SMASHING SUBCATEGORIES AND THE TELESCOPE CONJECTURE – AN ALGEBRAIC APPROACH

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Abstract. We prove a modified version of Ravenel's telescope conjecture. It is shown that every smashing subcategory of the stable homotopy category is generated by a set of maps between finite spectra. This result is based on a new characterization of smashing subcategories, which leads in addition to a classification of these subcategories in terms of the category of finite spectra. The approach presented here is purely algebraic; it is based on an analysis of pure-injective objects in a compactly generated triangulated category, and covers therefore also situations arising in algebraic geometry and representation theory.

Introduction

Smashing subcategories naturally arise in the stable homotopy category $\mathcal{S}$ from localization functors $l\colon \mathcal{S} \to \mathcal{S}$ which induce for every spectrum X a natural isomorphism $l(X) \simeq X \wedge l(S)$ between the localization of X and the smash product of X with the localization of the sphere spectrum S. In fact, a localization functor has this property if and only if it preserves arbitrary coproducts in $\mathcal{S}$. Therefore one calls a full subcategory $\mathcal{R}$ of $\mathcal{S}$ *smashing* if $\mathcal{R} = \{X \in \mathcal{S} \mid l(X) = 0\}$ for some localization functor $l\colon \mathcal{S} \to \mathcal{S}$ which preserves coproducts. In this paper we study smashing subcategories from an algebraic point of view. The main result is a new characterization of smashing subcategories which leads to a classification in terms of certain ideals in the category of finite spectra. One motivation for this work is the telescope conjecture of Ravenel and Bousfield which states that every smashing subcategory is generated by finite spectra. The approach presented here is purely algebraic and covers therefore also situations arising in algebraic geometry and representation theory where one studies certain triangulated categories having a number of formal properties in common with the stable homotopy category.

Let $\mathcal{C}$ be a compactly generated triangulated category, for example the stable homotopy category. Thus $\mathcal{C}$ is a triangulated category with arbitrary coproducts, and $\mathcal{C}$ is generated by a set of compact objects (an object X in $\mathcal{C}$ is *compact* if the representable functor $\operatorname{Hom}(X, -)$ preserves coproducts). Recall that a full triangulated subcategory $\mathcal{B}$ of $\mathcal{C}$ is *localizing* if $\mathcal{B}$ is closed under taking coproducts. We say that a localizing subcategory $\mathcal{B}$ is *strictly localizing* if the inclusion functor $\mathcal{B} \to \mathcal{C}$ has a right adjoint, and $\mathcal{B}$ is called *smashing* if there exists a right adjoint for the inclusion $\mathcal{B} \to \mathcal{C}$ which preserves coproducts. Note that a full subcategory $\mathcal{B}$ is strictly localizing if and only if there exists a localization functor $l\colon \mathcal{C} \to \mathcal{C}$ such that $\mathcal{B} = \{X \in \mathcal{C} \mid l(X) = 0\}$, and $\mathcal{B}$ is smashing if and only if the corresponding localization functor preserves coproducts.

Theorem A. *Let $\mathcal{B}$ be a localizing subcategory of $\mathcal{C}$, and denote by $\mathfrak{I}$ the ideal of maps between compact objects in $\mathcal{C}$ which factor through some object in $\mathcal{B}$. Then the following conditions are equivalent:*

(1) *$\mathcal{B}$ is smashing;*

(2) *an object X in $\mathcal{C}$ belongs to $\mathcal{B}$ if and only if every map $C \to X$ from a compact object C factors through a map $C \to D$ in $\mathfrak{I}$;*
(3) *an object X in $\mathcal{C}$ satisfies* $\operatorname{Hom}(\mathcal{B}, X) = 0$ *if and only if* $\operatorname{Hom}(\mathfrak{I}, X) = 0$.

Let us mention an immediate consequence: The smashing subcategories of $\mathcal{C}$ form a set of cardinality at most 2^{κ} where κ denotes the cardinality of the set of isomorphism classes of maps between compact objects in $\mathcal{C}$. For example, the stable homotopy category has precisely $2^{\aleph_0}$ smashing subcategories because, in this case $\kappa = \aleph_0$, and arithmetic localization gives rise to a smashing subcategory for every set of primes.

Given any class $\mathfrak{I}$ of maps in $\mathcal{C}$, we say that a localizing subcategory $\mathcal{B}$ is *generated* by $\mathfrak{I}$ if $\mathcal{B}$ is the smallest localizing subcategory of $\mathcal{C}$ such that every map in $\mathfrak{I}$ factors through some object in $\mathcal{B}$. For example, $\mathcal{B}$ is generated by a class $\mathfrak{I} = \{\mathrm{id}_{X_i} \mid i \in I\}$ of identity maps if and only if $\mathcal{B}$ is the smallest localizing subcategory containing X_i for all $i \in I$.

Corollary. *Every smashing subcategory is generated by a set of maps between compact objects.*

The statement of the corollary is a modified version of the following "telescope conjecture" which is based on conjectures of Ravenel [23, 1.33] and Bousfield [6, 3.4] for the stable homotopy category:

Every smashing subcategory is generated by a set of identity maps between compact objects.

In this generality, the conjecture is known to be false. In fact, Keller gives an example of a smashing subcategory which contains no non-zero compact object [14]. Despite some efforts of Ravenel [24], the conjecture remains open for the stable homotopy category.

The characterization of smashing subcategories leads to a classification in terms of certain ideals which we now explain. We denote by $\mathcal{C}_0$ the full triangulated subcategory of compact objects in $\mathcal{C}$ and call an ideal $\mathfrak{I}$ of maps in $\mathcal{C}_0$ *exact* if there exists an exact functor $f\colon \mathcal{C}_0 \to \mathcal{D}$ into a triangulated category $\mathcal{D}$ such that $\mathfrak{I} = \{\phi \in \mathcal{C}_0 \mid f(\phi) = 0\}$.

Theorem B. *Let $\mathcal{C}$ be a compactly generated triangulated category and suppose that every cohomological functor $\mathcal{C}_0^{\mathrm{op}} \to \mathrm{Ab}$ is isomorphic to $\operatorname{Hom}(-, X)|_{\mathcal{C}_0}$ for some object X in $\mathcal{C}$. Then the maps*

$$\mathcal{B} \mapsto \{\phi \in \mathcal{C}_0 \mid \phi \text{ factors through an object in } \mathcal{B}\} \quad \text{and}$$

$$\mathfrak{I} \mapsto \{X \in \mathcal{C} \mid \text{every map } C \to X,\ C \in \mathcal{C}_0, \text{ factors through a map } C \to D \text{ in } \mathfrak{I}\}$$

induce mutually inverse bijections between the set of smashing subcategories of $\mathcal{C}$ and the set of exact ideals in $\mathcal{C}_0$.

Note that the additional assumption on $\mathcal{C}$ in the preceding theorem is automatically satisfied if there are at most countably many isomorphism classes of maps between compact objects in $\mathcal{C}$; in particular the stable homotopy category has this property [21]. The classification of smashing subcategories has the following consequence.

Corollary. *A localizing subcategory $\mathcal{B}$ of $\mathcal{C}$ is smashing if and only if $\mathcal{B}$ is generated by a class of maps between compact objects in $\mathcal{C}$. Moreover, given any class $\mathfrak{I}$ of maps between compact objects in $\mathcal{C}$, there exists a localizing subcategory of $\mathcal{C}$ which is generated by $\mathfrak{I}$.*

The preceding corollary amounts to a classical result of Bousfield and Ravenel if $\mathfrak{I}$ is a class of identity maps. In fact, they showed for the stable homotopy category that every class of compact objects generates a localizing subcategory which is smashing [6, 23]. However, if $\mathfrak{I}$ is a class of arbitrary maps in $\mathcal{C}$, it is not clear that there exists a localizing subcategory which is generated by $\mathfrak{I}$.

Our analysis of smashing subcategories is based on the concept of purity for compactly generated triangulated categories. Let us call a map $X \to Y$ in $\mathcal{C}$ a *pure monomorphism* if the induced map $\operatorname{Hom}(C,X) \to \operatorname{Hom}(C,Y)$ is a monomorphism for all compact objects C. An object X is called *pure-injective* if every pure monomorphism $X \to Y$ splits. These definitions are motivated by analogous concepts for the category of modules over a ring [7]. In this context one frequently studies the indecomposable pure-injective modules; they form the Ziegler spectrum of the ring [28]. We shall see that the isomorphism classes of indecomposable pure-injective objects in $\mathcal{C}$ form a set which we denote by $\operatorname{Sp}\mathcal{C}$.

Theorem C. *Let $\mathcal{B}$ be a smashing subcategory of $\mathcal{C}$, and let $\mathbf{U}$ be the set of objects Y in $\operatorname{Sp}\mathcal{C}$ such that $\operatorname{Hom}(\mathcal{B},Y)=0$. Then the following holds for any object X in $\mathcal{C}$:*

1. *$X \in \mathcal{B}$ if and only if $\operatorname{Hom}(X,\mathbf{U})=0$;*
2. *$\operatorname{Hom}(\mathcal{B},X)=0$ if and only if there is a pure monomorphism $X \to \prod_{i\in I} Y_i$ with $Y_i \in \mathbf{U}$ for all i.*

We obtain the following consequence if we put $\mathcal{B}=0$.

Corollary. *Every object X in $\mathcal{C}$ admits a pure monomorphism $X \to \prod_{i\in I} Y_i$ with $Y_i \in \operatorname{Sp}\mathcal{C}$ for all i. In particular, $\operatorname{Hom}(X,Y)=0$ for all $Y \in \operatorname{Sp}\mathcal{C}$ implies $X=0$.*

The concept of purity is closely related to the occurence of phantom maps. Recall that a map $X \to Y$ is a *phantom map* if the induced map $\operatorname{Hom}(C,X) \to \operatorname{Hom}(C,Y)$ is zero for all compact objects C. From the existence of pure-injective envelopes in $\mathcal{C}$ we derive for every object X the existence of a universal phantom map ending in X and a universal pure monomorphism starting in X.

Theorem D. *For every object X in $\mathcal{C}$ there exists, up to isomorphism, a unique triangle*

$$X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \xrightarrow{\gamma} X'[1]$$

having the following properties:

(A1) *a map $\phi\colon Y \to X$ is a phantom map if and only if ϕ factors through α;*
(A2) *every endomorphism ϕ of X' satisfying $\alpha = \alpha\circ\phi$ is an isomorphism.*

The same triangle is characterized, up to isomorphism, by the following properties:

(B1) *a map $\phi\colon X \to Y$ is a pure monomorphism if and only if β factors through ϕ;*
(B2) *every endomorphism ϕ of X'' satisfying $\beta = \phi\circ\beta$ is an isomorphism.*

Our main tool in this paper is a functor $h\colon \mathcal{C} \to \mathcal{M}$ into a module category $\mathcal{M}$ which has the following universal property:

1. *$h\colon \mathcal{C} \to \mathcal{M}$ is a cohomological functor into an abelian* AB 5 *category which preserves coproducts;*
2. *any functor $h'\colon \mathcal{C} \to \mathcal{M}'$ as in* (1) *has a unique factorization $h' = f\circ h$ such that $f\colon \mathcal{M} \to \mathcal{M}'$ is exact and preserves coproducts.*

In Section 1 of this paper we exploit the fact that h induces an equivalence between the full subcategory of pure-injective objects in $\mathcal{C}$ and the full subcategory of injective objects in $\mathcal{M}$. We continue in Section 2 with the problem of extending cohomological functors. For instance, we prove the following result where $\mathcal{C}_0$ denotes the full triangulated subcategory which is formed by the compact objects in $\mathcal{C}$.

Theorem E. *Every cohomological functor $f\colon \mathcal{C}_0 \to \mathcal{A}$ into an abelian* AB 5 *category $\mathcal{A}$ extends, up to isomorphism, uniquely to a cohomological functor $f'\colon \mathcal{C} \to \mathcal{A}$ which preserves coproducts. Moreover, if $\mathcal{A}$ is the category of abelian groups, then f' preserves products if and only if $f' \simeq \operatorname{Hom}(X, -)$ for some compact object X in $\mathcal{C}$.*

In Section 3 we derive from the universal property of $h\colon \mathcal{C} \to \mathcal{M}$ a strong relation between localizing subcategories in $\mathcal{C}$ and localizing subcategories in $\mathcal{M}$. This interplay between triangulated and module categories is crucial for our characterization of smashing subcategories. The final Section 4 is devoted to the proofs for the main results of this paper.

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1. Purity

1.1. **Pure-exactness.** Let $\mathcal{C}$ be a triangulated category [26, 27] and suppose that arbitrary coproducts exist in $\mathcal{C}$. An object X in $\mathcal{C}$ is called *compact* if for every family $(Y_i)_{i\in I}$ in $\mathcal{C}$ the canonical map $\coprod_i \operatorname{Hom}(X, Y_i) \to \operatorname{Hom}(X, \coprod_i Y_i)$ is an isomorphism. We denote by $\mathcal{C}_0$ the full subcategory of compact objects in $\mathcal{C}$ and observe that $\mathcal{C}_0$ is a triangulated subcategory of $\mathcal{C}$. Following [20], the category $\mathcal{C}$ is called *compactly generated* provided that the isomorphism classes of objects in $\mathcal{C}_0$ form a set, and $\operatorname{Hom}(C, X) = 0$ for all C in $\mathcal{C}_0$ implies $X = 0$ for every object X in $\mathcal{C}$. Examples of compactly generated triangulated categories arise in stable homotopy theory, algebraic geometry, and representation theory.

Definition 1.1. Let $\mathcal{C}$ be a compactly generated triangulated category.

(1) A map $X \to Y$ in $\mathcal{C}$ is said to be a *pure monomorphism* if the induced map $\operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y)$ is a monomorphism for all compact objects C in $\mathcal{C}$.
(2) An object X in $\mathcal{C}$ is called *pure-injective* if every pure monomorphism $\phi\colon X \to Y$ splits, i.e. there exist a map $\phi'\colon Y \to X$ such that $\phi' \circ \phi = \mathrm{id}_X$.
(3) A triangle $X \to Y \to Z \to X[1]$ is called *pure-exact* if the induced sequence $0 \to \operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y) \to \operatorname{Hom}(C, Z) \to 0$ is exact for all compact objects C in $\mathcal{C}$.

The preceding definition is motivated by analogous definitions for the category of modules over a ring [7]. However, contrary to the concept for modules, a pure monomorphism in $\mathcal{C}$ is usually not a monomorphism in the categorical sense. For the sake of completeness we include the following definition.

Definition 1.2. Let $\mathcal{C}$ be a compactly generated triangulated category.

(1) A map $Y \to Z$ in $\mathcal{C}$ is said to be a *pure epimorphism* if the induced map $\operatorname{Hom}(C, Y) \to \operatorname{Hom}(C, Z)$ is an epimorphism for all compact objects C in $\mathcal{C}$.
(2) An object Z in $\mathcal{C}$ is called *pure-projective* if every pure epimorphism $\psi\colon Y \to Z$ splits, i.e. there exist a map $\psi'\colon Z \to Y$ such that $\psi \circ \psi' = \mathrm{id}_Z$.

The concept of purity is closely related to the occurence of phantom maps. Recall that a map $X \to Y$ is a *phantom map* provided that the induced map $\operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y)$ is zero for all compact objects C in $\mathcal{C}$.

