Covering Monomial Algebras

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1 Introduction

Although covering has been a standard technique in the representation theory of finite-dimensional algebras for several years now and it has been pointed out by various authors (cf eg [10], [7]) that using covering techniques, the question, whether or not a given algebra has up to isomorphism only finitely many indecomposable representations, is decidable, no attempt at implementing such an algorithm seems to have been made. One problem with coverings when working with a computer is the fact, that in general they tend to be infinite-dimensional, giving rise to infinite combinatorial structures.

Based on an approach of E.Green already presented in [10] that uses a criterion for finite type for tree algebras by Bongartz and Ringel [4], we will in this paper introduce an algorithm, how for a monomial algebra to construct a finite part of its universal covering just large enough to decide, whether the algebra given is of finite type.

The algorithm presented has been implemented and will be part of a future release of the computer algebra system CREP for representation theory of finite-dimensional algebras.

2 Some notation

Let Q be a directed graph, which we will always assume to be connected. We will denote by Q_0 the set of vertices of Q, and for each pair $x, y \in Q_0$ by $Q(x, y)$ the set of arrows in Q starting in x and ending in y. By a *path* of length $m \geq 0$ from a vertex x to a vertex y in Q we will understand a sequence $\varpi = \alpha_1\alpha_2...\alpha_m$ of arrows such that for $i = 1, \ldots, m-1$ the endpoint of α_i coincides with the starting point of α_{i+1}, with the starting point of α_1 being x, the endpoint of α_m being y. Especially, we will for $x \in Q_0$ by e_x denote the unique path of length 0 from x to x.

By a *walk* in Q we will understand a sequence $\omega = (p_m, p_{m+1}, \ldots, p_n)$ of paths p_i with $m \in \{0, 1\}$, $m \leq n$, such that for any even index $i \in \{m, \ldots, n\}$ the starting point of p_i coincides with that of p_{i+1} and the endpoint of p_i with that of p_{i-1}, provided p_{i+1}, respectively p_{i-1} exist. Introducing for each arrow α in Q say $\alpha \in Q(x, y)$ a formal inverse α^{-1} with starting point y and endpoint x, we may consider such a walk a sequence $\omega = \beta_1\beta_2 \ldots \beta_l$ of arrows and formal inverses of arrows in Q, such that for $i = 1, \ldots l - 1$ the starting point of β_{i+1} coincides with the endpoint of β_i. We will then call the starting point of β_1 the starting point of ω, and the endpoint of β_l the endpoint of ω.

Using the convention $(\alpha^{-1})^{-1} = \alpha$ for any arrow α in Q we call a walk $w = \beta_1\beta_2 \ldots \beta_m$ *reduced*, if $\beta_i \neq (\beta_{i+1})^{-1}$ for $i = 1, \ldots, m - 1$.

3 Monomial algebras and coverings

Let k be a field. For a directed graph Q, denote by kQ the free k-vector space over the set of all paths in Q. We define the product $p \cdot q$ of two paths p and q in Q to be the concatenation pq, if this exists (i.e., if the endpoint of p coincides with the starting point of q,) respectively otherwise to be 0. The k linear closure of this multiplication on its basis makes kQ a k-algebra, the *path algebra* of Q.

By a *monomial algebra* one understands any factor algebra $A = kQ/I$ of the path algebra of a directed graph Q by an ideal I of kQ that admits a generating set $P = \{\rho_i | i \in J\}$ consisting entirely of paths ρ_i in Q. We may assume all of these paths ρ_i to be of length at least 2, since a path of length 0 would 'delete' a vertex, a path of length 1 an arrow of Q, telling us, that we started with the wrong directed graph Q. We will call P a set of relations for Q.

Let now Q be a finite directed graph, P a set of paths in Q, I the ideal of kQ generated by P. Set $A = kQ/I$.

