DECOMPOSING THICK SUBCATEGORIES OF THE STABLE MODULE CATEGORY

HENNING KRAUSE

ABSTRACT. Let $\underline{\mathrm{mod}}\, kG$ be the stable category of finitely generated modular representations of a finite group G over a field k. We prove a Krull-Schmidt theorem for thick subcategories of $\underline{\mathrm{mod}}\, kG$. It is shown that every thick tensor-ideal $\mathcal{C}$ of $\underline{\mathrm{mod}}\, kG$ (i.e. a thick subcategory which is a tensor ideal) has a (usually infinite) unique decomposition $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$ into indecomposable thick tensor-ideals. This decomposition follows from a decomposition of the corresponding idempotent kG-module $E_{\mathcal{C}}$ into indecomposable modules. If $\mathcal{C} = \mathcal{C}_W$ is the thick tensor-ideal corresponding to a closed homogeneous subvariety W of the maximal ideal spectrum of the cohomology ring $H^*(G,k)$, then the decomposition of $\mathcal{C}$ reflects the decomposition $W = \bigcup_{i=1}^n W_i$ of W into connected components.

INTRODUCTION

In modular representation theory of finite groups, one frequently passes to the stable module category which is a triangulated category. Following ideas from stable homotopy theory, Benson, Carlson, and Rickard studied in a number of papers the lattice of thick subcategories of the stable module category. In [4], these authors obtain for p-groups a description of this lattice which is an analogue of a result for the stable homotopy category due to Devinatz, Hopkins, and Smith [8, 11]. In this paper we prove a Krull-Schmidt theorem for this lattice which we now explain.

Let $\Lambda = kG$ be the group algebra of a finite group G over a field k. The stable category $\underline{\mathrm{mod}}\,\Lambda$ of finitely generated Λ-modules is a triangulated category and carries a tensor product which is induced from the tensor product on $\mathrm{mod}\,\Lambda$. A full subcategory $\mathcal{C}$ of $\underline{\mathrm{mod}}\,\Lambda$ is called a *thick tensor-ideal* in $\underline{\mathrm{mod}}\,\Lambda$ if $\mathcal{C}$ is a thick subcategory and $X \otimes Y \in \mathcal{C}$ for all $X \in \mathcal{C}$ and $Y \in \underline{\mathrm{mod}}\,\Lambda$. Given a thick tensor-ideal $\mathcal{C}$, we say that a family $(\mathcal{C}_i)_{i\in I}$ of thick tensor-ideals in $\underline{\mathrm{mod}}\,\Lambda$ is a *decomposition* of $\mathcal{C}$ if

(1) the objects in $\mathcal{C}$ are the finite coproducts of objects from the $\mathcal{C}_i$;
(2) $\mathcal{C}_i \cap \mathcal{C}_j = 0$ for all $i \neq j$ in I.

A decomposition $(\mathcal{C}_i)_{i\in I}$ of $\mathcal{C}$ is denoted by $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$, and we say that $\mathcal{C}$ is *indecomposable* if $\mathcal{C} \neq 0$ and any decomposition $\mathcal{C} = \mathcal{C}_1 \coprod \mathcal{C}_2$ implies $\mathcal{C}_1 = 0$ or $\mathcal{C}_2 = 0$.

Theorem A. *Every thick tensor-ideal $\mathcal{C}$ in $\underline{\mathrm{mod}}\,\Lambda$ has a unique decomposition $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$ into indecomposable thick tensor-ideals. Conversely, given any family $(\mathcal{C}_i)_{i\in I}$ of thick tensor-ideals in $\underline{\mathrm{mod}}\,\Lambda$ satisfying $\mathcal{C}_i \cap \mathcal{C}_j = 0$ for all $i \neq j$ in I, there exists a thick tensor-ideal $\mathcal{C}$ such that $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$.*

Before we discuss this result in more detail let us mention that there seems to be no analogous statement for the stable homotopy category. Our analysis of thick subcategories in $\underline{\mathrm{mod}}\,\Lambda$ is based on the analysis of certain objects in the stable category $\underline{\mathrm{Mod}}\,\Lambda$ of all Λ-modules. In fact, we are able to compute the centre $Z[\mathcal{C}^{\amalg}]$ of the localizing subcategory $\mathcal{C}^{\amalg}$ of $\underline{\mathrm{Mod}}\,\Lambda$ which is generated by a thick tensor-ideal $\mathcal{C}$ in $\underline{\mathrm{mod}}\,\Lambda$. Recall that the centre of an additive category is the endomorphism ring of the identity functor.

We prove that $Z[\mathcal{C}^{\amalg}]$ is a product of commutative local rings, and the corresponding decomposition $\mathrm{id}_{\mathcal{C}} = (e_i)_{i\in I}$ of the identity functor into central idempotents gives the decomposition $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$ if we define $\mathcal{C}_i = e_i(\mathcal{C})$ for every i. Note that the centre of the stable homotopy category is $\mathbb{Z}$ which is not a local ring. Therefore one cannot expect an analogue of Theorem A for the stable homotopy category because any Krull-Schmidt theorem requires local endomorphism rings for the indecomposable objects.

In [18], Rickard introduced for every thick subcategory $\mathcal{C}$ of $\underline{\mathrm{mod}}\,\Lambda$ a particular Λ-module $E_{\mathcal{C}}$. It is obtained from the trivial representation k by applying a right adjoint $e\colon \underline{\mathrm{Mod}}\,\Lambda \to \mathcal{C}^{\amalg}$ of the inclusion $\mathcal{C}^{\amalg} \to \underline{\mathrm{Mod}}\,\Lambda$, i.e. $E_{\mathcal{C}} = e(k)$. If $\mathcal{C}$ is a tensor-ideal, then the canonical map $E_{\mathcal{C}} \to k$ induces an isomorphism $E_{\mathcal{C}} \otimes E_{\mathcal{C}} \to E_{\mathcal{C}}$ in $\underline{\mathrm{Mod}}\,\Lambda$, and therefore $E_{\mathcal{C}}$ is called *idempotent*. In recent years, the study of such modules has become one of the most successful methodologies in modular representation theory of finite groups. We obtain Theorem A from the following result since $Z[\mathcal{C}^{\amalg}] \simeq \underline{\mathrm{End}}_{\Lambda}(E_{\mathcal{C}})$.