Lemma 1.3. *For a triangle* $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\chi} X[1]$ *the following are equivalent:*

(1) ϕ *is a phantom map;*
(2) ψ *is a pure monomorphism;*
(3) χ *is a pure epimorphism;*
(4) *the shifted triangle* $Y \to Z \to X[1] \to Y[1]$ *is pure-exact.*

Proof. Clear, since the induced sequence $\mathrm{Hom}(C,X) \to \mathrm{Hom}(C,Y) \to \mathrm{Hom}(C,Z) \to \mathrm{Hom}(C,X[1])$ is exact for every $C \in \mathcal{C}_0$. □

Lemma 1.4. *The following conditions are equivalent for an object* X *in* $\mathcal{C}$*:*

(1) X *is pure-injective;*
(2) *if* $\phi\colon Y \to X$ *is a phantom map, then* $\phi = 0$*;*
(3) *if* $\phi\colon V \to W$ *is a pure monomorphism, then every map* $V \to X$ *factors through* ϕ*.*

Proof. (1) $\Leftrightarrow$ (2) follows immediately from the preceding lemma, and the direction (3) $\Rightarrow$ (1) is also clear. To prove (1) $\Rightarrow$ (3), let $\phi\colon V \to W$ be a pure monomorphism and let $\psi\colon V \to X$ be a map with X pure-injective. We obtain a commutative diagram

$$\begin{array}{ccccccc}
U & \longrightarrow & V & \xrightarrow{\phi} & W & \longrightarrow & U[1] \\
\| & & \downarrow{\scriptstyle\psi} & & \downarrow & & \| \\
U & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & U[1]
\end{array}$$

such that both rows are triangles. The map $U \to V$ is a phantom map since ϕ is a pure monomorphism, and it follows that $U \to X$ is a phantom map. Therefore the map $X \to Y$ is a pure monomorphism which splits since X is pure-injective. It follows that ψ factors through ϕ. □

1.2. **Modules.** Let $\mathcal{C}$ be any additive category. A $\mathcal{C}$*-module* is by definition an additive functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category Ab of abelian groups, and we denote for $\mathcal{C}$-modules M and N by $\mathrm{Hom}(M,N)$ the class of natural transformations $M \to N$. A sequence $L \to M \to N$ of maps between $\mathcal{C}$-modules is *exact* if the sequence $L(X) \to M(X) \to N(X)$ is exact for all X in $\mathcal{C}$. A $\mathcal{C}$-module M is *finitely generated* if there exists an exact sequence $\mathrm{Hom}(-,X) \to M \to 0$ for some X in $\mathcal{C}$, and M is *finitely presented* if there exists an exact sequence $\mathrm{Hom}(-,X) \to \mathrm{Hom}(-,Y) \to M \to 0$ with X and Y in $\mathcal{C}$. Note that $\mathrm{Hom}(M,N)$ is a set for every finitely generated $\mathcal{C}$-module M by Yoneda's lemma. The finitely presented $\mathcal{C}$-modules form an additive category with cokernels which we denote by $\mathrm{mod}\,\mathcal{C}$. It is well-known that $\mathrm{mod}\,\mathcal{C}$ is abelian if and only if every map $Y \to Z$ in $\mathcal{C}$ has a *weak kernel* $X \to Y$, i.e. the sequence $\mathrm{Hom}(-,X) \to \mathrm{Hom}(-,Y) \to \mathrm{Hom}(-,Z)$ is eaxct. In particular, $\mathrm{mod}\,\mathcal{C}$ is abelian if $\mathcal{C}$ is triangulated.

Suppose now that $\mathcal{C}$ is skeletally small. Then the $\mathcal{C}$-modules form together with the natural transformations an abelian category which we denote by $\mathrm{Mod}\,\mathcal{C}$. Note that $\mathrm{Mod}\,\mathcal{C}$ has arbitrary products and coproducts which are defined pointwise. For example, $(\prod_i M_i)(X) = \prod_i M_i(X)$ for a family $(M_i)_{i\in I}$ in $\mathrm{Mod}\,\mathcal{C}$ and X in $\mathcal{C}$. We denote for every X in $\mathcal{C}$ by $H_X = \mathrm{Hom}(-,X)$ the corresponding representable functor and recall that $\mathrm{Hom}(H_X, M) \simeq M(X)$ for every module M by Yoneda's lemma. It follows that H_X is a projective object in $\mathrm{Mod}\,\mathcal{C}$. We shall also need to use the fact that $\mathrm{Mod}\,\mathcal{C}$ is a *Grothendieck category*, which as far as we are concerned means that it has injective envelopes [9].

Our main tool for studying a compactly generated triangulated category $\mathcal{C}$ is the *restricted Yoneda functor*

$$h_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \operatorname{Mod}\mathcal{C}_0, \quad X \mapsto H_X = \operatorname{Hom}(-,X)|_{\mathcal{C}_0}.$$

1.3. **Brown representability.** Recall that a (covariant) functor $f\colon \mathcal{C} \to \mathcal{A}$ from a triangulated category $\mathcal{C}$ into an abelian category $\mathcal{A}$ is *cohomological* if for every triangle $X \to Y \to Z \to X[1]$ in $\mathcal{C}$ the sequence $f(X) \to f(Y) \to f(Z) \to f(X[1])$ is exact. Examples of cohomological functors are the representable functors $\operatorname{Hom}(X,-)\colon \mathcal{C} \to \operatorname{Ab}$ and $\operatorname{Hom}(-,X)\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Ab}$ for any X in $\mathcal{C}$. The Brown representability theorem characterizes the representable cohomological functors $\mathcal{C}^{\mathrm{op}} \to \operatorname{Ab}$ for a compactly generated triangulated category $\mathcal{C}$.

Theorem (Brown). *Let $f\colon \mathcal{C}^{\mathrm{op}} \to \operatorname{Ab}$ be a cohomological functor such that the canonical map $f(\coprod_i X_i) \to \prod_i f(X_i)$ is an isomorphism for every family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$. Then $f \simeq \operatorname{Hom}(-,X)$ for some object X in $\mathcal{C}$.*

Proof. See Theorem 3.1 in [20]. □

The existence of arbitrary products in $\mathcal{C}$ is a well-known consequence of the Brown representability theorem.

Lemma 1.5. *The category $\mathcal{C}$ has arbitrary products.*

Proof. Let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}$ and let $f = \prod_i \operatorname{Hom}(-,X_i)$. Clearly, f is a cohomological functor which sends coproducts to products. Thus $f \simeq \operatorname{Hom}(-,X)$ by the Brown representability theorem, and it is easily checked that $X = \prod_i X_i$ in $\mathcal{C}$. □

1.4. **Pure-injectives.** Our analysis of pure-injective objects in a compactly generated triangulated category $\mathcal{C}$ is based on some properties of the restricted Yoneda functor $h_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}_0$. We need two lemmas. Recall that a module M is *fp-injective* if $\operatorname{Ext}^1(N,M) = 0$ for every finitely presented module N.

Lemma 1.6. *The $\mathcal{C}_0$-module H_X is fp-injective for every X in $\mathcal{C}$.*

Proof. A finitely presented $\mathcal{C}_0$-module N has a projective presentation

$$H_A \longrightarrow H_B \longrightarrow H_C \longrightarrow N \longrightarrow 0$$

coming from a triangle $A \to B \to C \to A[1]$ in $\mathcal{C}$ with objects in $\mathcal{C}_0$. Thus one can compute Ext^1 as the cohomology of the complex

$$\operatorname{Hom}(H_C,H_X) \longrightarrow \operatorname{Hom}(H_B,H_X) \longrightarrow \operatorname{Hom}(H_A,H_X).$$

This is, however, isomorphic to $\operatorname{Hom}(C,X) \to \operatorname{Hom}(B,X) \to \operatorname{Hom}(A,X)$, so it is exact. Therefore $\operatorname{Ext}^1(N,H_X) = 0$ and H_X is fp-injective. □

Lemma 1.7. *Let M be an injective $\mathcal{C}_0$-module. Then there exists, up to isomorphism, a unique object X in $\mathcal{C}$ such that $M \simeq H_X$. Moreover, $h_{\mathcal{C}}$ induces an isomorphism $\operatorname{Hom}(Y,X) \to \operatorname{Hom}(H_Y,H_X)$ for all Y in $\mathcal{C}$.*

Proof. Let $f = \operatorname{Hom}(-,M) \circ h_{\mathcal{C}}$. Then f is a cohomological functor since $h_{\mathcal{C}}$ is cohomological and $\operatorname{Hom}(-,M)$ is exact. Moreover, $h_{\mathcal{C}}$ preserves coproducts and $\operatorname{Hom}(-,M)$ induces an isomorphism $\operatorname{Hom}(\coprod_i N_i,M) \simeq \prod_i \operatorname{Hom}(N_i,M)$ for every family $(N_i)_{i\in I}$ of $\mathcal{C}_0$-modules. Therefore $f \simeq \operatorname{Hom}(-,X)$ for some object X in $\mathcal{C}$ by the Brown representability theorem. The induced map $\operatorname{Hom}(X,X) \simeq f(X) = \operatorname{Hom}(H_X,M)$ sends id_X to a map $\phi\colon H_X \to M$ which is an isomorphism since

$$H_X(C) = \operatorname{Hom}(C,X) \simeq \operatorname{Hom}(H_C,M) \simeq M(C)$$

for every compact object C by Yoneda's lemma. The inverse $\phi^{-1}\colon M \to H_X$ induces an isomorphism

$$\operatorname{Hom}(Y,X) \simeq \operatorname{Hom}(H_Y,M) \simeq \operatorname{Hom}(H_Y,H_X)$$

which is precisely the map induced by $h_{\mathcal{C}}$. This finishes the proof. □

The following theorem collects a number of characterizing properties of pure-injective objects. We denote for every object X and every set I by X^I the product and by $X^{(I)}$ the coproduct of $\operatorname{card} I$ copies of X.

Theorem 1.8. *The following conditions are equivalent for an object X in $\mathcal{C}$:*

1. *X is pure-injective;*
2. *$H_X = \operatorname{Hom}(-,X)|_{\mathcal{C}_0}$ is an injective $\mathcal{C}_0$-module;*
3. *the map $\operatorname{Hom}(Y,X) \to \operatorname{Hom}(H_Y,H_X)$, $\phi \mapsto \operatorname{Hom}(-,\phi)|_{\mathcal{C}_0}$, is an isomorphism for all Y in $\mathcal{C}$;*
4. *if $\phi\colon Y \to X$ is a phantom map, then $\phi = 0$;*
5. *for every set I the summation map $X^{(I)} \to X$ factors through the canonical map $X^{(I)} \to X^I$.*

Proof. (1) ⇒ (2) Let $H_X \to M$ be an injective envelope in $\operatorname{Mod}\mathcal{C}_0$. It follows from Lemma 1.7 that $M \simeq H_Y$ for some object Y in $\mathcal{C}$, and the map $H_X \to M \simeq H_Y$ is of the form H_ϕ for some $\phi\colon X \to Y$. Clearly, ϕ is a pure monomorphism, and ϕ splits since X is pure-injective. Thus H_X is a direct summand of M and therefore injective.

(2) ⇒ (3) Use Lemma 1.7.

(3) ⇒ (4) If $\phi\colon Y \to X$ is a phantom map, then $\operatorname{Hom}(-,\phi)|_{\mathcal{C}_0} = 0$. Thus it follows from (3) that $\phi = 0$.

(4) ⇒ (1) Use Lemma 1.4.

(2) ⇒ (5) Suppose that $M = H_X$ is an injective $\mathcal{C}_0$-module. It follows that the summation map $M^{(I)} \to M$ factors through the canonical monomorphism $M^{(I)} \to M^I$. The corresponding map $H_{X^I} \to H_X$ is of the form H_ϕ for some map $\phi\colon X^I \to X$ by Lemma 1.7, and it follows that the composition of ϕ with $X^{(I)} \to X^I$ is the summation map.

(5) ⇒ (2) $M = H_X$ is an fp-injective $\mathcal{C}_0$-module by Lemma 1.6 which is injective if the summation map $M^{(I)} \to M$ factors through the canonical monomorphism $M^{(I)} \to M^I$ for every set I by [17, Theorem 2.6]. □

We discuss a number of consequences.

Corollary 1.9. *The restricted Yoneda functor $\mathcal{C} \to \operatorname{Mod}\mathcal{C}_0$ induces an equivalence between the full subcategory of pure-injective objects in $\mathcal{C}$ and the full subcategory of injective objects in $\operatorname{Mod}\mathcal{C}_0$.*

Proof. The restricted Yoneda functor sends pure-injectives to injectives by Theorem 1.8, and it is fully faithful and dense by Lemma 1.7. □

Recall that an object X in any additive category is *indecomposable* if $X \neq 0$ and every decomposition $X = X_1 \coprod X_2$ implies $X_1 = 0$ or $X_2 = 0$. The isomorphism classes of indecomposable injective objects in $\operatorname{Mod}\mathcal{C}_0$ form a set since every indecomposable injective $\mathcal{C}_0$-module arises as an injective envelope of a finitely generated $\mathcal{C}_0$-module. It follows that the indecomposable pure-injective objects in $\mathcal{C}$ form a set which we denote by $\operatorname{Sp}\mathcal{C}$.

Corollary 1.10. *Every object X in $\mathcal{C}$ admits a pure monomorphism $X \to \prod_{i\in I} Y_i$ with $Y_i \in \operatorname{Sp}\mathcal{C}$ for all i. In particular, $\operatorname{Hom}(X,Y) = 0$ for all $Y \in \operatorname{Sp}\mathcal{C}$ implies $X = 0$.*

Proof. We observe first that the indecomposable injective $\mathcal{C}_0$-modules cogenerate $\operatorname{Mod}\mathcal{C}_0$. In fact, one could take the injective envelopes of all simple modules. To see this, observe that every non-zero module M has a finitely generated non-zero submodule U which has a maximal submodule V by Zorn's lemma. This gives a non-zero map from M to the injective envelope of U/V. Now let X be an object in $\mathcal{C}$ and choose a monomorphism $H_X \to M$ in $\operatorname{Mod}\mathcal{C}_0$ such that $M = \prod_i M_i$ is a product of indecomposable injective $\mathcal{C}_0$-modules. It follows from Lemma 1.7 that this map comes from a pure monomorphism $X \to \prod_{i\in I} Y_i$ with $M_i \simeq H_{Y_i}$ for all i, and each Y_i is indecomposable pure-injective by Corollary 1.9. □

Remark 1.11. The set $\operatorname{Sp}\mathcal{C}$ carries two natural topologies. A subset $\mathbf{U}$ of $\operatorname{Sp}\mathcal{C}$ is *Ziegler-closed* if and only if $\mathbf{U} = \{X \in \operatorname{Sp}\mathcal{C} \mid \operatorname{Hom}(\phi, X) = 0 \text{ for all } \phi \in \mathfrak{I}\}$ for some class $\mathfrak{I}$ of maps in $\mathcal{C}_0$; see [15, Lemma 4.1]. A subset $\mathbf{U}$ of $\operatorname{Sp}\mathcal{C}$ is *Zariski-open* if and only if there exists some class $\mathfrak{I}$ of maps in $\mathcal{C}_0$ such that $\mathbf{U} = \{X \in \operatorname{Sp}\mathcal{C} \mid \operatorname{Hom}(\phi, X) = 0 \text{ for some } \phi \in \mathfrak{I}\}$; see [9, Chap. VI]. We refer to [18] for a detailed discussion of both topologies in the context of modules over a ring.

A map $\phi\colon X \to Y$ in $\mathcal{C}$ is said to be a *pure-injective envelope* of X if Y is pure-injective and a composition $\psi \circ \phi$ with a map $\psi\colon Y \to Z$ is a pure-monomorphism if and only if ψ is a pure monomorphism.

Lemma 1.12. *The following are equivalent for a pure monomorphism $\phi\colon X \to Y$ in $\mathcal{C}$:*

(1) *ϕ is a pure-injective envelope of X;*
(2) *Y is pure-injective and every endomorphism ψ of Y satisfying $\psi \circ \phi = \phi$ is an isomorphism;*
(3) *$H_\phi\colon H_X \to H_Y$ is an injective envelope in* $\operatorname{Mod}\mathcal{C}_0$.

Proof. Straightforward. □

Corollary 1.13. *Every object X in $\mathcal{C}$ admits a pure-injective envelope $\phi\colon X \to Y$. If $\phi'\colon X \to Y'$ is another pure-injective envelope, then there exists an isomorphism $\psi\colon Y \to Y'$ such that $\phi' = \psi \circ \phi$.*

Proof. The assertion is a consequence of Theorem 1.8 and the existence of injective envelopes in $\operatorname{Mod}\mathcal{C}_0$. □

We are now in a position to prove Theorem D. In fact, the existence of a universal phantom map $X' \to X$ ending in a fixed object X follows from the existence of a pure-injective envelope $X \to X''$. We recall Theorem D for the convenience of the reader.

Theorem 1.14. *For every object X in $\mathcal{C}$ there exists, up to isomorphism, a unique triangle*

$$X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \xrightarrow{\gamma} X'[1]$$

having the following properties:

(A1) *a map $\phi\colon Y \to X$ is a phantom map if and only if ϕ factors through α;*
(A2) *every endomorphism ϕ of X' satisfying $\alpha = \alpha \circ \phi$ is an isomorphism.*

The same triangle is characterized, up to isomorphism, by the following properties:

(B1) *a map $\phi\colon X \to Y$ is a pure monomorphism if and only if β factors through ϕ;*
(B2) *every endomorphism ϕ of X'' satisfying $\beta = \phi \circ \beta$ is an isomorphism.*

Proof. Let X be an object in $\mathcal{C}$ and complete the pure-injective envelope $\beta\colon X \to X''$ to a triangle

$$X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \xrightarrow{\gamma} X'[1].$$

The map α is a phantom map by Lemma 1.3 since β is a pure monomorphism, and the property (4) in Theorem 1.8 implies that α is a universal phantom map ending in X since X'' is pure-injective. On the other hand, β is a universal pure monomorphism starting in X by Lemma 1.4 since X'' is pure-injective. This establishes (A1) and (B1). Condition (B2) is an immediate consequence of Lemma 1.12, and (A2) then follows from (B2). It is easily checked that each pair of conditions characterizes the above triangle, and therefore the proof is complete. □

It is interesting to observe that the full subcategory of pure-injective objects in $\mathcal{C}$ is completely determined by the full subcategory $\mathcal{C}_0$ of compact objects in $\mathcal{C}$.