By a *covering* of Q we understand a connected directed graph $\tilde{Q}$ together with a map $F : \tilde{Q} \longrightarrow Q$, mapping vertices to vertices, arrows to arrows, and paths to paths (i.e. if $\alpha\beta$ is a path in $\tilde{Q}$, then $F(\alpha)F(\beta)$ is a path in Q.), having in addition the following properties:

a) $F_0 : \tilde{(Q)}_0 \longrightarrow Q_0$ is a surjection onto the vertices of Q.

b) For any vertex x of $\tilde{Q}$ and any vertex b of Q, F induces a bijection $\bigcup_{y \in F^{-1}(b)} \tilde{Q}(x, y) \longrightarrow Q(F(x), b)$.

c) For any vertex y of $\tilde{Q}$ and any vertex a of Q, F induces a bijection $\bigcup_{x \in F^{-1}(a)} \tilde{Q}(x, y) \longrightarrow Q(a, F(y))$.

We can then set $\tilde{P} = F^{-1}(P)$, i.e. $\tilde{P}$ consists of all paths p in $\tilde{Q}$ such that $F(p) \in P$. Denoting by $\tilde{I}$ the ideal of $k\tilde{Q}$ generated by $\tilde{P}$ and setting $\tilde{A} = k\tilde{Q}/\tilde{I}$, we again obtain an algebra, which we will consider a covering of A. (In fact, the map F extends to an algebra epimorphism $F : \tilde{A} \to A$.)

In the situation above, for a finite connected directed graph Q there is always a covering $F : \tilde{Q} \to Q$ for which $\tilde{Q}$ is a tree. This covering, which is essentially unique, is called the *universal covering* of Q. The corresponding algebra $\tilde{A} = k\tilde{Q}/\tilde{I}$ is called universal covering of the monomial algebra $A = kQ/I$.

4 Representation type

Let Q be a directed graph, $P = \{\rho_i | i \in J\}$ a set of paths in Q. A *representation* V of Q subject to the set of relations P is given by a family $(V(a))_{a \in Q_0}$ of k vector spaces together with k linear maps $V(\alpha) : V(x) \to V(y)$ for each arrow $\alpha : x \to y$ in Q such that for each relation $\alpha_1\alpha_2 \ldots \alpha_m = \rho_i \in P$ the composition $V(\alpha_1)V(\alpha_2) \ldots V(\alpha_m) = 0$. Such a representation V is called *finite - dimensional*, if the vector space $\coprod_{a \in Q_0} V(a)$ is finite-dimensional.

We denote by $\mathrm{rep}_k(Q, P)$ the category of all finite-dimensional representations of Q subject to P. For $A = kQ/{<}P{>}$ this category is equivalent to the category mod$A$ of finite-dimensional right A modules. (Here${<}P{>}$ denotes the ideal of kQ generated by P.)

The algebra A (respectively the pair (Q, P) is called *representation finite* if up to isomorphism there is only a finite number of indecomposable finite-dimensional representations $V \in \mathrm{rep}_k(Q, P)$. The algebra A is called *locally representation finite*, if for each vertex $a \in Q_0$, up to isomorphism, there is only a finite number of indecomposable finite-dimensional representations $V \in \mathrm{rep}_k(Q, P)$ with $V(a) \neq 0$.

As a special case of a result of Gabriel central to the theory of covering, we have the following ([8], cf also [10]):

Theorem Let $A = kQ/<P>$ be a finite - dimensional monomial algebra, $\tilde{A}$ its universal covering. Then A is representation finite if and only if $\tilde{A}$ is locally representation finite.

In addition, in the situation of the theorem for A representation finite, one may obtain the representations of A from those of $\tilde{A}$.

As in our case $\tilde{Q}$ is a tree, $\tilde{A}$ is often easier to study than is A itself.

Also, being a tree, $\tilde{Q}$ can easily be constructed from Q by the defining properties of the universal covering. However, as in most cases $\tilde{Q}$ will be infinite, the question remains, where to stop. Here we can use a result of Bongartz and Ringel ([4]) in order to construct a finite part of $\tilde{Q}$ just large enough to be able to decide, whether A is representation finite.

5 Nonzero walks

Let Q be a directed tree, which is locally finite (i.e. for any vertex x of Q there are only finitely many arrows starting or ending in x), and let $P = \{\rho_i | i \in J\}$ be a set of paths in in Q.

