Normal forms for representations of representation-finite algebras

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1 Introduction and main result

Why is representation theory of algebras important for physics? Besides well-known answers like the occurrence of symmetries, operator algebras or quantum groups in physical theories there is a less popular and more naive answer to this question. Namely, representation theory of finite-dimensional algebras deals with normal forms of matrices. But finding normal forms for matrices is an important technique all over physics. Probably a student encounters this for the first time when trying to solve a system of linear differential equations describing some harmonic oscillator or when trying to find the main axes of a solid body by diagonalising the inertia tensor. Representation theory of finite-dimensional algebras provides a general environment to deal with problems of this shape. The aim of this paper is to show how computer algebra is used to obtain information about a certain type of normal forms and to indicate how this could be made available for practical use.

Let us explain the way how representation theory of finite dimensional algebras deals with normal forms of matrices. For this purpose we need concepts like quivers and their representations introduced in [Ga1]. Recall that a quiver Q is nothing but a directed graph with a set of vertices Q_0 and a set of arrows Q_1 such that each arrow α in Q_1 has a unique initial point $s(\alpha)$ and a unique final point $t(\alpha)$ in Q_0. For a given field k the path algebra kQ is the vector space having as basis the paths in the quiver Q. A path of positive length p in Q is a sequence $\alpha_p \dots \alpha_1$ of arrows such that $s(\alpha_i) = t(\alpha_{i-1})$ for all $i = 2, \dots, p$. Moreover, to each vertex i there is attached a path of length 0 starting and ending in i. The path algebra kQ is endowed with the multiplication which is defined for two paths v and u as the concatenation of the two paths if the final point of u coincides with the initial point of v and 0 otherwise. Note, that kQ is not finite-dimensional if Q has an oriented cycle. Representations of a path algebra kQ are easy to describe as representations of Q (over the given field k). A representation X of Q is given by a vector spaces $X(i)$ for each vertex i in Q_0 and a linear map $X(\alpha) : X(s(\alpha)) \to X(t(\alpha))$ for each arrow α in Q_1.

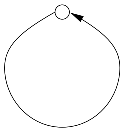
A homomorphism from a representation X to a representation Y of a quiver Q is a family $\phi = (\phi_i)_{i \in Q_0}$ of linear maps $\phi_i : X(i) \to Y(i)$ such that $Y(\alpha)\phi_{s(\alpha)} = \phi_{t(\alpha)}X(\alpha)$ for all arrows α. Such a homomorphism is called isomorphism if all the maps ϕ_i are bijective.

Two representations X, Y are said to be isomorphic if there exists an isomorphism from X to Y.

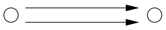
A representation X of Q is called finite-dimensional if all vector spaces $X(i)$ are finite-dimensional say of dimension $n(i)$. By choosing bases, each space $X(i)$ can be identified with $k^{n(i)}$. This forces each map $X(\alpha)$ to be a (left) multiplication by a $n(t(\alpha)) \times n(s(\alpha))$-matrix over k which we denote by $X(\alpha)$ as well. A representative of an isomorphism class of finite-dimensional representations having this shape is called a normal form.

The direct sum of a collection $X_1, \ldots, X_m$ of representations of a quiver Q is defined in the obvious way by building the direct sum of the vector spaces $X_1(i), \ldots, X_m(i)$ for each vertex i in Q_0. A representation X of Q is said to be indecomposable if it is not isomorphic to the direct sum of two non-zero representations. By the theorem of Krull-Remak-Schmidt any finite-dimensional representation is isomorphic to a direct sum of finitely many indecomposable representations which are unique up to permutation and isomorphism. Thus it is sufficient to find normal forms for all isomorphism classes of indecomposable representations.

The most popular normal forms are the Jordan block matrices which are the normal forms for the indecomposable representations of the one loop quiver.



Another classical example are the normal forms for matrix pencils (see [Gm]) which are the normal forms for the indecomposable representations of the Kronecker quiver below. For an application in perturbation theory we refer to [AF].