Corollary 1.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be compactly generated triangulated categories, and suppose that there exists an equivalence $f\colon \mathcal{C}_0 \to \mathcal{D}_0$ between the full subcategories of compact objects in $\mathcal{C}$ and $\mathcal{D}$. Then f induces an equivalence between the full subcategories of pure-injective objects in $\mathcal{C}$ and $\mathcal{D}$.*

Proof. The functor $h_{\mathcal{C}}\colon \mathcal{C} \to \operatorname{Mod}\mathcal{C}_0$ induces an equivalence between the full subcategory of pure-injectives in $\mathcal{C}$ and the full subcategory of injective $\mathcal{C}_0$-modules by Corollary 1.9. The assertion now follows since an equivalence $f\colon \mathcal{C}_0 \to \mathcal{D}_0$ induces an equivalence $\operatorname{Mod}\mathcal{C}_0 \to \operatorname{Mod}\mathcal{D}_0$. □

1.5. **Pure-injective modules.** The concept of purity has been studied extensively by algebraists. Pure-exactness and pure-injectivity for modules over a ring have been introduced by Cohn [7], and we refer to [13] for a modern treatment of this subject.

Let us recall briefly the relevant definitions. Let Λ be an associative ring with identity. We consider the category $\operatorname{Mod}\Lambda$ of (right) Λ-modules. A sequence $0 \to X \to Y \to Z \to 0$ of maps in $\operatorname{Mod}\Lambda$ is *pure-exact* if the induced sequence $0 \to \operatorname{Hom}(C, X) \to \operatorname{Hom}(C, Y) \to \operatorname{Hom}(C, Z) \to 0$ is exact for all finitely presented Λ-modules C. The map $X \to Y$ in such a sequence is called a *pure monomorphism.* Note that any pure-exact sequence is automatically an exact sequence in the usual sense. A module X is *pure-injective* if every pure monomorphism $X \to Y$ splits.

Suppose now that Λ is a *quasi-Frobenius ring*, i.e. projective and injective Λ-modules coincide. In this case the *stable category* $\underline{\operatorname{Mod}}\,\Lambda$ is triangulated; e.g. see [12]. Recall that the objects in $\underline{\operatorname{Mod}}\,\Lambda$ are those of $\operatorname{Mod}\Lambda$, and for two Λ-modules X, Y one defines $\underline{\operatorname{Hom}}(X, Y)$ to be $\operatorname{Hom}(X, Y)$ modulo the subgroup of maps which factor through a projective Λ-module. Note that the projection functor $\operatorname{Mod}\Lambda \to \underline{\operatorname{Mod}}\,\Lambda$ preserves products and coproducts. Thus $\underline{\operatorname{Mod}}\,\Lambda$ has arbitrary coproducts, and it is not difficult to check that an object X in $\underline{\operatorname{Mod}}\,\Lambda$ is compact if and only if $X \simeq Y$ in $\underline{\operatorname{Mod}}\,\Lambda$ for some finitely presented Λ-module Y. Therefore $\underline{\operatorname{Mod}}\,\Lambda$ is compactly generated.

Proposition 1.16. *A Λ-module X is pure-injective if and only if X is a pure-injective object in $\underline{\operatorname{Mod}}\,\Lambda$.*

Proof. We use the following characterization of pure-injectivity for Λ-modules which is due to Jensen and Lenzing [13, Proposition 7.32]: A Λ-module X is pure-injective if and only if for every set I the summation map $\sigma_I\colon X^{(I)} \to X$ factors through the canonical map $\iota_I\colon X^{(I)} \to X^I$. We now combine this characterization with the characterization of pure-injectivity in $\underline{\operatorname{Mod}}\,\Lambda$ from Theorem 1.8. Thus any pure-injective Λ-module is a pure-injective object in $\underline{\mathrm{Mod}}\,\Lambda$. To prove the converse, let X be a pure-injective object in $\underline{\mathrm{Mod}}\,\Lambda$ and fix a set I. Thus there exists a map $\phi\colon X^I \to X$ in $\mathrm{Mod}\,\Lambda$ such that $\sigma_I - \phi\circ\iota_I$ factors through a projective Λ-module P, i.e. $\sigma_I - \phi\circ\iota_I = \beta\circ\alpha$ for some map $\alpha\colon X^{(I)} \to P$. The map α factors through the monomorphism ι_I since P is injective, i.e. $\alpha = \alpha'\circ\iota_I$ for some map α', and therefore

$$\sigma_I = \beta\circ\alpha + \phi\circ\iota_I = (\beta\circ\alpha' + \phi)\circ\iota_I.$$

Thus σ_I factors through ι_I, and this finishes the proof. $\square$

For some further discussion of the relation between pure-injectives in $\mathrm{Mod}\,\Lambda$ and $\underline{\mathrm{Mod}}\,\Lambda$ we refer to [16, 5].

2. Cohomological and exact functors

2.1. **Extending functors.** Let $\mathcal{C}$ be any triangulated category. We recall the following well-known property of the Yoneda functor $h\colon \mathcal{C} \to \mathrm{mod}\,\mathcal{C}$, $X \mapsto \mathrm{Hom}(-,X)$.

Lemma 2.1. *Every additive functor $f\colon \mathcal{C} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ extends, up to isomorphism, uniquely to a right exact functor $f'\colon \mathrm{mod}\,\mathcal{C} \to \mathcal{A}$ such that $f = f'\circ h$. The functor f' is exact if and only if f is a cohomological functor.*

Proof. Any finitely presented $\mathcal{C}$-module M has a projective presentation

$$\mathrm{Hom}(-,X) \xrightarrow{\mathrm{Hom}(-,\phi)} \mathrm{Hom}(-,Y) \xrightarrow{\mathrm{Hom}(-,\psi)} \mathrm{Hom}(-,Z) \longrightarrow M \longrightarrow 0$$

coming from a triangle

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\chi} X[1]$$

in $\mathcal{C}$. We obtain a right exact functor $f'\colon \mathrm{mod}\,\mathcal{C} \to \mathcal{A}$ if we define $f'(M) = \mathrm{Coker}\, f(\psi)$. Clearly, $f = f'\circ h$ holds by construction. Exactness of f' implies that f is cohomological, since h is cohomological. Suppose now that f is cohomological. Taking projective presentations of the modules in an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ in $\mathrm{mod}\,\mathcal{C}$ as above, one obtains the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
f(X_1) & \longrightarrow & f(Y_1) & \longrightarrow & f(Z_1) & \longrightarrow & f'(M_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
f(X_1 \coprod X_3) & \longrightarrow & f(Y_1 \coprod Y_3) & \longrightarrow & f(Z_1 \coprod Z_3) & \longrightarrow & f'(M_2) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
f(X_3) & \longrightarrow & f(Y_3) & \longrightarrow & f(Z_3) & \longrightarrow & f'(M_3) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & & 0 & &
\end{array}$$

The rows are exact since f is cohomological, and therefore the exactness of the first three columns implies the exactness of the sequence $0 \to f'(M_1) \to f'(M_2) \to f'(M_3) \to 0$. Thus f' is exact and this finishes the proof. $\square$

Recall that an abelian category $\mathcal{A}$ satisfies *Grothendieck's* AB 5 *condition* if $\mathcal{A}$ has arbitrary coproducts and taking filtered colimits preserves exactness. For example, any module category is an AB 5 category. Suppose now that $\mathcal{C}$ is a skeletally small triangulated category and consider the Yoneda functor $h\colon \mathcal{C} \to \mathrm{Mod}\,\mathcal{C}$.

Lemma 2.2. *Every additive functor $f\colon \mathcal{C}\to\mathcal{A}$ into an abelian* AB 5 *category $\mathcal{A}$ extends, up to isomorphism, uniquely to a right exact functor $f'\colon \operatorname{Mod}\mathcal{C}\to\mathcal{A}$ which preserves coproducts and satisfies $f=f'\circ h$. The functor f' is exact if and only if f is a cohomological functor.*

Proof. Any $\mathcal{C}$-module M has a projective presentation

$$\coprod_i \operatorname{Hom}(-,X_i)\xrightarrow{(\operatorname{Hom}(-,\phi_{ij}))}\coprod_j \operatorname{Hom}(-,Y_j)\longrightarrow M\longrightarrow 0$$

which is given by a family of maps $\phi_{ij}\colon X_i\to Y_j$ in $\mathcal{C}$. We obtain a functor $f'\colon \operatorname{Mod}\mathcal{C}\to\mathcal{A}$ if we define $f'(M)$ as the cokernel of the map $(f(\phi_{ij}))\colon \coprod_i f(X_i)\to\coprod_j f(Y_j)$ in $\mathcal{A}$. It is easily checked that f' preserves colimits, and that $f=f'\circ h$. The restriction $f'|_{\operatorname{mod}\mathcal{C}}$ is exact if and only if f is cohomological by the preceding lemma. Now observe that any exact sequence $0\to L\xrightarrow{\phi} M\xrightarrow{\psi} N\to 0$ in $\operatorname{Mod}\mathcal{C}$ can be written as a filtered colimit of exact sequences $0\to L_i\xrightarrow{\phi_i} M_i\xrightarrow{\psi_i} N_i\to 0$ in $\operatorname{mod}\mathcal{C}$. To see this, write ϕ as a filtered colimit of maps $\phi_i'\colon L_i'\to M_i$ in $\operatorname{mod}\mathcal{C}$. Denoting by $\psi_i\colon M_i\to N_i$ the cokernel of each ϕ_i', we obtain a filtered system of exact sequences $0\to L_i\xrightarrow{\phi_i} M_i\xrightarrow{\psi_i} N_i\to 0$ in $\operatorname{mod}\mathcal{C}$ with colimit $0\to L\xrightarrow{\phi} M\xrightarrow{\psi} N\to 0$. It follows that f' is exact if and only if f is cohomological since $\mathcal{A}$ is an AB 5 category. ☐

We are now in a position to prove the first part of Theorem E. To this end suppose that $\mathcal{C}$ is compactly generated and consider the restricted Yoneda functor $h_{\mathcal{C}}\colon \mathcal{C}\to\operatorname{Mod}\mathcal{C}_0$.

Proposition 2.3. *Let $\mathcal{C}$ be a compactly generated triangulated category. Then every cohomological functor $f\colon \mathcal{C}_0\to\mathcal{A}$ into an abelian* AB 5 *category $\mathcal{A}$ extends, up to isomorphism, uniquely to a cohomological functor $f'\colon\mathcal{C}\to\mathcal{A}$ which preserves coproducts.*

Proof. We denote by $f^*\colon \operatorname{Mod}\mathcal{C}_0\to\mathcal{A}$ the exact colimit preserving functor which extends f, and define $f'=f^*\circ h_{\mathcal{C}}$. Clearly, f' is cohomological, preserves coproducts, and $f'|_{\mathcal{C}_0}=f$. Suppose there is another functor $f''\colon\mathcal{C}\to\mathcal{A}$ with these properties. We construct a natural transformation $\eta\colon f'\to f''$ as follows. If X is coproduct of compact objects in $\mathcal{C}$, then we obtain a unique isomorphism $\eta_X\colon f'(X)\to f''(X)$ since f' and f'' preserve coproducts. Now let $X=X_0$ be an arbitrary object in $\mathcal{C}$. We can choose pure-exact triangles $X_{i+1}\to P_i\to X_i\to X_{i+1}[1]$ with P_i being a coproduct of compact objects for $i=0,1$, and we obtain a sequence of maps $P_1\to P_0\to X$ in $\mathcal{C}$ such that $H_{P_1}\to H_{P_0}\to H_X\to 0$ is exact. This gives a commutative diagram

$$\begin{array}{ccccccc}
f'(P_1) & \longrightarrow & f'(P_0) & \longrightarrow & f'(X) & \longrightarrow & 0\\
\big\downarrow{\scriptstyle \eta_{P_1}} & & \big\downarrow{\scriptstyle \eta_{P_0}} & & & & \\
f''(P_1) & \longrightarrow & f''(P_0) & \longrightarrow & f''(X) & &
\end{array}$$

where the upper row is exact since f^* is exact. Thus there is a unique map $\eta_X\colon f'(X)\to f''(X)$ since the composition $P_1\to P_0\to X$ is zero. Now let $\mathcal{B}$ be the full subcategory formed by the objects X in $\mathcal{C}$ such that η_X is an isomorphism. Clearly, $\mathcal{B}$ contains $\mathcal{C}_0$, and it is triangulated since f' and f'' are cohomological. Furthermore, $\mathcal{B}$ is closed under taking coproducts since f' and f'' preserve coproducts. Thus $\mathcal{B}=\mathcal{C}$ by [20, Lemma 3.2], and therefore $\eta\colon f'\to f''$ is an isomorphism. ☐

The following consequence generalizes a result from [8].

Corollary 2.4. *Let $\mathcal{C}$ be a compactly generated triangulated category and let $f\colon \mathcal{C} \to \mathcal{A}$ be a cohomological functor into an abelian* AB 5 *category $\mathcal{A}$. Suppose also that f preserves coproducts. Then there exists, up to isomorphism, a unique exact functor $f'\colon \operatorname{Mod}\mathcal{C}_0 \to \mathcal{A}$ which preserves coproducts and satisfies $f = f' \circ h_{\mathcal{C}}$.*

Proof. Let $f'\colon \operatorname{Mod}\mathcal{C}_0 \to \mathcal{A}$ be the colimit preserving functor extending $f|_{\mathcal{C}_0}$ which exists by Lemma 2.2. We have $f \simeq f' \circ h_{\mathcal{C}}$ by the preceding theorem since both functors are cohomological and preserve coproducts. This gives the uniqeness of f'. □

Corollary 2.5. *The following are equivalent for a map $\phi\colon X \to Y$ in a compactly generated triangulated category $\mathcal{C}$:*
1. *ϕ is a phantom map;*
2. *$f(\phi) = 0$ for every cohomological functor $f\colon \mathcal{C} \to \mathcal{A}$ into an abelian* AB 5 *category $\mathcal{A}$ which preserves coproducts;*
3. *the induced map $\operatorname{Hom}(Y, Q) \to \operatorname{Hom}(X, Q)$ is zero for every* (*indecomposable*) *pure-injective object Q in $\mathcal{C}$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is an immediate consequence of Corollary 2.4. The equivalence (1) $\Leftrightarrow$ (3) follows from the fact that ϕ is a phantom map if and only if the map ψ in a triangle $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\chi} X[1]$ is a pure monomorphism. In addition, one uses the existence of a pure monomorphism $Y \to \prod_{i\in I} Z_i$ into a product of indecomposable pure-injectives which has been established in Corollary 1.10. □

2.2. **Adjoint functors.** We study pairs of adjoint functors between compactly generated triangulated categories. This is based on properties of adjoint functors between module categories. We start with some notation. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between skeletally small additive categories. Then we denote by $f_*\colon \operatorname{Mod}\mathcal{D} \to \operatorname{Mod}\mathcal{C}$, $X \mapsto X \circ f$ the corresponding restriction functor, and $f^*\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{D}$ denotes the unique functor which preserves colimits and sends $\operatorname{Hom}(-, X)$ to $\operatorname{Hom}(-, f(X))$ for every X in $\mathcal{C}$. Applying Yoneda's lemma, we get for every X in $\mathcal{C}$ and every $\mathcal{D}$-module M a functorial isomorphism

$$\operatorname{Hom}(f^*(\operatorname{Hom}(-, X)), M) \simeq M(f(X)) = f_*(M)(X) \simeq \operatorname{Hom}(\operatorname{Hom}(-, X), f_*(M))$$

which shows that f^* is a left adjoint for f_*.