We call a path $\varpi = \alpha_1\alpha_2\dots\alpha_m$ *nonzero* if $\eta(\varpi)$ its image under the canonical epimorphism $kQ \to kQ/<P>$ is not 0. This is equivalent to ϖ having no subpath contained in P.

We call a reduced walk ω *nonzero*, if it can be written in the form $\omega = (p_l, \dots, p_n)$, where for each $i = l, \dots n$ the path p_i is nonzero and of positive length.

Denote for a given vertex $s \in Q_0$ by V_s the set of all endpoints of nonzero walks starting in s, and by W_s the set of all such walks themselves. We shall call V_s the support of nonzero walks starting in s.

Q being a tree, each nonzero walk starting in s is uniquely determined by its endpoint. Thus there is a bijection between V_s and W_s. As shown by Bongartz and Ringel ([4], cf also [10]), W_s can be equipped with the structure of a partially ordered set, such that there holds:

Theorem Let V_s be a finite set for each vertex $s \in Q_0$, then $A = kQ/<P>$ is representation finite if and only if for each $s \in Q_0$, W_s is representation finite as partially ordered set.

Due to Klejner ([11]), there is a handy criterion for a partially ordered set to be representation finite. In fact, a partially ordered set is representation finite, if and only if it does not contain one of a list of five so-called critical partially ordered sets as full ordered subset.

As in addition each nonzero walk starting at a fixed vertex $s \in Q_0$ induces a different indecomposable representation M of $A = kQ/{<}P{>}$ with $M(s) \neq 0$, A cannot be representation finite, if the set V_s (or W_s) is infinite.

6 Restricting to a finite part

Let $\tilde{A} = k\tilde{Q}/{<}\tilde{P}{>}$ be the universal covering of a monomial algebra $A = kQ/{<}P{>}$, induced from he covering map $F : \tilde{Q} \to Q$. Then for any vertex $A \in Q_o$, the subgraphs $\tilde{Q}_{(s)}$ of $\tilde{Q}$ spanned by the sets V_s, for $s \in F^{-1}(x)$ are all isomorphic, and so are the induced partially ordered sets W_s. Thus in order to study A via $\tilde{A}$ it is sufficient to know, for each vertex $a \in Q_0$, $\tilde{Q}_{(s)}$ for only one vertex $s \in F^{-1}(x)$. Thus we arrive at the following algorithm, for constructing a sufficiently large finite subgraph of $\tilde{Q}$:

Algorithm Let Q be a finite directed graph, P a finite set of paths in Q.

- Start with a spanning tree Q' of Q. let $F' : Q' \mapsto Q$ be the corresponding inclusion map.

  Any such spanning tree will be part of the universal cover of Q, so we may start the construction with an arbitrary one.

- From the defining properties of the universal covering $F : \tilde{Q} \to Q$, we know that for any vertex $x \in \tilde{Q}$ there are as many arrow in $\tilde{Q}$ starting respectively ending in x, as there are arrows in Q starting respectively ending in $F(x)$.

  For any vertex of the original spanning tree, add these arrows, together with their endpoints, respectively their starting points to the graph Q', and extend the map F' accordingly.

  Also, construct a corresponding set P' of relations for Q' induced from P along with Q'.

- Proceed width first through the points of Q' already constructed in order to add all arrows in $\tilde{Q}$ starting respectively ending there, unless, for a vertex x of Q' one of the following situations occurs:

[1] There is no nonzero walk in Q' with respect to P' from x back to a point of the initially constructed spanning tree.

(In this case x will not be contained in any support V_s for a point s inside the initial spanning tree.)

Note that, as by construction Q' also is a tree, there is in fact exactly one reduced walk connecting the vertex x to the initial spanning tree – which is here to be considered.

[2] There is a nonzero path $\omega = \alpha_1\alpha_2 \dots \alpha_m$ in Q' between the vertex x and some vertex $y \in Q'$, such that $F'(x) = F'(y)$ and ω is of length $m \geq l$, where l denotes the maximal length of a path in the set of relations P, and for which the square $F'(\omega)F'(\omega)$ is not contained in the ideal $<P>$ in kQ.

