DOLD-KAN TYPE THEOREM FOR Γ-GROUPS

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0. Introduction

Γ-spaces were introduced by Segal [S], who proved that they are combinatorial models for connective spectra (see also [A], [BF]). Based on Kan-Thurston theorem we show that any Γ-space is stably weak equivalent to a discrete Γ-group. By a well-known theorem of Dold-Kan the Moore normalization establishes the equivalence between the category of simplicial abelian groups and the category of chain complexes (see [DP]). mimicking the construction of normalization of simplicial groups, we give a similar construction for Γ-groups. This construction is based on the notion of cross-effects of functors [BP], which is a generalizatin of the classical definition of Eilenberg and Mac Lane [EM] to the non-abelian setup. Finally a Dold-Kan type theorem for the category of Γ-groups is proved. In abelian case our theorem claims that the category of abelian Γ-groups is equivalent to the category of functors Ab^{Ω}, where Ω is the category of all nonempty finite sets and surjections. In the non-abelian case one needs to consider functors $T : \Omega \to Gr$ together with maps

$$\{,\} : T_n \times T_m \to T_{n+m}$$

satisfying additional relations. Here T_n is the value of T on the set

$$< n >: = \{1, \cdots, n\}.$$

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1. Γ-spaces and Γ-groups

1.1. Stable homotopy of Γ-spaces. Let Γ be the category of finite pointed sets. We will assume that objects of Γ are sets $[n]$, where

$$[n] := \{0, 1, ..., n\}$$

with basepoint 0, $n \geq 0$. Let $Sets_*$ be the category of all pointed sets. Let $\mathcal{C}$ be a category with zero object. A *Γ-object over* $\mathcal{C}$ is a functor $A : \Gamma \to \mathcal{C}$ such that $A([0]) = 0$. Let $\mathcal{C}^\Gamma$ be the category of Γ-objects over $\mathcal{C}$. By definition a *pointed Γ-set* is the same as a Γ-object over the category of pointed sets. Similarly an *abelian Γ-group* or a *Γ-group* is the same as a Γ-object over the category of abelian groups or groups respectively. A *Γ-space* is the same as a Γ-object over the category $s.Sets_*$ of pointed simplicial sets. Hence any Γ-set (and hence any Γ-group) yields a discrete Γ-space. A natural transformation $F \to T$ of Γ-spaces is called a *pointwise fibration* (resp. *pointwise weak equivalence*) if $F([n]) \to T([n])$ is a fibration (resp. weak equivalence) for any $n \geq 0$. Any Γ-space F gives rise to a binatural transformation

$$X \wedge F(Y) \to F(X \wedge Y)$$

as follows. For any $x \in X$ define $\hat{x} : Y \to X \wedge Y$, by $y \mapsto (x, y)$. Then apply F to get $F(\hat{x}) : F(Y) \to F(X \wedge Y)$. Now one defines $X \wedge F(Y) \to F(X \wedge Y)$, by $(x, z) \mapsto F(\hat{x})(z)$. There is a standard way to prolong a Γ-space F to a functor $s.Sets_* \to s.Sets_*$. First using direct limits one can prolong F to a functor $Sets_* \to s.Sets_*$, then by degreewise action one obtains a functor from pointed simplicial sets to the category of pointed bisimplicial sets and finally by taking the diagonal of a bisimplicial set one gets a functor $s.Sets_* \to s.Sets_*$. By abuse of notation we will still denote this functor by F. By Proposition 4.9 of [BF] F preserves weak equivalences. Any Γ-space gives rise to a spectrum $F(S^n), n \geq 0$, because of the maps $S^1 \wedge F(S^n) \to F(S^{n+1})$. Hence *stable homotopy groups* of a Γ-space F are defined by

$$\pi_*^{st}(F) = colim_n \pi_{*+n} F(S^n).$$

A natural transformation $F \to T$ is called a *stable weak equivalence* if it induces isomorphisms on π_*^{st}.

1.2. Γ-groups are enough. The goal of this section is to prove that any Γ-space is stably weak equivalent to a discrete Γ-group. I learned recently that this fact was also known to Manos Lydakis.

