E_∞-structure for $Q_*(R)$

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Abstract

MacLane homology of a ring R is the Hochschild homology of the so called cubical construction $Q_*(R)$, which is a chain-algebra. If we take a commutative ring R, the Dixmier product on $Q_*(R)$ is no longer commutative. The main result of this paper is that it is commutative up to higher homotopies, i.e. is a E_∞-algebra. The E_∞-operad which acts on $Q_*(R)$ is constructed by using the analogue of the Q_*-complex in the context of finite sets. For the correct notation of an operad action on these complexes the definition of an E_∞-monoidal functor is introduced.

1.1 Introduction

MacLane (co)homology of a ring R with coefficients in an R-bimodul M can be defined as the Hochschild (co)homology of the differential graded ring $Q_*(R)$ with coefficients in M:

$$\begin{array}{rcl} HML_*(R,M) & := & H_*(Q_*(R),M) \\ HML^*(R,M) & := & H^*(Q_*(R),M) \end{array}$$

We will recall the definition of $Q_*(R)$ below, but first we want to mention some of the remarkable properties of this (co)homology: The second MacLane cohomology $HML^2(R,M)$ classifies arbitrary extensions of R by the bimodule M, unlike the second Hochschild cohomology which classifies only split-extensions. If the ring contains the rational numbers then Hochschild-theory and MacLane-theory coincide. MacLane homology is isomorphic to stable K-theory K_*^s introduced by Waldhausen and to topological Hochschild homology THH_*. For proves of these statements see [P-W] and [D-McC].
For an arbitrary ring R the Dixmier product on the Eilenberg-MacLane cubical construction gives $Q_*(R)$ the structure of a ring. Let us recall that Hochschild homology of commutative algebras has a very rich structure with products and homology operations. As MacLane homology is a special case of Hochschild homology it would be nice to have similar structures for it. But unfortunately a commutative ring R will not lead to a commutative product in $Q_*(R)$. The main goal of this paper is to show that $Q_*(R)$ is equipped with the structure of an E_∞-algebra. Before proving this, we will introduce the notion of an E_∞-monoidal functor which describes an action of an E_∞-operad on the images of a functor between symmetric monoidal abelian categories. Instead of directly constructing an E_∞-operad that acts on $Q_*(R)$ we show that the analogue of

the Q-construction SQ_* in the set context is an E_∞- monoidal functor. The transfer of this structure to Q_* is done by using left Kan extensions.
Finally we prove that the fact that the Q-construction maps associative algebras to associative algebras and commutative ones to E_∞-algebras can be generalized to the case of A_∞- and E_∞-algebras. Applying the same methods as for proving that Q_* is an E_∞-monoidal functor yields an easy proof that the chain complex functor C_* from the category of simplicial abelian groups to the category of chain complexes is an E_∞-comonoidal functor.

The fact that $Q_*(R)$ is an E_∞-algebra for every commutative ring leads to new operations in MacLane homology. We will investigate these implications in a different paper.

1.2 Eilenberg-MacLane's cubical construction

Let us now recall the definition of the cubical construction. For a comprehensive overview see [MLa], [L], [J-P] and [F-P-S-V-W].

Given an abelian group A the Eilenberg-MacLane cubical construction assigns a chain complex $Q_*(A)$ to this group in a functorial way. This chain complex $Q_*(A)$ is a quotient of an auxiliary complex $Q'_*(A)$ which in turn is defined as the free abelian group generated by all maps from the vertices of an n-dimensional unit-cube C_n to A:

$$Q'_*(A) := \mathbf{Z}\,[A\,[C_n]]$$

Let $(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_i \in \{0,1\}$ denote a vertex of the n-cube for $n > 0$ and let $()$ denote the single element of C_0. Define two cubical face maps

$$0_i, 1_i : C_n \longrightarrow C_{n+1}$$

for $1 \le i \le n+1$ in the following manner:

$$\begin{aligned} 0_i(\varepsilon_1, \dots, \varepsilon_n) &= (\varepsilon_1, \dots, \varepsilon_{i-1}, 0, \varepsilon_i, \dots, \varepsilon_n) \\ 1_i(\varepsilon_1, \dots, \varepsilon_n) &= (\varepsilon_1, \dots, \varepsilon_{i-1}, 1, \varepsilon_i, \dots, \varepsilon_n) \end{aligned}$$

Define for $1 \le i \le n$ the maps

$$R'_i, S'_i, P'_i : A\,[C_n] \longrightarrow A\,[C_{n+1}]$$

as follows:

$$(R'_i f)(e) := f(0_i e), \quad (S'_i f)(e) := f(1_i e), \quad (P'_i f)(e) := f(0_i e) + f(1_i e)$$

Let P_i, S_i, R_i denote the linearizations of these maps from $Q'_n(A)$ to $Q'_{n+1}(A)$. With the help of these maps we can define the boundary map for Q'_*

$$\delta := \sum_{i=1}^{n} (-1)^i (P_i - R_i - S_i)$$

As $\delta^2 = 0$, $Q'_*(A)$ is a chain complex.

Now $Q_*(A)$ is obtained from $Q'_*(A)$ by a normalization. We divide out all maps from C_n to A, which are zero on one face of the cube, i.e. maps $f : C_n \longrightarrow A$ with $f(0_i e) = 0$ or $f(1_i e) = 0 \quad \forall e \in C_{n-1}$ and $f() = 0$ for $n = 0$. Let $N_*(A)$ denote all elements in the subgroup generated by such maps. Then

$$Q_*(A) := Q'_*(A)/N_*(A)$$

Since the boundary is well-defined on the quotient, this defines a chain complex as well.

The homology of this complex is closely related to the stable homology of Eilenberg-MacLane spaces

$$H_n(Q_*(A)) \cong H_{n+k}(K(A,k)) \qquad \forall k \geq n$$

For two abelian groups A and B one can define the *Dixmier product* of $Q_*(A)$ with $Q_*(B)$

$$Q_*(A) \otimes Q_*(B) \longrightarrow Q_*(A \otimes B)$$

For two generators of Q'_* $f \in A\,[C_n]$ and $g \in B\,[C_m]$ the product is given by

$$(fg)(\varepsilon_1, \ldots, \varepsilon_{n+m}) := f(\varepsilon_1, \ldots, \varepsilon_n) \otimes g(\varepsilon_{n+1}, \ldots, \varepsilon_{n+m})$$

Obviously this product is well-defined on the normalization Q_*.