Proposition 2.6. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between compactly generated triangulated categories. Suppose also that f preserves coproducts, and that the right adjoint $g\colon \mathcal{D} \to \mathcal{C}$ of f preserves coproducts.*
1. *f induces a functor $f_0\colon \mathcal{C}_0 \to \mathcal{D}_0$ which makes the following diagrams commutative:*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow{\scriptstyle h_{\mathcal{C}}} & & \downarrow{\scriptstyle h_{\mathcal{D}}} \\ \operatorname{Mod}\mathcal{C}_0 & \xrightarrow{(f_0)^*} & \operatorname{Mod}\mathcal{D}_0 \end{array} \qquad \begin{array}{ccc} \mathcal{D} & \xrightarrow{g} & \mathcal{C} \\ \downarrow{\scriptstyle h_{\mathcal{D}}} & & \downarrow{\scriptstyle h_{\mathcal{C}}} \\ \operatorname{Mod}\mathcal{D}_0 & \xrightarrow{(f_0)_*} & \operatorname{Mod}\mathcal{C}_0 \end{array}$$

2. *The functors $(f_0)^*$ and $(f_0)_*$ are both exact.*
3. *The functor g sends pure-exact triangles to pure-exact triangles, and pure-injectives to pure-injectives.*

Proof. The existence of the right adjoint $g\colon \mathcal{D} \to \mathcal{C}$ is an immediate consequence of the Brown representability theorem, since for every object X in $\mathcal{D}$ there exists a unique object $Y = g(X)$ in $\mathcal{C}$ such that $\operatorname{Hom}(-, X) \circ f \simeq \operatorname{Hom}(-, Y)$.

(1) Given a compact object X in $\mathcal{C}$, it is well-known that $f(X)$ is compact since g preserves coproducts. This follows from the following sequence of canonical isomorphisms for every family $(Y_i)_{i\in I}$ of objects in $\mathcal{D}$:

$$\begin{aligned}\coprod_i \operatorname{Hom}(f(X),Y_i) &\simeq \coprod_i \operatorname{Hom}(X,g(Y_i)) \simeq \operatorname{Hom}(X,\coprod_i g(Y_i)) \\ &\simeq \operatorname{Hom}(X,g(\coprod_i Y_i)) \simeq \operatorname{Hom}(f(X),\coprod_i Y_i).\end{aligned}$$

Therefore f induces a functor $f_0\colon \mathcal{C}_0 \to \mathcal{D}_0$. The composition $h_{\mathcal{D}}\circ f$ is a cohomological functor which preserves coproducts. Thus there exists a unique exact functor $f'\colon \operatorname{Mod}\mathcal{C}_0 \to \operatorname{Mod}\mathcal{D}_0$ commuting with coproducts and satisfying $h_{\mathcal{D}}\circ f = f'\circ h_{\mathcal{C}}$ by Corollary 2.4. We claim that $(f_0)^* = f'$. In fact, both functors are right exact, preserve coproducts, and coincide on the full subcategory of finitely generated projective objects in $\operatorname{Mod}\mathcal{C}_0$. The assertion follows since every object M in $\operatorname{Mod}\mathcal{C}_0$ has a projective presentation

$$\coprod_i H_{X_i} \xrightarrow{(\phi_{ij})} \coprod_j H_{Y_j} \longrightarrow M \to 0$$

with X_i and Y_j in $\mathcal{C}_0$ for all i and j.

To prove $h_{\mathcal{C}}\circ g = (f_0)_*\circ h_{\mathcal{D}}$, observe that for every C in $\mathcal{C}_0$ and X in $\mathcal{D}$ we have

$$(h_{\mathcal{C}}\circ g)(X)(C) = \operatorname{Hom}(C,g(X)) \simeq \operatorname{Hom}(f(C),X) = H_X(f(C)) = ((f_0)_*\circ h_{\mathcal{D}})(X)(C).$$

(2) The exactness of $(f_0)^*$ has already been noticed, and the restriction $(f_0)_*$ is automatically exact.

(3) The first assertion follows directly from the adjointness formula and the fact that f preserves compactness. The second assertion follows from the characterization of pure-injectivity in part (5) of Theorem 1.8, and the fact that g preserves products and coproducts. ☐

2.3. **Flat modules.** Let $\mathcal{C}$ be a skeletally small additive category. Recall that there exists a *tensor product*

$$\operatorname{Mod}\mathcal{C}\times\operatorname{Mod}\mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Ab}, \quad (M,N)\to M\otimes_{\mathcal{C}} N$$

where for any $\mathcal{C}$-module M, the tensor functor $M\otimes_{\mathcal{C}} -$ is determined by the fact that it preserves colimits and $M\otimes_{\mathcal{C}}\operatorname{Hom}(X,-)\simeq M(X)$ for all X in $\mathcal{C}$. Observe that the existence of such a tensor product is an immediate consequence of Lemma 2.2. A $\mathcal{C}$-module M is *flat* if the tensor functor $M\otimes_{\mathcal{C}} -$ exact, and we denote by $\operatorname{Flat}\mathcal{C}$ the full subcategory of flat $\mathcal{C}$-modules. Recall that a $\mathcal{C}$-module M is flat if and only if M is a filtered colimit of representable functors [22, Theorem 3.2]. Therefore $\operatorname{Flat}\mathcal{C}$ is equivalent to the category of *ind-objects* over $\mathcal{C}$ in the sense of Grothendieck and Verdier [11]. In particular, $\operatorname{Flat}\mathcal{C}$ is a category with filtered colimits, and every functor $f\colon \mathcal{C}\to\mathcal{D}$ into a category $\mathcal{D}$ with filtered colimits extends uniquely to a functor $f'\colon \operatorname{Flat}\mathcal{C}\to\mathcal{D}$ preserving filtered colimits and satisfying $f'(\operatorname{Hom}(-,X)) = f(X)$ for all X in $\mathcal{C}$.

Suppose now that $\mathcal{C}$ is triangulated. Then we have the following characterization of flat $\mathcal{C}$-modules which has been observed independently by Beligiannis [3].

Lemma 2.7. *The following are equivalent for an additive functor $M\colon \mathcal{C}^{\mathrm{op}}\to\mathrm{Ab}$:*

(1) *M is a flat $\mathcal{C}$-module;*
(2) *M is a cohomological functor;*
(3) *M is a fp-injective $\mathcal{C}$-module.*

Proof. (1) $\Leftrightarrow$ (2) M is flat if and only if the restriction $M \otimes_{\mathcal{C}} -|_{\operatorname{mod}\mathcal{C}^{\mathrm{op}}}$ is exact since every exact sequence in $\operatorname{Mod}\mathcal{C}^{\mathrm{op}}$ can be written as a filtered colimit of exact sequences in $\operatorname{mod}\mathcal{C}^{\mathrm{op}}$. Thus M is flat if and only if $M \otimes_{\mathcal{C}} -|_{\mathcal{C}^{\mathrm{op}}}$ is a cohomological functor by Lemma 2.1. The assertion now follows since $M \simeq M \otimes_{\mathcal{C}} -|_{\mathcal{C}^{\mathrm{op}}}$.

(2) $\Leftrightarrow$ (3) Use the argument from the proof of Lemma 1.6. $\square$

We combine the preceding lemma with our results about cohomological functors on compactly generated triangulated categories. Note that the following theorem generalizes a result of Christensen and Strickland in [8].

Theorem 2.8. *Let $\mathcal{C}$ be a compactly generated triangulated category. Then the following categories are pairwise equivalent:*

1. *the category of cohomological functors $\mathcal{C} \to \mathrm{Ab}$ which preserve coproducts;*
2. *the category of cohomological functors $\mathcal{C}_0 \to \mathrm{Ab}$;*
3. *the category of ind-objects over $(\mathcal{C}_0)^{\mathrm{op}}$.*

Proof. Combine Proposition 2.3 and Lemma 2.7. $\square$

The finitely presented modules over a ring Λ are characterized by the fact that the corresponding tensor functor $M \otimes_\Lambda -$ preserves arbitrary products of Λ^{op}-modules. In fact, it is sufficient to assume that $M \otimes_\Lambda -$ preserves products of finitely generated projective modules; e.g. see [25, Lemma I.13.2]. This result generalizes to rings with several objects and leads to a characterization of cohomological functors $\mathcal{C} \to \mathrm{Ab}$ which preserve products; it is the second part of Theorem E.

Proposition 2.9. *Let $\mathcal{C}$ be a compactly generated triangulated category. Then the following are equivalent for a cohomological functor $f \colon \mathcal{C} \to \mathrm{Ab}$ which preserves coproducts:*

1. *$f(\prod_i X_i) \simeq \prod_i f(X_i)$ for every family $(X_i)_{i\in I}$ of objects in $\mathcal{C}$;*
2. *$f(\prod_i X_i) \simeq \prod_i f(X_i)$ for every family $(X_i)_{i\in I}$ of compact objects in $\mathcal{C}$;*
3. *$f \simeq \operatorname{Hom}(C, -)$ for some compact object C in $\mathcal{C}$.*

Proof. The directions (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are clear. Therefore suppose that f preserves products of compact objects. The functor f extends uniquely to a colimit preserving functor $f' \colon \operatorname{Mod}\mathcal{C}_0 \to \mathrm{Ab}$ by Corollary 2.4. We have $f' \simeq - \otimes_{\mathcal{C}_0} M$ for $M = f|_{\mathcal{C}_0}$ and M is flat by Lemma 2.7. Moreover, M is finitely presented since f' preserves products of finitely generated projective $\mathcal{C}_0$-modules. Any flat module is finitely presented if and only if it is finitely generated projective (e.g. see [25, Corollary I.11.5]), and therefore $M \simeq \operatorname{Hom}(C, -)$ for some C in $\mathcal{C}_0$. We obtain

$$f(Y) = f'(H_Y) \simeq H_Y \otimes_{\mathcal{C}_0} \operatorname{Hom}(C, -) \simeq H_Y(C) = \operatorname{Hom}(C, Y)$$

for every Y in $\mathcal{C}$, and therefore $f \simeq \operatorname{Hom}(C, -)$. $\square$

2.4. **Pure-semisimplicity.** A compactly generated triangulated category $\mathcal{C}$ is *pure-semisimple* if every pure monomorphism in $\mathcal{C}$ splits; equivalently if every object in $\mathcal{C}$ is pure-injective. Our aim is a characterization of pure-semisimplicity, using the fact that this property is equivalent to a number of familiar properties of the module category $\operatorname{Mod}\mathcal{C}_0$. For instance, Bass has characterized the rings for which every flat module is projective. This can be generalized to rings with several objects and then describes when every flat $\mathcal{C}_0$-module is a projective $\mathcal{C}_0$-module, see [13, Theorem B.12]. On the other hand, noetherian rings can be characterized by the fact that every fp-injective module is injective. Moreover, Matlis showed that a ring is noetherian if and only if every injective module is a coproduct of indecomposable modules. These results generalize to rings with several objects as well, see [13, Theorem B.17]. We obtain therefore the following characterization of pure-semisimplicity, since the restricted Yoneda functor $\mathcal{C} \to \operatorname{Mod} \mathcal{C}_0$ identifies every object in $\mathcal{C}$ with a $\mathcal{C}_0$-module which is flat and fp-injective by Lemma 1.6 and Lemma 2.7.

Theorem 2.10. *The following are equivalent for a compactly generated triangulated category $\mathcal{C}$:*

1. *$\mathcal{C}$ is pure-semisimple;*
2. *every object in $\mathcal{C}$ is a coproduct of indecomposable objects with local endomorphism rings;*
3. *every compact object is a finite coproduct of indecomposable objects with local endomorphism rings, and, given a sequence*
$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} \dots$$
of non-isomorphisms between indecomposable compact objects, the composition $\phi_n \circ \dots \circ \phi_2 \circ \phi_1$ is zero for n sufficiently large;
4. *the restricted Yoneda functor $h_{\mathcal{C}} \colon \mathcal{C} \to \operatorname{Mod} \mathcal{C}_0$, $X \mapsto \operatorname{Hom}(-, X)|_{\mathcal{C}_0}$, is fully faithful;*
5. *$\mathcal{C}$ has filtered colimits.*

This characterization, and indeed a host of other equivalent statements have been obtained independently by Beligiannis in [4].

3. Localization

3.1. **Cohomological ideals.** Let $\mathcal{C}$ be an additive category. An *ideal* $\mathfrak{I}$ in $\mathcal{C}$ consists of subgroups $\mathfrak{I}(X, Y)$ in $\operatorname{Hom}(X, Y)$ for every pair of objects X, Y in $\mathcal{C}$ such that for all ϕ in $\mathfrak{I}(X, Y)$ and all maps $\alpha \colon X' \to X$ and $\beta \colon Y \to Y'$ in $\mathcal{C}$ the composition $\beta \circ \phi \circ \alpha$ belongs to $\mathfrak{I}(X', Y')$.

Definition 3.1. An ideal $\mathfrak{I}$ in a triangulated category $\mathcal{C}$ is called *cohomological* if there exists a cohomological functor $f \colon \mathcal{C} \to \mathcal{A}$ into an abelian category $\mathcal{A}$ such that $\mathfrak{I} = \{\phi \in \mathcal{C} \mid f(\phi) = 0\}$.

Given an ideal $\mathfrak{I}$ in $\mathcal{C}$, we denote by $\mathcal{S}_{\mathfrak{I}}$ the full subcategory of objects M in $\operatorname{mod} \mathcal{C}$ such that $M \simeq \operatorname{Im} H_\phi$ for some ϕ in $\mathfrak{I}$. If $\mathcal{C}$ is skeletally small, then $\mathcal{T}_{\mathfrak{I}}$ denotes the full subcategory of filtered colimits $\varinjlim M_i$ in $\operatorname{Mod} \mathcal{C}$ such that M_i belongs to $\mathcal{S}_{\mathfrak{I}}$ for all i. Recall that a full subcategory $\mathcal{S}$ of an abelian category $\mathcal{A}$ is a *Serre subcategory* provided that for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ the object X belongs to $\mathcal{S}$ if and only if X' and X'' belong to $\mathcal{S}$.

Lemma 3.2. *Let $\mathfrak{I}$ be a cohomological ideal in a triangulated category $\mathcal{C}$.*

1. *$\mathcal{S}_{\mathfrak{I}}$ is a Serre subcategory of* $\operatorname{mod} \mathcal{C}$.
2. *If $\mathcal{C}$ is skeletally small, then $\mathcal{T}_{\mathfrak{I}}$ is a Serre subcategory of* $\operatorname{Mod} \mathcal{C}$.

Proof. (1) Let $f \colon \mathcal{C} \to \mathcal{A}$ be a cohomological functor such that $\mathfrak{I} = \{\phi \in \mathcal{C} \mid f(\phi) = 0\}$, and denote by $f' \colon \operatorname{mod} \mathcal{C} \to \mathcal{A}$ the exact functor extending f which exists by Lemma 2.1. The full subcategory $\mathcal{S} = \{M \in \operatorname{mod} \mathcal{C} \mid f'(M) = 0\}$ is a Serre subcategory of $\operatorname{mod} \mathcal{C}$ since f' is exact. Now observe that every finitely presented $\mathcal{C}$-module M with projective presentation $H_X \to H_Y \to M \to 0$ is isomorphic to $\operatorname{Im} H_\phi$ where ϕ is the map occuring in the triangle $X \to Y \xrightarrow{\phi} Z \to X[1]$. Given an arbitrary map ϕ in $\mathcal{C}$, we have $f(\phi) = 0$

if and only if $f(\operatorname{Im} H_\phi) = 0$, and therefore $\mathcal{S}_{\mathfrak{I}} = \mathcal{S}$. Thus $\mathcal{S}_{\mathfrak{I}}$ is a Serre subcategory of $\operatorname{mod} \mathcal{C}$.

(2) See [15, Theorem 2.8]. □

Let $f\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. We denote by $f^*\colon \operatorname{mod} \mathcal{C} \to \operatorname{mod} \mathcal{D}$ the unique right exact functor which sends $\operatorname{Hom}(-, X)$ to $\operatorname{Hom}(-, f(X))$ for all X in $\mathcal{C}$. If $\mathcal{C}$ and $\mathcal{D}$ are skeletally small, then f^* extends uniquely to a colimit preserving functor $\operatorname{Mod} \mathcal{C} \to \operatorname{Mod} \mathcal{D}$ which we also denote by f^*.

Lemma 3.3. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. Then $\mathfrak{I} = \{\phi \in \mathcal{C} \mid f(\phi) = 0\}$ is a cohomological ideal in $\mathcal{C}$. Moreover, the following holds:*

(1) *$\mathcal{S}_{\mathfrak{I}} = \{M \in \operatorname{mod} \mathcal{C} \mid f^*(M) = 0\}$.*
(2) *If $\mathcal{C}$ and $\mathcal{D}$ are skeletally small, then $\mathcal{T}_{\mathfrak{I}} = \{M \in \operatorname{Mod} \mathcal{C} \mid f^*(M) = 0\}$.*

Proof. Let $f'\colon \mathcal{C} \to \operatorname{mod} \mathcal{D}$ be the composition of f with the Yoneda functor $\mathcal{D} \to \operatorname{mod} \mathcal{D}$. This functor is cohomological, and $f(\phi) = 0$ if and only if $f'(\phi) = 0$ for every map $\phi \in \mathcal{C}$ since the Yoneda functor is faithful. Thus $\mathfrak{I}$ is a cohomological ideal.