1.2. Theorem. *Any Γ-space has the stable weak homotopy type of a discrete Γ-group.*

Proof. This is a consequence of Lemma 1.3 and Lemma 1.4.

Let $\mathcal{F} : Sets_* \to Gr$ be the functor from the category of pointed sets to the category of groups, which is left adjoint to the forgetful functor $Gr \to Sets_*$. For any Γ-space A precomposition with $\mathcal{F}$ gives a simplicial Γ-group $\mathcal{F}(A)$ and a morphism of Γ-spaces

$$j : A \to \mathcal{F}(A).$$

1.3. Lemma. *The morphism j is a stable weak equivalence.*

Proof. Thanks to the Milnor theorem [M] we know that $\mathcal{F}(A)(S^n)$ has the same homotopy type as $\Omega\Sigma A(S^n)$. Since A sends points to points, one sees that the simplicial object $A(S^n)$ is trivial in dimensions $< n$. Thus $A(S^n)$ is $(n-1)$-connected and the result follows from the Freudenthal suspension theorem.

For a simplicial group X_*, let $\bar{W}X_*$ be the Eilenberg-Mac Lane classifying space and for any reduced simplicial set K_* let GK_* be the Kan loop simplicial group (see [Kan]). It is well known that, for any simplicial group X_* the canonical homomorphism $G\bar{W}(X_*) \to X_*$ is a weak equivalence (see [Kan]).

1.4. Lemma. *Let A be a Γ-object over the category of simplicial groups, then there exists a Γ-group A' which has the same weak stable homotopy type as A.*

Proof. According to the well known theorem of Kan and Thurston, for any connected space X there exists a group G and a map $BG \to X$ which induces isomorphism on homology. The proof of this theorem given in [KT] shows that one can choose G functorially in X. Therefore there exist a Γ-group A' and a map of Γ-spaces $\eta : \bar{W}A' \to \bar{W}A$ which pointwise yields an isomorphism on homology. By usual argument with bisimplicial objects, it follows that for any pointed simplicial set K, the map $\eta(K)$ yields also an isomorphism on homology. Therefore $\eta(S^n)$ is a weak equivalence for $n > 1$. It follows from Lemma 5.2 of [W], that for any pointwise fibration of Γ-spaces $B' \to B \to B''$ and for any pointed connected simplicial set K the induced sequence

$$B'(K) \to B(K) \to B''(K)$$

is still a fibration up to homotopy. We apply this remark to the pointwise fibration $G\bar{W} \to E\bar{W} \to \bar{W}$ in order to conclude that $G\bar{W}A' \to G\bar{W}A$ is a stable weak equivalence. Since the canonical homomorphisms

$$G\bar{W}(A) \to A \text{ and } G\bar{W}(A') \to A'$$

are pointwise weak equivalences, the result follows.

2. The main construction

In this section we construct the functor

$$cr : Gr^{\Gamma} \to Gr^{\Omega},$$

which is the main step in our version of the Dold-Kan type theorem for Γ-groups. Here Ω is the category of all nonempty finite sets and surjections. We will assume that the objects of Ω are the sets

$$< n >: = \{1, ..., n\}, \ \ n \geq 0.$$

Let us recall that objects of the category Gr^{Γ} are functors $T : \Gamma \to Gr$ with the property: $T([0]) = 0$. For a functor $F : \Omega \to Gr$, let F_n denote the value of F at $< n >$.

For any $n \geq 1$ and any $1 \leq i \leq n$, one defines the pointed maps

$$r_i : [n] \to [n-1]$$

by $r_i(i) = 0$, $r_i(j) = j$ if $j < i$ and $r_i(j) = j - 1$, if $j > i$. For a functor $T : \Gamma \to Gr$, let $r_{i*} : T([n]) \to T([n-1])$ be the homomorphism induced by r_i. We put

$$T_n : = \bigcap_{i=1}^{n} ker \ (r_{i*}).$$

Let Ω_* be the category of finite pointed sets and surjections. For any set S we let S_+ be the pointed set obtained from S by adding a distinguished point. There are functors

$$\Omega \xrightarrow{?_+} \Omega_* \subset \Gamma,$$

where $?_+$ assigns S_+ to a set S. We always assume that $?_+(< n >) = [n]$. For any Γ-group T one can consider the composition

$$T_+ : = \ T \circ ?_+ : \Omega \to Gr.$$

2.1. Proposition. *For any Γ-group T, the functor T_+ has a unique subfunctor*

$$cr(T) : \Omega \to Gr,$$

which assigns T_n to $< n >$.