If R is a ring, the Dixmier product equips $Q_*(R)$ with the structure of a differential graded ring. If M is an R-module then $Q_*(M)$ is a $Q_*(R)$-module by extending the Dixmier product with the module action

$$Q_*(R) \otimes Q_*(M) \longrightarrow Q_*(R \otimes M) \longrightarrow Q_*(M)$$

In addition we have an augmentation map

$$\varepsilon : Q_*(A) \longrightarrow A$$

for any abelian group A. On C_n with $n > 0$, ε is zero and on C_0 it just evaluates the function on $()$

$$\varepsilon(f) = f()$$

This map provides M with the structure of a $Q_*(R)$-module.

1.3 E_∞-monoidal functors

It is well known that for any abelian category which allows arbitrary exact direct sums one can define a generalization of the usual tensorproduct. Let Ab denote the category of abelian groups and let $\mathcal{A}$ be an abelian category with arbitrary direct sums. Then there exists a natural bifunctor

$$\begin{array}{ccc} Ab \times \mathcal{A} & \longrightarrow & \mathcal{A} \\ (A, X) & \mapsto & A \otimes X \end{array}$$

with the following properties

1) $\mathbf{Z} \otimes X \cong X$
2) $(\oplus_i A_i) \otimes X \cong \oplus (A_i \otimes X)$
3) $\_\_ \otimes X : Ab \longrightarrow \mathcal{A}$ is a right-exact functor.

Such a bifunctor is unique up to isomorphism. Define the bifunctor for free abelian groups with the help of 1) and 2) and for an arbitrary abelian group $A \cong F/R$ we can define it as

$$A \otimes X = \text{cokernel } (R \otimes X \longrightarrow F \otimes X)$$

If we choose a different presentation $A \cong F'/R'$ we obtain an isomorphic object.

The aim of the rest of this section is to build up a reasonable notion of an E_∞-monoidal functor betweeen monoidal categories. Since operads in our context consist of chain complexes of abelian groups, we should have an action of the category $Ch(Ab)$ on our monoidal category.

Definition 1.1 *An abelian category $\underline{B}$* is a $Ch(Ab)$-category, *if there is a bifunctor*

$$Ch(Ab) \times \underline{B} \longrightarrow \underline{B}$$

$$(A_*, X) \mapsto A_* \otimes X$$

with the following properties:

1) If $\mathbf{Z}_*$ denotes the chain-complex which is $\mathbf{Z}$ in degree zero and zero in other degrees, we have

$$\mathbf{Z}_* \otimes X \cong X \qquad \forall X \in \underline{B}$$

2)For any $A_*, B_* \in Ch(Ab)$ and $\forall X, Y \in \underline{B}$ we have

$$(A_* \otimes B_*) \otimes X \quad \cong \quad A_* \otimes (B_* \otimes X)$$

3) $A \otimes \_\_ : \underline{B} \longrightarrow \underline{B}$ and $\_\_ \otimes X : Ch(Ab) \longrightarrow \underline{B}$ commute with direct sums and are right-exact.

Example
For any abelian category $\mathcal{A}$ with direct sums, the category of chain complexes $Ch(\mathcal{A})$ has a natural structure of a $Ch(Ab)$-category which extends the action of Ab on $\mathcal{A}$.

Let $(\underline{B}, \square)$ be a symmetric monoidal abelian category which is a $Ch(Ab)$-category.

Definition 1.2 *The action of $Ch(Ab)$ on $\underline{B}$ is said to be* compatible *if it fits to the monoidal structure in $\underline{B}$*

$$A_* \otimes (X \square Y) \cong (A_* \otimes X) \square Y$$

Now let $(\underline{A}, \triangle)$ and $(\underline{B}, \square)$ denote two symmetric monoidal categories.

Definition 1.3 *A functor $T : \underline{A} \longrightarrow \underline{B}$ is a* lax monoidal functor *if there exists a map*

$$T(A_1) \square T(A_2) \longrightarrow T(A_1 \triangle A_2)$$

that satisfies the usual associativity and unit coherence conditions (see [T] and the references therein).

If a functor $F : \underline{A} \longrightarrow \underline{B}$ is equipped with a map

$$F(A_1 \triangle A_2) \longrightarrow F(A_1) \square F(A_2)$$

having the dual properties, then F is called a lax comomoidal functor.

Remark
A lax monoidal functor sends a monoid M in $\underline{A}$ to a monoid in $\underline{B}$ and the image of a comonoid under a lax comonoidal functor is again a comonoid. If both the categories are symmetric monoidal then the image of a commutative (co)monoid under a lax (co)monoidal functor is not commutative in general. Commutativity is preserved if one assumes that the functor is in addition symmetric, that means that the following diagramm commutes for all objects in $\underline{A}$:

$$\begin{array}{ccc} T(A_1) \square T(A_2) & \longrightarrow & T(A_1 \triangle A_2) \\ \downarrow & & \downarrow \\ T(A_2) \square T(A_1) & \longrightarrow & T(A_2 \triangle A_1) \end{array}$$

We will deal with two functors which are not symmetric, but which fulfill an E_∞-condition.