(1) The functor $f^*\colon \operatorname{mod} \mathcal{C} \to \operatorname{mod} \mathcal{D}$ is the unique exact functor extending f'. Therefore $\mathcal{S}_{\mathfrak{I}} = \{M \in \operatorname{mod} \mathcal{C} \mid f^*(M) = 0\}$ by the argument given in the proof of Lemma 3.2.

(2) We denote by $\mathcal{T}$ the full subcategory of $\mathcal{C}$-modules M such that $f^*(M) = 0$. It follows from (1) that $\mathcal{T}_{\mathfrak{I}} \subseteq \mathcal{T}$ since f^* preserves filtered colimits. To prove the other inclusion, we use the right adjoint $f_*\colon \operatorname{Mod} \mathcal{D} \to \operatorname{Mod} \mathcal{C}$, $M \mapsto M \circ f$ for f^*. We denote by $t\colon \operatorname{Mod} \mathcal{C} \to \operatorname{Mod} \mathcal{C}$ the functor which is obtained from the functorial exact sequence

$$0 \longrightarrow t(M) \longrightarrow M \xrightarrow{\mu_M} (f_* \circ f^*)(M).$$

Note that t induces a right adjoint for the inclusion $\mathcal{T} \to \operatorname{Mod} \mathcal{C}$ since $f^*(\mu_M)$ is an isomorphism for all M. Moreover, t preserves filtered colimits since f^* and f_* have this property. Now let $M \in \operatorname{mod} \mathcal{C}$, and write $t(M) = \varinjlim M_i$ as a filtered colimit of finitely generated submodules. For all i, we have $M_i \in \mathcal{T}$ since $\mathcal{T}$ is closed under taking submodules, and $M_i \in \operatorname{mod} \mathcal{C}$ since $\mathcal{C}$ has weak kernels and therefore finitely generated submodules of finitely presented modules are again finitely presented. It follows that $t(M)$ is a filtered colimit of modules in $\mathcal{S} = \mathcal{T} \cap \operatorname{mod} \mathcal{C}$. Given any module M in $\mathcal{T}$, we can write $M = \varinjlim M_i$ as a filtered colimit of finitely presented modules. Thus $M = t(\varinjlim M_i) \simeq \varinjlim t(M_i)$ is a filtered colimit of modules in $\mathcal{S}$, and $\mathcal{T} \subseteq \mathcal{T}_{\mathfrak{I}}$ follows since $\mathcal{S} = \mathcal{S}_{\mathfrak{I}}$ by (1). □

3.2. **Localization for triangulated categories.** Let $\mathcal{C}$ be a compactly generated triangulated category. Recall that a full triangulated subcategory $\mathcal{B}$ of $\mathcal{C}$ is *localizing* if $\mathcal{B}$ is closed under taking coproducts. The *quotient category* $\mathcal{C}/\mathcal{B}$ is, by definition, the category of fractions $\mathcal{C}[\Sigma^{-1}]$ (in the sense of [10]) with respect to the class Σ of maps $Y \to Z$ which admit a triangle $X \to Y \to Z \to X[1]$ with X in $\mathcal{B}$. Thus the corresponding *quotient functor* $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ is the universal functor which inverts every map in Σ. Note that $\mathcal{C}[\Sigma^{-1}]$ is a *large* category which means that the maps between fixed objects are not assumed to form a set. Let us mention a few basic facts about the formation of the quotient category $\mathcal{C}/\mathcal{B}$ which we shall use frequently without further reference.

Lemma 3.4. *The quotient functor $f\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ has the following properties:*

(1) *The triangulation of $\mathcal{C}$ induces a triangulation for $\mathcal{C}/\mathcal{B}$ and f is an exact functor.*
(2) *Let X be an object in $\mathcal{C}$. Then $f(X) = 0$ if and only if $X \in \mathcal{B}$.*
(3) *Let ϕ be a map in $\mathcal{C}$. Then $f(\phi) = 0$ if and only if ϕ factors through some object in $\mathcal{B}$.*

Proof. See [27, Corollaire 2.2.11]. □

The following lemma characterizes the existence of a right adjoint for the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$.

Lemma 3.5. *Let $\mathcal{B}$ be a localizing subcategory of a compactly generated triangulated category $\mathcal{C}$. Then the following are equivalent:*

(1) *the maps between fixed objects in $\mathcal{C}/\mathcal{B}$ form a set;*
(2) *the quotient functor $f\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ has a right adjoint $g\colon \mathcal{C}/\mathcal{B} \to \mathcal{C}$;*
(3) *the inclusion functor $\mathcal{B} \to \mathcal{C}$ has a right adjoint $e\colon \mathcal{C} \to \mathcal{B}$.*

Moreover, in this case there is for every object X in $\mathcal{C}$ a triangle

$$(g \circ f)(X)[-1] \xrightarrow{\alpha_X} e(X) \xrightarrow{\beta_X} X \xrightarrow{\gamma_X} (g \circ f)(X)$$

which is functorial in X.

A localizing subcategory $\mathcal{B}$ which satisfies the equivalent conditions of the preceding lemma admits a *localization functor* $\mathcal{C} \to \mathcal{C}$ which is, by definition, the composition of the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$ with a right adjoint $\mathcal{C}/\mathcal{B} \to \mathcal{C}$. To prove Lemma 3.5 we shall need the following lemma about $\mathcal{C}[\Sigma^{-1}]$.

Lemma 3.6. *Let $\mathcal{C}$ be any category with coproducts. Suppose that Σ is a class of maps in $\mathcal{C}$ which admits a calculus of left fractions. If $\coprod_i \sigma_i \in \Sigma$ for every family $(\sigma_i)_{i\in I}$ in Σ, then the quotient category $\mathcal{C}[\Sigma^{-1}]$ has coproducts and the quotient functor $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves coproducts.*

Proof. Recall from [10] that the objects in $\mathcal{C}[\Sigma^{-1}]$ are those of $\mathcal{C}$, and that the maps $X \to Y$ in $\mathcal{C}[\Sigma^{-1}]$ are equivalence classes of *left fractions* $X \overset{\phi}{\to} Z \overset{\sigma}{\leftarrow} Y$ with $\sigma \in \Sigma$. Now let $(X_i)_{i\in I}$ be a family of objects in $\mathcal{C}[\Sigma^{-1}]$. We claim that the coproduct $\coprod_i X_i$ in $\mathcal{C}$ is also a coproduct in $\mathcal{C}[\Sigma^{-1}]$. Thus we need to show that for every object Y, the canonical map $\alpha\colon \operatorname{Hom}(\coprod_i X_i, Y) \to \prod_i \operatorname{Hom}(X_i, Y)$ between Hom-sets in $\mathcal{C}[\Sigma^{-1}]$ is bijective.

To check surjectivity, let $(X_i \overset{\phi_i}{\to} Z_i \overset{\sigma_i}{\leftarrow} Y)_{i\in I}$ be a family of left fractions. We obtain a commutative diagram

$$\begin{array}{ccccc} \coprod_i X_i & \xrightarrow{\coprod_i \phi_i} & \coprod_i Z_i & \xleftarrow{\coprod_i \sigma_i} & \coprod_i Y \\ & & \downarrow & & \downarrow{\scriptstyle \pi_Y} \\ & & Z & \xleftarrow{\sigma} & Y \end{array}$$

where $\pi_Y\colon \coprod_i Y \to Y$ is the summation map and $\sigma \in \Sigma$. It is easily checked that

$$(X_i \to Z \overset{\sigma}{\leftarrow} Y) \sim (X_i \overset{\phi_i}{\to} Z_i \overset{\sigma_i}{\leftarrow} Y)$$

for all $i \in I$, and therefore α sends $\coprod_i X_i \to Z \overset{\sigma}{\leftarrow} Y$ to the family $(X_i \overset{\phi_i}{\to} Z_i \overset{\sigma_i}{\leftarrow} Y_i)_{i\in I}$.

To check injectivity, let $\coprod_i X_i \overset{\phi'}{\to} Z' \overset{\sigma'}{\leftarrow} Y$ and $\coprod_i X_i \overset{\phi''}{\to} Z'' \overset{\sigma''}{\leftarrow} Y$ be left fraction such that

$$(X_i \overset{\phi'_i}{\to} Z' \overset{\sigma'}{\leftarrow} Y) \sim (X_i \overset{\phi''_i}{\to} Z'' \overset{\sigma''}{\leftarrow} Y)$$

for all i. We may assume that $Z' = Z = Z''$ and $\sigma' = \sigma = \sigma''$ since we can choose maps $\tau'\colon Z' \to Z$ and $\tau''\colon Z'' \to Z$ with $\tau' \circ \sigma' = \tau'' \circ \sigma'' \in \Sigma$. Thus there are maps $\psi_i\colon Z \to Z_i$ with $\psi_i \circ \phi'_i = \psi_i \circ \phi''_i$ and $\psi_i \circ \sigma \in \Sigma$ for all i. Each ψ_i belongs to the *saturation* $\overline{\Sigma}$ of Σ which is the class of all maps in $\mathcal{C}$ which become an isomorphism in $\mathcal{C}[\Sigma^{-1}]$. Note that a map α in $\mathcal{C}$ belongs to $\overline{\Sigma}$ if and only if there are maps α' and α'' such that $\alpha \circ \alpha'$ and

$\alpha''\circ\alpha$ belong to Σ. Therefore $\overline{\Sigma}$ is also closed under taking coproducts. Moreover, $\overline{\Sigma}$ admits a calculus of left fractions, and we obtain therefore a commutative diagram

$$\begin{array}{ccccccc}
\coprod_i X_i & \longrightarrow & \coprod_i Z & \xrightarrow{\pi_Z} & Z & \xleftarrow{\sigma} & Y \\
 & & \downarrow{\scriptstyle \coprod_i \psi_i} & & \downarrow{\scriptstyle\tau} & & \\
 & & \coprod_i Z_i & \longrightarrow & Z^* & &
\end{array}$$

with $\tau\in\overline{\Sigma}$. Thus $\tau\circ\sigma\in\overline{\Sigma}$, and we have

$$(\coprod_i X_i \xrightarrow{\phi'} Z \xleftarrow{\sigma} Y) \sim (\coprod_i X_i \xrightarrow{\phi''} Z \xleftarrow{\sigma} Y)$$

since $\pi_Z\circ\coprod_i\phi_i' = \phi'$ and $\pi_Z\circ\coprod_i\phi_i'' = \phi'$. Therefore α is also injective, and this completes the proof. $\square$

Proof of Lemma 3.5. (1) $\Rightarrow$ (2) The quotient functor preserves coproducts by Lemma 3.6, since Σ is closed under taking coproducts. Given an object X in $\mathcal{C}/\mathcal{B}$, the composition $\operatorname{Hom}(-,X)\circ f$ is a cohomological functor which sends coproducts to products. Thus there exists Y in $\mathcal{C}$ with $\operatorname{Hom}(-,X)\circ f\simeq\operatorname{Hom}(-,Y)$ by the Brown representability theorem. We put $g(X)=Y$, and it is easily checked that this gives a right adjoint $g\colon\mathcal{C}/\mathcal{B}\to\mathcal{C}$ for f.

(2) $\Rightarrow$ (1) Let $X=f(X')$ and Y be objects in $\mathcal{C}/\mathcal{B}$. Then $\operatorname{Hom}(X,Y)\simeq\operatorname{Hom}(X',g(Y))$ since g is a right adjoint of f. Thus the maps between objects in $\mathcal{C}/\mathcal{B}$ form a set.

(2) $\Rightarrow$ (3) Suppose that f has a right adjoint g. Completing the canonical map $\gamma_X\colon X\to(g\circ f)(X)$ to a triangle

$$(g\circ f)(X)[-1]\longrightarrow Y\longrightarrow X\xrightarrow{\gamma_X}(g\circ f)(X)$$

for every X in $\mathcal{C}$ gives a functor $e\colon\mathcal{C}\to\mathcal{B}$ if we put $e(X)=Y$. In fact, $f(\gamma_X)$ is an isomorphism and therefore $f(Y)=0$ which implies $Y\in\mathcal{B}$. Given $Y'\in\mathcal{B}$, one applies $\operatorname{Hom}(Y',-)$ to the above triangle and gets an isomorphism $\operatorname{Hom}(Y',Y)\to\operatorname{Hom}(Y',X)$. Thus e is a right adjoint for the inclusion $\mathcal{B}\to\mathcal{C}$.

(3) $\Rightarrow$ (2) Suppose that the inclusion $\mathcal{B}\to\mathcal{C}$ has a right adjoint e, and let $X=f(X')$ be an object in $\mathcal{C}/\mathcal{B}$. Completing the canonical map $\beta_{X'}\colon e(X')\to X'$ to a triangle

$$Y[-1]\longrightarrow e(X')\xrightarrow{\beta_{X'}}X'\longrightarrow Y$$

gives a functor $g\colon\mathcal{C}/\mathcal{B}\to\mathcal{C}$ if we put $g(X)=Y$. It is not hard to check that this defines a right adjoint for the quotient functor $\mathcal{C}\to\mathcal{C}/\mathcal{B}$, but we leave the details to the reader.

The last assertion is an immediate consequence of the construction given in (2) $\Rightarrow$ (3). $\square$

We continue with a series of lemmas which collect some basic properties of the quotient functor and its right adjoint, assuming that it exists. The notation of Lemma 3.5 remains fixed.

Lemma 3.7. *The natural transformation* $\mathrm{id}_{\mathcal{C}}\to g\circ f$ *induces a functorial isomorphism* $\operatorname{Hom}((g\circ f)(X),Y)\to\operatorname{Hom}(X,Y)$ *for all* X *and* Y *such that* $\operatorname{Hom}(\mathcal{B},Y)=0$.

Proof. Apply $\operatorname{Hom}(-,Y)$ to the triangle in Lemma 3.5. $\square$

Given any class $\mathcal{B}$ of objects in $\mathcal{C}$, we say that an object Y in $\mathcal{C}$ is *$\mathcal{B}$-local* if $\operatorname{Hom}(X,Y)=0$ for all X in $\mathcal{B}$. The full subcategory of $\mathcal{B}$-local objects is denoted by $\mathcal{B}^{\perp}$. The definition of *$\mathfrak{I}$-local* objects for a class $\mathfrak{I}$ of maps in $\mathcal{C}$ is analogous.

Lemma 3.8. *The functor* g *induces an equivalence between* $\mathcal{C}/\mathcal{B}$ *and* $\mathcal{B}^{\perp}$.

Proof. An inverse is the composition of the inclusion $\mathcal{B}^\perp \to \mathcal{C}$ with the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$; use Lemma 3.5. $\square$

3.3. **Cohomological ideals and localization.** Let $\mathcal{C}$ be a compactly generated triangulated category. Given an ideal $\mathfrak{I}$ in $\mathcal{C}_0$, we define a full subcategory $\mathcal{C}_\mathfrak{I}$ as follows:

$$\mathcal{C}_\mathfrak{I} = \{X \in \mathcal{C} \mid \text{every map } C \to X,\ C \in \mathcal{C}_0, \text{ factors through a map } C \to D \text{ in } \mathfrak{I}\}.$$

Given a full additive subcategory $\mathcal{B}$ of $\mathcal{C}$, we define an ideal $\mathfrak{I}_\mathcal{B}$ as follows:

$$\mathfrak{I}_\mathcal{B} = \{\phi \in \mathcal{C}_0 \mid \phi \text{ factors through an object in } \mathcal{B}\}.$$

We are interested in properties of the category $\mathcal{C}_\mathfrak{I}$ and collect them in two technical lemmas.

Lemma 3.9. *Let $\mathfrak{I}$ be an ideal in $\mathcal{C}_0$ and $X \in \mathcal{C}$. Then $X \in \mathcal{C}_\mathfrak{I}$ if and only if $H_X \in \mathcal{T}_\mathfrak{I}$.*

Proof. Suppose first that $H_X \in \mathcal{T}_\mathfrak{I}$. Thus $H_X = \varinjlim \operatorname{Im} H_{\phi_i}$ with $\phi_i \in \mathfrak{I}$ for all i. Now let $\phi\colon C \to X$ be any map with $C \in \mathcal{C}_0$. We have $\operatorname{Hom}(C, X) = H_X(C) = \varinjlim \operatorname{Im} H_{\phi_i}(C)$ and obtain therefore a factorization $C \to C_i \xrightarrow{\phi_i} D_i \to X$ of ϕ for some i. The composition $C \to C_i \xrightarrow{\phi_i} D_i$ belongs to $\mathfrak{I}$ since $\phi_i \in \mathfrak{I}$, and this implies $X \in \mathcal{C}_\mathfrak{I}$.