Proof. Assume $f : [n] \to [k]$ is a surjection, then for any $1 \leq j \leq k$ the composition $r_j \circ f$ factors trough r_i for $i \in f^{-1}(j)$. Therefore $T(f)$ sends T_n to the kernel of $T(r_j)$, which shows that such subfunctor exists and clearly it is unique. Let us remark, that if f is not in the image of $?_+$, then the induced map is zero.

2.2. Remark. Clearly a surjection $f : [n] \to [m]$ in Ω_* has the form g_+ for some $g \in Hom_\Omega(< n >, < m >)$ iff $f^{-1}(0) = 0$. Let Ω_0 be the collection of all surjections f in Ω_* which are not in the image of $?_+$. Clearly Ω_0 is a two-sided ideal in Ω_*. Hence one can identify the category of functors $\Omega \to Gr$ with the subcategory of those functors $T : \Omega_* \to Gr$, for which $T(f) = 0$, as soon as $f \in \Omega_0$.

2.3. Cross-effects of functors. Here we recall the definition of cross-effects of functors in the nonabelian case (see [BP]). Let $\mathcal{C}$ be a category with a zero object and finite sums. Denote the sum of objects X and Y by $X \vee Y$. Let $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ be the inclusions. We have also the retractions $r_1 = (1_X, 0) : X \vee Y \to X$ and $r_2 = (0, 1_Y) : X \vee Y \to Y$. Let $F : \mathcal{C} \to Gr$ be a functor with $F(0) = 0$. The *second cross-effect* of F is given by

$$F(X \mid Y) := ker(F(X \vee Y) \overset{(r_1,r_2)}{\longrightarrow} F(X) \times F(Y)).$$

One easily shows that one has the short exact sequence

$$0 \to F(X \mid Y) \to F(X \vee Y) \to F(X) \times F(Y) \to 0. \tag{2.3.1}$$

Indeed, the projection $(r_{1*}, r_{2*}) : F(X \vee Y) \to F(X) \times F(Y)$ has a set level section, given by $(x, y) \mapsto i_{1*}(x)i_{2*}(y)$. Here $x \in F(X)$ and $y \in F(Y)$.

A functor F is called *abelian* if it has values in the category of abelian groups. If F is an abelian functor then the above short exact sequence splits and the splitting is natural. Clearly the cross-effect of functors defines an exact functor from the category of functors $Gr^{\mathcal{C}}$ to the category of bifunctors $Gr^{\mathcal{C}\times\mathcal{C}}$. In order to define the third cross-effect of F, one can consider the second cross-effect $F(X \mid Y)$ as a functor on X and then take the second cross-effect of it. In this way one gets all higher cross-effects. One easily shows that the n-th cross-effect of F can also be defined by

$$F(X_1 \mid ... \mid X_n) :=$$
$$ker(F(X_1 \vee ... \vee X_n) \to F(X_2 \vee ... \vee X_n) \times ... \times F(X_1 \vee ... \vee X_{n-1})).$$

A functor F is called *of degree* $\leq n$, if the $(n+1)$-th cross-effect functor vanishes. One remarks that, in the case when $\mathcal{C}$ is an additive category and values of F are abelian groups, then this is the classical notion by Eilenberg and Mac Lane (see [EM]). Let T be a Γ-group. By comparing the definitions, one sees that T_n is the same as the value of the n-th cross-effect of T on $([1] \mid \cdots \mid [1])$. Then by induction one easily shows that for any Γ-group T one has a bijection

$$T([n]) \cong \prod_{\gamma} T_k, \tag{2.3.2}$$

where the product is taken over all pointed and order preserving injections $\gamma : [k] \to [n]$. Moreover if F is an abelian Γ-group then one has the natural isomorphism of groups. One can ask to redefine the group structure on the right hand side of (2.3.2) in order to get the isomorphism of groups. We will do this in section 4.