To what extent are quivers and their path algebras typical in the representation theory of algebras? If we wish to study a finite-dimensional representation X of a given k-algebra A, then we clearly can consider X as representation of the factor algebra of A by the annihilator ideal of X. Since this factor algebra is finite-dimensional, it suffices to consider finite-dimensional algebras when looking for normal forms for finite-dimensional indecomposable representations. It was emphasised by Gabriel (see [Ga1] and [Ga3]) that any finite-dimensional algebra over an algebraically closed field k (like $\mathbb{C}$) can be considered (up to Morita equivalence) as a factor algebra kQ/I of the path algebra kQ associated with a finite quiver Q. The ideal I of kQ can be chosen to be admissible which means that it is contained in the square of the ideal of kQ generated by the arrows but contains some power of this ideal. Note, that such an ideal can always be generated by finitely many generators $\rho_1, \ldots, \rho_r$ such that each of these 'relations' ρ_l is a linear combination $\rho_l = \sum_{q=1}^{m} \lambda_q u_q$ where $\lambda_q \in k$ and all the u_q are paths from a vertex i to a vertex j of Q_0. We observed above that representations of a quiver algebra kQ can be identified with the representations of Q over k. This transfers easily to representations of kQ/I. Namely, the representations of this factor algebra are just those representations X of Q satisfying all the relations ρ_l meaning that $0 = \sum_{q=1}^{m} \lambda_q X(u_q)$. Here we use the abbreviation $X(u) := X(\alpha_p) \cdots X(\alpha_1)$ when considering a path $u = \alpha_p \ldots \alpha_1$ in Q. Thus classifying the indecomposable representations of a finite-dimensional algebra $A = kQ/I$ amounts to finding normal forms for the indecomposable representations of Q satisfying the relations $\rho_1, \ldots, \rho_r$ which generate the ideal I.

In the last 30 years a lot of theoretical efforts were made to understand the representations of a finite-dimensional algebra. But due to the combinatorial nature of quivers also concrete algorithms were developed and implemented to classify certain classes of algebras. However, the normal forms of indecomposable representations of a given algebra will usually not only depend on discrete parameters. An exception is furnished by the representation-finite algebras defined by the property that there are only finitely many isomorphism classes of indecomposable representations or in other words the list of normal forms of indecomposable representations is finite. There are a lot of results about representation-finite algebras culminating in the theorem on the existence of multiplicative bases proved in [BGRS]. In this paper to each representation-finite finite-dimensional algebra A a so-called standard form kQ/I is associated which is representation-finite as well. The ideal I is especially nice having generators consisting of paths and differences of paths. Of particular interest for us is, that the standard form is isomorphic to A (up to Morita equivalence) provided that the characteristic of k is different from 2. Moreover, if one is only interested in one particular indecomposable representation, then by passing to the annihilator factor algebra one may assume that the representation is faithful. In this case by [Bo2] the algebra A is isomorphic to its standard form also if the field k is of characteristic 2.

Not so much is known about normal forms for indecomposable representations of representation-finite algebras. In this paper we want to improve this situation by presenting the following result about standard forms.

Theorem. *Let $A = kQ/I$ be the standard form of a representation-finite algebra over an algebraically closed field k. Then for each isomorphism class of indecomposable representations of kQ/I there is a normal form X such that for all arrows α in the quiver Q the matrix $X(\alpha)$ has only the entries* 0 *and* 1.

Note, that a similar result is proved in [Ri2] for normal forms of exceptional representations of hereditary algebras.

We tried to point out at the beginning of the paper that representation theory of algebras could be helpful for physicists because it provides normal forms for matrices. In this sense we hope that the above theorem is of interest. But how is computer algebra involved? An essential step in the proof of the above theorem is a case by case inspection in a data base containing the list of all exceptional simply connected algebras algebras. This data base is provided by the program system CREP (Combinatorial REPresentation theory) developed in Bielefeld and available from the ftp-server `ftp.uni-bielefeld.de` in the directory `pub/math/f-d-alg`. In this directory one will also find the manual [DN]. CREP behaves like a collection of packages of procedures for the computer algebra system Maple. The required data base is contained in a package called `esrdalg`.

