Iterated Monoidal Categories II

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Introduction

In recent years many examples of a new kind of algebraic structure on a category have been discovered: braided monoidal categories, such as categories of representations of quantum groups (cf. [11] & [13]). It is striking to note that there appears to be a connection between braided monoidal categories and 2-fold loop spaces similar to the one between monoidal categories and 1-fold loop spaces or between symmetric monoidal categories and infinte loop spaces. It is shown in [9] that the group completion of the nerve of a braided monoidal category is a 2-fold loop space. This result raises an obvious question: what algebraic structure on a category corresponds to an n-fold loop structure for $3 \leq n < \infty$? Unfortunately the proof of [9] sheds no light on this matter.

In [10] we proposed a solution to this problem based on pursuing an analogy to the tautology that an n-fold loop space is a loop space in the category of $(n-1)$-fold loop spaces. Noting the correspondence between loop spaces and monoidal categories, we iteratively defined the notion of n-fold monoidal category as a monoid in the category of $(n-1)$-fold monoidal categories. There are some subtleties involved in making this definition work: one has to define "monoidal" up to a requisite degree of what category theorists call "laxness". We showed that the group completion of the nerve of an n-fold monoidal category is an n-fold loop space. But [10] raised more questions than it answered: what is the precise relation before group completion of an n-fold monoidal category to an algebra over the little n-cubes operad, which as shown in [4] and [15] characterizes the notion of n-fold loop space? A priori, it is conceivable that after group completion the nerve of an n-fold monoidal category has more structure, it might even be an infinite loop space. E.g. the version of n-fold monoidal category investigated by Joyal and Street [11] gives an infinite loop space for $n \geq 3$.

The present paper answers this question. Our main result is that our notion of iterated monoidal category precisely corresponds to the notion of an algebra over the little n-cubes operad for all n. Firstly the group completion of the nerve of such a category is an n-fold loop space. Secondly one can form an operad in the category of small categories which parametrizes the algebraic structure of an n-fold monoidal category. We show that the nerve of this categorical operad is a topological operad which is equivalent, as an operad, to the little n-cubes operad. Thus our result can be regarded as an algebraic characterization of the notion of n-fold loop space. We also note that this algebraically defined operad is a finite simplicial operad and is closely related to the Milgram construction [18] for $\Omega^n S^n X$.

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1 n-fold Monoidal Categories

In this section we gradually develop the notion of iterated monoidal category. We start by recalling the standard notion of monoidal category and defining a slightly nonstandard variant of the notion of monoidal functor.

Definition 1.1. A *(strict) monoidal category* is a category $\mathcal{C}$ together with a functor $\square : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object 0 such that

1. $\square$ is strictly associative.
2. 0 is a strict 2-sided unit for $\square$.

A monoidal functor $(F,\eta) : \mathcal{C} \to \mathcal{D}$ between monoidal categories consists of a functor F such that $F(0) = 0$ together with a natural transformation

$$\eta_{A,B} : F(A)\Box F(B) \to F(A\Box B),$$

which satisfies the following conditions

1. Internal Associativity: The following diagram commutes

$$\begin{tikzcd} F(A)\Box F(B)\Box F(C) \arrow[r, "\eta_{A,B}\Box id_{F(C)}"] \arrow[d, "id_{F(A)}\Box\eta_{B,C}"] & F(A\Box B)\Box F(C) \arrow[d, "\eta_{A\Box B,C}"] \\ F(A)\Box F(B\Box C) \arrow[r, "\eta_{A,B\Box C}"] & F(A\Box B\Box C) \end{tikzcd}$$

2. Internal Unit Conditions: $\eta_{A,0} = \eta_{0,A} = id_{F(A)}$.

Given two monoidal functors $(F,\eta) : \mathcal{C} \to \mathcal{D}$ and $(G,\zeta) : \mathcal{D} \to \mathcal{E}$, we define their composite to be the monoidal functor $(GF,\xi) : \mathcal{C} \to \mathcal{E}$, where ξ denotes the composite

$$GF(A)\Box GF(B) \xrightarrow{\zeta_{F(A),F(B)}} G\big(F(A)\Box F(B)\big) \xrightarrow{G(\eta_{A,B})} GF(A\Box B).$$

(It is an exercise to check that ξ satisfies the associativity condition above.) We denote by **MonCat** the category of monoidal categories and monoidal functors. Note that the usual product in **Cat** defines a product in **MonCat**.

Remark 1.2. It is usually required in standard definitions of the notion of monoidal functor that η be an isomorphism. As we will discuss below, it is crucial for us not to make this requirement.

Definition 1.3. A *2-fold monoidal category* is a monoid in **MonCat**. This means that we are given a monoidal category $(\mathcal{C}, \Box_1, 0)$ and a monoidal functor $(\Box_2, \eta) : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ which satisfies

1. External Associativity: the following diagram commutes in **MonCat**

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C}\times\mathcal{C} \arrow[r, "(\Box_2,\eta)\times id_{\mathcal{C}}"] \arrow[d, "id_{\mathcal{C}}\times(\Box_2,\eta)"] & \mathcal{C}\times\mathcal{C} \arrow[d, "(\Box_2,\eta)"] \\ \mathcal{C}\times\mathcal{C} \arrow[r, "(\Box_2,\eta)"] & \mathcal{C} \end{tikzcd}$$

2. External Unit Conditions: the following diagram commutes in **MonCat**

$$\begin{tikzcd} \mathcal{C}\times 0 \arrow[r, "\subseteq"] \arrow[d, "\cong"] & \mathcal{C}\times\mathcal{C} \arrow[d, "(\Box_2,\eta)"] & 0\times\mathcal{C} \arrow[l, "\supseteq"'] \arrow[d, "\cong"] \\ \mathcal{C} \arrow[r, "="] & \mathcal{C} & \mathcal{C} \arrow[l, "="'] \end{tikzcd}$$

Explicitly this means that we are given a second associative binary operation $\Box_2 : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$, for which 0 is also a two-sided unit. Moreover we are given a natural transformation

$$\eta_{A,B,C,D} : (A\Box_2 B)\Box_1(C\Box_2 D) \to (A\Box_1 C)\Box_2(B\Box_1 D).$$

The internal unit conditions give $\eta_{A,B,0,0} = \eta_{0,0,A,B} = id_{A\Box_2 B}$, while the external unit conditions give $\eta_{A,0,B,0} = \eta_{0,A,0,B} = id_{A\Box_1 B}$. The internal associativity condition gives the commutative diagram

$$\begin{tikzcd}
(U\Box_2 V)\Box_1(W\Box_2 X)\Box_1(Y\Box_2 Z) \arrow[r, "\eta_{U,V,W,X}\Box_1 id_{Y\Box_2 Z}"] \arrow[d, "id_{U\Box_2 V}\Box_1\eta_{W,X,Y,Z}"] & \big((U\Box_1 W)\Box_2(V\Box_1 X)\big)\Box_1(Y\Box_2 Z) \arrow[d, "\eta_{U\Box_1 W,V\Box_1 X,Y,Z}"] \\
(U\Box_2 V)\Box_1\big((W\Box_1 Y)\Box_2(X\Box_1 Z)\big) \arrow[r, "\eta_{U,V,W\Box_1 Y,X\Box_1 Z}"] & (U\Box_1 W\Box_1 Y)\Box_2(V\Box_1 X\Box_1 Z)
\end{tikzcd}$$

The external associativity condition gives the commutative diagram

$$\begin{tikzcd}
(U\Box_2 V\Box_2 W)\Box_1(X\Box_2 Y\Box_2 Z) \arrow[r, "\eta_{U\Box_2 V,W,X\Box_2 Y,Z}"] \arrow[d, "\eta_{U,V\Box_2 W,X,Y\Box_2 Z}"] & \big((U\Box_2 V)\Box_1(X\Box_2 Y)\big)\Box_2(W\Box_1 Z) \arrow[d, "\eta_{U,V,X,Y}\Box_2 id_{W\Box_1 Z}"] \\
(U\Box_1 X)\Box_2\big((V\Box_2 W)\Box_1(Y\Box_2 Z)\big) \arrow[r, "id_{U\Box_1 X}\Box_2\eta_{V,W,Y,Z}"] & (U\Box_1 X)\Box_2(V\Box_1 Y)\Box_2(W\Box_1 Z)
\end{tikzcd}$$

Remark 1.4. Notice that we have natural transformations

$$\eta_{A,0,0,B} : A\Box_1 B \to A\Box_2 B \qquad \text{and} \qquad \eta_{0,A,B,0} : A\Box_1 B \to B\Box_2 A.$$

If we had insisted a 2-fold monoidal category be a monoid in the category of monoidal categories and *strictly monoidal* functors, this would amount to requiring that $\eta = id$. In view of the above, this would imply $A\Box_1 B = A\Box_2 B = B\Box_1 A$ and similarly for morphisms. Thus the nerve of such a category would be a commutative topological monoid and its group completion would be equivalent to a product of abelian Eilenberg-MacLane spaces.

Remark 1.5. Recall that a braided monoidal category (also known as braided tensor category) is a category $\mathcal{C}$ together with a functor $\Box_1 : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ which is strictly associative, has a strict 2-sided unit object 0 and with a natural commutativity isomorphism $c_{A,B} : A\Box_1 B \longrightarrow B\Box_1 A$ satisfying the following properties:

1. Unit Condition: $c_{A,0} = c_{0,A} = id_A$.
2. Associativity Conditions: For any three objects A, B, C the following diagrams commute:

$$\begin{tikzcd}
A\Box_1 B\Box_1 C \arrow[rr, "c_{A\Box_1 B,C}"] \arrow[dr, "id_A\Box_1 c_{B,C}"'] & & C\Box_1 A\Box_1 B \\
& A\Box_1 C\Box_1 B \arrow[ur, "c_{A,C}\Box_1 id_B"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
A\Box_1 B\Box_1 C \arrow[rr, "c_{A,B\Box_1 C}"] \arrow[dr, "c_{A,B}\Box_1 id_C"'] & & B\Box_1 C\Box_1 A \\
& B\Box_1 A\Box_1 C \arrow[ur, "id_B\Box_1 c_{A,C}"'] &
\end{tikzcd}$$

We claim that a braided monoidal category is exactly the same thing as a 2-fold monoidal category with $\Box_1 = \Box_2$, η an isomorphism, and with

$$\eta_{A,B,0,C} = \eta_{A,0,B,C} = id_{A\Box_1 B\Box_1 C}.$$

Assuming that $\Box_1 = \Box_2$ and that the natural isomorphism $\eta_{A,B,C,D}$ satisfies $\eta_{A,B,0,C} = \eta_{A,0,B,C} = id_{A\Box_1 B\Box_1 C}$, one proceeds as follows to show that we have a braided monoidal category. In the internal associativity diagram take $V = W = 0$ and obtain that $\eta_{U,X,Y,Z} = id_U \Box_1 \eta_{0,X,Y,Z}$. Then take $X = Y = 0$ and obtain that $\eta_{U,V,W,Z} = \eta_{U,V,W,0} \Box_1 id_Z$. Combining these two facts, one obtains that

$$\eta_{A,B,C,D} = id_A \Box_1 c_{B,C} \Box_1 id_D, \tag{$*$}$$

where $c_{B,C} = \eta_{0,B,C,0}$. Then take $U = Z = W = 0$ in the internal associativity law to get the first associativity law for c, and take $U = Z = X = 0$ to get the other one. With the additional conditions we have here the external associativity law is superfluous.

Conversely given a braided monoidal category, we can define a 2-fold monoidal structure by $(*)$.

Remark 1.6. Joyal and Street [11] considered a very similar concept to our notion of 2-fold monoidal category. They loosened our requirement that the two operations $\Box_1$ and $\Box_2$ be strictly associative with a strict unit by only requiring these conditions to hold up to coherent natural isomorphisms. More significantly they required the natural transformation $\eta_{A,B,C,D}$ to be an isomorphism. They then showed that such a category is naturally equivalent to a braided monoidal category. Briefly given such a category one obtains an equivalent braided monoidal category by discarding one of the two operations, say $\Box_2$, and defining the commutativity isomorphism for the remaining operation $\Box_1$ to be the composite

$$A \Box_1 B \xrightarrow{\eta_{0,A,B,0}} B \Box_2 A \xrightarrow{\eta^{-1}_{B,0,0,A}} B \Box_1 A.$$

Our requirement that the operations be strictly associative and unital are not significant restrictions and were adopted for convenience and simplicity. One can always replace categories with operations which are associative and unital up to coherent natural isomorphisms by equivalent categories with strictly associative and unital operations.

There is now a pretty obvious way to define the notion of a 2-fold monoidal functor between 2-fold monoidal categories $F : \mathcal{C} \to \mathcal{D}$. It is a functor together with two natural transformations:

$$\lambda^1_{A,B} : F(A) \Box_1 F(B) \to F(A \Box_1 B)$$

$$\lambda^2_{A,B} : F(A) \Box_2 F(B) \to F(A \Box_2 B)$$

satisfying the same associativity and unit conditions as in the case of monoidal functors. In addition we require that the following hexagonal interchange diagram commutes:

$$\begin{tikzcd}
(F(A)\Box_2 F(B))\Box_1(F(C)\Box_2 F(D)) \arrow[rr, "\eta_{F(A),F(B),F(C),F(D)}"] \arrow[d, "\lambda^2_{A,B}\Box_1\lambda^2_{C,D}"] & & (F(A)\Box_1 F(C))\Box_2(F(B)\Box_1 F(D)) \arrow[d, "\lambda^1_{A,C}\Box_2\lambda^1_{B,D}"] \\
F(A\Box_2 B)\Box_1 F(C\Box_2 D) \arrow[d, "\lambda^1_{A\Box_2 B, C\Box_2 D}"] & & F(A\Box_1 C)\Box_2 F(B\Box_1 D) \arrow[d, "\lambda^2_{A\Box_1 C, B\Box_1 D}"] \\
F((A\Box_2 B)\Box_1(C\Box_2 D)) \arrow[rr, "F(\eta_{A,B,C,D})"] & & F((A\Box_1 C)\Box_2(B\Box_1 D))
\end{tikzcd}$$

We can now define the category **2-MonCat** of 2-fold monoidal categories and 2-fold monoidal functors, and then define a 3-fold monoidal category as a monoid in **2-MonCat**. From this point on, the iteration of this notion is quite straightforward and we arrive at the following definitions.

Definition 1.7. An *n-fold monoidal category* is a category $\mathcal{C}$ with the following structure.

1. There are n distinct multiplications

$$\Box_1, \Box_2, \ldots, \Box_n : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

which are strictly associative and $\mathcal{C}$ has an object 0 which is a strict unit for all the multiplications.