The following fact is easy consequence of (2.3.2).

Lemma 2.4. *Let F be a Γ-group, with $F_n = 0$ for any $n \geq 1$. Then $F = 0$.*

Proposition 2.5. *Let F be a Γ-group. Then F is of degree $\leq n$ iff $F_i = 0$ for any $i \geq n+1$.*

Proof. One needs only to show that for any $S_1, \cdots, S_{n+1}$, the cross-effect $F(S_1 \mid \cdots \mid S_{n+1}) = 0$ as soon as $F_i = 0$ for $i > n$. For a set S one puts $T(S) := F(S_1 \mid \cdots \mid S_{n+1})$, where $S_1 = S, S_2 = \cdots = S_{n+1} = [1]$. In this way we get a Γ-group T. By definition $T_i = F_{i+n}$. Hence $T = 0$ by Lemma 2.4. Similarly for a fix S_1, one defines a Γ-group U by $U(S) := F(S_1 \mid \cdots \mid S_{n+1})$, where $S_2 = S$ and $S_3 = \cdots = S_{n+1} = [1]$. By definition $U_i = F(S_1 \mid \cdots \mid S_{n+i})$, where $S_2 = \cdots = S_{n+1} = [1]$. Thus $U_i = 0$ for $i > 1$ and by Lemma 2.4 we have $U = 0$. Proceding in the same way we get that $F(S_1 \mid \cdots \mid S_{n+1}) = 0$, for any $S_1, \cdots, S_{n+1}$.

2.6. Generators and relations for Ω. We recall the presentation of Ω (see for example Section II.1 of [Wh], compare also with E.6.1.7 of [L2]). In what follows, Σ_n denotes the symmetric group on n letters. We set $\tau_k = (k, k+1) \in \Sigma_n$, where $k = 1, \cdots, n-1$. We need also $\omega_i = (i \cdots n) \in \Sigma_n$. Moreover we define for $i = 1, \cdots, n-1$ a non-decreasing and surjective map

$$P_i : \{1, \cdots, n\} \to \{1, \cdots, n-1\},$$

which sends both i and $i+1$ to i. Furthermore we let P, τ and ω be P_1, τ_1 and ω_1 respectively. Let us remark that part b) of the next Lemma easily follows from a), because $P_{i+1} = \omega^i P \omega^{-i}$.

Lemma 2.7. a) *A presentation of Ω is given by the generators τ_k and P_i and the relations*

$$\text{those of } \Sigma_n,$$

$$P_i P_j = P_{j-1} P_i \ for\ i < j,$$

$$P_i \tau_i = P_i$$

$$\begin{aligned} \tau_k P_j &= P_j \tau_k \text{ for } k < j-1, \\ &= P_{j-1} \tau_j \tau_{j-1} \text{ for } k = j-1, \end{aligned}$$

$$= P_{j+1}\tau_j\tau_{j+1} \text{ for } k = j,$$

$$= P_j\tau_{k+1} \text{ for } k > j$$

b) *A presentation of* Ω *is given by the generators* τ_i, ω_i *and* P *and the relations*

$$\text{those of } \Sigma_n,$$

$$P\omega^k P\omega^{-k} = \omega^{k-1}P\omega^{-k+1}P, \text{ for } k \geq 1$$

$$P\tau = P$$

$$\tau_i P = P\tau_{i+1} \text{ for } 2 \leq i \leq n-2$$

$$\omega_2 P\omega_3^{-1} = P$$

3. A Dold-Kan type theorem for abelian Γ-groups

It is clear that if $T : \Gamma \to Ab$ is an abelian Γ-group, then $cr(T)$ has values in the category of abelian groups. Hence it yields a functor $cr : Ab^\Gamma \to Ab^\Omega$.

3.1. Theorem. *The functor*

$$cr : Ab^\Gamma \to Ab^\Omega$$

is an equivalence of categories.