Theorem B. *The Λ-module $E_{\mathcal{C}}$ has, up to isomorphism, a unique decomposition $E_{\mathcal{C}} = \coprod_{i\in I} E_i$ into indecomposable Λ-modules with local endomorphism ring. Moreover, the endomorphism ring of $E_{\mathcal{C}}$ has the following properties:*

(1) $\bigcap_{n\in\mathbb{N}} \mathfrak{r}^n = 0$ *for the Jacobson radical* $\mathfrak{r}$ *of* $\underline{\mathrm{End}}_{\Lambda}(E_{\mathcal{C}})$*;*
(2) *if $\mathcal{C}$ is a tensor-ideal, then* $\underline{\mathrm{End}}_{\Lambda}(E_{\mathcal{C}}) \simeq \prod_{i\in I} \underline{\mathrm{End}}_{\Lambda}(E_i)$

The decomposition of the module $E_{\mathcal{C}}$ is the consequence of a more general result. In fact, we introduce the concept of an endofinite object in a compactly generated triangulated category and prove that any endofinite object decomposes uniquely into indecomposable objects. Theorem B then follows since $E_{\mathcal{C}}$ is an endofinite object in the localizing subcategory of $\underline{\mathrm{Mod}}\,\Lambda$ which is generated by $\mathcal{C}$.

Interesting examples of endofinite objects also arise in stable homotopy theory. For instance, the classifying space BG of G is endofinite in the category of spectra [15]. Therefore we obtain a new proof for the existence of a stable splitting of BG which does not involve Segal's conjecture; see Benson's survey [2] for more details.

Let us end this introduction with a geometric interpretation of Theorem A in terms of the projective prime ideal spectrum of the cohomology ring $H^*(G,k)$. More precisely, we denote by $\mathcal{V}_G(k)$ the set of closed homogeneous irreducible non-zero subvarieties of the maximal ideal spectrum $V_G(k)$ of $H^*(G,k)$. Generalizing the definition of a closed homogeneous subvariety of $V_G(k)$, Benson, Carlson, and Rickard consider in [3] subsets $\mathcal{W}$ of $\mathcal{V}_G(k)$ which are closed under *specialization*, i.e. $W \in \mathcal{W}$ and $V \subseteq W$ implies $V \in \mathcal{W}$. For instance, any closed homogeneous subvariety W of $V_G(k)$ is determined by $\mathcal{W} = \{V \in \mathcal{V}_G(k) \mid V \subseteq W\}$. Given any subset $\mathcal{W}$ of $\mathcal{V}_G(k)$ closed under specialization, we say that $\mathcal{W}$ is *projectively connected* if any decomposition $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$ into subsets of $\mathcal{V}_G(k)$ closed under specialization such that $\mathcal{W}_1 \cap \mathcal{W}_2 = \emptyset$ implies $\mathcal{W} = \mathcal{W}_1$ or $\mathcal{W} = \mathcal{W}_2$. The following result generalizes the decomposition $W = \bigcup_{i=1}^n W_i$ of a closed homogeneous subvariety of $V_G(k)$ into connected components.

Theorem C. *Every subset $\mathcal{W}$ of $\mathcal{V}_G(k)$ closed under specialization has a unique decomposition $\mathcal{W} = \bigcup_{i\in I} \mathcal{W}_i$ into projectively connected non-empty subsets of $\mathcal{V}_G(k)$ closed under specialization such that $\mathcal{W}_i \cap \mathcal{W}_j = \emptyset$ for all $i \neq j$ in I.*

1. Endofiniteness

Let $\mathcal{T}$ be a triangulated category [19] and suppose that arbitrary coproducts exist in $\mathcal{T}$. An object X in $\mathcal{T}$ is called *compact* if for every family $(Y_i)_{i\in I}$ in $\mathcal{T}$ the canonical map $\coprod_i \mathrm{Hom}(X, Y_i) \to \mathrm{Hom}(X, \coprod_i Y_i)$ is an isomorphism. We denote by $\mathcal{T}_0$ the full subcategory of compact objects in $\mathcal{T}$ and observe that $\mathcal{T}_0$ is a triangulated subcategory of $\mathcal{T}$. Following [17], the category $\mathcal{T}$ is called *compactly generated* provided that the isomorphism classes of objects in $\mathcal{T}_0$ form a set, and $\operatorname{Hom}(C, X) = 0$ for all C in $\mathcal{T}_0$ implies $X = 0$ for every object X in $\mathcal{T}$.

In [7], Crawley-Boevey introduced the concept of endofiniteness for locally finitely presented categories. We make the following analogous definition for a compactly generated triangulated category $\mathcal{T}$.

Definition 1.1. An object X in $\mathcal{T}$ is called *endofinite* if for every compact object C in $\mathcal{T}$ the $\operatorname{End}(X)$-module $\operatorname{Hom}(C, X)$ has finite composition length.

The following result collects some useful properties of endofinite objects.

Theorem 1.2. *An endofinite object X has the following properties:*

(1) *there exists, up to isomorphism, a unique decomposition $X = \coprod_{i \in I} X_i$ into indecomposable objects with local endomorphism ring;*
(2) *$\bigcap_{n \in \mathbb{N}} \mathfrak{r}^n = 0$ for the Jacobson radical $\mathfrak{r}$ of $\operatorname{End}(X)$;*
(3) *if $\phi\colon Y \to X$ is a phantom map, i.e. the induced map $\operatorname{Hom}(C, Y) \to \operatorname{Hom}(C, X)$ is zero for all compact C, then $\phi = 0$.*

Proof. The proof uses the fact that the category $\mathcal{T}$ can be embedded (modulo the phantom maps in $\mathcal{T}$) into an abelian Grothendieck category. We recall briefly this construction and refer to [14] for more details. Let $\operatorname{Mod} \mathcal{T}_0$ be the category of additive functors $\mathcal{T}_0^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups, and denote by $\operatorname{mod} \mathcal{T}_0$ the full subcategory of finitely presented functors. We shall use the *Yoneda functor*

$$h\colon \mathcal{T} \longrightarrow \operatorname{Mod} \mathcal{T}_0, \quad X \mapsto H_X = \operatorname{Hom}(-, X)|_{\mathcal{T}_0}.$$

If X is endofinite, then $\operatorname{Hom}(F, H_X)$ is a $\operatorname{End}(H_X)$-module of finite length for all F in $\operatorname{mod} \mathcal{T}_0$, and therefore H_X is an endofinite object of $\operatorname{Mod} \mathcal{T}_0$ in the sense of [7]. We claim that H_X is an injective object. To this end let $\varepsilon\colon 0 \to H_X \to M \to N \to 0$ be an exact sequence. Every map $F \to N$ with F in $\operatorname{mod} \mathcal{T}_0$ factors through $M \to N$ since H_X is *fp-injective*, i.e. $\operatorname{Ext}^1(-, H_X)$ vanishes on finitely presented objects, by [14, Lemma 1.6]. Thus ε is a pure-exact sequence which is split exact since every endofinite object is pure-injective [7, 3.6]. The injectivity of H_X implies that h induces an isomorphism $\operatorname{End}(X) \to \operatorname{End}(H_X)$ by [14, Lemma 1.7]. Any endofinite object in $\operatorname{Mod} \mathcal{T}_0$ has a decomposition into indecomposable objects with local endomorphism ring by [7, Theorem 3.5.2], and this gives the desired decomposition of X in $\mathcal{T}$.

