{j-1}$, where the number of factors from $U_\ell \setminus U'_\ell$ with $\ell \in \{2,4,6\}$ is even. From this and the fact that $(U_\ell \setminus U'_\ell) \cdot (U_\ell \setminus U'_\ell) \subseteq U'_\ell$, for all $\ell \in \{2,4,6\}$ in our case ($\mathbb{K} = \mathbf{GF}(2)$), one easily deduces that $\widetilde{U}$ is closed under multiplication and hence a subgroup of U. This subgroup is obviously generated by (and we use the notation introduced above) $1,3,5,7,8,2',4',6',8',24$ and 46. Using again the formulae (1) up to (15) in TITS [15](1.7.1), we shall show that all these elements are contained in U', thus completing the proof of the lemma. Indeed, we obtain successively:

$$\begin{aligned}
[8',[1,8']] &= [8',2'34566'7] = [8',2'3] = [8',2'][2',[8',3]][8',3] = \\
&= (6'54')[2',6']6' = 54' \in U', \\
[8',[1,8]] &= [8',2344'56'7] = [8',23] = [8',2][2,[8',3]][8',3] = \\
&= (76'4')[2,6']6' = 74' \in U', \\
[1,74'] &= [1,7] = 35 \in U', \\
[8',35] &= [8',3] = 6' \in U', \\
[1,6'] &= 4' \in U', \\
(54')4' &= 5 \in U', \\
(74')4' &= 7 \in U', \\
(35)5 &= 3 \in U', \\
3[1,8]76'54' &= 3(2344'56'7)76'54' = 24 \in U', \\
[1,24] &= [2,[1,4]][1,4] = [2,2']2' = 2' \in U', \\
32'[1,8']76'5 &= 32'(2'34566'7)76'5 = 46 \in U'.
\end{aligned}$$

The lemma is proved. $\square$

Lemma 9 *If* $|\mathbb{K}| > 2$, *then* $U' = U$.

Proof. We have to show $U_i \subseteq U'$, for all i, $2 \le i \le 7$. Combining Lemma 8 with the formulae (1) up to (15) in TITS [15](1.7.1), we first obtain

$$[U_1, 42] = [U_1, 4] = U_2' \subseteq U',$$
$$[U_1, 6'] = U_4' \subseteq U',$$
$$[3, U_8'] = U_6' \subseteq U',$$
$$[64, U_8] = [6, U_8] = U_7 \subseteq U'.$$

From the identities $[2', a_8] = a_3(\sigma(a)a)_{4'}(\sigma(a)a^2)_{6'}$ and $[2', a_{8'}] = a_{4'}a_5\sigma(a)_{6'}$, for all $a \in \mathbb{K}$, we then deduce $U_3 \subseteq U'$ and $U_5 \subseteq U'$, respectively. Formula (7) of Tits [15](1.7.1) shows that

$$[t_1, u_8] \in U_3(tu)_2(\sigma(t)\sigma(u)u)_4 U_4' U_5 U_6' U_7, \qquad \forall t, u \in \mathbb{K}. \qquad (*)$$

Replacing t by $a\sigma(a)b^{-1}$ and u by $b\sigma(a)^{-1}$ in $(*)$, we deduce that $a_2 b_4 \in U'$, for all $a, b \in \mathbb{K}^\times$ $(= \mathbb{K}\setminus\{0\})$. Since $(a_2b_4)(\tilde{a}_2b_4) \in (a+\tilde{a})_2 U_2' U_3 U_4'$, this implies $(a+\tilde{a})_2 \in U'$, for all $a, \tilde{a} \in \mathbb{K}^\times$. Since $|\mathbb{K}| > 2$, we obtain $x_2 \in U'$, for all $x \in \mathbb{K}$ and consequently $U_2 \subseteq U'$. Similarly, $U_4 \subseteq U'$. Finally, Formula (6) of Tits [15](1.7.1) implies $[a_1, 8'] \in U_2' U_3 U_4 U_5 U_6' U_7 a_6$, for all $a \in \mathbb{K}$. Hence we have $U_6 \subseteq U'$ as well. $\square$

Combining Lemma 8 and Lemma 9 with Corollary 5, we obtain:

Corollary 10 *The opposite-flag geometry $\Gamma^{(F)}$ is connected for any Moufang octagon with lines of length > 5. In the (up to duality, unique) excluded case, $\Gamma^{(F)}$ has exactly 2 connected components.*

References

[1] P. Abramenko, *Twin Buildings and Applications to S-Arithmetic Groups*, Lecture Notes in Math. **1641**, Springer, Berlin Heidelberg, 1996.

[2] P. Abramenko and B. Mühlherr, Présentations de certaines BN-paires jumelées comme sommes amalgamées, *C. R. Acad. Sci. Paris Sér. I. Math.* **325** (1997), 701 – 706.

[3] A. E. Brouwer, The complement of a geometric hyperplane in a generalized polygon is usually connected, **in** *Finite Geometry and Combinatorics*, ed. F. De Clerck *et al.*, Cambridge University Press, *London Math. Soc. Lect. Notes Ser.* **191** (1993), 53 – 57.

[4] R. W. Carter, *Simple Groups of Lie Type*, John Wiley & Sons, 1972.

[5] J. R. Faulkner, Groups with Steinberg relations and coordinatization of polygonal geometries, *Mem. Am. Math. Soc.* (10) **185** (1977).

[6] B. Mühlherr and M. A. Ronan, Local to global structure in twin buildings, *Invent. Math.* **122** (1995), 71 – 81.

[7] M. A. Ronan, *Generalized Hexagons*, Thesis, University of Oregon, 1978.

[8] M. A. Ronan, A geometric characterization of Moufang hexagons, *Invent. Math.* **57** (1980), 227 – 262.

[9] M. A. Ronan, *Lectures on Buildings*, Academic Press, 1989.

[10] J. Tits, Sur la trialité et certains groupes qui s'en déduisent, *Inst. Hautes Études Sci. Publ. Math.* **2** (1959), 13 – 60.

[11] J. Tits, *Buildings of Spherical Type and Finite BN-pairs*, Lecture Notes in Math. **386**, Springer, Berlin–Heidelberg–New York, 1974.

[12] J. Tits, Non-existence de certains polygones généralisés, I, *Invent. Math.* **36** (1976), 275 – 284.

[13] J. Tits, Endliche Spiegelungsgruppen, die als Weylgruppen auftreten, *Invent. Math.* **43** (1977), 283 – 295.

[14] J. Tits, Non-existence de certains polygones généralisés, II, *Invent. Math.* **51** (1979), 267 – 269.

[15] J. Tits, Moufang octagons and the Ree groups of type ${}^2F_4$, *Amer. J. Math.* **105** (1983), 539 – 594.

[16] J. Tits, Ensembles ordonnés, immeubles et sommes amalgamées, *Bull. Soc. Math. Belg.* **38** (1986), 367 – 387.

[17] J. Tits, Moufang polygons, I. Root data, *Bull. Belg. Math. Soc. Simon Stevin* **1** (1994), 455 – 468.

[18] J. Tits and R. Weiss, The classification of Moufang polygons, *in preparation.*

[19] H. Van Maldeghem, *Generalized Polygons*, Birkhäuser, Basel, *to appear.*

[20] R. Weiss, The nonexistence of certain Moufang polygons, *Invent. Math.* **51** (1979), 261 – 266.

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