In the data base the indecomposable modules are only represented by their dimension vectors. Since these vectors provide unique invariants, this is satisfactory from the theoretical point of view. In the next section we will show how this information can be used to get the above theorem which gives a more explicite information. In fact, it turns out that with some more computational effort we would really be able to calculate the normal forms precisely. We will return to this in the final section.

2 The proof of the theorem

From [BGRS] we know that for the standard form kQ/I there is a universal Galois covering such that each finite convex subcategory is a completely separating algebra (see [Dr4]). The push-down functor associated with the universal covering (see [Ga4]) uses the matrices attached to the arrows of the covering as blocks for the matrices attached to the arrows of Q. Therefore, in order to prove our theorem, we may assume that $A = kQ/I$ itself is completely separating.

A completely separating algebra can be rewritten in a way which is more suitable for our purposes. By [Dr4] there is a finite partially ordered set S such that $A = kS/J$ where J is an admissible ideal of the incidence algebra kS of S over k. Recall that the pairs (y, x) of elements of S satisfying $x \geq y$ form a k-basis of kS. The multiplication is defined on the base vectors by $(z, y)(y, x) := (z, x)$ and $(z, y')(y, x) := 0$ for $y \neq y'$. That J is admissible means that J is generated by elements $\rho = (y, x)$ such that $x \neq y$ are not direct neighbours in S. Representations of kS can be identified with representations of S over k. Such a representation X is given by a vector space $X(x)$ for each element x of S and a linear map $X(y, x) : X(x) \to X(y)$ for each $y \leq x$. In addition, these maps have to satisfy $X(z, x) = X(z, y)X(y, x)$ for all $z \leq y \leq x$. Representations of the factor algebra kS/J are representations X of S where $X(y, x) = 0$ for every generator $\rho = (y, x)$ of J.

How are Q and S related? The quiver Q for the completely separating algebra $A = kS/J$ is nothing but the usual Hasse diagram of S which means that the vertices of Q are the elements of S and there is an arrow from x to y in Q provided that $y < x$ are direct neighbours in S (i.e. there is no element z of S such that $y < z < x$). Thus, in order to prove our theorem, for each isomorphism class of finite-dimensional indecomposable representations of kS/J we have to find a representative Y such that $Y(y, x)$ is given by a matrix whose coefficients are only 0 or 1 for all direct neighbours $y < x$ in S. This will be done by constructing suitable bases in the spaces $X(x)$ for an arbitrarily chosen representative X.

Let us first introduce a class of representations of kS/J which has by definition the desired bases. A subset T of S is said to be strictly convex if it is a convex connected subset of S which does not contain y and x simultaneously for any generator $\rho = (y, x)$ of J. It is rather obvious that from a strictly convex subset T one gets an indecomposable representation X_T of kS/J by putting $X_T(x) := k$ for $x \in T$, $X_T(x) := 0$ for $x \notin T$, $X_T(y, x) := 1$ for $x, y \in T$ and $X_T(y, x) := 0$ in all other cases. An example for a strictly convex subset of S is the set $\check{s}$ which for a given element s of S is the set of all elements $y \leq s$ in S such that (y, s) is not contained in J. Note, that $P_s := X_{\check{s}}$ is the indecomposable projective representation of kS/J attached to s.

Unfortunately, not all indecomposable representations are isomorphic to some X_T. But it is shown in [Dr3] that any X can be written as fiber sum over thin start modules with respect to an element element s of S such that the dimension of $X(s)$ is maximal among all the dimensions of the spaces $X(x)$. This result is obtained from the data base in the package `esrdalg` of CREP by calculating the Auslander-Reiten quivers of all exceptional sincere completely separating algebras kS/J (see [Dr1], [Dr2]). For the algebras in the regular families classified in [Bo1] the result is easy to see. For our purposes it is not necessary to review details about fiber sum functors (see e.g. [Dr3] and [Dr5]). We only need that writing X as fiber sum over thin start modules with respect to s means the existence of an exact sequence

$$P_s^m \xrightarrow{\delta} \oplus_{i=1}^n X_{T_i} \xrightarrow{\sigma} X \to 0$$