Let us give an alternative proof for (1). To this end consider the localizing subcategory $\mathcal{L}$ of $\operatorname{Mod} \mathcal{T}_0$ which is generated by all F in $\operatorname{mod} \mathcal{T}_0$ with $\operatorname{Hom}(F, H_X) = 0$. We can form the quotient category $\operatorname{Mod} \mathcal{T}_0/\mathcal{L}$ in the sense of [9], and obtain a locally finite Grothendieck category since the objects in $\operatorname{mod} \mathcal{T}_0$ form a generating set of finite length objects. Moreover, the quotient functor $q\colon \operatorname{Mod} \mathcal{T}_0 \to \operatorname{Mod} \mathcal{T}_0/\mathcal{L}$ identifies H_X with an injective object in $\operatorname{Mod} \mathcal{T}_0/\mathcal{L}$. More precisely, $q(H_X)$ is fp-injective in $\operatorname{Mod} \mathcal{T}_0/\mathcal{L}$ since H_X is fp-injective in $\operatorname{Mod} \mathcal{T}_0$ by [14, Lemma 1.6]. However, any fp-injective object in a locally noetherian category is injective by an analogue of Baer's criterion for Grothendieck categories. It follows from [12, Corollary 2.11] that H_X is *$\mathcal{L}$-closed*, i.e. $\operatorname{Hom}(\mathcal{L}, H_X) = 0 = \operatorname{Ext}^1(\mathcal{L}, H_X)$, since $\mathcal{L}$ is generated by finitely presented objects. Therefore H_X is an injective object in $\operatorname{Mod} \mathcal{T}_0$ since $q(H_X)$ is injective, and $\operatorname{End}(H_X) \simeq \operatorname{End}(q(H_X))$; see [9]. Any injective object in a locally finite category decomposes uniquely into indecomposables with local endomorphism ring [9, IV.2], and this gives an alternative proof for (1) since

$$\operatorname{End}(X) \simeq \operatorname{End}(H_X) \simeq \operatorname{End}(q(H_X)).$$

Part (2) follows from the general fact that $\bigcap_{n\in\mathbb{N}} \operatorname{rad}^n \operatorname{End}(Q) = 0$ for every injective object Q in a locally finite category [9, IV.4]. To prove (3) consider a phantom map $\phi\colon Y \to X$. The functor h induces an isomorphism $\operatorname{Hom}(Y,X) \to \operatorname{Hom}(H_Y, H_X)$, by [14, Lemma 1.7], since H_X is injective, and therefore $\phi = 0$. This finishes the proof. $\square$

We include for later reference the following lemma.

Lemma 1.3. *Let $\mathcal{S}$ be a localizing subcategory of $\mathcal{T}$ which is generated by compact objects from $\mathcal{T}$, and denote by $e\colon \mathcal{T} \to \mathcal{S}$ a right adjoint of the inclusion $\mathcal{S} \to \mathcal{T}$.*

(1) *$\mathcal{S}$ is a compactly generated triangulated category.*
(2) *If X is an endofinite object in $\mathcal{T}$, the $e(X)$ is endofinite in $\mathcal{S}$.*

Proof. Suppose that $\mathcal{S}$ is generated by a class $\mathcal{D}$ of objects in $\mathcal{T}_0$ and denote by $\mathcal{C}$ the thick subcategory of $\mathcal{T}_0$ which is generated by $\mathcal{D}$. Clearly, $\mathcal{C} \subseteq \mathcal{S}_0$. Suppose now that $\operatorname{Hom}(\mathcal{S}_0, X) = 0$ for some X in $\mathcal{S}$, and denote by $\mathcal{R}$ the localizing subcategory of all Y in $\mathcal{T}$ such that $\operatorname{Hom}(Y, X[n]) = 0$ for all $n \in \mathbb{Z}$. Clearly, $\mathcal{R}$ contains $\mathcal{S}$ and therefore $X = 0$. It is not hard to check that $\mathcal{S}_0 = \mathcal{C}$ (e.g., see [16, Lemma 2.2]), and we conclude that $\mathcal{S}$ is compactly generated.

Now let X be an endofinite object in $\mathcal{T}$ and fix C in $\mathcal{S}_0$. The functorial isomorphism $\operatorname{Hom}(C,X) \simeq \operatorname{Hom}(C, e(X))$ shows that $\operatorname{Hom}(C, e(X))$ has finite length over $\operatorname{End}(e(X))$ since C is also compact in $\mathcal{T}$. Therefore $e(X)$ is endofinite in $\mathcal{S}$. $\square$

Example 1.4. (1) A spectrum X is endofinite in the stable homotopy category if and only if the stable homotopy groups $\pi_n^s(X)$ have finite length over the endomorphism ring $\{X;X\}$ of X for all $n \in \mathbb{Z}$; see [15].

(2) A Λ-module X has *finite endolength* (i.e. the length of X as module over $\operatorname{End}_\Lambda(X)$ is finite) if and only if X is an endofinite object in the stable module category $\underline{\operatorname{Mod}}\,\Lambda$.

2. Decomposing thick subcategories

Let $\Lambda = kG$ be the group algebra of a finite group G over a field k. We consider the category $\operatorname{Mod}\Lambda$ of (right) Λ-modules, and $\operatorname{mod}\Lambda$ denotes the full subcategory of all finitely generated Λ-modules. The algebra Λ is *quasi-Frobenius*, i.e. projective and injective Λ-modules coincide. Therefore the *stable category* $\underline{\operatorname{Mod}}\,\Lambda$ is triangulated; e.g. see [10]. Recall that the objects in $\underline{\operatorname{Mod}}\,\Lambda$ are those of $\operatorname{Mod}\Lambda$, and for two Λ-modules X, Y one defines $\underline{\operatorname{Hom}}_\Lambda(X,Y)$ to be $\operatorname{Hom}_\Lambda(X,Y)$ modulo the subgroup of maps which factor through a projective Λ-module. Note that the projection functor $\operatorname{Mod}\Lambda \to \underline{\operatorname{Mod}}\,\Lambda$ preserves coproducts. Thus $\underline{\operatorname{Mod}}\,\Lambda$ has arbitrary coproducts, and it is not difficult to check that an object X in $\underline{\operatorname{Mod}}\,\Lambda$ is compact if and only if $X \simeq Y$ in $\underline{\operatorname{Mod}}\,\Lambda$ for some finitely generated Λ-module Y. Therefore we shall not distinguish between $(\underline{\operatorname{Mod}}\,\Lambda)_0$ and $\underline{\operatorname{mod}}\,\Lambda$. The description of the compact objects shows that $\underline{\operatorname{Mod}}\,\Lambda$ is compactly generated.