Definition 1.4 *Let $\underline{A}$ and $\underline{B}$ denote two $Ch(Ab)$-categories.*
A functor $T : \underline{A} \longrightarrow \underline{B}$ is a Ch(Ab)-functor *if there is a natural isomorphism*

$$\zeta_{C,T(X)} : C_* \otimes T(X) \cong T(C_* \otimes X) \qquad \forall C_* \in Ch(Ab) \qquad \forall X \in \underline{A}.$$

If $\underline{A}$ and $\underline{B}$ are in additions symmetric monoidal and have a compatible $Ch(Ab)$-action, then this isomorphism has to be coherent with the structure maps:
Let $\alpha_{X,Y,Z} : X\triangle(Y\triangle Z) \cong (X\triangle Y)\triangle Z$ denote the associativity isomorphism in $\underline{A}$, let $\lambda_X : 1\triangle X \cong X$ and $\varrho_X : X\triangle 1 \cong X$ denote the two unit isomorphisms and let finally $\gamma_{X,Y} : X\square Y \cong Y\square X$ be the twist isomorphism in $\underline{A}$ (compare [MLb]). Then ζ has to satisfy

$$\begin{array}{lrcl}
1) & T(id\otimes\alpha_{X,Y,Z})\circ\zeta_{C,T(X\triangle(Y\triangle Z))} & = & \zeta_{C,T((X\triangle Y)\triangle Z)}\circ id\otimes T(\alpha_{X,Y,Z}) \\
2) & T(id\otimes\varrho_X)\circ\zeta_{C,T(X\triangle 1)} & = & \zeta_{C,T(X)}\circ id\otimes T(\varrho_X) \\
3) & T(id\otimes\lambda_X)\circ\zeta_{C,T(1\triangle X)} & = & \zeta_{C,T(X)}\circ id\otimes T(\lambda_X) \\
4) & T(id\otimes\gamma_{X,Y})\circ\zeta_{C,T(X\square Y)} & = & \zeta_{C,T(Y\square X)}\circ id\otimes T(\gamma_{X,Y})
\end{array}$$

Lemma 1.5 *Let $\underline{C}$ and $\underline{D}$ be two abelian categories with direct sums and let $T : \underline{C} \longrightarrow \underline{D}$ be a right-exact functor, which preserves sums. Then the degreewise extension of T as a functor from Ch($\underline{C}$) to Ch($\underline{D}$) is a Ch(Ab)-functor between these standard Ch(Ab)-categories.*

Proof
The fact that T preserves sums guarantees that such a ζ exists. The coherence conditions are automatically satisfied, because of the properties of the tensor-product.

$\square$

In the rest of the paper we will use the standard notations and results concerning operads, as they can be found for instance in [K-M]. According to their terminology we will use the term "operad" for an operad with an action of the symmetric group. Operads without such an action are called non-Σ-operads.

Definition 1.6 *Let $(\underline{A}, \triangle)$ and $(\underline{B}, \square)$ be two symmetric monoidal abelian categories with a compatible Ch(Ab)-action.*
A functor $T : \underline{A} \longrightarrow \underline{B}$ is said to be an E_∞-monoidal functor, *if there exists an E_∞-operad O in Ch(Ab) with the following property:*
There are natural maps

$$O(n)\otimes T(A_1)\square\ldots\square T(A_n) \longrightarrow T(A_1\triangle\ldots\triangle A_n)$$

which fulfill the usual identities of an operad action on an algebra (Compare [K-M]).

1) The unit element in the operad acts in a compatible way:

$$\begin{tikzcd}
\mathbf{Z}\otimes T(A) \arrow[d] \arrow[r, phantom, "\cong"] & T(A) \\
O(1)\otimes T(A) \arrow[ur] &
\end{tikzcd}$$

2) The action of the operad is equivariant:

$$\begin{tikzcd}
O(n) \otimes T(A_{\sigma^{-1}(1)}) \square \cdots \square T(A_{\sigma^{-1}(n)}) \arrow[r] & T(A_{\sigma^{-1}(1)} \triangle \ldots \triangle A_{\sigma^{-1}(n)}) \\
O(n) \otimes T(A_1) \square \cdots \square T(A_n) \arrow[r] \arrow[u, "\sigma^{-1} \otimes \sigma"] & T(A_1 \triangle \ldots \triangle A_n) \arrow[u, "\sigma"']
\end{tikzcd}$$

3) The following associativity diagram commutes for $\sum_{i=1}^{n} j_i = j$ and $\sum_{i=1}^{n-1} j_i + 1 = j'$:

$$\begin{tikzcd}
O(n) \otimes O(j_1) \otimes \cdots \otimes O(j_n) \otimes T(A_1) \square \cdots \square T(A_j) \arrow[r] \arrow[d] & O(j) \otimes T(A_1) \square \cdots \square T(A_j) \arrow[d] \\
O(n) \otimes O(j_1) \otimes (T(A_1) \square \cdots \square T(A_{j_1})) \square \cdots \square O(j_n) \otimes (T(A_{j'}) \square \cdots \square T(A_j)) \arrow[d] & T(A_1 \triangle \ldots \triangle A_j) \\
O(n) \otimes T(A_1 \triangle \ldots \triangle A_{j_1}) \square \cdots \square T(A_{j'} \triangle \ldots \triangle A_j) \arrow[ur] &
\end{tikzcd}$$

Lemma 1.7 *Assume $T : \underline{A} \longrightarrow \underline{B}$ is an E_∞-monoidal functor and assume M is a commutative monoid in $\underline{A}$. Then $T(M)$ is an E_∞-monoid in $\underline{B}$. That means there are maps*

$$O(n) \otimes T(M) \square \cdots \square T(M) \longrightarrow T(M)$$

which fulfills all the axioms for operad action.

Proof
The restriction of the operad action to the image of M gives a map

$$O(n) \otimes T(M) \square \cdots \square T(M) \longrightarrow T(M \triangle \cdots \triangle M)$$

Composing this map with the map which is induced by the monoid map

$$M \triangle \cdots \triangle M \xrightarrow{\mu} M$$

we obtain

$$O(n) \otimes T(M) \Box \cdots \Box T(M) \longrightarrow T(M).$$

As the monoid map is associative and respects the unit, we get the associativity of the operad action and the unit condition. The commutativity of the monoid map μ makes the following diagramm commute

$$\begin{tikzcd}
O(n) \otimes T(M) \Box \cdots \Box T(M) \arrow[r] & T(M \triangle \dots \triangle M) \arrow[dr] & \\
O(n) \otimes T(M) \Box \cdots \Box T(M) \arrow[r] \arrow[u, "\sigma^{-1} \otimes \sigma"] & T(M \triangle \dots \triangle M) \arrow[r] \arrow[u, "\sigma"'] & T(M)
\end{tikzcd}$$

Therefore $T(M)$ is an E_∞-monoid.