where all the sets T_i are strictly convex subsets of S containing s. In addition, δ is an indecomposable matrix representation (see [NR]) of the set $\{T_1, \dots, T_n\}$ which is partially ordered by putting $T_i \geq T_j$ if there is a homomorphism $\phi : X_{T_i} \to X_{T_j}$ such that $\phi_s \neq 0$. Let us put $Y := \oplus_{i=1}^n X_{T_i}$. If we equip the spaces $P_s^m(x)$ resp. $Y(x)$ with the canonical bases induced from the summands P_s resp. X_{T_i}, then the linear map $\delta_s : k^m \to k^n$ is given by one of Kleiner's exact matrix representations of partially ordered sets of finite type displayed in [Kl].

The crucial observation is that all matrices in Kleiner's list only have the coefficients 0 and 1. Moreover, it is easy to see that also the cokernel map $\sigma_s : k^n \to X(s)$ can be taken from Kleiner's list after choosing appropriate bases in $X(s)$ and possibly replacing δ by $\psi\delta$ using some autmorphism ψ of Y. This is the information we will need to complete our proof.

A $m \times n$-matrix E over the field k is said to be elementary if the non-zero columns form a family of pairwise different canonical base vectors. Examples of elementary matrices are unit matrices. An elementary matrix E is said to be a projection if the rank of E is m and an injection if the rank of E is n. We will need that for a matrix C having only the entries 0 and 1 also the products EC and $E'C$ with elementary matrices E and E' have only the entries 0 and 1. From the definition of X_T we obtain that all $Y(y, x)$ are elementary matrices.

Let us now consider the elements y of S such $y < s$ and $X(y) \neq 0$. Since $s \in T_i$ we obtain that $Y(y, s)$ is an epimorphism given by a projection matrix. Due to the following commutative diagram also $X(y, s)$ has to be an epimorphism.

$$\begin{array}{ccc} Y(s) & \xrightarrow{\sigma_s} & X(s) \\ {\scriptstyle Y(y,s)}\downarrow & & \downarrow{\scriptstyle X(y,s)} \\ Y(y) & \xrightarrow{\sigma_y} & X(y) \end{array}$$

Consequently, using $X(y, s) = X(y, x)X(x, s)$, we can choose bases simultaneously in all $X(y)$ with $y < s$ such that $X(y, x)$ is represented by a projection matrix for all $y < x \leq s$.

Let us look at the commutative diagram once more. As $Y(y, s)$ is given by a projection matrix, there is an injection matrix L such that $Y(y, s)L$ is the identity matrix. Hence, σ_y can be calculated as $X(y, s)\sigma_s L$. Because $X(y, s)$ and L are elementary matrices, also σ_y is given by a matrix which has only the entries 0 and 1.

If $y \leq x$ are elements of S such that $x \not\leq s$ and $y \not\leq s$, then we can choose σ_x and σ_y to be identities and $X(y, x) = Y(y, x)$ is represented by an elementary matrix.

It remains to deal with the case that x is an upper neighbour of y in S and $x \not\leq s$, $y \leq s$. We obtain the following commutative diagram:

$$\begin{array}{ccc} Y(x) & = & X(x) \\ {\scriptstyle Y(y,x)}\downarrow & & \downarrow{\scriptstyle X(y,x)} \\ Y(y) & \xrightarrow{\sigma_y} & X(y) \end{array}$$

With respect to the chosen bases $Y(y,x)$ is given by an elementary matrix and σ_y by a matrix which has only the entries 0 and 1. Hence, also the matrix $X(y,x)$ has this property and our proof is complete.

3 Final remarks

In the introduction we promised to explain that our methods could be used to obtain even more precise results on the normal forms by more programming effort. In fact, CREP can be used to obtain explicitely for each indecomposable representation X of a completely separating algebra the strictly convex sets $T_1, \ldots, T_n$. But this is the only information one would need to construct the representing matrices for the $X(y,x)$ explicitely.

One still needs implementations of algorithms producing the standard form of a representation-finite algebra and its universal Galois covering. This already exists in CREP for the special case of monomial algebras. We refer to [No] for details. A more general version is in preparation.

We would like to know if researchers in physics would be interested in these programs.

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