2.1. **Thick subcategories.** Let $\mathcal{C}$ be a *thick subcategory* of $\underline{\operatorname{mod}}\,\Lambda$, i.e. $\mathcal{C}$ is a full triangulated subcategory of $\underline{\operatorname{mod}}\,\Lambda$ which is closed under taking direct summands. We denote by $\mathcal{C}^{\amalg}$ the localizing subcategory of $\underline{\operatorname{Mod}}\,\Lambda$ which is generated by $\mathcal{C}$. Recall that a full triangulated subcategory of $\underline{\operatorname{Mod}}\,\Lambda$ is *localizing* if it is closed under taking coproducts. The inclusion $\mathcal{C}^{\amalg} \to \underline{\operatorname{Mod}}\,\Lambda$ has a right adjoint $e\colon \underline{\operatorname{Mod}}\,\Lambda \to \mathcal{C}^{\amalg}$ (e.g., see [16, 18]), and we denote by $\varepsilon\colon E_{\mathcal{C}} = e(k) \to k$ the corresponding map for the trivial representation k. Using the results about endofinite objects we can now prove the first portion of Theorem B.

Theorem 2.1. *The Λ-module $E_{\mathcal{C}}$ has, up to isomorphism, a unique decomposition $E_{\mathcal{C}} = \coprod_{i\in I} E_i$ into indecomposable Λ-modules with local endomorphism ring. Moreover, $\bigcap_{n\in\mathbb{N}} \mathfrak{r}^n = 0$ for the Jacobson radical $\mathfrak{r}$ of $\underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}})$.*

Proof. The trivial representation k is certainly an endofinite object in $\underline{\mathrm{Mod}}\, kG$, and therefore $E_{\mathcal{C}}$ is endofinite in $\mathcal{C}^{\amalg}$ by Lemma 1.3. We can apply Theorem 1.2 and obtain a decomposition $E_{\mathcal{C}} = \coprod_{i\in I} E_i$ into indecomposables in $\underline{\mathrm{Mod}}\,\Lambda$ plus the assertion about the Jacobson radical of $\underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}})$. It is well-known that any Λ-module X has a decomposition $X = X_P \coprod P_X$ such that X_P has no non-zero projective direct sumand and P_X is projective [13, Proposition 5.10]. Clearly, any projective Λ-module is a coproduct of indecomposable modules since Λ is artinian. On the other hand, the endomorphisms of X_P which factor through projectives belong to the Jacobson radical of $\mathrm{End}_\Lambda(X_P)$, and therefore the decomposition of $E_{\mathcal{C}}$ in $\underline{\mathrm{Mod}}\,\Lambda$ gives the desired decomposition in $\mathrm{Mod}\,\Lambda$. □

Remark 2.2. Let $\mathfrak{s}$ be the Jacobson radical of $\mathrm{End}_\Lambda(E_{\mathcal{C}})$. Then $(\bigcap_{n\in\mathbb{N}} \mathfrak{s}^n)^l = 0$ where l denotes the Loewy length of Λ.

2.2. **Thick tensor-ideals.** Let $\mathcal{C}$ be a thick subcategory of $\underline{\mathrm{mod}}\,\Lambda$ and assume in addition that $X \otimes Y \in \mathcal{C}$ for all $X \in \mathcal{C}$ and $Y \in \underline{\mathrm{mod}}\,\Lambda$. We call a subcategory $\mathcal{C}$ with these properties a *thick tensor-ideal* in $\underline{\mathrm{mod}}\,\Lambda$. We shall use the following description of the objects in $\mathcal{C}^{\amalg}$ which is due to Rickard.

Proposition 2.3. *Suppose that $\mathcal{C}$ is a thick tensor-ideal in $\underline{\mathrm{mod}}\,\Lambda$. Then the following are equivalent for an object X in $\underline{\mathrm{Mod}}\,\Lambda$:*
(1) *X belongs to $\mathcal{C}^{\amalg}$;*
(2) *$\varepsilon \otimes X \colon E_{\mathcal{C}} \otimes X \to k \otimes X \simeq X$ is an isomorphism.*

Proof. Combine Proposition 5.6 and Proposition 5.13 from [18]. □

The following lemma will be useful for decomposing a thick tensor-ideal.

Lemma 2.4. *Let $\mathcal{C}$ and $\mathcal{D}$ be thick tensor-ideals in $\underline{\mathrm{mod}}\,\Lambda$. Then the following are equivalent:*
(1) *$\mathcal{C} \cap \mathcal{D} = 0$;*
(2) *$\mathcal{C} \otimes \mathcal{D} = 0$, i.e. $X \otimes Y = 0$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$;*
(3) *$\underline{\mathrm{Hom}}_\Lambda(\mathcal{C}, \mathcal{D}) = 0$, i.e. $\underline{\mathrm{Hom}}_\Lambda(X, Y) = 0$ for all $X \in \mathcal{C}$ and $Y \in \mathcal{D}$;*
(4) *$\underline{\mathrm{Hom}}_\Lambda(\mathcal{C}^{\amalg}, \mathcal{D}^{\amalg}) = 0$, i.e. $\underline{\mathrm{Hom}}_\Lambda(X, Y) = 0$ for all $X \in \mathcal{C}^{\amalg}$ and $Y \in \mathcal{D}^{\amalg}$;*
(5) *$\mathcal{C}^{\amalg} \otimes \mathcal{D}^{\amalg} = 0$, i.e. $X \otimes Y = 0$ for all $X \in \mathcal{C}^{\amalg}$ and $Y \in \mathcal{D}^{\amalg}$;*
(6) *$E_{\mathcal{C}} \otimes E_{\mathcal{D}} = 0$.*
(7) *$\mathcal{C}^{\amalg} \cap \mathcal{D}^{\amalg} = 0$;*

Proof. (1) ⇒ (2) Clear, since $\mathcal{C} \otimes \mathcal{D} \subseteq \mathcal{C} \cap \mathcal{D}$.

(2) ⇒ (3) Let $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Then the k-dual X^* is a direct summand of $X^* \otimes X \otimes X^*$ and belongs therefore to $\mathcal{C}$. It follows that $\mathrm{Hom}_k(X, Y) \simeq X^* \otimes Y$ is a projective Λ-module since $\mathcal{C} \otimes \mathcal{D} = 0$. Any Λ-map $X \to Y$ factors through $X \otimes \mathrm{Hom}_k(X, Y)$, and therefore $\underline{\mathrm{Hom}}_\Lambda(X, Y) = 0$.