2. For each pair (i,j) such that $1 \le i < j \le n$ there is a natural transformation

$$\eta^{ij}_{A,B,C,D} : (A\Box_j B)\Box_i(C\Box_j D) \to (A\Box_i C)\Box_j(B\Box_i D).$$

These natural transformations η^{ij} are subject to the following conditions:

(a) Internal unit condition: $\eta^{ij}_{A,B,0,0} = \eta^{ij}_{0,0,A,B} = id_{A\Box_j B}$

(b) External unit condition: $\eta^{ij}_{A,0,B,0} = \eta^{ij}_{0,A,0,B} = id_{A\Box_i B}$

(c) Internal associativity condition: The following diagram commutes

$$\begin{tikzcd}
(U\Box_j V)\Box_i(W\Box_j X)\Box_i(Y\Box_j Z) \arrow[r, "\eta^{ij}_{U,V,W,X}\Box_i id_{Y\Box_j Z}"] \arrow[d, "id_{U\Box_j V}\Box_i \eta^{ij}_{W,X,Y,Z}"] & \big((U\Box_i W)\Box_j(V\Box_i X)\big)\Box_i(Y\Box_j Z) \arrow[d, "\eta^{ij}_{U\Box_i W,V\Box_i X,YZ}"] \\
(U\Box_j V)\Box_i\big((W\Box_i Y)\Box_j(X\Box_i Z)\big) \arrow[r, "\eta^{ij}_{U,V,W\Box_i Y,X\Box_i Z}"] & (U\Box_i W\Box_i Y)\Box_j(V\Box_i X\Box_i Z)
\end{tikzcd}$$

(d) External associativity condition: The following diagram commutes

$$\begin{tikzcd}
(U\Box_j V\Box_j W)\Box_i(X\Box_j Y\Box_j Z) \arrow[r, "\eta^{ij}_{U\Box_j V,W,X\Box_j Y,Z}"] \arrow[d, "\eta^{ij}_{U,V\Box_j W,X,Y\Box_j Z}"] & \big((U\Box_j V)\Box_i(X\Box_j Y)\big)\Box_j(W\Box_i Z) \arrow[d, "\eta^{ij}_{U,V,X,Y}\Box_j id_{W\Box_i Z}"] \\
(U\Box_i X)\Box_j\big((V\Box_j W)\Box_i(Y\Box_j Z)\big) \arrow[r, "id_{U\Box_i X}\Box_j \eta^{ij}_{V,W,Y,Z}"] & (U\Box_i X)\Box_j(V\Box_i Y)\Box_j(W\Box_i Z)
\end{tikzcd}$$

Finally it is required that for each triple (i,j,k) satisfying $1 \le i < j < k \le n$ the (big!) hexagonal interchange diagram commutes.

$$\begin{tikzcd}[column sep=tiny]
 & ((A_1\Box_k A_2)\Box_j(B_1\Box_k B_2))\Box_i((C_1\Box_k C_2)\Box_j(D_1\Box_k D_2)) \arrow[dl, "\eta^{jk}_{A_1,A_2,B_1,B_2}\Box_i \eta^{jk}_{C_1,C_2,D_1,D_2}"'] \arrow[dr, "\eta^{ij}_{A_1\Box_k A_2,B_1\Box_k B_2,C_1\Box_k C_2,D_1\Box_k D_2}"] & \\
((A_1\Box_j B_1)\Box_k(A_2\Box_j B_2))\Box_i((C_1\Box_j D_1)\Box_k(C_2\Box_j D_2)) \arrow[d, "\eta^{ik}_{A_1\Box_j B_1,A_2\Box_j B_2,C_1\Box_j D_1,C_2\Box_j D_2}"] & & ((A_1\Box_k A_2)\Box_i(C_1\Box_k C_2))\Box_j((B_1\Box_k B_2)\Box_i(D_1\Box_k D_2)) \arrow[d, "\eta^{ik}_{A_1,A_2,C_1,C_2}\Box_j \eta^{ik}_{B_1,B_2,D_1,D_2}"'] \\
((A_1\Box_j B_1)\Box_i(C_1\Box_j D_1))\Box_k((A_2\Box_j B_2)\Box_i(C_2\Box_j D_2)) \arrow[dr, "\eta^{ij}_{A_1,B_1,C_1,D_1}\Box_k \eta^{ij}_{A_2,B_2,C_2,D_2}"'] & & ((A_1\Box_i C_1)\Box_k(A_2\Box_i C_2))\Box_j((B_1\Box_i D_1)\Box_k(B_2\Box_i D_2)) \arrow[dl, "\eta^{jk}_{A_1\Box_i C_1,A_2\Box_j C_2,B_1\Box_i D_1,B_2\Box_j D_2}"] \\
 & ((A_1\Box_i C_1)\Box_j(B_1\Box_i D_1))\Box_k((A_2\Box_i C_2)\Box_j(B_2\Box_i D_2)) &
\end{tikzcd}$$

Definition 1.8. An *n-fold monoidal functor* $(F, \lambda^1, \ldots, \lambda^n) : \mathcal{C} \to \mathcal{D}$ between n-fold monoidal categories consists of a functor F such that $F(0) = 0$ together with natural transformations

$$\lambda^i_{A,B} : F(A)\Box_i F(B) \to F(A\Box_i B) \quad i = 1, 2, \ldots, n$$

satisfying the same associativity and unit conditions as monoidal functors. In addition the following hexagonal interchange diagram commutes:

$$\begin{tikzcd}
(F(A)\Box_j F(B))\Box_i(F(C)\Box_j F(D)) \arrow[rr, "\eta^{ij}_{F(A),F(B),F(C),F(D)}"] \arrow[d, "\lambda^j_{A,B}\Box_i\lambda^j_{C,D}"] & & (F(A)\Box_i F(C))\Box_j(F(B)\Box_i F(D)) \arrow[d, "\lambda^i_{A,C}\Box_j\lambda^i_{B,D}"] \\
F(A\Box_j B)\Box_i F(C\Box_j D) \arrow[d, "\lambda^i_{A\Box_j B,C\Box_j D}"] & & F(A\Box_i C)\Box_j F(B\Box_i D) \arrow[d, "\lambda^j_{A\Box_i C,B\Box_i D}"] \\
F((A\Box_j B)\Box_i(C\Box_j D)) \arrow[rr, "F(\eta^{ij}_{A,B,C,D})"] & & F((A\Box_i C)\Box_j(B\Box_i D))
\end{tikzcd}$$

Composition of n-fold monoidal functors is defined in exactly the same way as for monoidal functors. However there is an additional exercise to check that the resulting composite satisfies the hexagonal interchange diagram.

It is pretty straightforward to check that an $(n+1)$-fold monoidal category is exactly the same thing as a monoid in **n-MonCat**, the category of n-fold monoidal categories and functors. Note that the hexagonal interchange diagrams for the $(n+1)$-st monoidal operation regarded as an n-fold monoidal functor is what gives rise to the giant hexagonal diagrams involving $\Box_i$, $\Box_j$ and $\Box_{n+1}$.

Remark 1.9. Recall that a symmetric monoidal category is defined in the same way as a braided monoidal category, subject to the additional requirement that the commutativity isomorphism

$$c_{A,B} : A\Box B \xrightarrow{\cong} B\Box A$$

satisfy the symmetry condition

$$c_{B,A} = c^{-1}_{A,B}$$

It is easy to see a symmetric monoidal category is n-fold monoidal for all n. One merely has to take

$$\Box_1 = \Box_2 = \cdots = \Box_n = \Box$$

and define

$$\eta^{ij}_{A,B,C,D} = id_A \Box c_{B,C} \Box id_D$$

for all $i < j$.

Remark 1.10. Joyal and Street [11] arrived at pretty much the same definitions as we do in their context. Because of their insistence that the interchange natural transformations $\eta^{ij}_{A,B,C,D}$ be isomorphisms, however as they observed, for $n \geq 3$ such a notion is equivalent to the notion of symmetric monoidal category, by an argument similar to that of Remark 1.6. Thus the nerves of such categories have group completions which are infinite loop spaces rather than n-fold loop spaces. In Remark 3.15 we will give a homotopy theoretic interpretation of this phenomenon.

2 Connection with n-fold Loop Spaces

In this section we sketch a proof of our assertion that the group completion of the nerve of an n-fold monoidal category is a n-fold loop space. The proof closely mimics Thomason's [27] proof for the analogous connection between symmetric monoidal categories and infinite loop spaces. That proof in turn is based on Segal's ideas [23]. Our proof sketch omits some important details which depend on the coherence theorem for n-fold monoidal categories which we will discuss in Section 4. Later on in section Section 6 we will give an alternative proof of our assertion based on the operad approach to n-fold loop spaces due to May [15].

Segal showed that a space Y is homotopy equivalent to a 1-fold loop space if and only if one can construct a "bar construction on Y up to homotopy." This means a simplicial space $X_* : \Delta^{op} \to \mathbf{Top}$ (where $\mathbf{Top}$ is the category of compactly generated spaces) with $X_1 = Y$ and satisfying

1. There is a homotopy equivalence $X_n \xrightarrow{\simeq} (X_1)^n$ induced by certain iterated face maps and X_0 is contractible.
2. The multiplication induced by $(X_1)^2 \xleftarrow{\simeq} X_2 \xrightarrow{d_1} X_1$ admits a homotopy inverse. (This holds if $\pi_0(X_1)$ is a group and if X_1 is numerably contractible, eg. a CW-complex.)

Moreover he showed that the geometric realization $|X_*|$ is an up-to-homotopy delooping of $X_1 = Y$, ie. $\Omega|X_*| \simeq X_1 = Y$. It was subsequently shown [17] that if condition (2) is omitted, then under some mild additional homotopy commutativity assumption $H_*(\Omega|X_*|)$ is obtained from $H_*(X_1)$ by inverting the elements of $\pi_0(X_1) \subset H_0(X_1)$. This relation is usually referred to as saying that $\Omega|X_*|$ is the *group completion* of X_1. Simplicial spaces satisfying condition (1) are referred to as *special* Δ*-spaces*.

Segal also noted that one could formulate categorical versions of these concepts. For instance a *special* Δ*-category* is a simplicial category $\mathcal{C}_* : \Delta^{op} \to \mathbf{Cat}$ satisfying

1. There is an equivalence of categories $\mathcal{C}_n \xrightarrow{\simeq} (\mathcal{C}_1)^n$ induced by certain iterated face maps and C_0 has a initial/terminal object.

Since the nerve construction preserves products and sends categorical equivalences to homotopy equivalences, the nerve of a special Δ-category is a special Δ-space.

Segal noted that a strictly monoidal category $\mathcal{C}$ naturally gives rise to a special Δ-category $\mathcal{C}_*$ with $\mathcal{C}_n = (\mathcal{C})^n$ via the bar construction. If the monoidal structure is not strictly associative, then one can still construct a special Δ-category $\mathcal{C}_*$ but with $\mathcal{C}_n \simeq (\mathcal{C})^n$. Here one has to use the extra flexibility of allowing categorical equivalences rather than isomorphisms. (This is not critical in this case, since monoidal categories are equivalent to strictly associative ones. When one attempts to put symmetric monoidal categories in this framework one encounters the problem that commutativity can not be made strict.)

Segal's construction of special Δ-categories in the absence of strict algebraic relations (like associativity) was incomplete and ad hoc. This was remedied by Thomason [27], who noted that this construction could be done in two steps. First one can construct a *lax functor* $\mathcal{C}_* : \Delta^{op} \to \mathbf{Cat}$ such that $\mathcal{C}_n = (C_1)^n$. Next one could use the result of Street [26], which states that for any category $\mathcal{I}$ and any lax functor $F : \mathcal{I} \to \mathbf{Cat}$, one can construct an equivalent strict functor $\widehat{F} : \mathcal{I} \to \mathbf{Cat}$. This functor $\widehat{F}$ is called the *Street rectification* of the original lax functor F. Applying this to the lax functor $\mathcal{C}_* : \Delta^{op} \to \mathbf{Cat}$, one obtains a strict functor $\widehat{\mathcal{C}_*} : \Delta^{op} \to \mathbf{Cat}$, which is the desired special Δ-category.

While Segal never explicitly considered n-fold loop spaces except in the special cases $n = 1$ and $n = \infty$, as noted by Dunn [7], his ideas can easily be adapted to this case. One needs to consider special $(\Delta)^n$-spaces. These are the same thing as n-simplicial spaces $X_{**\cdots*} : \Delta^{op} \times \Delta^{op} \times \cdots \times \Delta^{op} \to \mathbf{Top}$ satisfying the condition

1. There is a homotopy equivalence $X_{p_1,p_2,\ldots,p_n} \xrightarrow{\simeq} (X_{11\ldots1})^{p_1 p_2 \cdots p_n}$ induced by certain iterated face maps.

We call such functors *special* $(\Delta)^n$*-spaces*. From Segal's results in the 1-fold loop case, we easily see that for a special $(\Delta)^n$-space $X_{**\cdots*}$ that $\Omega^n|X_{**\cdots*}|$ is a group completion of $X_{**\cdots*}$. The notion of special $(\Delta)^n$-category can be formulated similarly.

Theorem 2.1 *An n-fold monoidal category $\mathcal{C}$ determines a lax functor $\mathcal{C}_{**\cdots*} : \Delta^{op} \times \Delta^{op} \times \cdots \times \Delta^{op} \to \mathbf{Cat}$ such that $\mathcal{C}_{p_1,p_2,\ldots,p_n} = \mathcal{C}^{p_1 p_2 \cdots p_n}$.*

Proof: The lax functor $\mathcal{C}_{**\cdots*}$ is already specified on objects of $(\Delta^{op})^n$. We begin to define the lax functor on morphisms of $(\Delta^{op})^n$ by first considering morphisms of the special form

$$(id,\ldots,id,\alpha,id,\ldots,id):(p_1,\ldots,p_{i-1},q_i,p_{i+1},\ldots,p_n)\longrightarrow(p_1,\ldots,p_{i-1},p_i,p_{i+1},\ldots,p_n)$$

which have only one nontrivial component $\alpha:q_i\to p_i$ in Δ.

Recall that given a morphism $\alpha:p_i\to q_i$ in Δ^{op} and a strict monoidal category $\mathcal{A}$, the bar construction defines a corresponding functor $\mathcal{A}^{p_i}\to\mathcal{A}^{q_i}$. Now consider the category $\mathcal{A}=\mathcal{C}^{p_{i+1}p_{i+2}\cdots p_n}$ as a monoidal category with respect to the i-th operation $\Box_i$ applied componentwise. This defines a functor

$$\mathcal{C}^{p_ip_{i+1}\cdots p_n}=\mathcal{A}^{p_i}\xrightarrow{\alpha^*}\mathcal{A}^{q_i}=\mathcal{C}^{q_ip_{i+1}\cdots p_n}$$

Now taking the $p_1p_2\ldots p_{i-1}$-fold product of this functor with itself gives a functor

$$\mathcal{C}_{p_1,p_2,\cdots,p_n}=\mathcal{C}^{p_1p_2\cdots p_n}\xrightarrow{(\alpha^*)^{p_1p_2\cdots p_{i-1}}}\mathcal{C}^{p_1\cdots q_i\cdots p_n}=\mathcal{C}_{p_1,\cdots,q_i,\cdots,p_n}$$

which we define to be the value of the lax functor $\mathcal{C}_{**\cdots*}$ on the morphism $(id,\ldots,id,\alpha,id,\ldots,id)$.

Now given an arbitrary morphism $(\alpha_1,\alpha_2,\ldots,\alpha_n):(p_1,p_2,\ldots,p_n)\longrightarrow(q_1,q_2,\ldots,q_n)$ in $(\Delta^{op})^n$ we define its value under $\mathcal{C}_{**\cdots*}$ to be the resulting composite functor.

$$\mathcal{C}_{p_1,p_2,\ldots,p_n}\longrightarrow\mathcal{C}_{q_1,p_2,\ldots,p_n}\longrightarrow\cdots\longrightarrow\mathcal{C}_{q_1,q_2,\ldots,q_{n-1},p_n}\longrightarrow\mathcal{C}_{q_1,q_2,\ldots,q_{n-1},q_n}$$

We claim that this defines a lax functor $\mathcal{C}_{**\cdots*}:\Delta^{op}\times\Delta^{op}\times\cdots\times\Delta^{op}\to\mathbf{Cat}$. To see this suppose we are given two morphisms

$$\underline{\alpha}=(\alpha_1,\alpha_2,\ldots,\alpha_n)$$
$$\underline{\beta}=(\beta_1,\beta_2,\ldots,\beta_n)$$

in $(\Delta^{op})^n$ and consider the composite

$$\underline{\beta}\underline{\alpha}=(\beta_1\alpha_1,\beta_2\alpha_2,\ldots,\beta_n\alpha_n)$$

By definition the value of $\mathcal{C}_{**\cdots*}$ on $\underline{\beta}\underline{\alpha}$ is the functor given by the composite of the induced functor

$$(\beta_1\alpha_1,id,\ldots,id)^*$$

followed by

$$(id,\beta_2\alpha_2,id,\ldots,id)^*$$

etc. Since $\mathcal{C}_{**\cdots*}$ is a functor when restricted to morphisms having the form that all but one component is trivial, $(\underline{\beta}\underline{\alpha})^*$ can be further decomposed as the composite

$$(\beta_1,,id,\ldots,id)^*(\alpha_1,,id,\ldots,id)^*$$

followed by

$$(id,\beta_2,id,\ldots,id)^*(id,\alpha_2,id,\ldots,id)^*$$

etc. Similarly $\underline{\beta}^*\underline{\alpha}^*$ breaks up as a composite of exactly the same functors, but composed in a different order.

Thus to construct a natural transformation $\underline{\beta}^*\underline{\alpha}^*\to(\underline{\beta}\underline{\alpha})^*$ it suffices to construct natural transformations

$$(id,\ldots,id,\kappa_i,id,\ldots,id)^*(id,\ldots,id,\lambda_j,id,\ldots,id)^*\longrightarrow(id,\ldots,id,\lambda_j,id,\ldots,id)^*(id,\ldots,id,\kappa_i,id,\ldots,id)^*$$

where the indices i,j indicating the location of the nontrivial components satisfy $i<j$ and $\kappa_i:p_i\to q_i$, $\lambda_j:p_j\to q_j$ are arbitrary morphisms in Δ^{op}.