□

Dually we obtain the notion of an E_∞-*comonoidal functor*: A functor F between two symmetric monoidal categories with a compatible $Ch(Ab)$-action is an E_∞-comonoidal functor, if there exist an E_∞-operad P, which acts on the images of F in the following way:
There is a natural map

$$P(n) \otimes F(A_1 \triangle \dots \triangle A_n) \longrightarrow F(A_1) \Box \cdots \Box F(A_n)$$

which fulfills the analogue conditions for an operad action, only the associativity condition needs to be modified.

The following diagram has to be commutative for $\sum_{i=1}^{n} j_i = j$ and $\sum_{i=1}^{n-1} j_i + 1 = j'$:

$$\begin{tikzcd}
O(n) \otimes O(j_1) \otimes \cdots \otimes O(j_n) \otimes F(A_1 \triangle \dots \triangle A_j) \arrow[r] \arrow[d] & O(j) \otimes F(A_1 \triangle \dots \triangle A_j) \arrow[d] \\
O(j_1) \otimes F(A_1 \triangle \dots \triangle A_{j_1}) \Box \cdots \Box O(j_n) \otimes F(A_{j'} \triangle \dots \triangle A_j) \arrow[r] & F(A_1) \Box \cdots \Box F(A_j)
\end{tikzcd}$$

For the left downleading arrow we use the action of $O(n)$ and the usual shuffle maps to split the image of the product.

Remark
In our applications our functors will already be lax (co)monoidal but for the definition of an E_∞-(co)monoidal functor this assumption is not necessary.

1.4 SQ_* as an E_∞-monoidal functor

In order to show that $Q_*(R)$ is an E_∞-algebra for every commutative ring R we will first show that the analogue functor in the set context SQ_* is an E_∞-monoidal functor. This functor has much nicer properties than Q_*, for instance $SQ_* \longrightarrow t$ is a projective resolution in the category of all contravariant functors from finite pointed sets to abelian groups (see [P], [L]). Here t is the functor $S \mapsto Sets_*(S, \mathbf{Z})$. With the help of the left Kan extension of t we can transfer this statement to Q_*, because $t_! SQ_* = Q_*$ (see [P]).

1.4.1 E_∞-operad in the set-context

Let us recall how the analogue of the Q_*-complex in the set context SQ_* is defined : For each finite pointed set X the chain-complex $SQ'_*(X)$ in degree n is the free abelian group generated by all families $\mathcal{X}_{(a_1,\ldots a_n)}$ of pairwise disjoint subsets of X indexed by n-tupels of elements $a_i \in \{0,1\}$. As in Q_* we divide out all elements that map a face of the cube to the empty set, the result of this normalization process is $SQ_*(X)$. The boundary is analogous to δ for Q_*. For a detailed description see [P]. The aim of the following section is to show that $SQ_* : (Fin_*, \wedge) \longrightarrow (Ch(Ab), \otimes)$ is an E_∞-monoidal functor.

We are interested in the functors $SQ_*^{\boxtimes n}$ and $SQ_*^{\wedge n}$ from $Fin_* \times \ldots \times Fin_*$ to Ab, which are defined as

$$\begin{array}{rcl} SQ_*^{\boxtimes n}(S_1,\ldots,S_n) & = & SQ_*(S_1)\otimes\cdots\otimes SQ_*(S_n) \\ SQ_*^{\wedge n}(S_1,\ldots,S_n) & = & SQ_*(S_1\wedge\ldots\wedge S_n) \end{array}.$$

The operad that is supposed to act on SQ_* will consist of the homomorphism complex for two chain-complexes. For arbitrary chain-complexes A_* and B_* this complex is

$$\mathcal{H}om_j(A_*, B_*) := \prod_{k\in\mathbf{Z}} Hom(A_k, B_{k+j}) \qquad j \in \mathbf{Z}$$

The differential d is defined componentwise in the following way

$$(df)_i := f_{i-1}\circ\delta + (-1)^{n+1}\delta'\circ f_i \quad \forall f \in \mathcal{H}om_j(A_*, B_*)$$

With $\mathcal{F}_n$ we denote the category of functors from $Fin_* \times \ldots \times Fin_*$ to Ab. Now we are prepared to define the operad which is supposed to act on $SQ_*^{\boxtimes n}$.

We define

$$O_{SQ_*}(n) := \mathcal{H}om_{\mathcal{F}_n}(SQ_*^{\boxtimes n}, SQ_*^{\wedge n})$$

Hence O_{SQ_*} consists of chains of abelian groups. The action of the symmetric group is given in the following way:

For any permutation $\sigma \in \Sigma_n$ and for any element f in $O_{SQ_*}(n)$ we define

$$f.\sigma := SQ_*(\sigma^{-1}) \circ f \circ \sigma.$$

We permutate the entries first, then we apply the corresponding component of f and finally we apply $SQ_*(\sigma^{-1})$ in order to end up in the right component again.
The product in this operad is just defined to be the composition

$$\gamma(f, g_1, \dots, g_n) := f \circ (g_1 \otimes \dots \otimes g_n)$$

whenever f is in $O_{SQ_*}(n)$ and the g_i are elements of $O_{SQ_*}(k_i)$ with $\sum_{i=1}^n k_i = n$.

This operad is acting on SQ_* in the following way:
We have natural transformations

$$O_{SQ_*}(n) \otimes SQ_*(S_1) \otimes \dots \otimes SQ_*(S_n) \longrightarrow SQ_*(S_1 \wedge \dots \wedge S_n).$$

satisfying the usual equations for operad actions.

Let $Sets_*$ denote the category of pointed sets and define

$$t : Fin_*^{op} \longrightarrow Ab, \quad t(X) = Hom_{Sets_*}(X, \mathbf{Z})$$

We define $t^{\wedge n}$ and $t^{\boxtimes n}$ in the obvious way. Now we can compute the homology of O_{SQ_*}.