(3) ⇒ (4) Let $\mathcal{D}'$ be the localizing subcategory of all X in $\underline{\mathrm{Mod}}\,\Lambda$ such that $\underline{\mathrm{Hom}}_\Lambda(\mathcal{C}, X) = 0$. Clearly, $\mathcal{D}^{\amalg} \subseteq \mathcal{D}'$ since $\mathcal{D} \subseteq \mathcal{D}'$. Now let $\mathcal{C}'$ be the localizing subcategory of all X in $\underline{\mathrm{Mod}}\,\Lambda$ such that $\underline{\mathrm{Hom}}_\Lambda(X, \mathcal{D}') = 0$. We have $\mathcal{C}^{\amalg} \subseteq \mathcal{C}'$ since $\mathcal{C} \subseteq \mathcal{C}'$, and therefore $\underline{\mathrm{Hom}}_\Lambda(\mathcal{C}^{\amalg}, \mathcal{D}^{\amalg}) = 0$.

(4) $\Rightarrow$ (5) If $\underline{\mathrm{Hom}}_\Lambda(\mathcal{C}^\amalg, \mathcal{D}^\amalg) = 0$, then $\mathcal{C}^\amalg \otimes \mathcal{D}^\amalg = 0$ since $\mathcal{C}^\amalg \otimes \mathcal{D}^\amalg \subseteq \mathcal{C}^\amalg \cap \mathcal{D}^\amalg$ by Proposition 2.3.

(5) $\Rightarrow$ (6) Clear.

(6) $\Rightarrow$ (7) Clear, since $E_\mathcal{C} \otimes E_\mathcal{D} \otimes X \simeq X$ for every $X \in \mathcal{C}^\amalg \cap \mathcal{D}^\amalg$ by Proposition 2.3.

(7) $\Rightarrow$ (1) Clear. ☐

We continue with two results which establish the strong connection between decompositions of $\mathcal{C}$ and $E_\mathcal{C}$. To this end the following definition is needed. Let $\varepsilon \colon E \to k$ be a map in $\underline{\mathrm{Mod}}\,\Lambda$. We denote by $\mathcal{L}_\varepsilon$ the localizing subcategory of all X in $\underline{\mathrm{Mod}}\,\Lambda$ such that $\varepsilon \otimes X$ is an isomorphism. The intersection $\mathcal{C}_\varepsilon = \mathcal{L}_\varepsilon \cap \underline{\mathrm{mod}}\,\Lambda$ gives a thick subcategory of $\underline{\mathrm{mod}}\,\Lambda$, and we observe that $\mathcal{C}_\varepsilon$ is a thick tensor-ideal if $\varepsilon \otimes E$ is an isomorphism.

Proposition 2.5. *Let $\mathcal{C}$ be a thick tensor-ideal in $\underline{\mathrm{mod}}\,\Lambda$. Every decomposition $E_\mathcal{C} = \coprod_i E_i$ satisfying $E_i \otimes E_j = 0$ for all $i \neq j$, induces a decomposition $\mathcal{C} = \coprod_i \mathcal{C}_i$ of $\mathcal{C}$ such that $E_i \simeq E_{\mathcal{C}_i}$ for all i.*

Proof. Let $(\varepsilon_i) \colon \coprod_i E_i \to k$ be the decomposition of $\varepsilon \colon E_\mathcal{C} \to k$. The assumption $E_i \otimes E_j = 0$ for all $i \neq j$ implies that $\varepsilon_i \otimes E_i$ is an isomorphism for all i. Therefore we get a decomposition $\mathcal{C} = \coprod_i \mathcal{C}_i$ of $\mathcal{C}$ if we define $\mathcal{C}_i = \mathcal{C}_{\varepsilon_i}$ for all i. It remains to show that $E_i \simeq E_{\mathcal{C}_i}$ for all i. Observe first that $E_i \otimes E_{\mathcal{C}_j} = 0$ for all $i \neq j$ since $E_i \otimes \mathcal{C}_j = 0$. Now consider the canonical map $\phi \colon \coprod_j E_{\mathcal{C}_j} \to k$. We have $E_{\mathcal{C}_i} \otimes E_{\mathcal{C}_j} = 0$ for all $i \neq j$ by Lemma 2.4. Therefore $\mathcal{C} \subseteq \mathcal{L}_\phi$ and consequently $\mathcal{C}^\amalg \subseteq \mathcal{L}_\phi$. It follows that

$$E_i \simeq E_i \otimes (\coprod_j E_{\mathcal{C}_j}) \simeq \coprod_j (E_i \otimes E_{\mathcal{C}_j}) \simeq E_i \otimes E_{\mathcal{C}_i}.$$

On the other hand, $E_i \otimes E_{\mathcal{C}_i} \simeq E_{\mathcal{C}_i}$ since $\mathcal{C}_i^\amalg \subseteq \mathcal{L}_{\varepsilon_i}$. Thus $E_i \simeq E_{\mathcal{C}_i}$ for all i. ☐

Proposition 2.6. *Let $\mathcal{C}$ be a thick tensor-ideal in $\underline{\mathrm{mod}}\,\Lambda$. Every decomposition $\mathcal{C} = \coprod_i \mathcal{C}_i$ of $\mathcal{C}$ induces a decomposition $E_\mathcal{C} = \coprod_i E_i$ such that $E_i \otimes E_j = 0$ for all $i \neq j$ and $E_{\mathcal{C}_i} \simeq E_i$ for all i.*

Proof. We define $E_i = E_{\mathcal{C}_i}$ for all i and observe that $E_i \otimes E_j = 0$ for all $i \neq j$ by Lemma 2.4. We claim that $E_\mathcal{C} \simeq \coprod_i E_i$. To this end consider the canonical map $\phi \colon \coprod_i E_i \to k$. Observe that $\mathcal{C}^\amalg \subseteq \mathcal{L}_\phi$ since $\mathcal{C} \subseteq \mathcal{L}_\phi$. Therefore $E_\mathcal{C} \simeq (\coprod_i E_i) \otimes E_\mathcal{C}$. On the other hand,

$$(\coprod_i E_i) \otimes E_\mathcal{C} \simeq \coprod_i (E_i \otimes E_\mathcal{C}) \simeq \coprod_i E_i$$

since $\mathcal{C}_i \subseteq \mathcal{C}$ for all i. Thus $E_\mathcal{C} \simeq \coprod_i E_i$. ☐

We are now in a position to prove the decomposition theorem for thick tensor-ideals.