(In this case, the algebra $A = kQ/<P>$ cannot be finite-dimensional, as it is supposed to be. Conversely, if A is not finite-dimensional, this situation will inevitably occur.)

[3] There is a nonzero walk $\omega = \beta_1, \beta_2 \dots \beta_m$ of length $m \geq 1$ from the vertex x to some vertex y of Q', such that $F'(x) = F'(y)$, β_1 and β_m are not both arrows or both formal inverses of an arrow, and $\beta_1 \neq \beta_m^{-1}$.

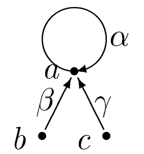
(In this cases, concatenation of copies of ω yields paths

$\beta_1, \beta_2 \dots \beta_m\beta_1, \beta_2 \dots \beta_m \dots \beta_1, \beta_2 \dots \beta_m$

of arbitrarily big length, so that V_x will not be finite and thus $\tilde{A}$ not locally representation finite, which also answers the question for A.)

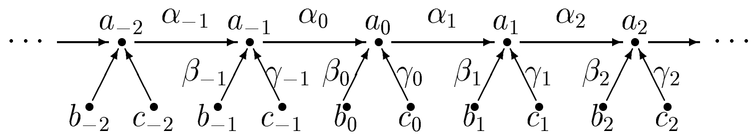
The algorithm will always stop after a finite number of steps, providing a finite directed graph Q', and a corresponding set of relations P'. In case $A = kQ/<P>$ has not turned out to be infinite-dimensional, or has already been recognized not to be representation finite by condition [3], having started with a spanning tree for Q encertains, that Q' will contain the support sets V_s for a point $s \in F^{-1}(x)$ for each vertex $x \in Q_0$. This allows to apply the criterion of Bongartz and Ringel.

Note, that though the support V_s of nonzero walks in a universal covering $\tilde{A}$ of a monomial algebra A may be finite for all vertices $s \in \tilde{Q}_0$, $\tilde{A}$ may still admit indecomposable modules of arbitrarily large support. For example, consider the monomial algebra $A = kQ/<P>$ given by the following directed graph Q

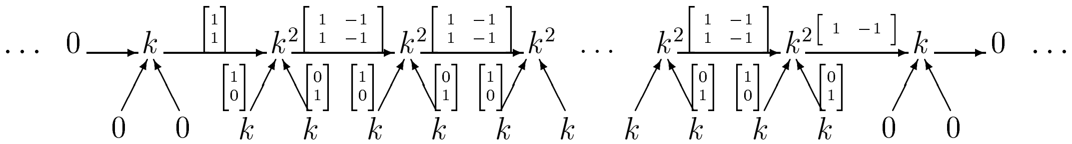


and the one-elementary set of relations $P = \{\alpha\alpha\}$.

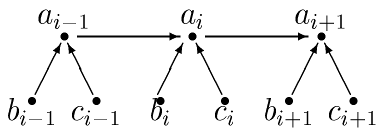
The universal covering $\tilde{A} = k\tilde{Q}/<\tilde{P}>$ is given by the graph $\tilde{Q}$:



and the set of relations $\tilde{P} = \{\alpha_i\alpha_{i+1} | \alpha \in \mathbb{Z}\}$. For this, there are arbitrarily large indecomposable representations of the form:

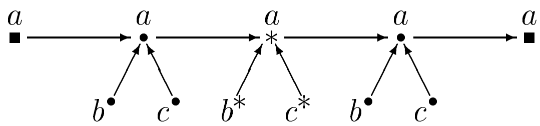


On the other hand, for fixed $i \in \mathbb{Z}$, the supports V_s for $s \in \{a_i, b_i, c_i\}$ all are:



This finite part is sufficient, to show, that $\tilde{A}$ is not locally representation finite, and thus A is not representation finite.

Our implementation of the above algorithm yields the following directed graph Q' (together with the corresponding restriction P' of the set $\tilde{P}$ to Q'):



Here the vertices marked by $'*'$ are those belonging to the initial spanning tree of Q. The vertices marked by '▪' are those vertices constructed, for which there is no nonzero walk back to the initial spanning tree.

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