To prove the converse, suppose that $X \in \mathcal{C}_\mathfrak{I}$. Every module is a filtered colimit of finitely presented ones. More precisely, $H_X = \varinjlim_{i \in I} M_i$ where I denotes the filtered category of maps $\mu_i\colon M_i \to H_X$ with $M_i \in \operatorname{mod} \mathcal{C}_0$. We claim that the full subcategory J of maps $\mu_i\colon M_i \to X_X$ with $M_i \in \mathcal{S}_\mathfrak{I}$ is *cofinal*, i.e., for every $i \in I$ there exists a map $\nu\colon M_i \to M_j$ for some $j \in J$ such that $\mu_i = \mu_j \circ \nu$. To prove this claim, let $\phi_i\colon C_i \to D_i$ be a map in $\mathcal{C}_0$ with $M_i \simeq \operatorname{Im} H_{\phi_i}$ which exists by the argument given in the proof of Lemma 3.2. We get a factorization $M_i \to H_{D_i} \to H_X$ of μ_i since H_X is fp-injective by Lemma 1.6, and the corresponding map $D_i \to X$ has a factorization $D_i \to E \to X$ for some $\psi\colon D_i \to E$ in $\mathfrak{I}$ since $X \in \mathcal{C}_\mathfrak{I}$. Thus μ_i factors through the map $\operatorname{Im} H_\psi \to H_X$ with $\operatorname{Im} H_\psi \in \mathcal{S}_\mathfrak{I}$. Therefore J is cofinal in I, and the inclusion $J \to I$ induces an isomorphism $\varinjlim_{i \in J} M_i \simeq \varinjlim_{i \in I} M_i \simeq H_X$ which proves $H_X \in \mathcal{T}_\mathfrak{I}$. $\square$

Lemma 3.10. *Let $\mathfrak{I}$ be a cohomological ideal in $\mathcal{C}_0$ such that $\phi[n] \in \mathfrak{I}$ for all $\phi \in \mathfrak{I}$ and $n \in \mathbb{Z}$.*

1. *$\mathcal{C}_\mathfrak{I}$ is a localizing subcategory of $\mathcal{C}$.*
2. *If $\mathfrak{I} = \mathfrak{I}_{(\mathcal{C}_\mathfrak{I})}$, then the inclusion $\mathcal{C}_\mathfrak{I} \to \mathcal{C}$ has a right adjoint and $(\mathcal{C}_\mathfrak{I})^\perp = \mathfrak{I}^\perp$.*

Proof. (1) Clearly, $\mathcal{C}_\mathfrak{I}$ is closed under the shift in $\mathcal{C}$ since $\mathfrak{I}$ is closed under the shift. Now let $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\chi} X[1]$ be a triangle in $\mathcal{C}$ with $X, Y \in \mathcal{C}_\mathfrak{I}$. We need to show that $Z \in \mathcal{C}_\mathfrak{I}$. We apply the description of $\mathcal{C}_\mathfrak{I}$ given in Lemma 3.9. The triangle induces an exact sequence $0 \to \operatorname{Im} H_\psi \to H_Z \to \operatorname{Im} H_\chi \to 0$ in $\operatorname{Mod} \mathcal{C}_0$. The category $\mathcal{T}_\mathfrak{I}$ is a Serre subcategory of $\operatorname{Mod} \mathcal{C}_0$ by Lemma 3.2 since $\mathfrak{I}$ is cohomological, and therefore H_Z belongs to $\mathcal{T}_\mathfrak{I}$. Thus $Z \in \mathcal{C}_\mathfrak{I}$. Furthermore, $\mathcal{C}_\mathfrak{I}$ is closed under taking coproducts because $\mathcal{T}_\mathfrak{I}$ has this property, and we conclude that $\mathcal{C}_\mathfrak{I}$ is localizing.

(2) In order to show that the inclusion $\mathcal{C}_\mathfrak{I} \to \mathcal{C}$ has a right adjoint, it is by Lemma 3.5 sufficient to show that for two objects X and Y in $\mathcal{C}$ the maps $X \to Y$ in $\mathcal{C}/\mathcal{C}_\mathfrak{I}$ form a set. In fact, it is sufficient to check this for all $X \in \mathcal{C}_0$ and $Y \in \mathcal{C}$ since $\mathcal{C}_0$ generates $\mathcal{C}$. To prove this claim, we consider the exact quotient functor $q\colon \operatorname{Mod} \mathcal{C}_0 \to \operatorname{Mod} \mathcal{C}_0/\mathcal{T}_\mathfrak{I}$ with respect to the Serre subcategory $\mathcal{T}_\mathfrak{I}$ and observe that the maps in $\operatorname{Mod} \mathcal{C}_0/\mathcal{T}_\mathfrak{I}$ form a set [9, Proposition III.8]. The composition of q with the Yoneda functor $h\colon \mathcal{C} \to$

$\operatorname{Mod}\mathcal{C}_0$ annihilates $\mathcal{C}_{\mathfrak{I}}$ by Lemma 3.9, and therefore $q\circ h$ induces a cohomological functor $h'\colon \mathcal{C}/\mathcal{C}_{\mathfrak{I}} \to \operatorname{Mod}\mathcal{C}_0/\mathcal{T}_{\mathfrak{I}}$ making the following diagram of functors commutative:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}/\mathcal{C}_{\mathfrak{I}} \\ \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle h'} \\ \operatorname{Mod}\mathcal{C}_0 & \xrightarrow{q} & \operatorname{Mod}\mathcal{C}_0/\mathcal{T}_{\mathfrak{I}} \end{array}$$

We claim that h' induces an injective map $\operatorname{Hom}(X,Y)\to\operatorname{Hom}(h'(X),h'(Y))$ for all $X\in\mathcal{C}_0$ and $Y\in\mathcal{C}$. To this end choose a map $\alpha\colon X\to Y$ in $\mathcal{C}/\mathcal{C}_{\mathfrak{I}}$ which is by definition a left fraction $X\xrightarrow{\phi} Z \xleftarrow{\sigma} Y$, and assume that $h'(\alpha)=0$. It follows that $q(H_\phi)=0$ since $q(H_\sigma)$ is an isomorphism, and therefore $\operatorname{Im}H_\phi\in\mathcal{T}_{\mathfrak{I}}$ since q is exact. Thus $\operatorname{Im}H_\phi$ is a filtered colimit of objects in $\mathcal{S}_{\mathfrak{I}}$. An argument similar to that given in the proof of Lemma 3.9 shows that the map $\phi\colon X\to Z$ factors through a map $\psi\colon X\to X'$ in $\mathfrak{I}$ since $X\in\mathcal{C}_0$. Thus ϕ factors through an object in $\mathcal{C}_{\mathfrak{I}}$ by our assumption on $\mathfrak{I}$, and therefore $\alpha=0$. We conclude that the maps between fixed objects in $\mathcal{C}/\mathcal{C}_{\mathfrak{I}}$ form a set, and therefore the inclusion $\mathcal{C}_{\mathfrak{I}}\to\mathcal{C}$ has a right adjoint.

It remains to show that $(\mathcal{C}_{\mathfrak{I}})^\perp=\mathfrak{I}^\perp$. Clearly, $(\mathcal{C}_{\mathfrak{I}})^\perp\subseteq\mathfrak{I}^\perp$ since every map in $\mathfrak{I}$ factors through an object in $\mathcal{C}_{\mathfrak{I}}$. To prove the other inclusion, let $\mathcal{B}=\mathcal{C}_{\mathfrak{I}}$ and consider for any object X in $\mathcal{C}$ the triangle

$$(g\circ f)(X)[-1]\xrightarrow{\alpha_X} e(X)\xrightarrow{\beta_X} X\xrightarrow{\gamma_X}(g\circ f)(X)$$

as in Lemma 3.5. Now suppose that $\operatorname{Hom}(\mathfrak{I},X)=0$. Every map $\phi\colon C\to e(X)$ with C in $\mathcal{C}_0$ has a factorization $C\xrightarrow{\phi'} D\xrightarrow{\phi''} e(X)$ with $\phi'\in\mathfrak{I}$, and therefore $\beta_X\circ\phi=0$. Thus ϕ factors through α_X. The same argument shows that ϕ'' factors through α_X, and therefore $\phi=0$ since $\operatorname{Hom}(\phi',(g\circ f)(X)[-1])=0$ by Lemma 3.7. We conclude that $e(X)=0$ and therefore $\operatorname{Hom}(\mathcal{C}_{\mathfrak{I}},X)=0$. ◻

3.4. **Approximations.** We need to recall the following definition from [2]. Let $\mathcal{Y}$ be a class of objects in a category $\mathcal{C}$. Then a map $X\to Y$ is a *left $\mathcal{Y}$-approximation* of X if Y belongs to $\mathcal{Y}$ and if the induced map $\operatorname{Hom}(Y,Y')\to\operatorname{Hom}(X,Y')$ is surjective for every Y' in $\mathcal{Y}$. For example, if we view $\mathcal{Y}$ as a full subcategory of $\mathcal{C}$ and assume the existence of a left adjoint $f\colon\mathcal{C}\to\mathcal{Y}$ for the inclusion $\mathcal{Y}\to\mathcal{C}$, then the canonical map $X\to f(X)$ is a left $\mathcal{Y}$-approximation for every X in $\mathcal{C}$. In general, a left $\mathcal{Y}$-approximation is far from being unique.

Suppose now that $\mathfrak{I}$ is a class of maps in a triangulated category $\mathcal{C}$ such that their isomorphism classes form a set. Recall that $\mathfrak{I}^\perp$ denotes the full subcategory of objects X in $\mathcal{C}$ satisfying $\operatorname{Hom}(\phi,X)=0$ for all $\phi\in\mathfrak{I}$. We construct for any object X in $\mathcal{C}$ a left $\mathfrak{I}^\perp$-approximation $\gamma_{X,\mathfrak{I}^\perp}\colon X\to X_{\mathfrak{I}^\perp}$. To this end we define inductively maps $\alpha_n\colon X_n\to X_{n+1}$ for every $n\geq 0$. By definition, set $X_0=X$. Let Ψ_n be a representative set of non-zero maps $\psi\colon C\to X_n$ which factor through some map $C\to D$ in $\mathfrak{I}$. We obtain α_n if we complete the canonical map $\coprod_{\psi\in\Psi_n}C\to X_n$ to a triangle

$$\coprod_{\psi\in\Psi_n}C\longrightarrow X_n\xrightarrow{\alpha_n}X_{n+1}\longrightarrow(\coprod_{\psi\in\Psi_n}C)[1].$$

We denote by $X_{\mathfrak{I}^\perp}$ the homotopy colimit $\operatorname{hocolim}X_n$ of the sequence

$$X=X_0\xrightarrow{\alpha_0}X_1\xrightarrow{\alpha_1}X_2\xrightarrow{\alpha_2}\dots.$$

More precisely, $X_{\mathfrak{I}^\perp}$ is obtained from the triangle

$$\coprod_n X_n \xrightarrow{\mathrm{id}-\alpha} \coprod_n X_n \longrightarrow \operatorname{hocolim} X_n \longrightarrow (\coprod_n X_n)[1].$$

We denote by $\gamma_{X,\mathfrak{I}^\perp}\colon X \to X_{\mathfrak{I}^\perp}$ the canonical map from X_0 into $\operatorname{hocolim} X_n$, but this map is only unique up to a non-unique isomorphism since the construction involves the completion of various maps to triangles. Given any map $\psi\colon X \to Y$ in $\mathcal{C}$, we obtain a sequence of commuting diagrams

$$\begin{array}{ccccccccc} X & \xrightarrow{\mathrm{id}} & X_0 & \xrightarrow{\alpha_0} & X_1 & \xrightarrow{\alpha_1} & X_2 & \xrightarrow{\alpha_2} & \cdots \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi_0} & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_2} & & \\ Y & \xrightarrow{\mathrm{id}} & Y_0 & \xrightarrow{\beta_0} & Y_1 & \xrightarrow{\beta_1} & Y_2 & \xrightarrow{\beta_2} & \cdots \end{array}$$

and we denote by $\psi_{\mathfrak{I}^\perp}\colon X_{\mathfrak{I}^\perp} \to Y_{\mathfrak{I}^\perp}$ a map which makes the following diagram commutative

$$\begin{array}{ccccccc} \coprod_n X_n & \xrightarrow{\mathrm{id}-\alpha} & \coprod X_n & \longrightarrow & \operatorname{hocolim} X_n & \longrightarrow & (\coprod_n X_n)[1] \\ \downarrow{\scriptstyle\coprod\psi_n} & & \downarrow{\scriptstyle\coprod\psi_n} & & \downarrow{\scriptstyle\psi_{\mathfrak{I}^\perp}} & & \downarrow{\scriptstyle(\coprod\psi_n)[1]} \\ \coprod_n Y_n & \xrightarrow{\mathrm{id}-\beta} & \coprod Y_n & \longrightarrow & \operatorname{hocolim} Y_n & \longrightarrow & (\coprod_n Y_n)[1] \end{array}$$

Note that the map $\psi_{\mathfrak{I}^\perp}$ is not unique.

Proposition 3.11. *Let $\mathcal{C}$ be a triangulated category and suppose that $\mathfrak{I}$ is a class of maps between compact objects in $\mathcal{C}$ such that their isomorphism classes form a set. Then the map $\gamma_{X,\mathfrak{I}^\perp}\colon X \to X_{\mathfrak{I}^\perp}$ is a left $\mathfrak{I}^\perp$-approximation.*

Proof. We need to show that $X_{\mathfrak{I}^\perp}$ is $\mathfrak{I}$-local. To this end let $\phi\colon C \to D$ be a map in $\mathfrak{I}$. The canonical maps $\xi_n\colon X_n \to \operatorname{hocolim} X_n$ induce an isomorphism

$$\varinjlim \operatorname{Hom}(D, X_n) \longrightarrow \operatorname{Hom}(D, \operatorname{hocolim} X_n)$$

since D is compact; e.g. see [19, Lemma 1.5]. Therefore any map $\psi\colon D \to X_{\mathfrak{I}^\perp}$ has a factorization $\psi = \xi_n \circ \psi'$ for some $n \in \mathbb{N}$. The construction of α_n implies $\alpha_n \circ \psi' \circ \phi = 0$, and therefore $\psi \circ \phi = 0$. Thus $X_{\mathfrak{I}^\perp}$ is $\mathfrak{I}$-local. We have $Y_{\mathfrak{I}^\perp} = Y$ for every $\mathfrak{I}$-local object Y, and therefore every map $\psi\colon X \to Y$ with $Y \in \mathfrak{I}^\perp$ factors through $\gamma_{X,\mathfrak{I}^\perp}$ via $\psi_{\mathfrak{I}^\perp}\colon X_{\mathfrak{I}^\perp} \to Y_{\mathfrak{I}^\perp} = Y$. Thus $\gamma_{X,\mathfrak{I}^\perp}$ is a left $\mathfrak{I}^\perp$-approximation. $\square$

We include the following lemma for later reference.

Lemma 3.12. *Let $f\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories which preserves coproducts. Suppose that $\mathfrak{I}$ is a class of maps in $\mathcal{C}$ such that their isomorphism classes form a set. If $f(\mathfrak{I}) = 0$, then the map $f(\gamma_{X,\mathfrak{I}^\perp})\colon f(X) \to f(X_{\mathfrak{I}^\perp})$ is a split monomorphism.*

Proof. The construction of each α_n implies that $f(\alpha_n)$ is a split monomorphism for every n. It follows that $\mathrm{id}_{f(X)}$ factors through $f(\operatorname{hocolim} X_n) \simeq \operatorname{hocolim} f(X_n)$. $\square$

4. Smashing subcategories

4.1. **A characterization of smashing subcategories.** Let $\mathcal{C}$ be a compactly generated triangulated category and suppose that $\mathcal{B}$ is a localizing subcategory of $\mathcal{C}$. We denote by $\mathcal{D} = \mathcal{C}/\mathcal{B}$ the quotient category and $f\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ denotes the corresponding quotient functor. A right adjoint of f is denoted by $g\colon \mathcal{C}/\mathcal{B} \to \mathcal{C}$, provided it exists. Recall that $\mathcal{B}$ is *smashing* if the inclusion $\mathcal{B} \to \mathcal{C}$ has a right adjoint which preserves coproducts. Note that this is equivalent to the fact that the quotient functor has a right adjoint which preserves coproducts.