Proof of Theorem A. Let $\mathcal{C}$ be a thick tensor-ideal in $\underline{\mathrm{mod}}\,\Lambda$ and fix the decomposition $E_\mathcal{C} = \coprod_{i \in I} E_i$ in $\underline{\mathrm{Mod}}\,\Lambda$ into indecomposable objects which exists by Theorem 2.1. We claim that $E_i \otimes E_j = 0$ for all $i \neq j$ in I. In fact, for every $i \in I$ we obtain a decomposition

$$E_i \simeq E_i \otimes (\coprod_{s \in I} E_s) \simeq \coprod_{s \in I} (E_i \otimes E_s),$$

and we have $E_i \simeq E_i \otimes E_t$ for precisely one $t \in I$ since E_i is indecomposable. The same argument gives $E_t \simeq E_t \otimes E_i$, and therefore $i = t$. Thus $E_i \otimes E_j = 0$ for all $i \neq j$ in I. We obtain a decomposition $\mathcal{C} = \coprod_{i \in I} \mathcal{C}_i$ from Proposition 2.5, and it follows from Proposition 2.6 that every $\mathcal{C}_i$ is indecomposable. Moreover, the uniqueness of the decomposition $\mathcal{C} = \coprod_{i \in I} \mathcal{C}_i$ (up to permutation of the $\mathcal{C}_i$) follows from the corresponding uniqueness of the decomposition $E_\mathcal{C} = \coprod_{i \in I} E_i$.

Suppose now that $(\mathcal{C}_i)_{i\in I}$ is a family of thick tensor ideals in $\underline{\mathrm{mod}}\,\Lambda$ with $\mathcal{C}_i \cap \mathcal{C}_j = 0$ for all $i \neq j$ in I. We define $\mathcal{C}$ to be the full subcategory of all finite coproducts of objects from the $\mathcal{C}_i$. Clearly, $\mathcal{C}$ is a tensor-ideal. By Lemma 2.4, $\underline{\mathrm{Hom}}_\Lambda(\mathcal{C}_i, \mathcal{C}_j) = 0$ for all $i \neq j$. Therefore $\mathcal{C}$ is a thick subcategory of $\underline{\mathrm{mod}}\,\Lambda$ since any map $\phi\colon X \to Y$ in $\mathcal{C}$ has a decomposition $(\phi_i)\colon \coprod_i X_i \to \coprod_i Y_i$ with $\phi_i \in \mathcal{C}_i$ for all i. It follows that $\mathcal{C}$ is a thick tensor-ideal with $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$. This finishes the proof of Theorem A. □

The following corollary summarizes the basic properties of a decomposition $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$.

Corollary 2.7. *Let $\mathcal{C} = \coprod_{i\in I} \mathcal{C}_i$ be a decomposition of a thick tensor-ideal in $\underline{\mathrm{mod}}\,\Lambda$. Then $E_{\mathcal{C}} \simeq \coprod_{i\in I} E_{\mathcal{C}_i}$ and $\mathcal{C}$ is indecomposable if and only if $E_{\mathcal{C}}$ is indecomposable. Moreover, $\underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}}) \simeq \prod_{i\in I} \underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}_i})$.*

2.3. **The centre of a localizing subcategory.** Given an additive category $\mathcal{A}$, we denote by $Z[\mathcal{A}]$ the *centre* of $\mathcal{A}$ which is the endomorphism ring of the identity functor $\mathrm{id}_{\mathcal{A}}$. More precisely, $Z[\mathcal{A}]$ is the ring of all natural transformation $\mathrm{id}_{\mathcal{A}} \to \mathrm{id}_{\mathcal{A}}$. Note that $Z[\mathcal{A}]$ is a commutative ring. Suppose now that $\mathcal{C}$ is a thick tensor-ideal in $\underline{\mathrm{mod}}\,\Lambda$ and let $\mathcal{C}^{\amalg}$ be the localizing subcategory which is generated by $\mathcal{C}$. Then we have the following description of $Z[\mathcal{C}^{\amalg}]$.

Proposition 2.8. *The map $Z[\mathcal{C}^{\amalg}] \to \underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}})$, $\phi \mapsto \phi_{E_{\mathcal{C}}}$, is a ring isomorphism. In particular, $\underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}})$ is a commutative ring.*

Proof. We construct an inverse for $Z[\mathcal{C}^{\amalg}] \to \underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}})$. To this end consider the canonical map $\varepsilon\colon E_{\mathcal{C}} \to k$ and recall from Proposition 2.3 that for every X in $\mathcal{C}^{\amalg}$ the map $\varepsilon_X = \varepsilon \otimes X\colon E \otimes X \to X$ is an isomorphism. If ψ is an endomorphism of $E_{\mathcal{C}}$, then we obtain for every X in $\mathcal{C}^{\amalg}$ an endomorphism $\varepsilon_X \circ (\psi \otimes X) \circ \varepsilon_X^{-1}$ of X. In fact, the map

$$\underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}}) \longrightarrow Z[\mathcal{C}^{\amalg}], \quad \psi \mapsto (\varepsilon_X \circ (\psi \otimes X) \circ \varepsilon_X^{-1})_X$$

gives a ring homomorphism which is an inverse for $Z[\mathcal{C}^{\amalg}] \to \underline{\mathrm{End}}_\Lambda(E_{\mathcal{C}})$. □

Corollary 2.9. *$Z[\mathcal{C}^{\amalg}]$ is a product of commutative local rings.*

3. Decomposing Varieties

3.1. **Varieties for finitely generated modules.** Let $H^*(G,k) = \mathrm{Ext}^*_\Lambda(k,k)$ be the cohomology ring of G over the field k, and we assume for simplicity that k is algebraically closed. We denote by $V_G(k)$ the maximal ideal spectrum of $H^*(G,k)$ and for every finitely generated Λ-module X we denote by $V(X)$ the corresponding closed homogeneous subvariety of $V_G(k)$ (see [1, Chapter 5] for precise definitions). Given a closed homogeneous subvariety W of $V_G(k)$, the finitely generated Λ-modules whose variety is contained in W form a thick tensor-ideal $\mathcal{C}_W$ in $\underline{\mathrm{mod}}\,\Lambda$. We shall see that the decomposition $\mathcal{C}_W = \coprod_i \mathcal{C}_i$ into indecomposable thick tensor-ideals corresponds to the decomposition $W = \bigcup_i W_i$ of W into connected components, i.e. $\mathcal{C}_i = \mathcal{C}_{W_i}$ for all i. We proceed in two steps.