Lemma 1.8

$$H_i(O_{SQ_*}(n)) = \begin{cases} 0 & : \quad i > 0 \\ Ext^{-i}_{\mathcal{F}_n}(t^{\boxtimes n}, t^{\wedge n}) & : \quad i \leq 0 \end{cases}$$

Proof
We know that $SQ_* \longrightarrow t$ is a projective resolution in Ab^{Fin_*}, consequently $SQ_*^{\boxtimes n} \longrightarrow t^{\boxtimes n}$ and $SQ_*^{\wedge n} \longrightarrow t^{\wedge n}$ are also resolutions in $\mathcal{F}_n$. In addition $SQ_*^{\boxtimes n} \longrightarrow t^{\boxtimes n}$ are still projective resolutions because the exterior tensor product $\boxtimes$ of two standard projective functors is again projective (see 1.11.1). Therefore

$$\mathcal{H}om_{\mathcal{F}_n}(SQ_*^{\boxtimes n}, SQ_*^{\wedge n}) \longrightarrow \mathcal{H}om_{\mathcal{F}_n}(SQ_*^{\boxtimes n}, t^{\wedge n})$$

is a weak equivalence and hence we get the result in homology.

$\square$

Obviously $Hom_{\mathcal{F}_n}(t^{\boxtimes n}, t^{\wedge n})$ forms a new operad. If we truncate the chain-complex $\mathcal{H}om_{\mathcal{F}_n} * (SQ_*^{\boxtimes n}, SQ_*^{\wedge n})$ and use only its non-negative degrees we get a map to $Hom_{\mathcal{F}_n}(t^{\boxtimes n}, t^{\wedge n})$ via the augmentation map.

For an arbitrary chain complex C_* we define the truncated complex

$$\tau(C)_* := \begin{cases} 0 & : \quad k < 0 \\ C_k & : \quad k > 0 \\ \text{cycles}(C_0) & : \quad k = 0 \end{cases}$$

we have maps

$$\tau(C)_* \otimes \tau(C')_* \longrightarrow \tau(C \otimes C')_*$$

for all chain-complexes C_* and C'_*. These maps are associative and commutative. Hence for every operad P, $\tau(P)$ is again an operad. Thus by truncating O_{SQ_*} we still have an operad and by the natural inclusion $\tau(O_{SQ_*}) \hookrightarrow O_{SQ*}$ the truncated operad $\tau(O_{SQ_*})$ acts on SQ_* as well. As $\tau(O_{SQ_*})$ consists only of the cycles in degree zero, we get the projection map

$$\tau(O_{SQ_*})_0 \longrightarrow Hom_{\mathcal{F}_n}(t^{\boxtimes n}, t^{\wedge n})$$

We define the operad $O_t(n)$ to be this abelian group of morphisms

$$O_t(n) = Hom_{\mathcal{F}_n}(t^{\boxtimes n}, t^{\wedge n}).$$

The composition and the action of the symmetric group are defined in the same way as for O_{SQ_*}.

In addition we are able to get a map from the commutative operad Com to O_t. In $Hom(i_n t^{\boxtimes n}, i_n t^{\wedge n})$ we have a multiplication map

$$m(t(S_1) \otimes \cdots \otimes t(S_n)) = t(S_1 \wedge \ldots \wedge S_n)$$

which commutes with the action of the symmetric group on O_t:

$$(m.\sigma)(t(S_1) \otimes \cdots \otimes t(S_n)) = t(\sigma^{-1}).m(t(S_{\sigma^{-1}(1)}) \otimes \cdots \otimes t(S_{\sigma^{-1}(n)}))$$
$$= t(S_1 \wedge \ldots \wedge S_n) = m(S_1 \otimes \cdots \otimes S_n)$$

By sending the identity map to this multiplication m we obtain a map from $Com(n)$ to $O_t(n)$.

Hence we get the following diagramm of chains of abelian groups:

$$\begin{array}{ccc} & & \tau(O_{SQ_*})(n) \\ & & \downarrow \\ Com(n) & \longrightarrow & O_t(n) \end{array}$$

We can build the degreewise pullback of these operads. This yields a new operad O.

Since the map from $\tau(O_{SQ_*})$ to O_t is surjective and is a quasi-isomorphism by lemma 1.8 we can conclude that the right downleading arrow is a quasi-isomorphism, hence our operad O is acyclic, that means

$$H_i(O(n)) = \begin{cases} 0 & : \quad i \neq 0 \\ \mathbf{Z} & : \quad i = 0 \end{cases}$$

In addition, this operad acts on $SQ_*^{\boxtimes n}$ via the induced pullback-map. With this construction we get

Theorem 1.9 *The lax monoidal functor SQ_* from the monoidal category $(Fin_*, \wedge)$ to the monoidal category $(Ch(Ab), \otimes)$ is an E_∞-monoidal functor.*

Proof
The only thing that is left to show is the fact that any acyclic operad which consists of free abelian groups in every degree can be a target of a map from an E_∞-operad, that means from an operad which is not only acyclic but also Σ-free. With such a map we will be able to transfer the action of O on SQ_* to an action of an E_∞-operad on it. The missing link is proved in the following lemma:

Lemma 1.10 *For an acyclic operad P whose components $P(n)$ are free abelian groups there is an E_∞-operad P' and a map of operads $P' \longrightarrow P$.*

Proof
Take an arbitrary E_∞-operad A and build the tensor product of P with A

$$(P \otimes A)(j) = P(j) \otimes A(j)$$

This operad is still acyclic, because both the components are. Using the augmentation of A we get maps $P \otimes A \longrightarrow P$. Furthermore $(P \otimes A)(n)$ is $\mathbf{Z}[\Sigma_n]$-free, because $A(n)$ is $\mathbf{Z}[\Sigma_n]$-free and $P(n)$ is free.