Thus we have to construct a natural transformation from the top-right to the left-bottom of the following diagram:

$$\begin{tikzcd}
\left((\mathcal{C}^{p_{j+1}\cdots p_n})^{p_j}\right)^{p_1p_2\cdots p_{j-1}} \arrow[r, "(\lambda_j^*)^{p_1p_2\cdots p_{j-1}}"] \arrow[d, equal] & \left((\mathcal{C}^{p_{j+1}\cdots p_n})^{q_j}\right)^{p_1p_2\cdots p_{j-1}} \arrow[d, equal] \\
\left((\mathcal{C}^{p_{i+1}\cdots p_n})^{p_i}\right)^{p_1p_2\cdots p_{i-1}} \arrow[d, "(\kappa_i^*)^{p_1p_2\cdots p_{i-1}}"] & \left((\mathcal{C}^{p_{i+1}\cdots q_j\cdots p_n})^{p_i}\right)^{p_1p_2\cdots p_{i-1}} \arrow[d, "(\kappa_i^*)^{p_1p_2\cdots p_{i-1}}"] \\
\left((\mathcal{C}^{p_{i+1}\cdots p_n})^{q_i}\right)^{p_1p_2\cdots p_{i-1}} \arrow[d, equal] & \left((\mathcal{C}^{p_{i+1}\cdots q_j\cdots p_n})^{q_i}\right)^{p_1p_2\cdots p_{i-1}} \arrow[d, equal] \\
\left((\mathcal{C}^{p_{j+1}\cdots p_n})^{p_j}\right)^{p_1\cdots q_j\cdots p_{j-1}} \arrow[r, "(\lambda_j^*)^{p_1\cdots q_j\cdots p_{j-1}}"] & \left((\mathcal{C}^{p_{j+1}\cdots p_n})^{q_j}\right)^{p_1\cdots q_j\cdots p_{j-1}}
\end{tikzcd}$$

(The vertical equality signs in the diagram are actually canonical permutations.)

Now to construct a natural transformation between two functors taking values in a product of categories, it suffices to construct a natural transformation separately in each component of the product. Thus we may as well assume that

$$p_1 = \cdots = p_{i-1} = q_i = p_{i+1} = \cdots = p_{j-1} = q_j = p_{j+1} = \cdots = p_n = 1$$

To simplify the notation a little bit, we denote $p_i = r$ and $p_j = s$. Then the previous diagram simplifies to

$$\begin{tikzcd}
(\mathcal{C}^r)^s \arrow[r, "\cong"] \arrow[d, "(\kappa_i^*)^s"] & (\mathcal{C}^s)^r \arrow[r, "(\lambda_j^*)^r"] & \mathcal{C}^r \arrow[d, "\kappa_i^*"] \\
\mathcal{C}^s \arrow[rr, "\lambda_j^*"] & & \mathcal{C}
\end{tikzcd}$$

Now κ_i^* and λ_j^*, being induced maps in the bar construction, have to have the general form:

$$\kappa_i^*(A_1, \ldots, A_r) = A_k \Box_i A_{k+1} \Box_i \ldots \Box_i A_{k+u} := \prod\nolimits_{k \le x \le k+u}^{\Box_i} A_x$$

$$\lambda_j^*(B_1, \ldots, B_s) = B_l \Box_j B_{l+1} \Box_j \ldots \Box_j B_{l+v} := \prod\nolimits_{l \le y \le l+v}^{\Box_j} B_y$$

Now if we track an arbitrary object across the top and right of this diagram we obtain

$$\begin{tikzcd}
(C_{xy})_{\substack{1\le x\le r \\ 1\le y\le s}} \arrow[r, mapsto] & \left(\prod\nolimits_{l\le y\le l+v}^{\Box_j} C_{xy}\right)_{1\le x\le r} \arrow[d, mapsto] \\
& \prod\nolimits_{k\le x\le k+u}^{\Box_i}\left(\prod\nolimits_{l\le y\le l+v}^{\Box_j} C_{xy}\right)
\end{tikzcd}$$

On the other hand if we track it across the left and bottom we obtain:

$$\begin{tikzcd}
(C_{xy})_{\substack{1\le x\le r \\ 1\le y\le s}} \arrow[d, mapsto] & \\
\left(\prod\nolimits_{k\le x\le k+u}^{\Box_i} C_{xy}\right)_{1\le y\le s} \arrow[r, mapsto] & \prod\nolimits_{l\le y\le l+v}^{\Box_j}\left(\prod\nolimits_{k\le x\le k+u}^{\Box_i} C_{xy}\right)
\end{tikzcd}$$

It is clear that there is a natural transformation, built out of repeated applications of η^{ij},

$$\prod^{\Box_i}_{k\leq x\leq k+u}\left(\prod^{\Box_j}_{l\leq y\leq l+v} C_{xy}\right) \longrightarrow \prod^{\Box_j}_{l\leq y\leq l+v}\left(\prod^{\Box_i}_{k\leq x\leq k+u} C_{xy}\right)$$

Lastly we must verify that the natural transformations $\underline{\beta}^*\underline{\alpha}^* \to (\underline{\beta}\underline{\alpha})^*$ we have just constructed satisfy a certain associativity condition. To see this we rely on the coherence theorem for n-fold monoidal categories, which we will state and prove in the following two sections. That theorem states that any diagram built out of the natural transformations η^{ij} must commute.

Theorem 2.2 *The group completion of the nerve of an n-fold monoidal category is an n-fold loop space.*

Proof Sketch: By the preceding theorem, we have a lax functor

$$\mathcal{C}_{**\cdots*} : \Delta^{op}\times\Delta^{op}\times\cdots\times\Delta^{op}\to\mathbf{Cat}$$

such that $\mathcal{C}_{p_1,p_2,\ldots,p_n} = \mathcal{C}^{p_1p_2\cdots p_n}$. Now apply Street rectification to obtain a genuine functor

$$\widehat{\mathcal{C}}_{**\cdots*} : \Delta^{op}\times\Delta^{op}\times\cdots\times\Delta^{op}\to\mathbf{Cat},$$

with $\widehat{\mathcal{C}}_{p_1,p_2,\ldots,p_n}\simeq\mathcal{C}^{p_1p_2\cdots p_n}$. Taking nerves we obtain a functor $B\widehat{\mathcal{C}}_{**\cdots*} : \Delta^{op}\times\Delta^{op}\times\cdots\times\Delta^{op}\to\mathbf{Top}$, with $B\widehat{\mathcal{C}}_{p_1,p_2,\ldots,p_n}\simeq(B\mathcal{C})^{p_1,p_2,\ldots,p_n}$. Thus $B\widehat{\mathcal{C}}_{**\cdots*}$ is a special $(\Delta)^n$-space, and the result follows.

3 Free n-fold Monoidal Categories and Their Associated Operad: Statement of Main Results

In this section we consider an alternative and more precise way of relating n-fold monoidal categories to n-fold loop spaces: via operads. First of all we consider free n-fold monoidal categories and construct an associated operad which acts on nerves of n-fold monoidal categories. We then discuss the relation of this operad to Milgram's permutohedral construction used to approximate free loop spaces, and to the little n-cubes operad of Boardman and Vogt.

Definition 3.1. Let $\mathcal{C}$ be a small category. By $\mathcal{F}_n\mathcal{C}$ we will denote the free n-fold monoidal category generated by $\mathcal{C}$. $\mathcal{F}_n\mathcal{C}$ may be constructed as follows. As objects one takes all finite expressions generated by the objects of $\mathcal{C}$ using associative operations $\Box_1$, $\Box_2$, ... $\Box_n$. For example

$$(((C_1\Box_1C_2\Box_1C_3)\Box_2C_4\Box_2(C_5\Box_3C_6))\Box_2C_7)\Box_3(C_8\Box_2C_9)$$

Included among such possible expressions is the vacuous expression, denoted 0, which serves as the unit object. The morphisms of $\mathcal{F}_n\mathcal{C}$ are finite composites of all possible finite formal expressions generated by the morphisms of $\mathcal{C}$ and symbols $\eta^{ij}_{A,B,C,D}$ with $1\leq i<j\leq n$ and A, B, C, D objects of $\mathcal{F}_n\mathcal{C}$, using the associative operations $\Box_1$, $\Box_2$, ... , and $\Box_n$. Two such composites of formal expressions are identified if and only if one can be converted to the other by repeated use of various functoriality, naturality and associativity diagrams. (This is a special case of forming a colimit in theories, cf. [4, p. 33 Prop.2.5].)

As a special case we may take $\mathcal{C}$ to be a finite set whose elements are taken to be the objects, with the morphisms understood to be just the identities of these objects. We will denote by $\mathcal{M}_n(k)$ the full subcategory of $\mathcal{F}_n\{1,2,\ldots,k\}$ whose objects are expressions in which each element 1, 2, ... , k occurs exactly once. For example $(2\Box_1 1)\Box_2 3$ is an object of $\mathcal{M}_n(3)$ but not of $\mathcal{M}_n(4)$, whereas $(1\Box_2 2)\Box_1 1$ is not in any $\mathcal{M}_n(k)$. The symmetric group Σ_k acts freely on $\mathcal{M}_n(k)$ via functors, by permuting labels on both objects and morphisms. It is easy to see that for any category $\mathcal{C}$

$$\mathcal{F}_n\mathcal{C}\cong\coprod_{k\geq 0}\mathcal{M}_n(k)\times_{\Sigma_k}\mathcal{C}^k$$

In particular

$$\mathcal{F}_n\{1\}\cong\coprod_{k\geq 0}\mathcal{M}_n(k)/\Sigma_k$$

If $\mathcal{C}$ is already n-fold monoidal, then we have a natural evaluation functor $\mathcal{F}_n\mathcal{C} \to \mathcal{C}$ which gives rise to functors

$$\mathcal{M}_n(k) \times_{\Sigma_k} \mathcal{C}^n \longrightarrow \mathcal{C}$$

As a special case we get maps

$$\mathcal{M}_n(k) \times \mathcal{M}_n(i_1) \times \mathcal{M}_n(i_2) \times \cdots \times \mathcal{M}_n(i_k) \longrightarrow \mathcal{M}_n(i_1 + i_2 + \cdots + i_k)$$

by replacing the labels $\{1, 2, \ldots, i_j\}$ in $\mathcal{M}_n(i_j)$ with the labels $\{i_1+i_2+\cdots+i_{j-1}+1, \ldots, i_1+i_2+\cdots+i_{j-1}+i_j\}$. This gives $\{\mathcal{M}_n(k)\}_{k\geq 0}$ the structure of an operad in the category of small categories, with a natural action on n-fold monoidal categories. Since the nerve construction preserves products, the nerve of this categorical operad is a topological operad, which we also abusively denote $\mathcal{M}_n$, and this operad acts on nerves of n-fold monoidal categories.

Definition 3.2. It will be convenient to be a bit more general and consider categories $\mathcal{M}_n(S)$, where S is an arbitrary finite set. Again we define $\mathcal{M}_n(S)$ to be the full subcategory of the free n-fold monoidal category $\mathcal{F}_n(S)$ whose objects are expressions in which each element of S occurs precisely once. Obviously any bijection $S \cong S'$ extends to an isomorphism of categories $\mathcal{M}_n(S) \cong \mathcal{M}_n(S')$. If $S \subset T$, there is a restriction functor $\mathcal{M}_n(T) \to \mathcal{M}_n(S)$, induced by the functor $\mathcal{F}_n(T) \to \mathcal{F}_n(S)$ which sends the elements of $T - S$ to 0.

The following is an amusing exercise for the reader:

Exercise 3.3. Let a_k^n denote the number of objects in $\mathcal{M}_n(k)/\Sigma_k$. Then $a_0^n = a_1^n = 1$, $a_2^n = n$, $a_3^n = 2n^2 - n$, $a_4^n = 5n^3 - 5n^2 + n$ and we have the recurrence relation

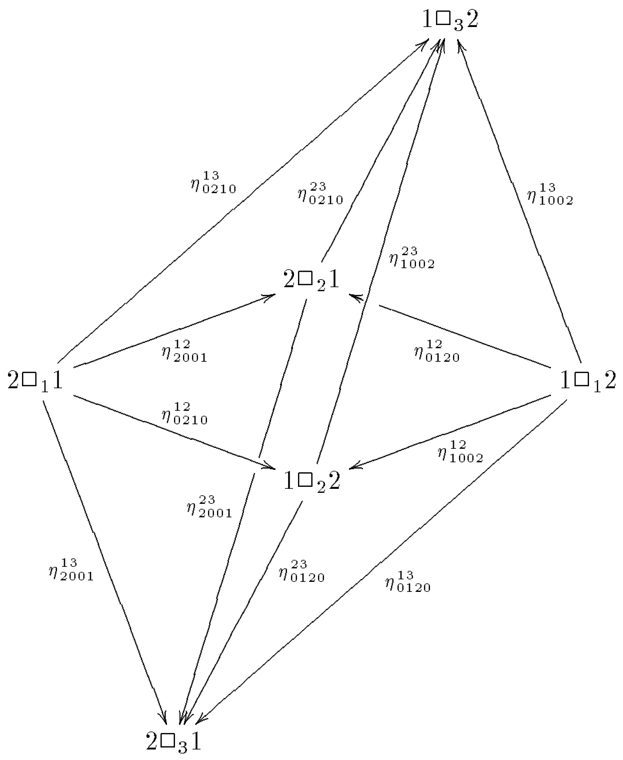
$$a_k^n = na_1^n a_{n-1}^n + \sum_{i=2}^{n-1}(n-1)a_i^n a_{k-i}^n$$

The ratios $\frac{a_{k+1}^n}{a_k^n}$ slowly increase to a limit of $2n - 1 + 2\sqrt{n^2 - n}$. Thus the number of objects in $\mathcal{M}_n(k)$ is $k!a_k^n$.

While it may seem from the definition that the operads $\mathcal{M}_n(k)$ are some infinite-dimensional abstract algebraic monstrosities, this is not the case. They are actually nice compact polyhedra.

Example 3.4. It is not difficult to see that $\mathcal{M}_n(2)$ is the $(n-1)$-dimensional octahedron and thus homeomorphic to S^{n-1}. (By $(n-1)$-dimensional octahedron we mean the boundary of the convex hull of $\{\pm\boldsymbol{e_1}, \pm\boldsymbol{e_2}, \ldots, \pm\boldsymbol{e_n}\}$, where $\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}$ is the standard basis for $\mathbb{R}^n$.) Clearly $\mathcal{M}_n(2)$ is generated by the morphisms $\eta^{ij}_{a,0,0,b}$ and $\eta^{ij}_{0,a,b,0}$, where $a \neq b \in \{1, 2\}$ and $1 \leq i < j \leq n$. The "Giant Hexagon" shows that this set of morphisms is closed under composition. We thus obtain the following picture for the nerve

$\mathcal{M}_3(2)$



and this picture obviously generalizes to $\mathcal{M}_n(2)$ for all n.

If we hope to get similar nice pictures of $\mathcal{M}_n(k)$ for $k > 2$, we need a better description of the categories $\mathcal{M}_n(k)$ than that given in the definition. It is a priori very difficult to determine when two different formal expressions describe the same morphism in the category. What we are dealing with, in effect, is the word problem for a category described by generators and relations. To present the solution to this word problem we need the following preliminary definition.

Definition 3.5. If a and b are distinct elements of $\{1, 2, \ldots, k\}$ and A is an object of $\mathcal{M}_n(k)$, we say that $a\Box_i b$ *in* A if the restriction functor $\mathcal{M}_n(k) \to \mathcal{M}_n(\{a, b\})$ sends A to $a\Box_i b$.

Theorem 3.6 (Coherence Theorem for n-fold Monoidal Categories) *Let A and B be objects of $\mathcal{M}_n(k)$. Then*

1. *There is at most one morphism $A \to B$*
2. *A necessary and sufficient condition for the existence of a morphism $A \to B$ is that for any two elements a, b in $\{1, 2, \ldots, k\}$ if $a\Box_i b$ in A, then in B either $a\Box_j b$ for some $j \geq i$ or $b\Box_j a$ for some $j > i$.*

Example 3.7. There is a morphism $A = (2\Box_2 3)\Box_1 1 \longrightarrow 2\Box_2 1\Box_2 3 = B$ in $\mathcal{M}_2(3)$ since

- $2\Box_1 1$ in A and $2\Box_2 1$ in B
- $3\Box_1 1$ in A and $1\Box_2 3$ in B
- $2\Box_2 3$ in A and $2\Box_2 3$ in B

but there is no morphism from A to $C = 1\Box_2 3\Box_2 2$ since $2\Box_2 3$ in A, while $3\Box_2 2$ in C.