Proposition 3.1. *Let $W = \bigcup_{i=1}^n W_i$ be a union of closed homogeneous subvarieties of $V_G(k)$ such that $W_i \cap W_j = \{0\}$ for all $i \neq j$. Then $\mathcal{C}_W = \coprod_{i=1}^n \mathcal{C}_{W_i}$.*

Proof. Clearly, $\mathcal{C}_{W_i} \subseteq \mathcal{C}_W$ and $\mathcal{C}_{W_i} \cap \mathcal{C}_{W_j} = 0$ for all $i \neq j$. Suppose now that $X \in \mathcal{C}_W$ and we may assume that X is indecomposable. Then $V(X) = \bigcup_{i=1}^n (V(X) \cap W_i)$ and therefore $V(X) = V(X) \cap W_i$ for some i since the variety of an indecomposable module is connected [6]. Thus $X \in \mathcal{C}_{W_i}$ and therefore $\mathcal{C}_W = \coprod_{i=1}^n \mathcal{C}_{W_i}$. □

Proposition 3.2. *Let W be a closed homogeneous subvariety of $V_G(k)$. Every decomposition $\mathcal{C}_W = \coprod_i \mathcal{C}_i$ into thick tensor-ideals induces a decomposition $W = \bigcup_i W_i$ into closed homogeneous subvarieties such that $W_i \cap W_j = \{0\}$ for all $i \neq j$ and $\mathcal{C}_{W_i} = \mathcal{C}_i$ for all i.*

Proof. Assume first that $\mathcal{C}_i$ is indecomposable for all i. We can choose a finitely generated Λ-module X such that $W = V(X)$ (e.g., see [1, Corollary 5.9.2]) and we assume in addition that the number of indecomposable direct summands of X is minimal. The decomposition $\mathcal{C}_W = \coprod_i \mathcal{C}_i$ gives a decomposition $W = \bigcup_i W_i$ if we define $W_i = V(X_i)$ for all i where $X = \coprod_i X_i$ denotes the decomposition of X such that $X_i \in \mathcal{C}_i$ for all i. In fact, $W = V(\coprod_i X_i) = \bigcup_i V(X_i)$ (e.g., see [1, Proposition 5.7.5]), and $W_i \cap W_j = \{0\}$ for all $i \neq j$ since $V(X_i) \cap V(X_j) = V(X_i \otimes X_j)$ (e.g., see [1, Theorem 5.7.11]) and $V(P) = \{0\}$ for any projective Λ-module P (e.g., see [1, Proposition 5.7.2]). Note that $\mathcal{C}_{W_i} \cap \mathcal{C}_i \neq 0$ for all i, since $0 \neq X_i \in \mathcal{C}_{W_i} \cap \mathcal{C}_i$. We obtain from Proposition 3.1 a decomposition $\mathcal{C}_W = \coprod_i \mathcal{C}_{W_i}$, and the uniqueness of the decomposition $\mathcal{C}_W = \coprod_i \mathcal{C}_i$ into indecomposable thick tensor-ideals from Theorem A implies $\mathcal{C}_{W_i} = \mathcal{C}_i$ for all i. If $\mathcal{C}_W = \coprod_i \mathcal{C}_i$ is an arbitrary decomposition, then we can pass to a refinement $\mathcal{C}_W = \coprod_i (\coprod_{j \in I_i} \mathcal{C}_{ij})$ with $\mathcal{C}_i = \coprod_{j \in I_i} \mathcal{C}_{ij}$ and $\mathcal{C}_{ij}$ indecomposable for all i and j, by Theorem A. Using the assertion for indecomposable decompositions, one obtains a decomposition $W = \bigcup_i (\bigcup_{j \in I_i} W_{ij})$ such that $\mathcal{C}_{W_{ij}} = \mathcal{C}_{ij}$ for all i, j. If we define $W_i = \bigcup_{j \in I_i} W_{ij}$ for every i, then Proposition 3.1 gives

$$\mathcal{C}_{W_i} = \coprod_{j \in I_i} \mathcal{C}_{W_{ij}} = \coprod_{j \in I_i} \mathcal{C}_{ij} = \mathcal{C}_i.$$

This finishes the proof. □

Recall that a closed homogeneous subvariety W of $V_G(k)$ is *projectively connected* if any decomposition $W = W_1 \cup W_2$ into closed homogeneous subvarieties with $W_1 \cap W_2 = \{0\}$ implies $W = W_1$ or $W = W_2$. We obtain the following immediate consequences of Proposition 3.1 and Proposition 3.2.

Corollary 3.3. *For a closed homogeneous subvariety W of $V_G(k)$ the following conditions are equivalent:*

(1) *W is non-zero and projectively connected;*
(2) *$\mathcal{C}_W$ is an indecomposable thick tensor-ideal in $\underline{\mathrm{mod}}\,\Lambda$;*
(3) *$E_{\mathcal{C}_W}$ is an indecomposable object in $\underline{\mathrm{Mod}}\,\Lambda$.*

Corollary 3.4. *Let W be a closed homogeneous subvariety of $V_G(k)$, and let $W = \bigcup_{i=1}^n W_i$ be the decomposition of W into projectively connected non-zero components. Then $\mathcal{C}_W = \coprod_{i=1}^n \mathcal{C}_{W_i}$ is the decomposition of $\mathcal{C}_W$ into indecomposable thick tensor ideals. Moreover, the corresponding idempotent Λ-module $E_{\mathcal{C}_W}$ has precisely n non-projective indecomposable direct summands.*

We give now an example of a thick tensor-ideal $\mathcal{C}$ such that the decomposition $\mathcal{C} = \coprod_{i \in I} \mathcal{C}_i$ into indecomposable thick tensor-ideals is infinite. However, it is clear that $\operatorname{card} I \leq \operatorname{card} \mathcal{S}$ whenever there is a set $\mathcal{S}$ of indecomposable Λ-modules such that $\mathcal{C}$ is the smallest thick tensor-ideal containing $\mathcal{S}$.

Example 3.5. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ be the Klein four group and $\operatorname{char} k = 2$. The finitely generated Λ-modules of complexity at most 1 form a thick tensor-ideal $\mathcal{C}$, and $\mathcal{C} = \coprod_{\lambda \in \mathbb{P}^1(k)} \mathcal{C}_\lambda$ where $\{\mathcal{C}_\lambda \mid \lambda \in \mathbb{P}^1(k)\}$ is the set of indecomposable thick tensor-ideal of the form $\mathcal{C}_{V_G\langle\zeta\rangle}$ for some $0 \neq \zeta \in H^1(G,k)$, and $V_G\langle\zeta\rangle$ denotes the closed homogenous subvariety of maximal ideals containing ζ.