□

1.5 E_∞-structure for Q_*

Having constructed an E_∞-operad acting on SQ_*, we want to pass to the Q-construction now. To this end we use the left Kan extension $t_! : Ab^{Fin_*^{op}} \longrightarrow Ab^{ab}$ of the functor t. Here ab denotes the category of finitely generated abelian groups. The functors $t_!$ and $(t \times \cdots \times t)_!$ are both right exact, because they are left-adjoint to the functors which are the precomposition with t and $(t \times \cdots \times t)$, respectively. With the help of the following two lemmas we can transform the operad-action on SQ_* to an operad-action on Q_*.

Lemma 1.11 *Left Kan extensions commute with taking tensor products of functors, i.e. for two functors* $f : \underline{A} \longrightarrow \underline{B}$ *and* $g : \underline{C} \longrightarrow \underline{D}$*, where* $\underline{A}, \underline{C}$ *are small categories, and for two functors* $F : \underline{A} \longrightarrow Ab$ *and* $G : \underline{C} \longrightarrow Ab$ *there exists a natural isomorphism*

$$(f \times g)_!(F \boxtimes G) \xrightarrow{\cong} f_! F \boxtimes g_! G$$

Here $(F \boxtimes G)(a, c) = F(a) \otimes G(c)$.

Proof
First we construct a transformation

$$(f \boxtimes g)_!(F \boxtimes G) \longrightarrow (f_! F) \boxtimes (g_! G)$$

By the definition of the Kan extension we have morphisms $F(a) \to f_! F(f(a))$ and $G(c) \to g_! G(g(c))$ for every $a \in \underline{A}$ and every $c \in \underline{C}$, which are natural in a and c. By tensoring these morphisms we get

$$F(a) \otimes G(c) \longrightarrow f_! F(f(a)) \otimes g_! G(g(c))$$

That means we have

$$(F \boxtimes G)(a, c) \longrightarrow (f_! F \boxtimes g_! G)((f \times g)(a, c))$$

Consequently, by the universal property of the Kan extension there is a natural transformation

$$\Upsilon_{F,G} : (f \times g)_!(F \boxtimes G) \longrightarrow f_! F \boxtimes g_! G$$

Since $(f \times g)_!$ is right-exact and preserves direct sums we get

1) If $F = \oplus_{i \in I} F_i$ and $\Upsilon_{F_i, G}$ is an isomorphism $\forall i$ then $\Upsilon_{F,G}$ is an isomorphism.
2) If we have a short sequence of functors

$$0 \to F' \to F \to F'' \to 0$$

and $\Upsilon_{F', G}, \Upsilon_{F,G}$ are isomorphisms then $\Upsilon_{F'', G}$ is an isomorphism as well.
In addition we know that for any $T : \underline{A} \longrightarrow Ab$ there is an epimorphism

$$\oplus_{a \in \underline{A}} h_a \longrightarrow T.$$

Here h_a is the functor $h_a(x) = \mathbf{Z}[Hom_{\underline{A}}(a, x)]$. Thus it is sufficient to prove the claim for $F = h_a$ and $G = h_c$.

But

$$h_a \boxtimes h_c = h_{(a,c)} \tag{1.11.1}$$

Indeed

$$
\begin{array}{rcl}
(h_a \boxtimes h_c)(x,y) & = & h_a(x) \otimes h_c(y) \\
& = & \mathbf{Z}[Hom_{\underline{A}}(a,x)] \otimes \mathbf{Z}[Hom_{\underline{C}}(c,y)] \\
& = & \mathbf{Z}[Hom_{\underline{A}}(a,x) \times Hom_{\underline{C}}(c,y)] \\
& = & \mathbf{Z}[Hom_{\underline{A}\times\underline{C}}((a,c),(x,y))] \\
& = & h_{(a,c)}(x,y)
\end{array}
$$

Thus we have

$$(f \times g)_!(h_a \boxtimes h_c) = (f \times g)_!(h_{(a,c)}) = h_{(f(a),g(c))}$$

On the other hand

$$f_! h_a \boxtimes g_! h_c = h_{f(a)} \boxtimes h_{g(c)} = h_{(f(a),g(c))}$$

This proves the lemma.

□

Lemma 1.12 *The functor* $(t \times \cdots \times t)_!$ *is a* $Ch(Ab)$*-functor.*

Proof
For the same reasons as in the last proof it is sufficient to prove the claim only for the functors h_S where S is a finite set.
We have to calculate

$$(t \times \ldots \times t)_!(A \otimes h_{S_1}(X_1) \otimes \cdots \otimes h_{S_n}(X_n))$$

for all abelian groups A and all finite pointed sets S_i. Applying the lemma above we see that this expression is isomorphic to

$$t_!(A \otimes h_{S_1}(X_1)) \otimes \cdots \otimes t_!(h_{S_n}(X_n))$$

But $t_!(A \otimes h_{S_1}(X_1))$ is isomorphic to $A \otimes t_!(h_{S_1}(X_1))$, because this is true for free A. For arbitrary A this property is a consequence of the right-exactness of $t_!$. Thus we obtain

$$
\begin{array}{l}
(t \times \ldots \times t)_!(A \otimes h_{S_1}(X_1) \otimes \cdots \otimes h_{S_n}(X_n)) \\
\quad \cong A \otimes t_! h_{S_1}(X_1) \otimes \cdots \otimes t_!(h_{S_n}(X_n)) \\
\quad = A \otimes h_{t(S_1)}(X_1) \otimes \cdots \otimes h_{t(S_n)}(X_n) \\
\quad \cong A \otimes h_{(t\times\ldots\times t)(S_1,\ldots,S_n)}(X_1,\ldots,X_n) \\
\quad = A \otimes (t \times \ldots \times t)_!(h_{S_1}(X_1) \otimes \cdots \otimes h_{S_n}(X_n))
\end{array}
$$

□

Now we have all means to state the following

Theorem 1.13 *The functor $Q_* : (Ab, \otimes) \longrightarrow (Ch(Ab), \otimes)$ is an E_∞-monoidal functor and therefore the Eilenberg-MacLane cubical chain algebra $Q_*(R)$ is an E_∞-algebra for any commutative ring R*