Remark 3.8. The first part of the Coherence Theorem asserts that any diagram built out of the natural transformations η^{ij} must commute. The necessity of the conditions in the second part of the Coherence Theorem is forced by existence of the restriction functors $R_{\{a,b\}} : M_n(k) \to M_n(\{a,b\})$, ie. if there is a morphism $A \to B$ in $M_n(k)$, then there must be a morphism $R_{\{a,b\}}(A) \to R_{\{a,b\}}(B)$ in $M_n(\{a,b\})$. It is far from obvious however, that these conditions are sufficient to insure the existence of a morphism $A \to B$.

Remark 3.9. The coherence theorem implies that the topological operad spaces $\mathcal{M}_n(k)$ are nerves of finite posets, and hence are compact polyhedra.

Definition 3.10. We define the Milgram subspace $\overline{\mathcal{J}}_n(k)$ to be the full subcategory of $\mathcal{M}_n(k)$ whose objects are contained in the free monoid with respect to $\Box_1$ on the free monoid with respect to $\Box_2$ ... on the free monoid with respect to $\Box_n$ on the set $\{1, 2, \ldots, k\}$. Thus the objects of $\overline{\mathcal{J}}_n(k)$ look like

$$((\ldots \Box_3 \ldots)\Box_2 \ldots)\Box_1 \ldots \Box_1((\ldots \Box_3 \ldots)\Box_2 \ldots)$$

ie. the operation $\Box_1$ can only occur at the outermost level, the operation $\Box_2$ can only occur at the next level, ... , the operation $\Box_n$ can only occur at the innermost level. Equivalently we can define the Milgram subspace to be the full subcategory of $\mathcal{M}_n(k)$ consisting of objects which can be written without parentheses using the operation precedence rules: $\Box_n$ has the highest precedence, $\Box_{n-1}$ has the next highest precedence, ... , $\Box_1$ has the lowest precedence.

Remark 3.11. The collection of Milgram subspaces $\{\overline{\mathcal{J}}_n(k)\}_{k \geq 0}$ is **not** a suboperad of the categorical operad $\mathcal{M}_n$. It is only closed under the actions of the symmetric groups and the unit maps

$$s_j : \overline{\mathcal{J}}_n(k) \longrightarrow \overline{\mathcal{J}}_n(k-1) \quad j = 1, 2, \ldots, k.$$

In other words $\overline{\mathcal{J}}_n(k)$, or rather its nerve which we also denote by $\overline{\mathcal{J}}_n(k)$, is a preoperad in the sense of Berger [2]. This structure is sufficient to define the premonad construction

$$\overline{J}_n(X) = \left(\coprod_{k \geq 0} \overline{\mathcal{J}}_n(k) \times_{\Sigma_k} X^k \right) \Big/ \approx$$

where X is any based space. If $\overline{\mathcal{J}}_n$ were an operad, this construction would be a monad, but this isn't the case here. The notion of preoperad and the associated premonad construction were introduced in [6], where preoperads are called "coefficient systems" (also cf. [15]).

In [18] Milgram defined a construction

$$J_n(X) = \left(\coprod_{k \geq 0} (P_k)^{n-1} \times X^k \right) \Big/ \approx$$

on based spaces X, where P_k denotes the permutohedron: the convex hull in $\mathbb{R}^k$ of the Σ_k orbit of a point such as $(1, 2, \ldots, k)$, all of whose coordinates are distinct. (P_k is a $k-1$-dimensional cell. Milgram uses the notation $C(k+1)$ to denote P_k.) He showed that if X is connected, then $J_n(X)$ has the weak homotopy type of $\Omega^n \Sigma^n(X)$.

Theorem 3.12 *For all spaces X, there is a natural homeomorphism*

$$\overline{J}_2(X) \cong J_2(X).$$

Unfortunately for $n > 2$ this does not hold. It turns out that our construction $\overline{J}_n(X)$ is a natural quotient of Milgram's construction, and may be thought of as a sort of *thin version* of the Milgram construction.

To understand the connection between $\overline{J}_n(X)$ and $J_n(X)$, we have to consider yet another variant form of the Milgram construction, which we will call the *thick Milgram construction* and denote $\widetilde{J}_n(X)$. This is defined as the premonad construction on a preoperad $\{\widetilde{\mathcal{J}}_n(k)\}_{k\geq 0}$ where

$$\widetilde{\mathcal{J}}_n(k) = P_k^{n-1} \times \Sigma_k / \approx$$

where the equivalence relation glues together the $k!$ copies of P_k^{n-1} along certain codimension 1 faces in the boundary.

Theorem 3.13 *1. There are natural quotient maps*

$$\widetilde{J}_n(X) \xrightarrow{q_1} J_n(X) \xrightarrow{q_2} \overline{J}_n(X)$$

which are homotopy equivalences.

2. Each of the variant forms of the Milgram construction arises from a preoperad having the generic form

$$D^{(n-1)(k-1)} \times \Sigma_k \Big/ \approx$$

where the equivalence relation glues together $k!$ copies of the $(n-1)(k-1)$-dimensional disk $D^{(n-1)(k-1)}$ along certain codimension 1 faces in the boundary. The quotient maps q_1 and q_2 induce equivalences of preoperads.

We would like to note that an earlier version of this paper suffered from some confusion about the relation between the Milgram construction and the preoperad $\{\overline{\mathcal{J}}_n(k)\}$. We would like to thank Clemens Berger for clearing up this point.

Our main result is the following

Theorem 3.14 *There is a chain of operad equivalences*

$$\mathcal{M}_n(k) \xleftarrow{\simeq} hocolim_{\mathcal{M}_n(k)} F \xrightarrow{\simeq} colim_{\mathcal{M}_n(k)} F \xhookrightarrow{\simeq} \mathcal{C}_n(k),$$

*where $\mathcal{C}_n(k)$ denotes the little n-cubes operad of Boardman and Vogt (and $F : \mathcal{M}_n(k) \to$ **Top** is a functor we construct in Chapter 6). Moreover the inclusion of the Milgram preoperad $\overline{\mathcal{J}}_n(k)$ in the operad $\mathcal{M}_n(k)$ is an equivalence of preoperads.*

This gives a more definitive way of showing that the group completion of the nerve of an n-fold monoidal category is an n-fold loop space. For the proof we have given in the preceding section leaves open the possibility that the group completion of the nerve of an n-fold monoidal category might have more structure than that of an n-fold loop space (eg. perhaps it might be an infinite loop space). This is a serious possibility, since as we have noted in Section 1, slightly variant definitions of the notion of n-fold monoidal category do indeed correspond to infinite loop spaces rather than n-fold loop spaces. The proof based on Theorem 3.14 rules out this possibility, since it shows that the free n-fold loop spaces $\Omega^n\Sigma^n X$, where X is a discrete space do arise as group completions of n-fold monoidal categories. In a subsequent paper we will show that in fact any n-fold loop space can be realized in this way.

Remark 3.15. Joyal and Street [11] noted that their theory of iterated monoidal categories collapses to the theory of symmetric monoidal categories when $n > 2$. The reason for this is that their theory requires that the interchange natural transformations $\eta^{ij}_{A,B,C,D}$ be isomorphisms. Hence the categorical operad for their theory is essentially obtained from our operad $\mathcal{M}_n$ by inverting all the morphisms. But inverting all the morphisms in a category has the effect of killing off all the higher homotopy groups of its nerve, leaving only the fundamental group intact (cf. Quillen [20]). But according to Theorem 3.14 the homotopy groups of $\mathcal{M}_n$ are isomorphic to those of $\mathcal{C}_n$. The spaces of $\mathcal{C}_2$ are $K(\pi, 1)$'s whereas the spaces of $\mathcal{C}_n$ are simply connected for $n > 2$. Thus inverting all the morphisms in $\mathcal{M}_2$ does not change its homotopy type, since all its higher homotopy groups are trivial anyway. But inverting the morphisms of $\mathcal{M}_n$ for $n > 2$ kills off all the homotopy, rendering them into trivial categories, which endows iterated monoidal categories on which they act with a symmetric monoidal structure.

A related result is

Theorem 3.16 *There is a chain of operad equivalences*

$$\mathcal{M}_n \xrightarrow{\simeq} \mathcal{K}^{(n)} \xrightarrow{\simeq} \Gamma^{(n)}$$

where $\mathcal{K}^{(n)}$ denotes the n-th filtration of Berger's complete graph operad (cf. [2]) and $\Gamma^{(n)}$ is the n-th Smith filtration of the operad which parametrizes (strict) symmetric monoidal categories (cf. [24]).

Remark 3.17. The homotopy type of the Milgram preoperad in the case $n = 2$ was determined by Salvetti [22], in the more general context of complements of hyperplane arrangements (also cf. [5]).

4 The Coherence Theorem for n-fold Monoidal Categories

This section is devoted to the proof of Theorem 3.6, the coherence theorem for n-fold monoidal categories. Before we proceed to the proof however, it will be convenient to reformulate the theorem.

Definition 4.1. Let $\widehat{\mathcal{M}}_n(k)$ denote the category with the same objects as $\mathcal{M}_n(k)$, but whose morphisms are as given in Theorem 3.6. That is, there is a (unique) morphism between objects $A \to B$ if and only if for any two elements a, b in $\{1, 2, \ldots, k\}$ if $a\Box_i b$ in A, then in B either $a\Box_j b$ for some $j \geq i$ or $b\Box_j a$ for some $j > i$. Note that by definition $\widehat{\mathcal{M}}_n(k)$ is a poset. More generally, following Definition 3.2, we can define a similar category $\widehat{\mathcal{M}}_n(S)$ for any finite set S. Note that if S and T are disjoint, then there are induced functors

$$\Box_i : \widehat{\mathcal{M}}_n(S) \times \widehat{\mathcal{M}}_n(T) \longrightarrow \widehat{\mathcal{M}}_n(S \amalg T)$$

for $i = 1, 2, \ldots, n$.

It follows immediately from Remark 3.8 that there is a functor

$$\Lambda^n_S : \mathcal{M}_n(S) \longrightarrow \widehat{\mathcal{M}}_n(S)$$

given by the identity on objects (which we will denote simply as Λ^n_k if $S = \{1, 2, \ldots, k\}$). Then the following is an obvious reformulation of Theorem 3.6

Theorem 4.2 (Reformulation of the Coherence Theorem for n-fold Monoidal Categories) *The functor*

$$\Lambda^n_S : \mathcal{M}_n(S) \longrightarrow \widehat{\mathcal{M}}_n(S)$$

is an isomorphism of categories.

As we noted in Definition 3.2, since $\mathcal{M}_n(S)$ only depends on the cardinality of S, it suffices to prove the coherence theorem for Λ^n_k. However it is convenient to recast our basic induction hypothesis in terms of Λ^n_S:

IH 1 *We assume that Λ^n_S is an isomorphism for every proper subset $S \subset \{1, 2, \ldots, k\}$.*

We note that the coherence theorem is trivially true when $k = 1$ and the octahedral picture of $\mathcal{M}_n(2)$ given in the preceding section shows that it is also true when $k = 2$. This starts our induction going.

Definition 4.3. If $A\Box_i B$ is an object in $\mathcal{M}_n(S)$, we denote by $|A|$ the subset of S consisting of all the generators present in A. Thus by definition

$$S = |A| \amalg |B|$$

We will say that $A \in \mathcal{M}_n(T)$ is a *partial* object of $\mathcal{M}_n(S)$ if $T \subset S$.

We begin with a few basic observations about the categories $\mathcal{M}_n(k)$.

Lemma 4.4 *Suppose that X is an object of $\mathcal{M}_n(S)$ and that there is a partition $S = S_1 \amalg S_2$ such that for any $x \in S_1$ and any $y \in S_2$, $x \square_i y$ in X. Then X has a decomposition*

$$X = X_1 \square_i X_2$$

with $|X_1| = S_1$ and $|X_2| = S_2$.

The proof is left as an easy exercise for the reader. (Hint: use induction on the cardinality of S.)

Definition 4.5. Let A and B be two partial objects in $\mathcal{M}_n(k)$.

1. The *difference* $A - |B|$ is the restriction of A to $\mathcal{M}_n(|A| - |B|)$, ie. it is the object obtained from A by zeroing out all generating objects of A which are also present in B.
2. The *intersection* $A \cap |B|$ is defined to be $A - |A - |B||$, ie. the object obtained from A by zeroing out all generating objects of A which are not present in B.

Proposition 4.6 *Let $f : A \square_i B \longrightarrow C \square_j D$ be a morphism in $\mathcal{M}_n(k)$.*

1. *If A, B, C and D are all different from 0, then $j \geq i$;*
2. *If $j = i$ and card$(|A|) =$ card$(|C|)$ then there exist two morphisms $g : A \longrightarrow C$ and $h : B \longrightarrow D$ in $\mathcal{M}_n(|A|)$ and $\mathcal{M}_n(|B|)$, respectively, such that $f = g \square_i h$ (we shall call such a morphism f a $\square_i$-split morphism).*
3. *If $j = i$ and card$(|A|) >$ card$(|C|)$ then there exist two morphisms $g : A \to C \square_i (A - |C|)$ and $h : (A - |C|) \square_i B \to D$ so that f factors as the composite*

$$A \square_i B \xrightarrow{g \square_i id_B} C \square_i (A - |C|) \square_i B \xrightarrow{id_C \square_i h} C \square_i D$$

4. *If $j = i$ and card$(|A|) <$ card$(|C|)$ then there exist two morphisms $g : B \to (C - |A|) \square_i D$ and $h : A \square_i (C - |A|) \to C$ so that f factors as the composite*

$$A \square_i B \xrightarrow{id_A \square_i g} A \square_i (C - |A|) \square_i D \xrightarrow{h \square_i id_D} C \square_i D$$

Proof: By definition any morphism in $\mathcal{M}_n(k)$ is a composition of nontrivial morphisms of the form $\eta^{ij}_{X,Y,Z,W}$ and $f_1 \square_i f_2$ where exactly one of f_1 or f_2 is an identity map (of a nonzero object). We shall refer to such morphisms as *indecomposable morphisms*. To prove part (1) it suffices to prove it for indecomposable morphisms. Now the assertion is evidently true for indecomposable morphisms of the form $f_1 \square_i f_2$. For nontrivial morphisms of the form

$$\eta^{ij}_{X,Y,Z,W} : (X \square_j Y) \square_i (Z \square_j W) \longrightarrow (X \square_i Z) \square_j (Y \square_i W)$$

the outer operation in the source object is i and the outer operation in the target object is $j > i$, since by the unit conditions $\eta^{ij}_{X,Y,Z,W}$ is the identity if any of the objects $X \square_j Y$, $Z \square_j W$, $X \square_i Z$, $Y \square_i W$ are equal to 0.

To check part (2), note first that the conditions $j = i$ and card$(|A|) =$ card$(|C|)$ imply that $|A| = |C|$ and $|B| = |D|$. For otherwise there would have to exist elements $x \in |A| \cap |D|$ and $y \in |B| \cap |C|$ and then we would have $x \square_i y$ in the source object $A \square_i B$ but $y \square_i x$ in the target object $C \square_i D$, which is precluded by the very existence of the functor Λ^n_k. If we then factor f into indecomposable morphisms

$$A \square_i B \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_{m-1} \longrightarrow C \square_i D,$$

it follows directly from Lemma 4.4 that each intermediate object X_r has a decomposition $X_r = X'_r \square_i X''_r$ with $|X'_r| = |A| = |C|$ and $|X''_r = |B| = |C|$. This reduces proving part (2) to the case when f is indecomposable.

By the argument of the preceding paragraph, f would then have to have the form $f = f_1 \Box_i f_2$, for some possibly different $\Box_i$ decomposition of the objects $A\Box_i B$ and $C\Box_i D$. But in that case an easy argument using induction hypothesis (IH.1) shows that the decomposition $f = f_1\Box_i f_2$ can be reparanthesized to a decomposition $f = g\Box_i h$ of the requisite form.