In [5], Benson and Gnacadja conjecture that for every closed homogeneous subvariety W of $V_G(k)$ the module $E_{\mathcal{C}_W}$ is pure-injective. This conjecture becomes true provided one works in the appropriate category. Recall from [14] the following concept of purity for a compactly generated triangulated category $\mathcal{T}$. A map $X \to Y$ in $\mathcal{T}$ is a *pure monomorphism* if the induced map $\operatorname{Hom}(C,X) \to \operatorname{Hom}(C,Y)$ is a monomorphism for every compact object C in $\mathcal{T}$, and X is *pure-injective* if every pure monomorphism $X \to Y$ splits. For example, a Λ-module X is pure-injective in $\operatorname{Mod}\Lambda$ if and only if X is a pure-injective object in $\underline{\operatorname{Mod}}\,\Lambda$. The statement in part (3) of Theorem 1.2 shows that every endofinite object X in $\mathcal{T}$ is pure-injective. We conclude that for every thick subcategory $\mathcal{C}$ of $\underline{\operatorname{mod}}\,\Lambda$ the corresponding Λ-module $E_{\mathcal{C}}$ is a pure-injective object in the localizing subcategory of $\underline{\operatorname{Mod}}\,\Lambda$ which is generated by $\mathcal{C}$.

3.2. **Varieties for infinitely generated modules.** In [3], Benson, Carlson, and Rickard generalize the definition of the variety $V(X)$ of a finitely generated Λ-module X. To this end they consider the set $\mathcal{V}_G(k)$ of closed homogeneous irreducible non-zero subvarieties of the maximal ideal spectrum $V_G(k)$ of $H^*(G,k)$. Note that $\mathcal{V}_G(k)$ can be identified with the projective prime ideal spectrum $\operatorname{Proj} H^*(G,k)$. Under this bijection a union of Zariski-closed subsets of $\operatorname{Proj} H^*(G,k)$ corresponds to a subset of $\mathcal{V}_G(k)$ which is closed under specialization. Recall that a subset $\mathcal{W}$ of $\mathcal{V}_G(k)$ is closed under *specialization* if $W \in \mathcal{W}$ and $V \subseteq W$ implies $V \in \mathcal{W}$. Given any Λ-module X, there is associated a subset $\mathcal{V}(X)$ of $\mathcal{V}_G(k)$ which is closed under specialization [3, Definition 10.2]. For example, $\mathcal{V}(X) = \{V \in \mathcal{V}_G(k) \mid V \subseteq V(X)\}$ if X is finitely generated. The following result is due to Benson, Carlson, and Rickard [4]. It is an analogue of Hopkins' classification of thick subcategories of the homotopy category of unbounded complexes of finitely generated projective modules over a commutative noetherian ring [11].

Proposition 3.6. *The map $\mathcal{W} \mapsto \mathcal{C}_{\mathcal{W}} = \{X \in \underline{\operatorname{mod}}\,\Lambda \mid \mathcal{V}(X) \subseteq \mathcal{W}\}$ defines a bijective correspondence between the subsets of $\mathcal{V}_G(k)$ which are closed under specialization and the thick tensor-ideals in $\underline{\operatorname{mod}}\,\Lambda$.*

Proof. See Theorem 3.4 in [4]. □

Clearly, the map $\mathcal{W} \mapsto \mathcal{C}_{\mathcal{W}}$ is inclusion preserving and identifies the projectively connected non-empty subsets of $\mathcal{V}_G(k)$ with the indecomposable thick tensor-ideals in $\underline{\operatorname{mod}}\,\Lambda$. Therefore Theorem C is a direct consequence of Theorem A and the preceding proposition. For the sake of completeness we state the following corollary of Theorem C.

Corollary 3.7. *Let $\mathcal{C}$ be a thick tensor-ideal in $\underline{\operatorname{mod}}\,\Lambda$, and suppose that $\mathcal{W}$ is the corresponding subset $\mathcal{W}$ of $\mathcal{V}_G(k)$ closed under specialization such that $\mathcal{C} = \mathcal{C}_{\mathcal{W}}$. Then the following conditions are equivalent:*

1. *$\mathcal{W}$ is non-empty and projectively connected;*
2. *$\mathcal{C}$ is an indecomposable thick tensor-ideal in $\underline{\operatorname{mod}}\,\Lambda$;*
3. *$E_{\mathcal{C}}$ is an indecomposable object in $\underline{\operatorname{Mod}}\,\Lambda$.*

References

[1] D.J. Benson, Representations and cohomology II, Cambridge University Press (1991).
[2] D.J. Benson, Stably splitting BG, Bull. Amer. Math. Soc. **33** (1996), 189–198.
[3] D.J. Benson, J.F. Carlson, and J. Rickard, Complexity and varieties for infinitely generated modules, II, Math. Proc. Cambridge Philos. Soc. **120** (1996), 597–615.

[4] D.J. Benson, J.F. Carlson, and J. Rickard, Thick subcategories of the stable module category, Fund. Math. **153** (1979), 59–80.
[5] D.J. Benson and G.P. Gnacadja, Phantom maps and purity in modular representation theory I, Preprint (1997).
[6] J.F. Carlson, The variety of an indecomposable module is connected, Invent. Math. **77** (1984), 291–299.
[7] W.W. Crawley-Boevey, Locally finitely presented additive categories, Comm. Algebra **22** (1994), 1644–1674.
[8] E.S. Devinatz, M.J. Hopkins, and J.H. Smith, Nilpotence and stable homotopy theory, I, Ann. Math. (2) **128** (1988), 207–241.
[9] P. Gabriel, Des catégories abéliennes, Bull. Soc. Math. France **90** (1962), 323–448.
[10] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Math. Soc. Lecture Note Ser. **119** (1988).
[11] M.J. Hopkins, Global methods in homotopy theory, in: Homotopy theory (Durham, 1985), London Math. Soc. Lecture Note Ser. **117** (1987), 73–96.
[12] H. Krause, The spectrum of a locally coherent category, J. Pure Appl. Algebra **114** (1997), 259–271.
[13] H. Krause, Stable equivalence preserves representation type, Comment. Math. Helv. **72** (1997), 266–284.
[14] H. Krause, Smashing subcategories and the telescope conjecture – an algebraic approach, Preprint, Universität Bielefeld (1998).
[15] H. Krause and U. Reichenbach, Endofiniteness in stable homotopy theory, in preparation.
[16] A. Neeman, The Connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel, Ann. Sci. École Norm. Sup. (4) **25** (1992), 547–566.
[17] A. Neeman, The Grothendieck duality theorem via Bousfield's techniques and Brown representability, J. Amer. Math. Soc. **9** (1996), 205–236.
[18] J. Rickard, Idempotent modules in the stable category, J. London Math. Soc. **56** (1997), 149–170.
[19] J.L. Verdier, Catégories derivées, état 0, Springer Lec. Notes **569** (1977), 262–311.

Fakultät für Mathematik, Universität Bielefeld, 33501 Bielefeld, Germany
E-mail address: `henning@mathematik.uni-bielefeld.de`