Proof
Applying the last lemma and using that $t_! SQ_* = Q_*$ (see [P], [L]) we get natural transformations

$$O(n) \otimes Q_*^{\boxtimes n} \longrightarrow (t \times \cdots \times t)_! SQ_*^{\wedge n}$$

In addition we obtain natural transformations

$$(t \times \cdots \times t)_! SQ_*^{\wedge n} \longrightarrow Q_*^{\otimes n}$$

by using the universal property of Kan extensions, where $Q_*^{\otimes n}$ is the functor which takes first the tensorpower of n abelian groups and then applies the Q_* functor, i.e.

$$Q_*^{\otimes n}(A_1, \ldots, A_n) = Q_*(A_1 \otimes \cdots \otimes A_n).$$

Indeed, by definition of the Kan extension we have homomorphisms

$$SQ_*(S) \longrightarrow (t_! SQ_*)(t(S))$$

which are natural in S. Replacing S by $S_1 \wedge \ldots \wedge S_n$ leads to

$$SQ_*^{\wedge n}(S_1, \ldots, S_n) = SQ_*(S_1 \wedge \ldots \wedge S_n) \longrightarrow (t_! SQ_*)(t(S_1 \wedge \ldots \wedge S_n))$$

We can calculate the right-hand side

$$\begin{aligned}
&t(S_1 \wedge \ldots \wedge S_n)\\
&= Hom_{Sets_*}(S_1 \wedge \ldots \wedge S_n, \mathbf{Z})\\
&= Hom_{Sets_*}(S_1; \mathbf{Z}) \otimes \cdots \otimes Hom_{Sets_*}(S_n, \mathbf{Z})\\
&= t(S_1) \otimes \cdots \otimes t(S_n)
\end{aligned}$$

and get a map

$$\begin{array}{ccc}
SQ_*^{\wedge n}(S_1, \ldots, S_n) & \longrightarrow & (t_! SQ_*)(t(S_1) \otimes \cdots \otimes t(S_n))\\
 & & \| \\
 & & (t_! SQ_*)^{\otimes n}(t(S_1), \ldots, t(S_n))
\end{array}$$

Hence by the universality of the Kan extension we obtain the desired natural transformation

$$(t \times \cdots \times t)_! SQ_*^{\wedge n} \longrightarrow (t_! SQ_*)^{\otimes n} = Q_*^{\otimes n}$$

Now we can apply the result that an E_∞-monoidal functor maps a commutative monoid, in our case a commutative ring, to an E_∞-monoid. Thus $Q_*(R)$ is an E_∞-algebra for every commutative ring R.

$\square$

1.6 Q_* and algebras over operads

In our definition of Q_* we considered the cubical construction for abelian groups. With the result of the theorem above we know that the functor Q_* maps commutative algebras to E_∞-algebras. It is easy to check that the cubical construction of an associative algebra is again associative if we use the Dixmier product as multiplication map. But what happens if we have an algebra over a given operad O? Of special interest are the cases of E_∞- and A_∞-algebras. In order to give a meaning to this question one needs to extend the source of the functor Q_* from the category of abelian groups to the category of chain complexes, which can be done by degreewise extension of Q_*. This extension preserves homotopy relations because the Q_* is additive up to homotopy. For more details see [J-McC].

Proposition 1.14 *The cubical construction Q_* maps algebras over a non-Σ-operad P again to algebras over the non-Σ-operad $Q_*(P)$.*

Proof
First of all we have to clarify the operad-structure of $Q_*(P)$: Define the new operad as $Q_*(P)(n) := Q_*(P(n))$. Since the $P(n)$ are already chain complexes, we have to take the degreewise prolongation of Q_* followed by building the total complex:

$$(Q_*(P(n)))_l := \oplus_{p+q=l} Q_p(P(n)_q)$$

The composition in P

$$P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n) \longrightarrow P(\sum_{i=1}^{n} k_i)$$

is taken to a composition in $Q_*(P)$ by using the Dixmier product:

$$\begin{array}{lcl} Q_*(P(n)) \otimes Q_*(P(k_1)) \otimes \cdots \otimes Q_*(P(k_n)) & & \\ \downarrow & & \\ Q_*(P(n) \otimes P(k_1) \otimes \cdots \otimes P(k_n)) & \to & Q_*(P(\sum_{i=1}^n k_i)) \end{array}$$

The associativity condition concerning this product is satisfied because the Dixmier product is associative and P is an operad.
The unit map of P

$$\eta : \mathbf{Z} \longrightarrow P(1)$$

is taken to a map

$$\tilde{\eta} : \mathbf{Z} \hookrightarrow Q_*(\mathbf{Z}) \stackrel{Q_*(\eta)}{\longrightarrow} Q_*(P(1))$$

As η is a map of complexes and $\mathbf{Z}$ is concentrated in degree zero, we know that

$$\tilde{\eta}.\mathbf{Z} \longrightarrow Q_0(P(1)_0)$$

But $Q_0(P(1)_0) \cong \mathbf{Z}[P(1)_0]$. Hence we can define the unit in $Q_*(P(1))$ to be the image of the unit of P under this isomorphism.
If A is a P-algebra, the algebra maps

$$P(n) \otimes A^{\otimes n} \xrightarrow{\nu} A$$

transfer to maps

$$Q_*(P(n)) \otimes Q_*(A)^{\otimes n} \to Q_*(P(n) \otimes A^{\otimes n}) \xrightarrow{Q_*(\nu)} Q_*(A).$$

Thus $Q_*(A)$ is a $Q_*(P)$-algebra.

□

Proposition 1.15 *If O is an A_∞-operad whose components are free abelian groups then $Q_*(O)$ is again an A_∞-operad.*

Proof
The only fact that is left to show is that Q_* sends acyclic operads over $\mathbf{Z}$ to acyclic ones. Since the augmentation $O \to \mathbf{Z}$ is a weak equivalence and O is degreewise free, we obtain that the augmentation is a homotopy equivalence. Thus $Q_*(O)$ has the same property by the result of Johnson and McCarthy ([J-McC]).