To check part (3), we first demonstrate that f factors through an object $X\Box_i Y\Box_i Z$ such that $|X| = |C|$, $|Y| = |A| - |C|$ and $|Z| = |B|$. Begin by factoring f as

$$A\Box_i B \xrightarrow{f_1} W \xrightarrow{f_2} C\Box_i D$$

with W having a maximal number of $\Box_i$ summands. Now factor f_2 into indecomposable morphisms as

$$W = U_0 \longrightarrow U_1 \longrightarrow U_2 \longrightarrow \cdots \longrightarrow U_m = C\Box_i D$$

We claim that for each U_p and any decomposition $U_p = U_p'\Box_i U_p''$ there is a corresponding decomposition $W = W'\Box_i W''$ with $|W'| = |U_p'|$ and $|W''| = |U_p''|$. If not, let U_p be the first object in the chain having a decomposition $U_p = U_p'\Box_i U_p''$ incompatible with W. Since the morphism

$$U_{p-1} \longrightarrow U_p$$

must be $\Box_i$-split, there is another decomposition $U_p = V_p'\Box_i V_p''$ which is compatible with U_{p-1} and hence with W. Let $W = W'\Box_i W''$ be the compatible decomposition with $|W'| = |V_p'|$ and $|W''| = |V_p''|$. Then according to part (2) f_2 factors as

$$W = W'\Box_i W'' \xrightarrow{f_2'\Box_i f_2''} U_p = V_p'\Box_i V_p'' \longrightarrow C\Box_i D$$

Then the incompatible decomposition $U_p = U_p'\Box_i U_p''$ must give either a decomposition of V_p' which is incompatible with that of W' or a decomposition of V_p'' which is incompatible with that of W''. In the first case IH.1 produces an intermediate object G' between W' and V_p' with more $\Box_i$ summands than W'. Hence f factors through $G'\Box_i W''$ which has more $\Box_i$ summands than W, which contradicts our choice of W. In the second case we obtain a similar contradiction. This proves the claim. Applying this to the decomposition $U_m = C\Box_i D$, we obtain a compatible decomposition $W = X\Box_i T$ with $|X| = |C|$ and $|T| = |D|$.

A similar argument on f_1 yields another decomposition $W = S\Box_i Z$ with $|S| = |A|$ and $|Z| = |B|$. Combining the two decompositions yields the desired decomposition $W = X\Box_i Y\Box_i Z$ with $|Y| = |A| - |C|$. Then (2) yields a factorization of f as

$$A\Box_i B \xrightarrow{g_1\Box_i g_2} X\Box_i Y\Box_i Z \xrightarrow{h_1\Box_i h_2} C\Box_i D$$

for some morphisms $g_1 : A \longrightarrow X\Box_i Y$, $g_2 : B \longrightarrow Z$, $h_1 : X \longrightarrow C$ and $h_2 : Y\Box_i Z \longrightarrow D$. This yields another factorization of f as

$$A\Box_i B \xrightarrow{g'\Box_i id_B} C\Box_i Y\Box_i B \xrightarrow{id_C\Box_i h'} C\Box_i D$$

where $g' = (h_1\Box_i id_Y)g_1 : A \longrightarrow C\Box_i Y$ and $h' = h_2(id_Y\Box_i g_2) : Y\Box_i B \longrightarrow D$. Then IH.1 yields a further factorization of g' as

$$A \xrightarrow{g} C\Box_i(A - |C|) \xrightarrow{id_C\Box_i l} C\Box_i Y \cdot$$

Then the desired factorization

$$A\Box_i B \xrightarrow{g\Box_i id_B} C\Box_i(A - |C|)\Box_i B \xrightarrow{id_C\Box_i h} C\Box_i D$$

is obtained by setting $h = h'(l\Box_i id_B)$. This concludes the proof of (3). The proof of (4) is similar. $\Box$

Remark 4.7. The results listed in Proposition 4.6 are also true in the category $\widehat{\mathcal{M}}_n(k)$, but in this case they follow immediately from the conditions that have to be satisfied by any two objects which are, respectively, the source and the target of a certain morphism.

Remark 4.8. By similar arguments one can show that given any morphism $f : A\square_i B \longrightarrow C\square_i D$ in $\mathcal{M}_n(k)$ with $i = 1$ or $i = n$, there are compatible $\square_i$ decompositions of the source and target, and hence by (2) f has a nontrivial decomposition $f = f_1\square_i f_2$.

Definition 4.9. Let $\mu : A\square_i B \longrightarrow C$ be a morphism in $\widehat{\mathcal{M}}_n(k)$ with $|A|$ having cardinality p and $|B|$ having cardinality q (so $p + q = k$). We say that μ is a *strong* (p, q)*–shuffle* if :

$$C - |B| = A \qquad \text{and} \qquad C - |A| = B$$

Note that this means that the order in which the generating objects appear in C is a (p, q)–shuffle (in the standard sense) of the order in which they appear in A and B. However it means that in addition the operations appearing in C are in some sense the operations appearing in A and B shuffled together.

Remark 4.10. The notion of strong shuffle defined above assumes implicitly the existence of *at most one* morphism between any two objects of the category $\widehat{\mathcal{M}}_n(k)$. This is why we cannot define it *a priori* in the category $\mathcal{M}_n(k)$.

Proposition 4.11 *Let $\mu : A\square_i B \longrightarrow C$ be a morphism in $\widehat{\mathcal{M}}_n(k)$. Then the following conditions are equivalent :*

1. *μ is a strong shuffle;*
2. *There is no nontrivial factorization of μ as*

   $$A\square_i B \xrightarrow{\mu_1\square_i\mu_2} X\square_i Y \xrightarrow{\quad\xi\quad} C$$

   with $\mu_1 : A \longrightarrow X$ and $\mu_2 : B \longrightarrow Y$.

Proof: Let A' and B' denote, respectively, the objects $C - |B|$ and $C - |A|$. Then μ obviously factors as

$$A\square_i B \longrightarrow A'\square_i B' \longrightarrow C.$$

Therefore conditions (1) and (2) are equivalent. $\square$

Definition 4.12. An object (or a partial object) A in $\mathcal{M}_n(k)$ is called :

1. $\square_i$*–reducible* if it can be expressed nontrivially as $A_1\square_i A_2$;
2. $\square_i$*–irreducible* if it is not $\square_i$–reducible.

Definition 4.13. A morphism $f : A\square_i B \longrightarrow C\square_r D$ in $\mathcal{M}_n(k)$ (or a morphism $\mu : A\square_i B \longrightarrow C\square_r D$ in $\widehat{\mathcal{M}}_n(k)$) is called :

1. *irreducible* if $r > i$ and all the objects through which f (or μ) factors nontrivially are $\square_j$–irreducible for all $j \in \{i,\ i+1,\ ...,\ r-1\}$;
2. *reducible* if it is not irreducible.

As we shall see below, we can't get very far with our basic induction hypothesis (IH.1). We have to use *double induction*, the second inductive hypothesis being related to the outermost operation in the targets of the morphisms to be considered. More precisely, we need :

IH 2 *Let $r \geq 2$ be a positive integer. Then*

$$\Lambda_S^n : Hom_{\mathcal{M}_n(k)}(A, B) \longrightarrow Hom_{\widehat{\mathcal{M}}_n(k)}(A, B)$$

is a bijection, whenever B is $\Box_j$–reducible with $j < r$.

Remark 4.14. Note that according to Remark 4.8 and Proposition 4.6 if $j = 1$ and $B = B_1 \Box_1 B_2$, then there is a compatible splitting $A = A_1 \Box_1 A_2$ and any morphism $f : A \to B$ in $\mathcal{M}_n(S)$ must also split $f = f_1 \Box_1 f_2$. Thus in this case, our first induction hypothesis (IH.1) implies that Λ_S^n is a bijection. This starts our second induction hypothesis. Note also that this argument proves more: namely that Λ_S^n is bijective on morphisms where the source and target have the same outermost operation $\Box_r$ and compatible $\Box_r$ splittings. Moreover this also holds when the source and target have the same outermost operation even when there are no compatible splittings. For by Proposition 4.6 parts (3) and (4), in that case one can insert a canonical intermediate object having splittings compatible with both source and target through which all morphisms must factor.

Lemma 4.15 *1. Let $\mu : A \Box_i B \longrightarrow C$ be a strong shuffle in $\widehat{\mathcal{M}}_n(k)$, with C $\Box_r$-reducible. Then for any splitting $C = C_1 \Box_r C_2$ there are (possibly trivial) splittings $A = A_1 \Box_r A_2$ and $B = B_1 \Box_r B_2$ and morphisms $g_1 : A_1 \Box_i B_1 \longrightarrow C_1$ and $g_2 : A_2 \Box_i B_2 \longrightarrow C_2$ in $\mathcal{M}_n(k)$ such that μ lifts to the composite*

$$A \Box_i B = (A_1 \Box_r A_2) \Box_i (B_1 \Box_r B_2) \xrightarrow{\eta^{ir}_{A_1, A_2, B_1, B_2}} (A_1 \Box_i B_1) \Box_r (A_2 \Box_i B_2) \xrightarrow{g_1 \Box_r g_2} C_1 \Box_r C_2$$

2. Any diagram of the form

$$\begin{tikzcd} A \Box_i B \arrow[r, "\eta^{ir}_{A_1, A_2, B_1, B_2}"] \arrow[d, "\eta^{ir}_{A'_1, A'_2, B'_1, B'_2}"'] & (A_1 \Box_i B_1) \Box_r (A_2 \Box_i B_2) \arrow[d, "g"] \\ (A'_1 \Box_i B'_1) \Box_r (A'_2 \Box_i B'_2) \arrow[r, "h"] & C \end{tikzcd}$$

with both $\eta^{ir}_{A_1, A_2, B_1, B_2}$ and $\eta^{ir}_{A'_1, A'_2, B'_1, B'_2}$ nontrivial, commutes in $\mathcal{M}_n(k)$.

Proof: Note first that the objects A, B, C can be decomposed into $\Box_r$–irreducible objects as follows :

$$\begin{aligned} A &= \overline{A}_1 \Box_r \overline{A}_2 \Box_r \ldots \Box_r \overline{A}_s \\ B &= \overline{B}_1 \Box_r \overline{B}_2 \Box_r \ldots \Box_r \overline{B}_t \\ C &= \overline{C}_1 \Box_r \overline{C}_2 \Box_r \ldots \Box_r \overline{C}_u \end{aligned}$$

with $s,\ t \geq 1$ and $u \geq 2$. Since μ is a strong shuffle there exist nondecreasing functions

$$\begin{aligned} \sigma &: \{1,\ 2, ...,\ s\} \longrightarrow \{1,\ 2, ...,\ u\} \\ \tau &: \{1,\ 2, ...,\ t\} \longrightarrow \{1,\ 2, ...,\ u\} \end{aligned}$$

defined, respectively, by the relations

$$\begin{aligned} |\overline{A}_j| &\subset |\overline{C}_{\sigma(j)}|, \text{ for all } j \in \{1,\ 2, ...,\ s\} \\ |\overline{B}_j| &\subset |\overline{C}_{\tau(j)}|, \text{ for all } j \in \{1,\ 2, ...,\ t\} \end{aligned}$$

Then

$$C_1 = \overline{C}_1 \Box_r \overline{C}_2 \Box_r \ldots \Box_r \overline{C}_v$$

$$C_2 = \overline{C}_{v+1} \Box_r \overline{C}_{v+2} \Box_r \ldots \Box_r \overline{C}_u$$

for some v. Now define the objects A_1, A_2, B_1, B_2 by

$$\begin{aligned} A_1 &:= \Box_r \{\overline{A}_j \mid \sigma(j) \leq v\} \qquad & B_1 &:= \Box_r \{\overline{B}_j \mid \tau(j) \leq v\} \\ A_2 &:= \Box_r \{\overline{A}_j \mid \sigma(j) > v\} \qquad & B_2 &:= \Box_r \{\overline{B}_j \mid \tau(j) > v\} \end{aligned}$$

Then clearly there are morphisms $A_1 \Box_i B_1 \longrightarrow C_1$ and $A_2 \Box_i B_2 \longrightarrow C_2$ in $\widehat{\mathcal{M}}_n(k)$. By IH.1 these morphisms have lifts $g_1 : A_1 \Box_i B_1 \longrightarrow C_1$ and $g_2 : A_2 \Box_i B_2 \longrightarrow C_2$ in $\mathcal{M}_n(k)$. It is immediate that

$$A \Box_i B = (A_1 \Box_r A_2) \Box_i (B_1 \Box_r B_2) \xrightarrow{\eta^{ir}_{A_1,A_2,B_1,B_2}} (A_1 \Box_i B_1) \Box_r (A_2 \Box_i B_2) \xrightarrow{g_1 \Box_r g_2} C_1 \Box_r C_2$$

is a lift of μ.

To prove part (2) we begin with a definition. We say that the chain in $\mathcal{M}_n(k)$

$$A \Box_i B = (A'_1 \Box_r A'_2) \Box_i (B'_1 \Box_r B'_2) \xrightarrow{\eta^{ir}_{A'_1,A'_2,B'_1,B'_2}} (A'_1 \Box_i B'_1) \Box_r (A'_2 \Box_i B'_2) \xrightarrow{\quad h \quad} C$$

is *subordinate* to the chain

$$A \Box_i B = (A_1 \Box_r A_2) \Box_i (B_1 \Box_r B_2) \xrightarrow{\eta^{ir}_{A_1,A_2,B_1,B_2}} (A_1 \Box_i B_1) \Box_r (A_2 \Box_i B_2) \xrightarrow{\quad g \quad} C$$

if there exists a splitting $A'_1 = A_1 \Box_r A''_1$ and $B'_1 = B_1 \Box_r B''_1$. We first show that the diagram in part (2) is commutative in this case. In other words, if one chain is subordinate to the other then their composites are equal.

Note that our hypothesis implies that $A_2 = A''_1 \Box_r A'_2$ and $B_2 = B''_1 \Box_r B'_2$. The existence of the morphisms

$$g : (A_1 \Box_i B_1) \Box_r \big((A''_1 \Box_r A'_2) \Box_i (B''_1 \Box_r B'_2)\big) \longrightarrow C$$

$$h : \big((A_1 \Box_r A''_1) \Box_i (B_1 \Box_i B''_1)\big) \Box_r (A'_2 \Box_i B'_2) \longrightarrow C$$

implies the existence of a morphism

$$(A_1 \Box_i B_1) \Box_r (A''_1 \Box_i B''_1) \Box_r (A'_2 \Box_i B'_2) \longrightarrow C$$

in $\widehat{\mathcal{M}}_n(k)$ which according to Remark 4.14 has a unique lift

$$l : (A_1 \Box_i B_1) \Box_r (A''_1 \Box_i B''_1) \Box_r (A'_2 \Box_i B'_2) \longrightarrow C$$

in $\mathcal{M}_n(k)$. This then yields the following diagram in $\mathcal{M}_n(k)$

$$\begin{tikzcd}
& (A_1 \Box_r A''_1 \Box_r A'_2) \Box_i (B_1 \Box_r B''_1 \Box_r B'_2) \arrow[dl] \arrow[dr] & \\
(A_1 \Box_i B_1) \Box_r \big((A''_1 \Box_r A'_2) \Box_i (B''_1 \Box_r B'_2)\big) \arrow[dr] \arrow[ddr, "g"'] \arrow[rr, phantom, "1"] & & \big((A_1 \Box_r A''_1) \Box_i (B_1 \Box_i B''_1)\big) \Box_r (A'_2 \Box_i B'_2) \arrow[dl] \arrow[ddl, "h"] \\
& (A_1 \Box_i B_1) \Box_r (A''_1 \Box_i B''_1) \Box_r (A'_2 \Box_i B'_2) \arrow[d, "l"] \arrow[dl, phantom, "2"] \arrow[dr, phantom, "3"] & \\
& C &
\end{tikzcd}$$

where the unlabelled arrows are those which occur in the external associativity diagram. Then the left hand side of the diagram is one of our given chains and the right hand side is the other (subordinate) one. This diagram commutes because all the inner diagrams commute: the square (1) by the external associativity law and the two triangles (2) and (3) by Remark 4.14.

Now given two arbitrary chains

$$A\Box_i B = (A_1\Box_r A_2)\Box_i(B_1\Box_r B_2) \xrightarrow{\eta^{ir}_{A_1,A_2,B_1,B_2}} (A_1\Box_i B_1)\Box_r(A_2\Box_i B_2) \xrightarrow{\quad g \quad} C$$

$$A\Box_i B = (A'_1\Box_r A'_2)\Box_i(B'_1\Box_r B'_2) \xrightarrow{\eta^{ir}_{A'_1,A'_2,B'_1,B'_2}} (A'_1\Box_i B'_1)\Box_r(A'_2\Box_i B'_2) \xrightarrow{\quad h \quad} C$$

one can construct a third chain

$$\begin{array}{c} A\Box_i B \\ \Big\downarrow {\scriptstyle \eta^{ir}_{A_1\cap|A'_1|,\,A-|A_1|\cap|A'_1|,\,B_1\cap|B'_1|,\,B-|B_1|\cap|B'_1|}} \\ \big((A_1\cap|A'_1|)\Box_i(B_1\cap|B'_1|)\big)\Box_r\big((A-|A_1|\cap|A'|)\Box_i(B-|B_1|\cap|B'_1|)\big) \xrightarrow{\quad l \quad} C \end{array}$$

provided that $A_1\cap|A'_1|$ and $B_1\cap|B'_1|$ are not simultaneously 0. (Again we use Remark 4.14 to construct l.) In this case both of the given chains are subordinate to this third one, hence the composites of the two are both equal to the composite of the third, and thus equal to each other.