Lemma 4.1. *Let $\mathcal{B}$ be a smashing subcategory of a compactly generated triangulated category $\mathcal{C}$. Then $\mathcal{C}/\mathcal{B}$ is a compactly generated triangulated category.*

Proof. $\mathcal{C}/\mathcal{B}$ has coproducts by Lemma 3.6, and the argument in the proof of Proposition 2.6 shows that $f(\mathcal{C}_0) \subseteq \mathcal{D}_0$. Suppose now that $\operatorname{Hom}(D, X) = 0$ for all D in $\mathcal{D}_0$ and some X in $\mathcal{D}$. Then $\operatorname{Hom}(C, g(X)) \simeq \operatorname{Hom}(f(C), X) = 0$ for all C in $\mathcal{C}_0$, and therefore $X = 0$, since $\mathcal{C}$ is compactly generated and g is faithful by Lemma 3.8. Thus $\mathcal{C}/\mathcal{B}$ is compactly generated. $\square$

We are now in a position to prove the characterization of smashing subcategories which is stated in Theorem A. We reformulate this theorem as follows.

Theorem 4.2. *Let $\mathcal{B}$ be a localizing subcategory of a compactly generated triangulated category $\mathcal{C}$, and denote by $\mathfrak{I}$ the ideal of maps between compact objects in $\mathcal{C}$ which factor through some object in $\mathcal{B}$. Then the following conditions are equivalent:*

(1) *$\mathcal{B}$ is smashing;*
(2) *$\mathcal{B} = \mathcal{C}_{\mathfrak{I}}$;*
(2') *$\mathcal{B} \subseteq \mathcal{C}_{\mathfrak{I}}$;*
(3) *$\mathfrak{I}^{\perp} = \mathcal{B}^{\perp}$;*
(3') *$\mathfrak{I}^{\perp} \subseteq \mathcal{B}^{\perp}$.*

Proof. (1) $\Rightarrow$ (2) We know from the preceding lemma that $\mathcal{D} = \mathcal{C}/\mathcal{B}$ is a compactly generated triangulated category. We have therefore by Proposition 2.6 an induced functor $(f_0)^* \colon \operatorname{Mod} \mathcal{C}_0 \to \operatorname{Mod} \mathcal{D}_0$ such that $(f_0)^* \circ h_{\mathcal{C}} = h_{\mathcal{D}} \circ f$. An object X in $\mathcal{C}$ belongs to $\mathcal{B}$ if and only if $(f_0)^*(H_X) = 0$ since $h_{\mathcal{D}}(Y) = 0$ if and only if $Y = 0$. Therefore $\mathcal{B} = \mathcal{C}_{\mathfrak{I}}$ by Lemma 3.3 and Lemma 3.9.

(2') $\Rightarrow$ (3) It is clear that $\mathfrak{I}^{\perp} \supseteq \mathcal{B}^{\perp}$. The condition (2') implies that $\mathfrak{I} = \mathfrak{I}_{(\mathcal{C}_{\mathfrak{I}})}$, and therefore $\mathfrak{I}^{\perp} = (\mathcal{C}_{\mathfrak{I}})^{\perp}$ by Lemma 3.10. Using again (2'), we have $(\mathcal{C}_{\mathfrak{I}})^{\perp} \subseteq \mathcal{B}^{\perp}$ and this implies $\mathfrak{I}^{\perp} \subseteq \mathcal{B}^{\perp}$.

(3') $\Rightarrow$ (2') Let X be an object in $\mathcal{C}$ and suppose $X \notin \mathcal{C}_{\mathfrak{I}}$. It follows from Lemma 3.9 that $H_X \notin \mathcal{T}_{\mathfrak{I}}$, and we find a maximal subobject $T \subseteq H_X$ with $T \in \mathcal{T}_{\mathfrak{I}}$ since $\mathcal{T}_{\mathfrak{I}}$ is a Serre subcategory of $\operatorname{Mod} \mathcal{C}_0$ which is closed under taking coproducts by Lemma 3.2. Choosing an injective envelope $H_X/T \to M$, we have $\operatorname{Hom}(H_X, M) \neq 0$ and $\operatorname{Hom}(\mathcal{T}_{\mathfrak{I}}, M) = 0$ by construction. Applying Lemma 1.7, we find an object Y in $\mathcal{C}$ such that $H_Y \simeq M$ and $\operatorname{Hom}(X, Y) \simeq \operatorname{Hom}(H_X, M) \neq 0$. Moreover, $\operatorname{Hom}(\mathfrak{I}, Y) = 0$ since $\operatorname{Hom}(\mathcal{T}_{\mathfrak{I}}, M) = 0$. Assuming (3'), it follows that $X \notin \mathcal{B}$. Thus (2') holds.

(2') $\Rightarrow$ (1) The condition (2') implies that $\mathfrak{I} = \mathfrak{I}_{(\mathcal{C}_{\mathfrak{I}})}$, and therefore $\mathfrak{I}^{\perp} = (\mathcal{C}_{\mathfrak{I}})^{\perp}$ by Lemma 3.10. We claim that $\mathcal{B} = \mathcal{C}_{\mathfrak{I}}$. To this end let $X \in \mathcal{C}_{\mathfrak{I}}$ and consider the $\mathfrak{I}^{\perp}$-approximation $\gamma_{X,\mathfrak{I}^{\perp}} \colon X \to X_{\mathfrak{I}^{\perp}}$ from Proposition 3.11. We have $\gamma_{X,\mathfrak{I}^{\perp}} = 0$ since $\mathfrak{I}^{\perp} = (\mathcal{C}_{\mathfrak{I}})^{\perp}$, and the quotient functor $f \colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ sends $\gamma_{X,\mathfrak{I}^{\perp}}$ to a split monomorphism by Lemma 3.12. Thus $f(X) = 0$ and therefore X belongs to $\mathcal{B}$. We conclude from Lemma 3.5 that the inclusion $\mathcal{B} \to \mathcal{C}$ has a right adjoint, and it remains to show that this right adjoint preserves coproducts. To this end consider the right adjoint $g \colon \mathcal{C}/\mathcal{B} \to \mathcal{C}$ of the quotient functor which identifies $\mathcal{C}/\mathcal{B}$ with $\mathcal{B}^{\perp}$ by Lemma 3.8. Now let $(X_i)_{i \in I}$ be a family of objects in $\mathcal{B}^{\perp}$. Using (3), we have $\operatorname{Hom}(\mathfrak{I}, X_i) = 0$ for all i, and therefore $\operatorname{Hom}(\mathfrak{I}, \coprod_i X_i) = 0$ since $\mathfrak{I}$ belongs to $\mathcal{C}_0$. Thus $\coprod_i X_i$ belongs to $\mathcal{B}^{\perp}$. It follows that g preserves coproducts and therefore $\mathcal{B}$ is smashing. $\square$

Remark 4.3. Localizations in $\mathcal{C}$ and $\operatorname{Mod}\mathcal{C}_0$ are closely related. In fact, if $\mathcal{B}$ is a smashing subcategory of $\mathcal{C}$, then one can use Proposition 2.6 to show that $f\circ g\simeq \mathrm{id}_{\mathcal{D}}$ implies $(f_0)^*\circ(f_0)_*\simeq \mathrm{id}_{\operatorname{Mod}\mathcal{D}_0}$ for $\mathcal{D}=\mathcal{C}/\mathcal{B}$. Therefore $(f_0)^*$ induces an equivalence $\operatorname{Mod}\mathcal{C}_0/\mathcal{T}_{\mathfrak{I}}\to\operatorname{Mod}(\mathcal{C}/\mathcal{B})_0$ where $\operatorname{Mod}\mathcal{C}_0/\mathcal{T}_{\mathfrak{I}}$ denotes the quotient category with respect to the localizing subcategory $\mathcal{T}_{\mathfrak{I}}=\{M\in\operatorname{Mod}\mathcal{C}_0\mid (f_0)^*(M)=0\}$; e.g. see [9, Proposition III.5]. This leads to the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{B} & \longrightarrow & \mathcal{C} & \xrightarrow{f} & \mathcal{C}/\mathcal{B}\\
\downarrow & & \downarrow{\scriptstyle h_{\mathcal{C}}} & & \downarrow\\
\mathcal{T}_{\mathfrak{I}} & \longrightarrow & \operatorname{Mod}\mathcal{C}_0 & \longrightarrow & \operatorname{Mod}\mathcal{C}_0/\mathcal{T}_{\mathfrak{I}}\\
 & & \| & & \downarrow{\scriptstyle\wr}\\
 & & \operatorname{Mod}\mathcal{C}_0 & \xrightarrow{(f_0)^*} & \operatorname{Mod}(\mathcal{C}/\mathcal{B})_0
\end{array}$$

Note that the composition $\mathcal{C}/\mathcal{B}\to\operatorname{Mod}\mathcal{C}_0/\mathcal{T}_{\mathfrak{I}}\to\operatorname{Mod}(\mathcal{C}/\mathcal{B})_0$ is just the restricted Yoneda functor $h_{\mathcal{C}/\mathcal{B}}$.

We proceed with the proof of Theorem C which we recall for the convenience of the reader.

Theorem 4.4. *Let $\mathcal{B}$ be a smashing subcategory of a compactly generated triangulated category $\mathcal{C}$, and let $\mathbf{U}$ be the set of objects Y in $\operatorname{Sp}\mathcal{C}$ such that $\operatorname{Hom}(\mathcal{B},Y)=0$. Then the following holds for any object X in $\mathcal{C}$:*

1. *$X\in\mathcal{B}$ if and only if $\operatorname{Hom}(X,\mathbf{U})=0$;*
2. *$\operatorname{Hom}(\mathcal{B},X)=0$ if and only if there is a pure monomorphism $X\to\prod_{i\in I}Y_i$ with $Y_i\in\mathbf{U}$ for all i.*

Proof. We identify $\mathcal{D}=\mathcal{C}/\mathcal{B}$ via g with the full subcategory of objects X in $\mathcal{C}$ such that $\operatorname{Hom}(\mathcal{B},X)=0$. This is possible by Lemma 3.8. In particular, this identifies $\operatorname{Sp}\mathcal{D}$ with $\mathbf{U}=\{X\in\operatorname{Sp}\mathcal{C}\mid\operatorname{Hom}(\mathcal{B},X)=0\}$ since g preserves pure-injectivity by Proposition 2.6.

(1) Clearly, $X\in\mathcal{B}$ implies $\operatorname{Hom}(X,\mathbf{U})=0$. Conversely, $\operatorname{Hom}(X,\mathbf{U})=0$ implies $\operatorname{Hom}((g\circ f)(X),\mathbf{U})=0$ by Lemma 3.7, and this implies $(g\circ f)(X)=0$ since $\mathbf{U}$ cogenerates $\mathcal{D}$ by Corollary 1.10. Thus $f(X)=0$ since g is faithful, and therefore X belongs to $\mathcal{B}$.

(2) Suppose first that $\operatorname{Hom}(\mathcal{B},X)=0$. We apply Corollary 1.10 and get a pure monomorphism $X\to\prod_i Y_i$ in $\mathcal{D}$ with $Y_i\in\mathbf{U}$ for all i. The inclusion $\mathcal{D}\to\mathcal{C}$ preserves pure monomorphisms by Proposition 2.6, and this proves one direction. Now suppose that we have a pure monomorphism $X\to Y$ in $\mathcal{C}$ with $\operatorname{Hom}(\mathcal{B},Y)=0$. It follows from part (3) in Theorem 4.2 that $\operatorname{Hom}(\mathcal{B},X)=0$, and therefore the proof of Theorem 4.4 is complete. ∎

Let $\mathcal{B}$ be a localizing subcategory of a triangulated category $\mathcal{C}$. Recall that a map $X\to Y$ in $\mathcal{C}$ is a *$\mathcal{B}$-localization* of X if Y is $\mathcal{B}$-local and the induced map $\operatorname{Hom}(Y,Y')\to\operatorname{Hom}(X,Y')$ is bijective for every $\mathcal{B}$-local object Y'. Let us describe an explicit construction of the $\mathcal{B}$-localization provided that $\mathcal{B}$ is smashing. We use the left $\mathfrak{I}^{\perp}$-approximation $\gamma_{X,\mathfrak{I}^{\perp}}\colon X\to X_{\mathfrak{I}^{\perp}}$ with respect to an ideal $\mathfrak{I}$ in $\mathcal{C}_0$ which has been constructed in Proposition 3.11. Recall from [2] that a map $\phi\colon X\to Y$ is *left minimal* if every endomorphism ψ of Y such that $\phi=\psi\circ\phi$ is an isomorphism.

Theorem 4.5. *Let $\mathcal{B}$ be a smashing subcategory of a compactly generated triangulated category $\mathcal{C}$ and let $\mathfrak{I} = \mathfrak{I}_{\mathcal{B}}$ be the corresponding ideal in $\mathcal{C}_0$. Then the left $\mathfrak{I}^{\perp}$-approximation $\gamma_{X,\mathfrak{I}^{\perp}} \colon X \to X_{\mathfrak{I}^{\perp}}$ of an object X in $\mathcal{C}$ has a decomposition*

$$(\gamma', \gamma'') \colon X \longrightarrow X_{\mathfrak{I}^{\perp}} = Y' \coprod Y''$$

such that γ' is left minimal and $\gamma'' = 0$. In this case, the map $\gamma' \colon X \to Y'$ is a $\mathcal{B}$-localization of X.

Proof. The $\mathfrak{I}^{\perp}$-approximation $\gamma_{X,\mathfrak{I}^{\perp}} \colon X \to X_{\mathfrak{I}^{\perp}}$ is also a $\mathcal{B}^{\perp}$-approximation since $\mathfrak{I}^{\perp} = \mathcal{B}^{\perp}$ by Theorem 4.2. There exists a $\mathcal{B}$-localization $\gamma_X \colon X \to Y$ of X by Lemma 3.5. We obtain therefore maps $\alpha \colon Y \to X_{\mathfrak{I}^{\perp}}$ and $\beta \colon X_{\mathfrak{I}^{\perp}} \to Y$ such that $\gamma_{X,\mathfrak{I}^{\perp}} = \alpha \circ \gamma_X$ and $\gamma_X = \beta \circ \gamma_{X,\mathfrak{I}^{\perp}}$. We have $\beta \circ \alpha = \mathrm{id}_Y$ since γ_X is a $\mathcal{B}$-localization of X, and this gives a decomposition $\gamma_{X,\mathfrak{I}^{\perp}} = (\gamma', \gamma'')$ such that γ' is isomorphic to γ_X and $\gamma'' = 0$. This finishes the proof. ☐

There are examples where the left $\mathfrak{I}^{\perp}$-approximation $\gamma_{X,\mathfrak{I}^{\perp}} \colon X \to X_{\mathfrak{I}^{\perp}}$ is different from the $\mathcal{B}$-localization of X. Take for instance a smashing subcategory $\mathcal{B} \neq 0$ with $\mathcal{B} \cap \mathcal{C}_0 = 0$.

4.2. **The modified telescope conjecture.** We are now in a position to prove the corollary of Theorem A; it will be an immediate consequence of the following proposition.

Proposition 4.6. *Let $\mathcal{B}$ be a smashing subcategory of a compactly generated triangulated category $\mathcal{C}$ and let $\mathfrak{I}_{\mathcal{B}}$ be the corresponding ideal in $\mathcal{C}_0$. Suppose that $f \colon \mathcal{C} \to \mathcal{D}$ is an exact functor into a triangulated category $\mathcal{D}$ which preserves coproducts. Then $f(\mathfrak{I}_{\mathcal{B}}) = 0$ if and only if $f(\mathcal{B}) = 0$.*

Proof. Let $\mathfrak{I} = \mathfrak{I}_{\mathcal{B}}$. Clearly, $f(\mathcal{B}) = 0$ implies $f(\mathfrak{I}) = 0$. Suppose now that $f(\mathfrak{I}) = 0$. Let X be an object in $\mathcal{B}$ and let $\gamma_{X,\mathfrak{I}^{\perp}} \colon X \to X_{\mathfrak{I}^{\perp}}$ be the left $\mathfrak{I}^{\perp}$-approximation from Proposition 3.11. Theorem 4.2 implies $\mathcal{B}^{\perp} = \mathfrak{I}^{\perp}$ and therefore $\gamma_{X,\mathfrak{I}^{\perp}} = 0$ since $X_{\mathfrak{I}^{\perp}}$ belongs to $\mathfrak{I}^{\perp}$. On the other hand, $f(\gamma_{X,\mathfrak{I}^{\perp}})$ is a split monomorphism by Lemma 3.12. Thus $f(X) = 0$. ☐

Corollary 4.7. *Let $\mathcal{B}$ be a smashing subcategory of a compactly generated triangulated category $\mathcal{C}$. Then $\mathcal{B}$ is generated by the corresponding ideal $\mathfrak{I} = \mathfrak{I}_{\mathcal{B}}$ in $\mathcal{C}_0$. More precisely,*

1. *$\mathcal{B}$ is a localizing subcategory of $\mathcal{C}$ and every map in $\mathfrak{I}$ factors through some object in $\mathcal{B}$;*
2. *if $\mathcal{B}'$ is any localizing subcategory of $\mathcal{C}$ such that every map in $\mathfrak{I}$ factors through some object in $\mathcal{B}'$, then $\mathcal{B} \subseteq \mathcal{B}'$.*

Proof. (1) follows immediately from the definitions of $\mathcal{B}$ and $\mathfrak{I}$. To prove (2), let $\mathcal{B}'$ be a localizing subcategory of $\mathcal{C}$ such that every map in $\mathfrak{I}$ factors through some object in $\mathcal{B}'$, and denote by $f \colon \mathcal{C} \to \mathcal{C}/\mathcal{B}'$ the corresponding quotient functor. Note that f preserves coproducts by Lemma 3.6. Clearly, $f(\mathfrak{I}) = 0$ and therefore $f(\mathcal{B}) = 0$ by the preceding proposition. Thus $\mathcal{B} \subseteq \mathcal{B}'$. ☐

4.3. **A classification of smashing subcategories.** In this section we consider a compactly generated triangulated category $\mathcal{C}$ such that the following additional property holds:

(B) Every cohomological functor $\mathcal{C}_0^{\mathrm{op}} \to \mathrm{Ab}$ of the form $\mathrm{Hom}(f(-), f(C))$ (where $f \colon \mathcal{C}_0 \to \mathcal{D}$ is any exact functor into a triangulated category $\mathcal{D}$ and C is any object in $\mathcal{C}_0$) is isomorphic to $\mathrm{Hom}(-, X)|_{\mathcal{C}_0}$ for some object X in $\mathcal{C}$.