□

MacLane homology of a ring with coefficients in a bimodule is defined as the Hochschild homology of Q_* of this ring with coefficients in the same bimodule. Since Q_* preserves A_∞-algebras over componentwise free operads, it is possible to extend the definition of MacLane homology to this kind of algebras, because Hochschild homology of A_∞-algebras has already been defined (see for instance [G-J]). For a non-Σ-operad, Q_* preserves all the operad structures. But we had to change the operad and apply Q_* to it. That this is not always necessary can be seen if we consider the case of associative algebras. Since Q_* preserves associativity we do not have to change the operad at all. The reason for this is the special structure of the operad As. For the following type of operads a stronger result can be gained.

Definition 1.16 *An operad K is called* topological *if it is an operad in the symmetric monoidal category of simplicial sets (s.sets, $\times$), i.e. each $K(n)$ is a simplicial set and we have the usual composition maps.*

Remark
Every topological operad K gives rise to an operad in the category of chain complexes of abelian groups, if we take the chain complex belonging to the simplicial abelian group $\mathbf{Z}[K]$.

For algebras over this type of operads we can prove the following result:

Proposition 1.17 *If K is a non-Σ topological operad and A is an algebra over $\mathbf{Z}[K]$, then $Q_*(A)$ is still a $\mathbf{Z}[K]$-algebra.*

Proof
We need to have maps

$$\mathbf{Z}[K(n)] \otimes Q_*(A)^{\otimes n} \longrightarrow Q_*(A)$$

which describes an action of the operad $\mathbf{Z}[K]$ on $Q_*(A)$. Let γ denote the action of $\mathbf{Z}[K]$ on A. For each element $\omega \in K(n)$ we obtain maps

$$Q_*(A)^{\otimes n} \longrightarrow Q_*(A^{\otimes n}) \stackrel{Q_*(\gamma(\omega))}{\longrightarrow} Q_*(A)$$

We can extend these maps by linearization and get

$$\mathbf{Z}[K(n)] \otimes Q_*(A)^{\otimes n} \longrightarrow Q_*(A)$$

These maps fulfill all axioms of an operad action, because of the properties of the Dixmier product and of γ.

□

Example
As we already saw, we do not have to change the operad if we apply Q_* to associative algebras. *As* considered as a non-Σ-operad can be viewed as the operad which comes from the topological operad $K(n) = *$.

For usual operads, i.e. operads which are equipped with an action of the symmetric group, we obtain a slightly different result:

Proposition 1.18 *If L is a topological operad and if A is a $\mathbf{Z}[L]$-algebra, then $Q_*(A)$ is an algebra over the operad $\mathbf{Z}[L] \otimes O$, where O is the E_∞-operad that we defined in 1.4.1 and that acts on $Q_*(B)$ for any abelian group B.*

Proof
Our goal is to define maps

$$\mathbf{Z}[L(n)] \otimes O(n) \otimes Q_*(A)^{\otimes n} \longrightarrow Q_*(A)$$

First we can use the action ψ of the operad O on Q_* to get maps

$$\mathbf{Z}[L(n)] \otimes O(n) \otimes Q_*(A)^{\otimes n} \stackrel{id \otimes \psi}{\longrightarrow} \mathbf{Z}[L(n)] \otimes Q_*(A^{\otimes n})$$

Using the same trick as in the proof above we can compose this first map with the action of $\mathbf{Z}[L]$ on A. The equivariance of ψ together with the equivariance of the action of $\mathbf{Z}[L]$ guarantees the equivariance of the whole map. All other axioms for an operad action are fulfilled because we use only the composition of two actions.

$\square$

Proposition 1.19 *For every E_∞-algebra A there is a quasi isomorphic E_∞-algebra A', such that $Q_*(A')$ is again an E_∞-algebra.*

Proof
In their book "Operads, algebras, modules and motives" [K-M], Kriz and May proved that one can replace the E_∞-operad that acts on our E_∞-algebra A by a different E_∞-operad, which we can choose. Our algebra A is then functorially replaced by a quasi isomorphic one A'. Especially we can choose this operad to be one that comes from a topological operad, i.e. an operad of the form $\mathbf{Z}[L]$. Then we can apply the results of the last proposition, and we obtain that $Q_*(A')$ is an algebra over the operad $\mathbf{Z}[L(n)] \otimes O(n)$. But the tensor product of two E_∞-operads is again E_∞. This proves the proposition.

$\square$

1.7 C_* is an E_∞-comonoidal functor

Using similar methods as for showing that SQ_* is an E_∞-monoidal functor we can show that the lax comonoidal and symmetric monoidal functor C_* that maps a simplicial abelian group to the belonging chain complex is an E_∞-comonoidal functor. All necessary statement that are used can be found in [D].
For all n-tupels of simplicial abelian groups $A_1, \ldots, A_n$ we have a map

$$C_*^{\otimes n}(A_1, \ldots, A_n) := C_*(A_1 \otimes \cdots \otimes A_n) \longrightarrow C_*(A_1) \otimes \cdots \otimes C_*(A_n)$$

There is a canonical choice for an operad that acts on the images of C_*, namely define

$$O(n) = \mathcal{H}om_*(C_*^{\otimes n}, C_*^{\boxtimes n})$$

Here $\mathcal{H}om$ means natural transformations of these two functors. As is shown in [D] the homology of this operad is trivial except in dimension zero

$$H_r O(n) \cong \begin{cases} \mathbf{Z} & , \quad r = 0 \\ 0 & , \quad \text{else} \end{cases}$$

The isomorphism from the zero homology to $\mathbf{Z}$ is given by the augmentation map. Hence we obtain a map from $\tau(O)$ to Com which is an isomorphism in homology, that means O is an E_∞-operad and C_* is an E_∞-comonoidal functor via the maps

$$\tau(O)(n) \otimes C_*(A_1 \otimes \cdots \otimes A_n) \longrightarrow C_*(A_1) \otimes \cdots \otimes C_*(A_n)$$

As a consequence we get that the chain-complex of a topological space is an E_∞-coalgebra. This result was already known to experts and was proved in [S] although this paper contains a lot of mistakes.

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