The remaining case is that both $A_1\cap|A'_1| = 0 = B_1\cap|B'_1|$. In this case we may assume without loss of generality that $A'_1 = 0 = B_1$. Then again using Remark 4.14 we can again construct a third chain

$$A\Box_i B \xrightarrow{\eta^{ir}_{A_1,A_2,B'_1,B'_2}} (A_1\Box_i B'_1)\Box_r(A_2\Box_i B'_2) \xrightarrow{\quad l \quad} C$$

This time the third chain is subordinate to both of the original ones, and again we get that their composites are equal. This completes the proof of part (2).

Lemma 4.16 *If $f : A\Box_i B \longrightarrow C$ is an irreducible morphism in $\mathcal{M}_n(k)$, with C $\Box_r$-reducible, then :*

1. *A and B are $\Box_i$–irreducible objects (in $\mathcal{M}_n(|A|)$ and $\mathcal{M}_n(|B|)$, respectively);*
2. *f factors as a composite :*

$$\begin{array}{ccc} (A_1\Box_r A_2)\Box_i(B_1\Box_r B_2) & \xrightarrow{\eta^{ir}_{A_1,A_2,B_1,B_2}} & (A_1\Box_i B_1)\Box_r(A_2\Box_i B_2) \\ & & \Big\downarrow g \\ & & C_1\Box_r C_2 \end{array}$$

   with $\Lambda^n_{|C_1|}(g_1)$ and $\Lambda^n_{|C_2|}(g_2)$ strong shuffles, where g_1, resp. g_2 are the restrictions of g to $A_1\Box_i B_1$, resp. $A_2\Box_i B_2$;

3. *$\Lambda^n_k(f)$ is a strong shuffle in $\widehat{\mathcal{M}}_n(k)$.*

Proof: Note first that the irreducible morphism f can be obviously written as a composite

$$\begin{array}{ccc} (A_1\Box_r A_2)\Box_i(B_1\Box_r B_2) & \xrightarrow{\eta^{ir}_{A_1,A_2,B_1,B_2}} & (A_1\Box_i B_1)\Box_r(A_2\Box_i B_2) \\ & & \Big\downarrow g \\ & & C \end{array}$$

due to the structure of its source.

Next we shall prove that A is $\Box_i$–irreducible. If not, then one of the objects A_1 and A_2 must be equal to 0 and the other one must be $\Box_i$–split. Without loss of generality we may assume $A_1 = 0$ and $A_2 = A_{21}\Box_i A_{22}$

with both A_{21} and A_{22} different from 0 (since the other case follows from a similar argument). Then the morphism $\eta^{ir}_{A_1,A_2,B_1,B_2} = \eta^{ir}_{0,A_{21}\Box_i A_{22},B_1,B_2}$ can be written as the composite

$$\begin{tikzcd}
A_{21}\Box_i A_{22}\Box_i(B_1\Box_r B_2) \arrow[r, "id_{A_{21}}\Box_i \eta^{ir}_{0,A_{22},B_1,B_2}"] & A_{21}\Box_i(B_1\Box_r(A_{22}\Box_i B_2)) \arrow[d, "\eta^{ir}_{0,A_{21},B_1,A_{22}\Box_i B_2}"] \\
 & B_1\Box_r(A_{21}\Box_i A_{22}\Box_i B_2)
\end{tikzcd}$$

according to the internal associativity law, contradicting the irreducibility of f. Thus A must be $\Box_i$–irreducible. In a similar way one can obtain the same property for B, finishing the proof of (1).

Next, suppose that at least one of the morphisms $\Lambda^n_{|C_1|}(g_1)$, $\Lambda^n_{|C_2|}(g_2)$ is not a strong shuffle. Then, as in the proof of Proposition 4.11 and using Remark 4.14, g can be factored as

$$(A_1\Box_i B_1)\Box_r(A_2\Box_i B_2) \xrightarrow{(a_1\Box_i b_1)\Box_r(a_2\Box_i b_2)} (X_1\Box_i Y_1)\Box_r(X_2\Box_i Y_2) \xrightarrow{\quad g' \quad} C$$

with at least one of the morphisms a_1, b_1, a_2, b_2 different from the corresponding identity, and we obtain the following commutative diagram (by the naturality of η^{ir}) :

$$\begin{tikzcd}
(A_1\Box_r A_2)\Box_i(B_1\Box_r B_2) \arrow[r, "\eta^{ir}_{A_1,A_2,B_1,B_2}"] \arrow[d, "(a_1\Box_r a_2)\Box_i(b_1\Box_r b_2)"] & (A_1\Box_i B_1)\Box_r(A_2\Box_i B_2) \arrow[d, "(a_1\Box_i b_1)\Box_r(a_2\Box_i b_2)"] \\
(X_1\Box_r X_2)\Box_i(Y_1\Box_r Y_2) \arrow[r, "\eta^{ir}_{X_1,X_2,Y_1,Y_2}"] & (X_1\Box_i Y_1)\Box_r(X_2\Box_i Y_2)
\end{tikzcd}$$

contradicting the irreducibility of f and completing the proof of (2).

To prove (3) assume that $\Lambda^n_k(f)$ is not a strong shuffle. Then we can factor $\Lambda^n_k(f)$ as

$$A\Box_i B \longrightarrow A'\Box_i B' \xrightarrow{\mu} C.$$

From the fact that $\Lambda^n_{|C_1|}(g_1)$ and $\Lambda^n_{|C_2|}(g_2)$ are strong shuffles and Proposition 4.6, it follows that any splitting $A' = A'_1\Box_r A'_2$ corresponds to a splitting $A = \tilde{A}_1\Box_r\tilde{A}_2$, and similarly that any splitting $B' = B'_1\Box_r B'_2$ corresponds to a splitting $B = \tilde{B}_1\Box_r\tilde{B}_2$. According to Lemma 4.15 (1), μ can be lifted to a composite

$$A'\Box_i B' = (A'_1\Box_r A'_2)\Box_i(B'_1\Box_r B'_2) \xrightarrow{\eta^{ir}_{A'_1,A'_2,B'_1,B'_2}} (A'_1\Box_i B'_1)\Box_r(A'_2\Box_i B'_2) \xrightarrow{\quad h \quad} C$$

Pick the corresponding splittings of $A = \tilde{A}_1\Box_r\tilde{A}_2$ and $B = \tilde{B}_1\Box_r\tilde{B}_2$, then use IH.1 to lift to morphisms

$$l_1 : \tilde{A}_1 \to A'_1, \qquad l_2 : \tilde{A}_2 \to A'_2, \qquad l_3 : \tilde{B}_1 \to B'_1, \qquad l_4 : \tilde{B}_2 \to B'_2.$$

Then we have the following diagram in $\mathcal{M}_n(k)$

$$\begin{tikzcd}
A\Box_i B \arrow[rr, "\eta^{ir}_{A_1,A_2,B_1,B_2}"] \arrow[dr, "\eta^{ir}_{\tilde{A}_1,\tilde{A}_2,\tilde{B}_1,\tilde{B}_2}"] \arrow[dd, "(l_1\Box_r l_2)\Box_i(l_3\Box_r l_4)"'] & & (A_1\Box_i B_1)\Box_r(A_2\Box_i B_2) \arrow[ddd, "g"] \\
 & (\tilde{A}_1\Box_i\tilde{B}_1)\Box_r(\tilde{A}_2\Box_i\tilde{B}_2) \arrow[dd, "(l_1\Box_i l_3)\Box_r(l_2\Box_i l_4)"] \arrow[dl, phantom, "\textcircled{1}"] \arrow[dr, phantom, "\textcircled{2}"] & \\
A'\Box_i B' \arrow[dr, "\eta^{ir}_{A'_1,A'_2,B'_1,B'_2}"] & & \\
 & (A'_1\Box_i B'_1)\Box_r(A'_2\Box_i B'_2) \arrow[r, "h"] & C
\end{tikzcd}$$

This diagram commutes since the two inner diagrams commute: (1) by naturality of η^{ir} and (2) according to Lemma 4.15(2). Since the composite across the top and right is f, this contradicts the supposed irreducibility of f. $\square$

Remark 4.17. Clearly if $f : A\square_i B \longrightarrow C$ is a morphism in $\mathcal{M}_n(k)$ such that $\Lambda^n_k(f)$ is irreducible, then f is also irreducible.

Lemma 4.18 *If $\mu : A\square_i B \longrightarrow C$ is an irreducible morphism in $\widehat{\mathcal{M}}_n(k)$, then μ has a unique preimage $f : A\square_i B \longrightarrow C$ in $\mathcal{M}_n(k)$*

Proof: By induction hypothesis (IH.2) we may as well assume that C is $\square_r$-reducible. By Proposition 4.11, μ is a strong shuffle. By Lemma 4.15(1), μ has at least one preimage f. But μ irreducible implies that any preimage f is also irreducible. This in turn implies that any preimage f must be a composite of the form:

$$A\square_i B = (A_1\square_r A_2)\square_i(B_1\square_r B_2) \xrightarrow{\eta^{ir}_{A_1,A_2,B_1,B_2}} (A_1\square_i B_1)\square_r(A_2\square_i B_2) \xrightarrow{\quad g \quad} C$$

By Lemma 4.15(2) it follows that f is unique. $\square$

Remark 4.19. The reader might wonder why the same argument doesn't show that any strong shuffle $\mu : A\square_i B \longrightarrow C$ has a unique preimage in $\mathcal{M}_n(k)$. According to Lemma 4.15, μ has a preimage of the form

$$\begin{tikzcd}
(A_1\square_r A_2)\square_i(B_1\square_r B_2) \arrow[r, "\eta^{ir}_{A_1,A_2,B_1,B_2}"] & (A_1\square_i B_1)\square_r(A_2\square_i B_2) \arrow[d, "g"] \\
 & C
\end{tikzcd}$$

and any two such preimages are equal. However at this point we can't rule out the possibility the μ has other preimages which do not decompose in this way. We will refer to the unique preimage of the first kind as the *standard lift* of the strong shuffle μ. For example, by Lemma 4.16 any irreducible morphism in $\mathcal{M}_n(k)$ is automatically a standard lift.

Lemma 4.20 *Suppose the following diagram is given in $\mathcal{M}_n(k)$:*

$$\begin{tikzcd}
A\square_i B\square_i C \arrow[r, "f\square_i id_C"] \arrow[d, "id_A\square_i g"] & D\square_i C \arrow[d, "h"] \\
A\square_i G \arrow[r, "l"] & F
\end{tikzcd}$$

with F $\square_r$-reducible, with $\Lambda^n_k(h)$, $\Lambda^n_k(l)$, $\Lambda^n_k(f)$ and $\Lambda^n_k(g)$ all strong shuffles, and with h and l being standard lifts. Then the diagram is commutative.

Proof: Let us first decompose the objects A, B, D C and F into $\square_r$–irreducible objects:

$$\begin{array}{ll}
A & = \overline{A}_1\square_r\overline{A}_2\square_r\ldots\square_r\overline{A}_s \\
B & = \overline{B}_1\square_r\overline{B}_2\square_r\ldots\square_r\overline{B}_t \\
D & = \overline{D}_1\square_r\overline{D}_2\square_r\ldots\square_r\overline{D}_u \\
C & = \overline{C}_1\square_r\overline{C}_2\square_r\ldots\square_r\overline{C}_v \\
F & = \overline{F}_1\square_r\overline{F}_2\square_r\ldots\square_r\overline{F}_w
\end{array}$$

Then a similar argument to the one used in Lemma 4.15 gives the nondecreasing functions

$$\begin{array}{ll}
\sigma & : \{1,\ 2, ...,\ u\} \longrightarrow \{1,\ 2, ...,\ w\} \\
\tau & : \{1,\ 2, ...,\ v\} \longrightarrow \{1,\ 2, ...,\ w\}
\end{array}$$

defined, respectively, by the relations

$$\begin{aligned} |\overline{D}_j| &\quad \subset |\overline{F}_{\sigma(j)}|, \text{ for all } j \in \{1,\ 2, ...,\ u\} \\ |\overline{C}_j| &\quad \subset |\overline{F}_{\tau(j)}|, \text{ for all } j \in \{1,\ 2, ...,\ v\} \end{aligned}$$

Since h is the standard lift, according to Lemma 4.15 it factors as the composite

$$(D_1 \Box_r D_2) \Box_i (C_1 \Box_r C_2) \xrightarrow{\eta^{ir}_{D_1,D_2,C_1,C_2}} (D_1 \Box_i C_1) \Box_r (D_2 \Box_i C_2) \xrightarrow{h_1 \Box_r h_2} F$$

with

$$\begin{aligned} D_1 &\ := \Box_r \{\overline{D}_j \mid j \in \sigma^{-1}(1)\} &\qquad C_1 &\ := \Box_r \{\overline{C}_j \mid j \in \tau^{-1}(1)\} \\ D_2 &\ := \Box_r \{\overline{D}_j \mid j \notin \sigma^{-1}(1)\} &\qquad C_2 &\ := \Box_r \{\overline{C}_j \mid j \notin \tau^{-1}(1)\} \end{aligned}$$

Moreover, the splitting of D as $D_1 \Box_r D_2$ gives the nondecreasing functions

$$\begin{aligned} \xi &\quad : \{1,\ 2, ...,\ s\} \longrightarrow \{1,\ 2\} \\ \zeta &\quad : \{1,\ 2, ...,\ t\} \longrightarrow \{1,\ 2\} \end{aligned}$$

defined, respectively, by the relations

$$\begin{aligned} |\overline{A}_j| &\quad \subset |D_{\xi(j)}|, \text{ for all } j \in \{1,\ 2, ...,\ s\} \\ |\overline{B}_j| &\quad \subset |D_{\zeta(j)}|, \text{ for all } j \in \{1,\ 2, ...,\ t\} \end{aligned}$$

Therefore by IH.1 f factors as the composite

$$(A_1 \Box_r A_2) \Box_i (B_1 \Box_r B_2) \xrightarrow{\eta^{ir}_{A_1,A_2,B_1,B_2}} (A_1 \Box_i B_1) \Box_r (A_2 \Box_i B_2) \xrightarrow{f_1 \Box_r f_2} D_1 \Box_r D_2$$

with

$$\begin{aligned} A_1 &\ := \Box_r \{\overline{A}_j \mid j \in \xi^{-1}(1)\} &\qquad B_1 &\ := \Box_r \{\overline{B}_j \mid j \in \zeta^{-1}(1)\} \\ A_2 &\ := \Box_r \{\overline{A}_j \mid j \notin \xi^{-1}(1)\} &\qquad B_2 &\ := \Box_r \{\overline{B}_j \mid j \notin \zeta^{-1}(1)\} \end{aligned}$$

Since $\Lambda^n_k(f)$ and $\Lambda^n_k(h)$ are strong shuffles, we have that

$$A_1 = A \cap |\overline{F}_1|, \quad B_1 = B \cap |\overline{F}_1|, \quad C_1 = C \cap |\overline{F}_1|$$

and that

$$A_2 = A \cap |F_2|, \quad B_2 = B \cap |F_2|, \quad C_2 = C \cap |F_2|$$

where $F_2 = \overline{F}_2 \Box_r \overline{F}_3 \Box_r \ldots \Box_r \overline{F}_w$.

Similar arguments give decompositions

$$(A_1 \Box_r A_2) \Box_i (G_1 \Box_r G_2) \xrightarrow{\eta^{ir}_{A_1,A_2,G_1,G_2}} (A_1 \Box_i G_1) \Box_r (A_2 \Box_i G_2) \xrightarrow{l_1 \Box_r l_2} F$$

and

$$(B_1 \Box_i B_2) \Box_i (C_1 \Box_i C_2) \xrightarrow{\eta^{ir}_{B_1,C_1,B_2,C_2}} (B_1 \Box_i C_1) \Box_r (B_2 \Box_i C_2) \xrightarrow{g_1 \Box_r g_2} G_1 \Box_r G_2$$

of l, respectively g.