Proof. We describe explicitly the quasi-inverse of the functor cr. According to remark 2.2 any functor $T : \Omega \to Ab$ can be considered as a functor $T : \Omega_* \to Ab$ with the property that any basepoint preserving surjection f which is not in the image of $?_+$ acts trivially on T. Let $i : \Omega_* \to \Gamma$ be the inclusion. It induces a functor $Ab^\Gamma \to Ab^{\Omega_*}$, which has a left adjoint, the so called left Kan extension. In this way one gets a functor

$$i_! : Ab^\Omega \to Ab^\Gamma.$$

According to the definition of the Kan extension, for any $m \geq 0$, the abelian group $i_!(T)([m])$ is generated by $\alpha \otimes x$, where $\alpha : [n] \to [m]$ is a pointed map and $x \in T_n$. Here T_n denotes the value of T on $< n >$. The defining relations are

$$\alpha \otimes x + \alpha \otimes y = \alpha \otimes (x + y)$$

$$\alpha\beta \otimes x = \alpha \otimes \beta x, \quad \alpha \in \Gamma, \ \beta \in \Omega_*.$$

For a pointed map $f : [m] \to [k]$ one defines $i_!T(f) : i_!T([m]) \to i_!T([k])$ by $\alpha \otimes x \mapsto f\alpha \otimes x$. Clearly $i_!T$ is Γ-abelian group. We need to show that cr and $i_!$ are mutually quasi-inverse. But this is a consequence of the isomorphism (2.3.2) and the following fact. For any $T \in Ab^{\Omega}$ one has a natural isomorphism

$$(3.1.1) \qquad (i_! T)([n]) \cong \bigoplus_{\gamma} T_k,$$

where the sum is taken over all pointed and order preserving injections $\gamma : [k] \to [n]$. Indeed, if $\gamma : [k] \to [n]$ is such a map and $x \in T_k$, then one assigns to this pair the element $\gamma \otimes x$ in $(i_* T)([n])$. This gives rise to the homomorphism from the right hand side to the left hand side. In order to construct the inverse homomorphism, let us remark that any pointed map α can be decomposed in Γ as $\alpha = \gamma\beta$, where $\beta : [k] \to [m]$ is a surjection in Γ and $\gamma : [m] \to [n]$ is an order preserving pointed injection, and such a decomposition is unique. Hence $\alpha \otimes x = \gamma \otimes \beta x$. Moreover $(\alpha, x) \mapsto (\gamma, \beta x)$ yields the expected inverse homomorphism and the proof is finished.

3.2. Remarks. 1). Here is an alternative way to define the functor $i_!(T)$. On objects it is defined via (3.1.1), which can be rewritten as follows

$$(i_! T)([n]) = \bigoplus_{\mu} T_{|\mu|}.$$

Here the sum is taken over all nonempty subsets $\mu \subset < n >$ and $\mid \mu \mid$ denotes the cardinality of the set μ. For a surjection $f : [n] \to [m]$, the restriction of the induced map f_* to the summand corresponding to a subset μ is zero if $0 \in f(\mu)$, otherwise it maps this summand to the summand corresponding to the subset $f(\mu) \subset < m >$.

This description shows that the theorem is still valid if one replaces Ab by any other abelian category. As a consequence we have

$$Ab^{\Gamma^{op}} \cong Ab^{\Omega^{op}}.$$

2) The same proof shows that *the category of all covariant (resp. contravariant) functors from the category of Γ to the category of abelian groups is equivalent to the category of all covariant (resp. contravariant) functors from the category of finite sets and surjections to the category of abelian groups.*

3.3. Corollary. *The category of all covariant (resp. contravariant) functors from Ω to the category of abelian groups is equivalent to the category, whose objects are sequences $(T_n, P), n \geq 1$, where T_n is a Σ_n-module and $P : T_n \to T_{n-1}$ (resp. $P : T_{n-1} \to T_n$) is a linear map $n \geq 2$. Moreover the following relations hold*

$$\omega_2 P \omega_3^{-1} = P, \quad P\omega^k P \omega^{-k} = \omega^{k-1} P \omega^{-k+1} P, \text{ for } k \geq 1$$

$$P\tau = P, \quad \tau_i P = P\tau_{i+1} \text{ for } 2 \leq i \leq n-2.$$

$$(\textit{resp. } \omega_3^{-1} P \omega_2 = P,\ \omega^{-k} P \omega^k P = P\omega^{-k+1} P \omega^{k-1}, \text{ for } k \geq 1,$$

$$\tau P = P, \quad P\tau_i = \tau_{i+1} P \text{ for } 2 \leq i \leq n-2).$$

Proof. This is a consequence of Lemma 2.7b) and Theorem 3.1.