This condition is a weak form of Brown representability. For example, (B) holds for the stable homotopy category [1]. More generally, (B) holds if the category $\mathcal{C}_0$ has a countable skeleton [21]. Our aim in this section is a classification of the smashing subcategories of $\mathcal{C}$. To this end we introduce the following class of ideals for a triangulated category.

Definition 4.8. An ideal $\mathfrak{I}$ in a triangulated category $\mathcal{C}$ is called *exact* if there exists an exact functor $f\colon \mathcal{C} \to \mathcal{D}$ into a triangulated category $\mathcal{D}$ such that $\mathfrak{I} = \{\phi \in \mathcal{C} \mid f(\phi) = 0\}$.

The following result gives a classification of smashing subcategories.

Theorem 4.9. *Let $\mathcal{C}$ be a compactly generated triangulated category and suppose that* (B) *holds. Then the maps*

$$\mathcal{B} \mapsto \{\phi \in \mathcal{C}_0 \mid \phi \textit{ factors through an object in } \mathcal{B}\} \quad \textit{and}$$

$$\mathfrak{I} \mapsto \{X \in \mathcal{C} \mid \textit{every map } C \to X,\ C \in \mathcal{C}_0, \textit{ factors through a map } C \to D \textit{ in } \mathfrak{I}\}$$

induce mutually inverse bijections between the set of smashing subcategories of $\mathcal{C}$ and the set of exact ideals in $\mathcal{C}_0$.

The proof of this theorem is based on the following lemma.

Lemma 4.10. *Let $\mathcal{C}$ be a compactly generated triangulated category and suppose that* (B) *holds. If $\mathfrak{I}$ is an exact ideal in $\mathcal{C}_0$, then $\mathfrak{I} = \mathfrak{I}_{(\mathcal{C}_\mathfrak{I})}$.*

Proof. Let $f\colon \mathcal{C}_0 \to \mathcal{D}_0$ be an exact functor such that $\mathfrak{I} = \{\phi \in \mathcal{C}_0 \mid f(\phi) = 0\}$. We may assume that $\mathcal{D}_0$ is a skeletally small triangulated category by taking the full subcategory formed by the objects in the image of f. Adding successively new objects arising from the completion of maps to triangles gives a full triangulated subcategory which needs to be skeletally small since $\mathcal{C}_0$ is skeletally small. Now observe that the inclusion $\mathfrak{I}_{(\mathcal{C}_\mathfrak{I})} \subseteq \mathfrak{I}$ is obvious from the definitions. To prove the other inclusion, we use the pair of adjoint functors $f^*\colon \operatorname{Mod}\mathcal{C}_0 \to \operatorname{Mod}\mathcal{D}_0$ and $f_*\colon \operatorname{Mod}\mathcal{D}_0 \to \operatorname{Mod}\mathcal{C}_0$ which have already been introduced. Let $\phi\colon C \to D$ be a map in $\mathfrak{I}$ and consider the canonical map $\mu\colon H_D \to (f_* \circ f^*)(H_D)$. Using our assumption on $\mathcal{C}$, there exists an object X in $\mathcal{C}$ such that $(f_* \circ f^*)(H_D) \simeq H_X$ since $(f_* \circ f^*)(H_D) \simeq \operatorname{Hom}(f(-), f(D))$. We obtain a map $\psi\colon D \to X$ with $\mu = H_\psi$ and consider the corresponding triangle

$$X[-1] \xrightarrow{\sigma} V \xrightarrow{\tau} D \xrightarrow{\psi} X$$

in $\mathcal{C}$ which induces an exact sequence

$$H_{D[-1]} \xrightarrow{\mu[-1]} H_{X[-1]} \longrightarrow H_V \longrightarrow H_D \xrightarrow{\mu} H_X$$

in $\operatorname{Mod}\mathcal{C}_0$. The maps $f^*(\mu)$ and $f^*(\mu[-1])$ are isomorphisms, and therefore $f^*(H_V) = 0$ since f^* is exact. Thus $V \in \mathcal{C}_\mathfrak{I}$ by Lemma 3.3 and Lemma 3.9. Now observe that $H_\psi \circ H_\phi = \mu \circ H_\phi = 0$ since the following diagram is commutative

$$\begin{array}{ccc} H_C & \longrightarrow & (f_* \circ f^*)(H_C) \\ \downarrow{\scriptstyle H_\phi} & & \downarrow{\scriptstyle (f_* \circ f^*)(H_\phi)} \\ H_D & \xrightarrow{\mu} & (f_* \circ f^*)(H_D) \end{array}$$

and $(f_* \circ f^*)(H_\phi) = f_*(H_{f(\phi)}) = 0$ by our assumption on ϕ. Thus $\psi \circ \phi = 0$ since C is compact, and therefore ϕ factors through V which is an object in $\mathcal{C}_\mathfrak{I}$. We conclude that $\mathfrak{I} \subseteq \mathfrak{I}_{(\mathcal{C}_\mathfrak{I})}$ and this finishes the proof. $\square$

We are now in a position to give the proof of the theorem which states the classification of the smashing subcategories of a compactly generated triangulated category $\mathcal{C}$.

Proof of Theorem 4.9. Let $\mathcal{B}$ be a smashing subcategory of $\mathcal{C}$ and denote by $f\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ the corresponding quotient functor. It is clear that $\mathfrak{I}_{\mathcal{B}}$ is an exact ideal in $\mathcal{C}_0$ since $\mathfrak{I}_{\mathcal{B}} = \{\phi \in \mathcal{C}_0 \mid f(\phi) = 0\}$. Suppose now that $\mathfrak{I}$ is an exact ideal in $\mathcal{C}_0$. We have $\mathfrak{I} = \mathfrak{I}_{(\mathcal{C}_{\mathfrak{I}})}$ by the preceding lemma, and a combination of Lemma 3.10 and Theorem 4.2 then shows that $\mathcal{C}_{\mathfrak{I}}$ is a smashing subcategory of $\mathcal{C}$. Given a smashing subcategory $\mathcal{B}$, we have $\mathcal{C}_{(\mathfrak{I}_{\mathcal{B}})} = \mathcal{B}$ by Theorem 4.2. Conversely, $\mathfrak{I}_{(\mathcal{C}_{\mathfrak{I}})} = \mathfrak{I}$ holds for every exact ideal in $\mathcal{C}_0$ by Lemma 4.10. Thus the maps $\mathcal{B} \mapsto \mathfrak{I}_{\mathcal{B}}$ and $\mathfrak{I} \mapsto \mathcal{C}_{\mathfrak{I}}$ are mutually inverse, and therefore the proof is complete. □

We continue with a number of applications of the above theorem. In fact, we are interested in the interplay between ideals in $\mathcal{C}_0$ and localizing subcategories of $\mathcal{C}$. The following lemma will be useful.

Lemma 4.11. *Let $(\mathfrak{I}_i)_{i\in I}$ be a family of exact ideals. Then $\bigcap_{i\in I} \mathfrak{I}_i$ is exact.*

Proof. Suppose that each $\mathfrak{I}_i$ is given by some exact functor $f_i\colon \mathcal{C}_0 \to \mathcal{D}_i$. The intersection $\mathfrak{I} = \bigcap_{i\in I} \mathfrak{I}_i$ is again exact since $\mathfrak{I} = \{\phi \in \mathcal{C}_0 \mid f(\phi) = 0\}$ where f denotes the exact functor $\mathcal{C}_0 \to \prod_i \mathcal{D}_i$, $X \mapsto (f_i(X))_i$. □

Recall that a lattice is *complete* if every subset has a least upper bound and a greatest lower bound.

Corollary 4.12. *Let $\mathcal{C}$ be a compactly generated triangulated category and suppose that* (B) *holds. Then the smashing subcategories of $\mathcal{C}$ form a partially ordered set which is a complete lattice.*

Proof. Theorem 4.9 translates the assertion of this corollary into a statement about the lattice of ideals in $\mathcal{C}_0$. Clearly, the cardinality of this lattice is bounded by 2^κ where κ denotes the cardinality of the set of isomorphism classes of maps in $\mathcal{C}_0$. The exact ideals in $\mathcal{C}_0$ form a complete lattice by the preceding lemma, and this finishes the proof. □

Given a localizing subcategory $\mathcal{B}$ of $\mathcal{C}$, it is not clear that the maps between fixed objects in the quotient category $\mathcal{C}/\mathcal{B}$ form a set. Therefore one calls a category *large* to point out that the maps between fixed objects are not assumed to form a set.

Lemma 4.13. *Let $\mathcal{B}$ be a skeletally small subcategory of a large triangulated category $\mathcal{C}$. Then there exists a skeletally small triangulated subcategory of $\mathcal{C}$ which contains $\mathcal{B}$.*

Proof. We construct inductively a chain $\mathcal{C}_1 \subseteq \mathcal{C}_2 \subseteq \mathcal{C}_3 \subseteq \dots$ of classes of maps in $\mathcal{C}$ and a chain $\mathcal{B} = \mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \dots$ of skeletally small subcategories of $\mathcal{C}$ as follows: Let $n \geq 1$ and assume that $\mathcal{B}_{n-1}$ is already defined. Let $\mathcal{C}_n$ be a class of maps in $\mathcal{C}$ satisfying the following conditions:

- if $\phi \in \mathcal{B}_{n-1}$ and $r \in \mathbb{Z}$, then $\phi[r] \in \mathcal{C}_n$;
- if $X \overset{\phi}{\to} Y \overset{\psi}{\to} Z \overset{\chi}{\to} X[1]$ is a triangle in $\mathcal{C}$ with $\phi \in \mathcal{B}_{n-1}$, then $\psi, \chi \in \mathcal{C}_n$;
- if there is a commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & & & \downarrow{\scriptstyle \alpha[1]} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & X'[1] \end{array}$$

in $\mathcal{B}_{n-1}$ such that the rows are triangles in $\mathcal{C}$, then there is a map $Z \to Z'$ in $\mathcal{C}_n$ making the diagram commutative;

• if there is a set of maps in $\mathcal{B}_{n-1}$ satisfying the assumptions of the octahedral axiom, then there are maps in $\mathcal{C}_n$ such that the octahedral axiom holds.
We may assume that the isomorphism classes of maps in $\mathcal{C}_n$ form a set since $\mathcal{B}_{n-1}$ is skeletally small. Now define $\mathcal{B}_n$ to be the smallest additive subcategory of $\mathcal{C}$ containing $\mathcal{C}_n$. It is easily checked that $\mathcal{B}_\infty = \bigcup_{n\in\mathbb{N}} \mathcal{B}_n$ is a skeletally small triangulated subcategory of $\mathcal{C}$ which contains $\mathcal{B}$. □

Corollary 4.14. *Let $\mathcal{C}$ be a compactly generated triangulated category and suppose that* (B) *holds. Suppose also that $\mathcal{B}$ is a localizing subcategory and denote by $\mathfrak{I}_\mathcal{B}$ the ideal of maps between compact objects in $\mathcal{C}$ which factor through some object in $\mathcal{B}$. Then there exists a unique smashing subcategory $\mathcal{B}'$ of $\mathcal{C}$ such that $\mathfrak{I}_{\mathcal{B}'} = \mathfrak{I}_\mathcal{B}$. Moreover, $\mathcal{B}' \subseteq \mathcal{B}$.*

Proof. Let $\mathfrak{I} = \mathfrak{I}_\mathcal{B}$ and let $f\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ be the quotient functor corresponding to $\mathcal{B}$. Clearly, $\mathfrak{I} = \{\phi \in \mathcal{C}_0 \mid f(\phi) = 0\}$ and we claim that $\mathfrak{I}$ is an exact ideal in $\mathcal{C}_0$. By Lemma 4.13, there exists a skeletally small triangulated subcategory $\mathcal{D}$ of $\mathcal{C}/\mathcal{B}$ containing the image of f, and we obtain therefore an exact functor $f'\colon \mathcal{C}_0 \to \mathcal{D}$, $X \mapsto f(X)$ with $\mathfrak{I} = \{\phi \in \mathcal{C}_0 \mid f'(\phi) = 0\}$. Thus $\mathfrak{I}$ is an exact ideal, and there exists a unique smashing subcategory $\mathcal{B}' = \mathcal{C}_\mathfrak{I}$ such that $\mathfrak{I}_{\mathcal{B}'} = \mathfrak{I}$ by Theorem 4.9. Finally $f(\mathfrak{I}) = 0$ implies $f(\mathcal{B}') = 0$ by Proposition 4.6, and therefore $\mathcal{B}' \subseteq \mathcal{B}$. □

The preceding corollary suggests the following definition.

Definition 4.15. A localizing subcategory $\mathcal{B}$ of a triangulated category $\mathcal{C}$ is said to be *generated* by a class $\mathfrak{I}$ of maps in $\mathcal{C}$ if the following holds:

(1) every map in $\mathfrak{I}$ factors through some object in $\mathcal{B}$;
(2) if $\mathcal{B}'$ is a localizing subcategory of $\mathcal{C}$ such that every map in $\mathfrak{I}$ factors through some object in $\mathcal{B}'$, then $\mathcal{B} \subseteq \mathcal{B}'$.

For example, $\mathcal{B}$ is generated by a class $\mathfrak{I} = \{\mathrm{id}_{X_i} \mid i \in I\}$ of identity maps if and only if $\mathcal{B}$ is the smallest localizing subcategory containing X_i for all $i \in I$. A classical result of Bousfield and Ravenel for the stable homotopy category says that every class of identity maps of compact objects generates a localizing subcategory which is smashing [6, 23]. This can be generalized as follows.

Corollary 4.16. *Let $\mathcal{C}$ be a compactly generated triangulated category and suppose that* (B) *holds. Then a localizing subcategory $\mathcal{B}$ of $\mathcal{C}$ is smashing if and only if $\mathcal{B}$ is generated by a class of maps between compact objects in $\mathcal{C}$. Moreover, given any class $\mathfrak{I}$ of maps between compact objects in $\mathcal{C}$, there exists a localizing subcategory of $\mathcal{C}$ which is generated by $\mathfrak{I}$.*

Proof. It has been shown in Corollary 4.7 that a smashing subcategory is generated by a class of maps in $\mathcal{C}_0$. To prove the converse, suppose that $\mathcal{B}$ is generated by a class $\mathfrak{I}$ of maps in $\mathcal{C}_0$. We have $\mathfrak{I} \subseteq \mathfrak{I}_\mathcal{B}$ and we may assume that $\mathfrak{I} = \mathfrak{I}_\mathcal{B}$. By Corollary 4.14, there exists a smashing subcategory $\mathcal{B}'$ with $\mathfrak{I}_{\mathcal{B}'} = \mathfrak{I}$ and $\mathcal{B}' \subseteq \mathcal{B}$. On the other hand, $\mathcal{B} \subseteq \mathcal{B}'$ since $\mathcal{B}$ is generated by $\mathfrak{I}$, and therefore $\mathcal{B}$ is smashing. Suppose now that $\mathfrak{I}$ is any class of maps in $\mathcal{C}_0$ and let $\mathfrak{J}$ be the intersection of all exact ideals in $\mathcal{C}_0$ containing $\mathfrak{I}$. It is an immediate consequence of the preceding corollary that $\mathcal{C}_\mathfrak{J}$ is a localizing subcategory which is generated by $\mathfrak{I}$. □

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