Thus we obtain the following diagram in $\mathcal{M}_n(k)$:

$$\begin{tikzcd}
(A_1 \Box_r A_2) \Box_i (B_1 \Box_i B_2) \Box_i (C_1 \Box_i C_2) \arrow[r, "\eta^{ir} \Box_i id_C"] \arrow[d, "id_A \Box_i \eta^{ir}"] \arrow[dr, phantom, "1"] & ((A_1 \Box_i B_1) \Box_r (A_2 \Box_i B_2)) \Box_i (C_1 \Box_i C_2) \arrow[r, "(f_1 \Box_r f_2) \Box_i id_C"] \arrow[d, "\eta^{ir}"] \arrow[dr, phantom, "2"] & (D_1 \Box_r D_2) \Box_i (C_1 \Box_i C_2) \arrow[d, "\eta^{ir}_{D_1,D_2,C_1,C_2}"'] \\
(A_1 \Box_r A_2) \Box_i ((B_1 \Box_i C_1) \Box_r (B_2 \Box_i C_2)) \arrow[r, "\eta^{ir}"] \arrow[d, "id_A \Box_i (g_1 \Box_r g_2)"] \arrow[dr, phantom, "3"] & (A_1 \Box_i B_1 \Box_i C_1) \Box_r (A_2 \Box_i B_2 \Box_i C_2) \arrow[r, "(f_1 \Box_i id_{C_1}) \Box_r (f_2 \Box_i id_{C_2})"] \arrow[d, "(id_{A_1} \Box_i g_1) \Box_r (id_{A_2} \Box_i g_2)"] \arrow[dr, phantom, "4"] & (D_1 \Box_i C_1) \Box_r (D_2 \Box_i C_2) \arrow[d, "h_1 \Box_r h_2"'] \\
(A_1 \Box_r A_2) \Box_i (G_1 \Box_r G_2) \arrow[r, "\eta^{ir}_{A_1,A_2,G_1,G_2}"] & (A_1 \Box_i G_1) \Box_r (A_2 \Box_i G_2) \arrow[r, "l_1 \Box_r l_2"] & F
\end{tikzcd}$$

The outer square of this diagram is the original diagram we want to show commutes. This follows from the fact that all the inner subdiagrams commute: (1) by the internal associativity diagram, (2) and (3) by naturality of η^{ir}, and (4) by Remark 4.14. $\square$

Lemma 4.21 *Suppose the following diagram is given in $\widehat{\mathcal{M}_n}(k)$:*

$$
\begin{tikzcd}
 & B\square_i C \arrow[d, "\varphi"] \\
D\square_j F \arrow[r, "\psi"] & G
\end{tikzcd}
$$

with both morphisms strong shuffles and $i \leq j$. Let X_s, $s = 1, 2, 3, 4$, be the objects defined, respectively, by

$$
\begin{array}{llll}
X_1 & = D - |C| & X_3 & = D - |B| \\
X_2 & = F - |C| & X_4 & = F - |B|
\end{array}
$$

Then :

1. *There exist two morphisms*

$$
\begin{array}{c}
(X_1\square_j X_2)\square_i(X_3\square_j X_4) \stackrel{\varphi_1\square_i\varphi_2}{\longrightarrow} B\square_i C \\
(X_1\square_i X_3)\square_j(X_2\square_i X_4) \stackrel{\psi_1\square_j\psi_2}{\longrightarrow} D\square_j F
\end{array}
$$

   in $\widehat{\mathcal{M}_n}(k)$ extending the given diagram to

$$
\begin{tikzcd}
 & (X_1\square_j X_2)\square_i(X_3\square_j X_4) \arrow[d, "\overline{\varphi}"] \\
(X_1\square_i X_3)\square_j(X_2\square_i X_4) \arrow[r, "\overline{\psi}"] & G
\end{tikzcd}
$$

2. *The extended diagram can be completed into a commutative triangle whenever either one of the following conditions are satisfied :*

   (a) $i \neq j$;

   (b) $i = j$ and at least one of the objects X_2, X_3 is equal to 0.

Proof: Note first that (b) follows immediately from (a). Indeed, the morphism

$$
(X_1\square_j X_2)\square_i(X_3\square_j X_4) \xrightarrow{\Lambda^n_k(\eta^{ij}_{X_1,X_2,X_3,X_4})} (X_1\square_i X_3)\square_j(X_2\square_i X_4)
$$

has the required property if $i \neq j$, while the morphism $id_{X_1\square_i X_2\square_i X_4}$ is taking care of the case $i = j$ (assuming, without loss of generality, $X_3 = 0$). So all we have to prove is (1).

The condition that both φ and ψ are strong shuffles yields the existence of the following morphisms :

$$
\begin{array}{c}
(D-|C|)\square_j(F-|C|) = (D\square_j F) - |C| \xrightarrow{\varphi_1} G - |C| = B \\
(D-|B|)\square_j(F-|B|) = (D\square_j F) - |B| \xrightarrow{\varphi_2} G - |B| = C \\
(B-|F|)\square_i(C-|F|) = (B\square_i C) - |F| \xrightarrow{\psi_1} G - |F| = D \\
(B-|D|)\square_i(C-|D|) = (B\square_i C) - |D| \xrightarrow{\psi_2} G - |D| = F
\end{array}
$$

and therefore the only thing still to prove is the following set of equalities :

$$
\begin{array}{ll}
D - |C| = B - |F|; & D - |B| = C - |F|; \\
F - |C| = B - |D|; & F - |B| = C - |D|.
\end{array}
$$

But this can be easily done – by using again the fact that both φ and ψ are strong shuffles – as follows :

$$\begin{aligned}
D-|C| &= (D-|F|)-|C| = ((D\Box_j F)-|F|)-|C| = (G-|F|)-|C| = \\
&= (G-|C|)-|F| = ((B\Box_i C)-|C|)-|F| = (B-|C|)-|F| = B-|F|; \\
F-|C| &= (F-|D|)-|C| = ((D\Box_j F)-|D|)-|C| = (G-|D|)-|C| = \\
&= (G-|C|)-|D| = ((B\Box_i C)-|C|)-|D| = (B-|C|)-|D| = B-|D|; \\
D-|B| &= (D-|F|)-|B| = ((D\Box_j F)-|F|)-|B| = (G-|F|)-|B| = \\
&= (G-|B|)-|F| = ((B\Box_i C)-|B|)-|F| = (C-|B|)-|F| = C-|F|; \\
F-|B| &= (F-|D|)-|B| = ((D\Box_j F)-|D|)-|B| = (G-|D|)-|B| = \\
&= (G-|B|)-|D| = ((B\Box_i C)-|B|)-|D| = (C-|B|)-|D| = C-|D|.
\end{aligned}$$

and the proof is completed. □

Lemma 4.22 *Suppose the following diagram is given in $\mathcal{M}_n(k)$:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B\Box_i C \arrow[d, "h"] \\
D\Box_j F \arrow[r, "l"] & G
\end{tikzcd}$$

with G $\Box_r$-reducible and with $i<j<r$. If $\Lambda^n_k(h)$ and $\Lambda^n_k(l)$ are both strong shuffles and h and l are standard lifts (cf. Remark 4.19), then the diagram is commutative.

Proof: Let G_1 and G_2 be the objects defined (uniquely) by the equality $G = G_1\Box_r G_2$ and the condition that G_1 is $\Box_r$–irreducible. By Lemma 4.15 we can replace the original given decompositions of h and l by new ones compatible with this splitting of G:

$$B\Box_i C = (X_1\Box_r X_2)\Box_i(X_3\Box_r X_4) \xrightarrow{\eta^{ir}_{X_1,X_2,X_3,X_4}} (X_1\Box_i X_3)\Box_r(X_2\Box_i X_4) \xrightarrow{h_1\Box_r h_2} G$$

$$D\Box_j F = (Z_1\Box_r Z_2)\Box_j(Z_3\Box_r Z_4) \xrightarrow{\eta^{ir}_{Z_1,Z_2,Z_3,Z_4}} (Z_1\Box_j Z_3)\Box_r(Z_2\Box_j Z_4) \xrightarrow{l_1\Box_r l_2} G.$$

Then the morphisms $\varphi_1 := \Lambda^n_{|G_1|}(h_1)$, $\varphi_2 := \Lambda^n_{|G_2|}(h_2)$, $\psi_1 := \Lambda^n_{|G_1|}(l_1)$ and $\psi_2 := \Lambda^n_{|G_2|}(l_2)$ are strong shuffles and we are within the hypotheses of Lemma 4.20 with the following two diagrams (in $\widehat{\mathcal{M}_n}(|G_1|)$ and $\widehat{\mathcal{M}_n}(|G_2|)$, respectively) :

$$\begin{tikzcd}
 & X_1\Box_i X_3 \arrow[d, "\varphi_1"] \\
Z_1\Box_j Z_3 \arrow[r, "\psi_1"] & G_1
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X_2\Box_i X_4 \arrow[d, "\varphi_2"] \\
Z_2\Box_j V_4 \arrow[r, "\psi_2"] & G_2
\end{tikzcd}$$

Therefore there exist the objects Y_u, $u = 1,\ 2,\ \ldots,\ 8$ together with the following morphisms (in the corresponding components of $\widehat{M_n}$) :

$$\begin{array}{ll}
Y_1\Box_j Y_3 \xrightarrow{\xi_1} X_1 & \qquad Y_2\Box_j Y_4 \xrightarrow{\xi_2} X_2 \\
Y_5\Box_j Y_7 \xrightarrow{\xi_3} X_3 & \qquad Y_6\Box_j Y_8 \xrightarrow{\xi_4} X_4 \\
Y_1\Box_i Y_5 \xrightarrow{\zeta_1} Z_1 & \qquad Y_2\Box_i Y_6 \xrightarrow{\zeta_2} Z_2 \\
Y_3\Box_i Y_7 \xrightarrow{\zeta_3} Z_3 & \qquad Y_4\Box_i Y_8 \xrightarrow{\zeta_4} Z_4
\end{array}$$

which – according to (IH.1) – can be lifted, respectively, to

$$\begin{array}{ll}
Y_1\Box_j Y_3 \xrightarrow{f_1} X_1 & \qquad Y_2\Box_j Y_4 \xrightarrow{f_2} X_2 \\
Y_5\Box_j Y_7 \xrightarrow{f_3} X_3 & \qquad Y_6\Box_j Y_8 \xrightarrow{f_4} X_4 \\
Y_1\Box_i Y_5 \xrightarrow{g_1} Z_1 & \qquad Y_2\Box_i Y_6 \xrightarrow{g_2} Z_2 \\
Y_3\Box_i Y_7 \xrightarrow{g_3} Z_3 & \qquad Y_4\Box_i Y_8 \xrightarrow{g_4} Z_4
\end{array}$$

in the corresponding components of $\mathcal{M}_n$ since the cardinalities of all the targets are smaller than k.

Also note that there exists a unique morphism $u : A \longrightarrow (Y^r_{12}\Box_j Y^r_{34})\Box_i(Y^r_{56}\Box_j Y^r_{78})$ in $\mathcal{M}_n(k)$, since such a morphism exists in $\widehat{\mathcal{M}_n}(k)$ and its target is $\Box_i$–split, with Y^r_{12} denoting the object $Y_1\Box_r Y_2$ and so on.

This gives rise to the following diagram in $\mathcal{M}_n(k)$:

$$\begin{tikzcd}
 & B\Box_i C \arrow[rrr, "\eta^{ir}"] & & & X^i_{13}\Box_r X^i_{24} \arrow[ddr, "h_1\Box_r h_2"] & \\
 & & Y^{jri}_{13245768} \arrow[ul, "f"'] \arrow[r, "\eta^{ir}"] \arrow[ur, phantom, "2"] \arrow[dr, phantom, "6"] & Y^{jir}_{13572468} \arrow[ur, "\hat{f}"] \arrow[dr] & & \\
A \arrow[uur] \arrow[r, "u"] \arrow[ddr] \arrow[ur, phantom, "1"] \arrow[dr, phantom, "5"] & Y^{rji}_{12345678} \arrow[ur] \arrow[dr] & & & Y^{ijr}_{15372648} \arrow[dd, "\hat{g}"'] \arrow[r, phantom, "3"] & G \\
 & & Y^{rij}_{12563478} \arrow[r] & Y^{irj}_{15263748} \arrow[ur, "\eta^{jr}"] \arrow[dll, "g"] \arrow[dr, phantom, "4"] & & \\
 & D\Box_j F \arrow[rrr, "\eta^{jr}"] & & & Z^j_{13}\Box_r Z^j_{24} \arrow[uur, "l_1\Box_r l_2"'] &
\end{tikzcd}$$

Here we denote

$$Y^{pqs}_{abcdxyzw} := (((Y_a\Box_p Y_b)\Box_q(Y_c\Box_p Y_d))\Box_s((Y_x\Box_p Y_y)\Box_q(Y_z\Box_p Y_w)))$$

$$X^i_{ab} := X_a\Box_i Y_b \qquad\qquad Z^j_{ab} := Z_a\Box_j Z_b$$

$$f := (f_1\Box_r f_2)\Box_i(f_3\Box_r f_4) \qquad\qquad \hat{f} := (f_1\Box_i f_3)\Box_r(f_2\Box_i f_4)$$

$$g := (g_1\Box_r g_2)\Box_j(g_3\Box_r g_4) \qquad\qquad \hat{g} := (g_1\Box_j g_3)\Box_r(g_2\Box_j g_4)$$

Then the outer hexagon is an expansion of the original diagram which we want to show commutes. To show this we observe that all the inner subdiagrams commute: diagrams (1) and (5) by IH.2, diagrams (2) and (4) by naturality of η^{ir} and η^{jr} respectively, diagram (6) is the "Giant Hexagon", and diagram (3) by Remark 4.14.

This concludes the proof. $\Box$

Lemma 4.23 *Let the following diagram be given in $\mathcal{M}_n(k)$:*

$$\begin{tikzcd}
A_1\Box_i A_2 \arrow[r, "\eta^{ij}_{A'_{11},A'_{12},A'_{21},A'_{22}}"] \arrow[d, "\eta^{ir}_{A''_{11},A''_{12},A''_{21},A''_{22}}"] & B_1\Box_j B_2 \arrow[d, "g"] \\
C_1\Box_r C_2 \arrow[r, "h"] & D_1\Box_r D_2
\end{tikzcd}$$

with $i<j<r$, $\eta^{ij}_{A'_{11},A'_{12},A'_{21},A'_{22}}$ and $\eta^{ir}_{A''_{11},A''_{12},A''_{21},A''_{22}}$ nontrivial and

$$\begin{array}{ll} A_1 & = A'_{11}\Box_j A'_{12} = A''_{11}\Box_r A''_{12} \\ A_2 & = A'_{21}\Box_j A'_{22} = A''_{21}\Box_r A''_{22} \end{array}$$

If $\Lambda^n_k(g\eta^{ij}_{A'_{11},A'_{12},A'_{21},A'_{22}}) = \Lambda^n_k(h\eta^{ir}_{A''_{11},A''_{12},A''_{21},A''_{22}})$ is a strong shuffle then the diagram is commutative.

Proof: Rewrite $g : B_1\Box_j B_2 \longrightarrow D_1\Box_r D_2$ in the form

$$B_1\Box_j B_2 \xrightarrow{g'} B'_1\Box_{j'}B'_2 \xrightarrow{g''} D_1\Box_r D_2$$

where g'' is irreducible. Then the result follows by applying Lemma 4.22 to the diagram:

$$\begin{tikzcd}
A_1\Box_i A_2 \arrow[r, "g'\eta^{ij}_{A'_{11},A'_{12},A'_{21},A'_{22}}"] \arrow[d, "id_{A_1\Box_i A_2}"] & B'_1\Box_{j'}B'_2 \arrow[d, "g''"] \\
A_1\Box_i A_2 \arrow[r, "h\eta^{ir}_{A''_{11},A''_{12},A''_{21},A''_{22}}"] & D_1\Box_r D_2
\end{tikzcd}$$

$\Box$

Lemma 4.24 *$f : A\Box_i B \longrightarrow C$ is irreducible in $\mathcal{M}_n(k)$ iff $\Lambda^n_k(f) : A\Box_i B \longrightarrow C$ is irreducible in $\widehat{\mathcal{M}}_n(k)$*

Proof: By (IH.2) we may as well assume that C is $\Box_r$-reducible. As noted in Remark 4.17, the implication

$$\Lambda^n_k(f) \text{ irreducible} \quad\Longrightarrow\quad f \text{ irreducible}$$

is trivially true.