3.4. Corollary. *For any covariant functor $T : \Gamma \to Ab$ and a natural number k there exists a subfunctor $T^{[k]} \subset T$ of degree $\leq k$, with the property that $T_i = (T^{[k]})_i$ for $i \leq k$. Moreover for any covariant functor $F : \Gamma \to Ab$ of degree $\leq k$, there exists a natural isomorphism*

$$\mathrm{Hom}_{\Gamma-mod}(F, T) \ \cong\ \mathrm{Hom}_{\Gamma-mod}(F, T^{[k]}).$$

Similarly, for any contravariant functor $T : \Gamma \to Ab$ and a natural number k there exists a quotient $T^{[k]}$ of T of degree $\leq k$, with the property that$T_i = (T^{[k]})_i$ for $i \leq k$. Moreover for contravariant functor $F : \Gamma \to Ab$ of degree $\leq k$, there exists a natural isomorphism

$$\mathrm{Hom}_{mod-\Gamma}(T, F) \ \cong\ \mathrm{Hom}_{mod-\Gamma}(T^{[k]}, F).$$

4. Nonabelian Γ-groups

Easy examples show that, unlike the abelian case, the functor

$$cr : Gr^{\Gamma} \to Gr^{\Omega}$$

is not anymore an equivalence of categories. What we will do is to enrich the category Gr^{Ω} in order to get still an equivalence of categories. It turns out that these additional data are very close to the notion of groups with projections (see [E], [St]). In this section we use additive notation for the group structure. We also fix some notation. For maps

$$f \in Hom_{\Omega}(< n >, < n' >), \quad g \in Hom_{\Omega}(< m >, < m' >)$$

and $h \in Hom_{\Omega}(< m >, < n' >)$, we define

$$f \vee g \in Hom_{\Omega}(< n + m >, < n' + m' >)$$

and

$$(f,h) \in Hom_\Omega(< n+m >, < n' >)$$

by $(f \vee g)(i) = f(i)$ if $1 \leq i \leq n$ and $(f \vee g)(i) = n' + g(i-n)$, if $n+1 \leq i \leq n+m$; $(f,h)(i) = f(i)$ if $i \leq n$ and $(f,h)(i) = h(i-n)$ if $n < i \leq n+m$.

Definition 4.1. *An Ω-group T is a functor $T : \Omega \to Gr,\ < n > \mapsto T_n$ together with maps*

$$\{,\} : T_n \times T_m \to T_{n+m}$$

satisfying the following conditions:

i) *For any $f \in Hom_\Omega(< n >, < n' >)$ and $g \in Hom_\Omega(< m >, < m' >)$ the following diagram*

$$\begin{array}{ccc} T_n \times T_m & \to & T_{n+m} \\ f_* \times g_* \downarrow & & \downarrow (f \vee g)_* \\ T_{n'} \times T_{m'} & \to & T_{n'+m'} \end{array}$$

is commutative.

ii) *For $x, y \in T_n$, one has*

$$-x - y + x + y = [x,y] = (1_{<n>}, 1_{<n>})_*\{x,y\}.$$

iii) *For $x \in T_n$ and $y \in T_m$, one has*

$$\{x,y\} = (t_{n,m})_*\{y,x\},$$

where $t_{n,m} \in Hom_\Omega(< n+m >, < n+m >)$ is given by $t_{n,m}(i) = i-n$ if $n < i$ and $t_{n,m}(i) = i+m$ if $i \leq n$.

iv) *for $x \in T_n$ and $y, z \in T_m$, one has*

$$\{x, y+z\} = \{x,z\} + \{x,y\} + (1_{<n>} \vee (1_{<m>}, 1_{<m>}))_*\{\{x,y\},z\}$$

v) *For $x \in T_n$, $y, \in T_m$, $z, \in T_k$ one has*

$$(x,y,z) + (y,z,x) + (z,x,y) = 0$$

where,

$$(x,y,z) = \{x,\{y,z\}\} + \alpha_*\{\{x,\{y,z\}\},\{x,y\}\} + \beta_*\{\{x,y\},\{y,z\}\}$$

while

$$\alpha \in Hom_\Omega(< 2n+2m+k >, < n+m+k >)$$

and

$$\beta \in Hom_\Omega(< n+2m+k >, < n+m+k >)$$

given by $\beta = 1_{<n>} \vee (1_{<m>}, 1_{<m>}) \vee 1_{<k>}$ *and* $\alpha(i) = i$ *if* $i \leq n+m+k$, $\alpha(n+m+k+i) = i$ *if* $i \leq n$, *and* $\alpha(2n+m+k+i) = n+i$ *if* $i \leq m$.

Lemma. 4.2. *For any* Γ*-group* T*, the functor* $cr(T) : \Omega \to Gr$ *has a natural structure of* Ω*-group.*

Proof. By definition $T_n \subset T([n])$. For any $m, n \geq 0$ one has morphisms in Γ:

$$i_{n,m} : [n] \to [n+m] \text{ and } j_{n,m} : [m] \to [n+m]$$

given by $i_{n,m}(k) = k$ and $j_{n,m}(k) = n+k$ respectively. For $x \in T_n$ and $y \in T_m$ we put

$$\{x, y\} := [(i_{n,m})_*(x), (j_{n,m})_*(y)] \in T([n+m]).$$

Let us calculate $r_{k*}(\{x, y\})$. By definition, one has

$$r_{k*}(\{x, y\}) = [(r_k i_{n,m})_*(x), (r_k j_{n,m})_*(y)].$$

If $k \leq n$, then $r_k i_{n,m}$ factors through r_k, and hence $(r_k i_{n,m})_*(x) = 0$. If $k > n$, then $r_k j_{n,m}$ factors through r_{k-n} and hence $(r_k j_{n,m})_*(y) = 0$. Hence in all cases $r_{k*}(\{x, y\}) = 0$. This shows that $\{x, y\} \in T_{n+m}$. So we have well-defined functions $\{,\} : T_n \times T_m \to T_{n+m}$. Straightforward calculation shows that the properties i) and ii) hold. One easily remarks that the relations iii)-v) are consequences of the standard commutator relations.

Theorem. 4.3. *The functor*

$$cr : Gr^\Gamma \to Gr^\Omega$$

factors through the category Ω*-*Gr *of* Ω*-groups. The induced functor* Cr : $Gr^\Gamma \to \Omega$*-*Gr *is an equivalence of categories.*

Proof. Let $T : \Omega \to Gr$, $< n > \mapsto T_n$ be a Ω-group. Mimicking the proof of Theorem 3.1 one defines a functor

$$i_!(T) : \Gamma \to Gr$$

as follows: For any $m \geq 0$ the group $i_!(T)([m])$ is generated by $\alpha * x$, where $\alpha : [n] \to [m]$ is a pointed map and $x \in T_n$. The defining relations are

$$\alpha * x + \alpha * y = \alpha * (x+y),$$

$$\alpha\beta * x = \alpha * \beta x,$$

$$-\alpha * x - \gamma * y + \alpha * x + \gamma * y = (\alpha \vee \gamma) * \{x, y\}, \quad \alpha, \gamma \in \Gamma, \ \beta \in \Omega_*.$$

Let us recall that if β does not belong to the image of $?_+$ then the action of β on x is zero. We need to show that Cr and $i_!$ are quasi-inverse to each other. For nonempty subsets μ and ν of $< m >$, let

$$f_{\mu,\nu} \in Hom_\Omega(<\mid \mu \mid + \mid \nu \mid>, <\mid \mu \cup \nu \mid>)$$

be the natural surjection. Then the proof of Theorem 3.1 shows that the group $i_!(T)([m])$ is the same as the free product of the groups $T_{|\mu|}$ modulo the relation

$$[x, y] = (f_{\mu,\nu})_* \{x, y\}, \quad x \in T_{|\mu|}, \ y \in T_{|\nu|},$$

where μ and ν run over all nonempty subsets of $< m >$. Now the theorem is a consequence of Theorem 1.1 and Theorem 1.2 of [ES]. In order to apply the result of [ES], one needs only to remark that for a Γ-group T and $n \geq 0$, the group $G = T([n])$ has n-commuting projections, which are induced by the maps $s_i : [n] \to [n], \ i = 1, \cdots, n$. Here $s_i(i) = 0, \ s_i(j) = j$ for $j \neq i$. Moreover for a subset $\mu \subset < n >$, the group

$$M_\mu = \bigcap_{i \in \mu} Ker \ s_i \bigcap_{i \notin \mu} Im \ s_i$$

of [ES] is the same as the image $(f_\mu)_*(T_{|\mu|})$. Here f_μ is the unique order preserving injection from $[\mid \mu \mid]$ to $[n]$, whose image is $\{0\} \cup \mu$.

4.4. Remark. I learned recently that based on ideas of Baues (see relation (1.13) on page 40 of [B]) Dreckmann in his thesis [D] proved a theorem from which one can deduce our Theorem 4.3.

4.5. Remarks on polynomial Γ-groups. We will say that *the degree* of a functor $T : \Omega \to Gr$ is less or equal to n if $T_k = 0$ for $k > n$. Here as usual T_k denotes the value of T on $< k >$. By Proposition 2.5 the degree of a functor $F : \Gamma \to Gr$ is less or equal to n iff same holds for $cr(F)$. It is easy consequence of Theorem 4.3 that if $deg \ T \leq n$, then the group T_k is abelian if $2k > n$. Moreover

$$\{,\} : T_k \times T_m \to T_{k+m}$$

is linear with respect of the second (resp. first) variable if $k + 2m > n$ (resp. $2k + m > n$). By Lemma 2.7 we see that the category of degree 1 functors is equivalent to the category of abelian groups (compare with [BP]). While the category of functors of degree less or equal than 2 is equivalent to the category of presquare groups. Here a *presquare group*

$$T = (T_2 \xrightarrow{P} T_1)$$

is given by a group T_1 and an abelian group T_2, together with an involution τ. Moreover P is a homomorphism and a biliniear map $\{,\} : T_1 \times T_1 \to T_2$ is given. In addition the following properties are satisfied

$$P\tau = P$$

$$\{Px, b\} = 0$$

$$P\{a, b\} = -a - b + a + b$$

$$\{a, b\} + \tau\{b, a\} = 0.$$

In [BP] we made the classification of quadratic endofunctors of the category of groups via square groups. By definition a *square group*

$$M = (M_e \overset{H}{\to} M_{ee} \xrightarrow{P} M_e)$$

is given by a group M_e and an abelian group M_{ee}. Moreover P is a homomorphism and H is a quadratic function, that is the cross effect

$$(a \mid b)_H = H(a + b) - H(b) - H(a)$$

is linear in $a, b \in Q_e$. In addition the following properties are satisfied $(a, b \in M_e; x, y \in M_{ee})$

$$(Px \mid b)_H = 0 \ \text{ and } \ (a \mid Py)_H = 0$$

$$P(a \mid b)_H = -a - b + a + b$$

$$PHP(x) = P(x) + P(x).$$

One shows that $\tau = HP - 1$ is an involution of M_{ee} and hence any square group gives rise to a presquare group. Therefore we proved the following

Proposition 4.6. *Let $T : \Gamma \to Gr$ be a Γ-group of degree ≤ 2. Let $\mathcal{F}ree$ be the category of finite generated free groups, and $\mathcal{F} : \Gamma \to \mathcal{F}ree$ be the functor which takes a pointed set S to free group generated by S, modulo to the relation $* = 0$. Then the following conditions are equivalent to each other*

i) *There exists a functor $\bar{T} : \mathcal{F}ree \to Gr$ for which $T = \bar{T} \circ \mathcal{F}$.*

ii) *there exist a function $H : T_1 \to T_2$, such that for any $a, b \in T_1$ the following holds*

$$(a \mid b)_H = \{a, b\}.$$

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