Now suppose f is irreducible. Then by Lemma 4.16, $\Lambda^n_k(f)$ is a strong shuffle and A and B are both $\Box_i$–irreducible. Thus we can't have a nontrivial factorization of $\Lambda^n_k(f)$ of the form

$$A\Box_i B \longrightarrow D\Box_i G \longrightarrow C$$

For if $\mathrm{card}(|D|) = \mathrm{card}(|A|)$, then this contradicts $\Lambda^n_k(f)$ being a strong shuffle. If $\mathrm{card}(|D|) < \mathrm{card}(|A|)$, then this contradicts $\Lambda^n_k(f)$ being a strong shuffle and A being $\Box_i$–irreducible (cf. Proposition 4.6(4) and Remark 4.7). Similarly we can rule out $\mathrm{card}(|D|) > \mathrm{card}(|A|)$.

Thus if $\Lambda^n_k(f)$ were not irreducible in $\widehat{\mathcal{M}}_n(k)$, then there would have to be a factorization of $\Lambda^n_k(f)$ of the form

$$A\Box_i B \xrightarrow{\mu} D\Box_j G \xrightarrow{\varphi} C$$

with $i < j < r$ and φ irreducible. Then by IH.2 we can lift μ to a morphism h and by Lemma 4.18 we can lift φ to an irreducible morphism l. But then by Lemma 4.22 we have the following commutative diagram in $\mathcal{M}_n(k)$

$$\begin{tikzcd}
A\Box_i B \arrow[r, "h"] \arrow[d, "id_{A\Box_i B}"] & D\Box_j G \arrow[d, "l"] \\
A\Box_i B \arrow[r, "f"] & C
\end{tikzcd}$$

contradicting the irreducibility of f. Thus $\Lambda^n_k(f)$ must be irreducible. $\Box$

Lemma 4.25 *Suppose the following diagram is given in $\mathcal{M}_n(k)$:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"] & B\Box_i C \arrow[d, "h"] \\
D\Box_i F \arrow[r, "l"] & G
\end{tikzcd}$$

with G $\Box_r$-reducible. If h and l are both irreducible then the diagram is commutative.

Proof: Note first that the given diagram can be projected in $\widehat{\mathcal{M}}_n(k)$ via the functor Λ^n_k, the result being the commutative diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B\Box_i C \arrow[d, "\varphi"] \\
D\Box_i F \arrow[r, "\psi"] & G
\end{tikzcd} \tag{1}$$

with $\varphi = \Lambda^n_k(h)$ and $\psi = \Lambda^n_k(l)$. According to Lemma 4.16, φ and ψ are strong shuffles in $\widehat{\mathcal{M}}_n(k)$. Therefore the lower right–hand side corner of (1) is exactly the diagram in Lemma 4.21 with $i = j$.

<u>*Case 1.*</u> Suppose the additional hypothesis in Lemma 4.21(b) is satisfied in our situation, namely one of the objects X_2, X_3 is equal to 0. Without loss of generality we can assume $X_3 = 0$. Then the extended diagram in Lemma 4.21 can be written as

$$\begin{tikzcd} X_1\Box_i X_2\Box_i X_4 \arrow[r, "\varphi_1\Box_i\varphi_2"] \arrow[d, "\psi_1\Box_i\psi_2"] & B\Box_i C \arrow[d, "\varphi"] \\ D\Box_i F \arrow[r, "\psi"] & G \end{tikzcd} \tag{2}$$

Next, the fact that B and C, on one hand, and D and F, on the other hand, have no common generating objects yields the following equivalences :

$$\begin{array}{c} X_3 = 0 \Longleftrightarrow C - |F| = 0 \Longleftrightarrow F - |B| = C \Longleftrightarrow X_4 = C \\ X_3 = 0 \Longleftrightarrow D - |B| = 0 \Longleftrightarrow B - |F| = D \Longleftrightarrow X_1 = D \end{array}$$

together with the equalities $\varphi_2 = id_C$ and $\psi_1 = id_D$ (in $\widehat{\mathcal{M}}_n(k)$). Therefore Lemma 4.18 and (IH.1) give the following (unique) lift of (2) in $\mathcal{M}_n(k)$:

$$\begin{tikzcd} D\Box_i X_2\Box_i C \arrow[r, "f_1\Box_i id_C"] \arrow[d, "id_D\Box_i g_2"] & B\Box_i C \arrow[d, "h"] \\ D\Box_i F \arrow[r, "l"] & G \end{tikzcd}$$

satisfying the hypotheses in Lemma 4.20; hence it is commutative :

$$h\circ(f_1\Box_i id_C) = l\circ(id_D\Box_i g_2) \tag{3}$$

Finally, there exists a morphism

$$\xi : A \longrightarrow D\Box_i X_2\Box_i C$$

in $\widehat{\mathcal{M}}_n(k)$. According to (IH.2), ξ has a unique lift a in $\mathcal{M}_n(k)$ and the morphisms f, g factor, respectively, as

$$\begin{array}{c} A \xrightarrow{\ a\ } D\Box_i X_2\Box_i C \xrightarrow{f_1\Box_i id_C} B\Box_i C \\ \\ A \xrightarrow{\ a\ } D\Box_i X_2\Box_i C \xrightarrow{id_D\Box_i g_2} D\Box_i F \end{array} \tag{4}$$

Now the conclusion follows immediately from (3) and (4).

<u>*Case 2.*</u> Let us assume now that both X_2 and X_3 are different from 0. In this situation the extended diagram in Lemma 4.21 cannot be closed to a commutative triangle. Nevertheless, we can consider the objects $B\cap D := B\cap|D| = D\cap|B|$ (since both B and D are restrictions of the object G as both φ and ψ are strong shuffles) and $C\cap F := C\cap|F| = F\cap|C|$.

Subcase 2.1. Suppose that at least one of the objects $B\cap D$ and $C\cap F$ is not equal to 0. (Without loss of generality we may consider $Y := B\cap D \neq 0$.)

Then the morphism $\varphi\circ\Lambda^n_k(f) = \psi\circ\Lambda^n_k(g)$ factors (in $\widehat{\mathcal{M}}_n(k)$) through the object $Y\Box_i Z$, with $Z := G - |Y|$. Suppose this factorization is

$$A \xrightarrow{\ \xi\ } Y\Box_i Z \xrightarrow{\ \mu\ } G$$

Then ξ has a unique lift a in $\mathcal{M}_n(k)$, according to (IH.2),while μ is a strong shuffle, by the definition of the objects Y and Z, and therefore it has a unique standard lift b in $\mathcal{M}_n(k)$, according to Lemma 4.15 and Remark 4.19.

Then we have the following diagrams in $\mathcal{M}_n(k)$ which can be shown to commute by the same argument as in Case 1 :

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "a"] & B\Box_i C \arrow[d, "h"] \\ Y\Box_i Z \arrow[r, "b"] & G \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "g"] \arrow[d, "a"] & D\Box_i F \arrow[d, "l"] \\ Y\Box_i Z \arrow[r, "b"] & G \end{tikzcd}$$

Subcase 2.2. The remaining situation is $B \cap D = C \cap F = 0$. In this case we must have $|B| = |F|$ and $|C| = |D|$. Since all the objects B, C, D and F are restrictions of the object G (because φ and ψ are strong shuffles), we must have $B = F$ and $C = D$. Therefore the given diagram can be written as

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B\Box_i C \arrow[d, "h"] \\ C\Box_i B \arrow[r, "l"] & G \end{tikzcd} \tag{5}$$

A closer look at the morphism f shows that – according to (IH.2) – it factors through a certain object

$$Z := \overline{B}_1\Box_{i-1}\overline{C}_1\Box_{i-1}\overline{B}_2\Box_{i-1}\overline{C}_2\Box_{i-1}\ldots\Box_{i-1}\overline{B}_m\Box_{i-1}\overline{C}_m \tag{6}$$

with m a positive integer and the objects $\overline{B}_t$, $\overline{C}_t (t \in \{1,\ 2,\ ...,\ m\})$ given by the procedure described below.

Consider the following subsets of $\{1,\ 2,\ ...,\ k\}$:

$$\begin{array}{ll}
\mathcal{B}_1 & := \{b \in |B| \mid \forall c \in |C|,\ \exists s < i \text{ such that } b\Box_s c \text{ in } A\} \\
\mathcal{C}_1 & := \{c \in |C| \mid \forall b \in |B| \setminus \mathcal{B}_1,\ \exists s < i \text{ such that } c\Box_s b \text{ in } A\} \\
\mathcal{B}_2 & := \{b \in |B| \setminus \mathcal{B}_1 \mid \forall c \in |C| \setminus \mathcal{C}_1,\ \exists s < i \text{ such that } b\Box_s c \text{ in } A\} \\
\mathcal{C}_2 & := \{c \in |C| \setminus \mathcal{C}_1 \mid \forall b \in |B| \setminus (\mathcal{B}_1 \cup \mathcal{B}_2),\ \exists s < i \text{ such that } c\Box_s b \text{ in } A\} \\
\ldots & \\
\mathcal{B}_{m-1} & := \{b \in |B| \setminus (\bigcup_{t=1}^{m-2} \mathcal{B}_t) \mid \forall c \in |C| \setminus (\bigcup_{t=1}^{m-2} \mathcal{C}_t),\ \exists s < i \text{ such that } b\Box_s c \text{ in } A\} \\
\mathcal{C}_{m-1} & := \{c \in |C| \setminus (\bigcup_{t=1}^{m-2} \mathcal{C}_t) \mid \forall b \in |B| \setminus (\bigcup_{t=1}^{m-1} \mathcal{B}_t),\ \exists s < i \text{ such that } c\Box_s b \text{ in } A\} \\
\mathcal{B}_m & := |B| \setminus (\bigcup_{t=1}^{m-1} \mathcal{B}_t) \\
\mathcal{C}_m & := |C| \setminus (\bigcup_{t=1}^{m-1} \mathcal{C}_t)
\end{array}$$

Then the objects $\overline{B}_t$ ($t \in \{1,\ 2,\ ...,\ m\}$) are defined as the results obtained by deleting in B all the generating objects from $|B| \setminus \mathcal{B}_t$, respectively. A similar definition gives the objects $\overline{C}_t$. Note that *only* $\overline{B}_1$ or $\overline{C}_m$ or both can be equal to 0. Moreover, if $\overline{B}_1 = 0$ then $m \geq 2$.

Subcase 2.2.1. If *only two* of the objects $\overline{B}_t$, $\overline{C}_t$ in the right–hand side of (6) are different from 0 then Z is given by one of the equalities

$$\begin{array}{ll} Z & = \overline{B}_1\Box_{i-1}\overline{C}_1 \\ Z & = \overline{C}_1\Box_{i-1}\overline{B}_2 \end{array}$$

and, in order to make a choice, we shall assume *the first one* to hold (the other situation being treated in a similar way). In this case we obviously have $B_1 = B$ and $C_1 = C$.

According to (IH.2), there exists a unique morphism $a : A \longrightarrow B\Box_{i-1}C$ in $\mathcal{M}_n(k)$. Then, again by (IH.2), both f and g factor through $B\Box_{i-1}C$ as

$$\begin{array}{ll} f & = \eta^{ji}_{B,0,0,C} \circ a \\ g & = \eta^{ji}_{0,B,C,0} \circ a \end{array}$$

with $j := i - 1$ and the commutativity of (7) is obviously reduced to the commutativity of the following diagram :

$$\begin{tikzcd} B\Box_j C \arrow[r, "\eta^{ji}_{B,0,0,C}"] \arrow[d, "\eta^{ji}_{0,B,C,0}"] & B\Box_i C \arrow[d, "h"] \\ C\Box_i B \arrow[r, "l"] & G \end{tikzcd} \tag{7}$$

Next, let us have a closer look at the irreducible morphism h. According to Lemma 4.15, the morphism h can be factored as

$$\begin{tikzcd}
(B_1\Box_r B_2)\Box_i(C_1\Box_r C_2) \arrow[r, "\eta^{ir}_{B_1,B_2,C_1,C_2}"] & (B_1\Box_i C_1)\Box_r(B_2\Box_i C_2) \arrow[d, "h_1\Box_r h_2"] \\
& G_1\Box_r G_2
\end{tikzcd}$$

for any decomposition $G_1\Box_r G_2$ of the $\Box_r$–reducible object G. But this fact implies the existence of a $\Box_r$–split morphism

$$\mu_1\Box_r\mu_2 : (B_1\Box_j C_1)\Box_r(B_2\Box_j C_2) \longrightarrow G_1\Box_r G_2$$

in $\widehat{\mathcal{M}}_n(k)$ which, according to (IH.1), has a unique lift $h'_1\Box_r h'_2$ in $\mathcal{M}_n(k)$. Hence we have obtained the following two diagrams in $\mathcal{M}_n(k)$:

$$\begin{tikzcd}
B\Box_j C \arrow[r, "\eta^{ji}_{B,0,0,C}"] \arrow[d, "\eta^{jr}_{B_1,B_2,C_1,C_2}"] & B\Box_i C \arrow[d, "h"] \\
D_1\Box_r D_2 \arrow[r, "h'_1\Box_r h'_2"] & G_1\Box_r G_2
\end{tikzcd}
\qquad
\begin{tikzcd}
B\Box_j C \arrow[r, "\eta^{ji}_{0,B,C,0}"] \arrow[d, "\eta^{jr}_{B_1,B_2,C_1,C_2}"] & C\Box_i B \arrow[d, "l"] \\
D_1\Box_r D_2 \arrow[r, "h'_1\Box_r h'_2"] & G_1\Box_r G_2
\end{tikzcd}$$

with $D_1\Box_r D_2$ denoting the object $(B_1\Box_j C_1)\Box_r(B_2\Box_j C_2)$. Now the conclusion follows easily by applying Lemma 4.23.

Subcase 2.2.2. If *at least three* of the objects $\overline{B}_t$, $\overline{C}_t$ in the right–hand side of (8) are different from 0 then Z is given by an equality having one of the forms

$$\begin{aligned} Z &= \overline{B}_1\Box_{i-1}\overline{C}_1\Box_{i-1}V \\ Z &= \overline{C}_1\Box_{i-1}\overline{B}_2\Box_{i-1}V \end{aligned}$$

corresponding to $\overline{B}_1 \neq 0$ and $\overline{B}_1 = 0$, respectively. Again we can assume *the first* equality to hold.

It follows that we have the following diagram in $\widehat{\mathcal{M}}_n(k)$

$$A \xrightarrow{\mu} \overline{B}_1\Box_{i-1}(G-|\overline{B}_1|) \xrightarrow{\nu} G.$$

Next factor ν as

$$\overline{B}_1\Box_{i-1}(G-|\overline{B}_1|) \xrightarrow{\nu_1} X\Box_j Y \xrightarrow{\nu_2} G$$

with $j < r$, ν_2 irreducible and $X\Box_j Y$ different from both $B\Box_i C$ and $C\Box_i B$ (since there are no morphisms from $\overline{B}_1\Box_{i-1}(G-|\overline{B}_1|)$ into either $B\Box_i C$ or $C\Box_i B$ in $\widehat{\mathcal{M}}_n(k)$). Now use (IH.2) to lift μ and ν_1 and Lemma 4.18 to lift ν_2. We denote the lifts by u, v_1 and v_2 respectively. This gives us a morphism $A \longrightarrow G$ in $\mathcal{M}_n(k)$ given by

$$A \xrightarrow{a} X\Box_j Y \xrightarrow{b} G$$

where $a = v_1 u$ and $b = v_2$.

Finally, let us consider the diagrams

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "a"] & B\Box_i C \arrow[d, "h"] \\
X\Box_j Y \arrow[r, "b"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "a"] & C\Box_i B \arrow[d, "l"] \\
X\Box_j Y \arrow[r, "b"] & G
\end{tikzcd}$$

which are commutative, either by Lemma 4.22 (for $i \neq j$) or by one of the cases already discussed during this proof (for $i = j$). Now the conclusion is immediate and the lemma is completely proven. $\Box$

Finally, we have all the necessary preliminaries for the proof of Theorem 4.2.

Proof of the Coherence Theorem for n-fold Monoidal Categories. It remains to show that

$$\Lambda_k^n : \mathrm{Hom}_{\mathcal{M}_n(k)}(A, B) \longrightarrow \mathrm{Hom}_{\widehat{\mathcal{M}}_n(k)}(A, B)$$

is a bijection when A is $\Box_r$–irreducible and B is $\Box_r$–reducible, since (IH.2) and Remark 4.14 take care of all the other possibilities.

Note first that any morphism $\mu : A \longrightarrow B$ in $\widehat{\mathcal{M}}_n(k)$ with A $\Box_r$–irreducible and B $\Box_r$–reducible can be factored as

$$A \xrightarrow{\mu'} A_1'\Box_i A_2' \xrightarrow{\mu_0} B$$

with μ_0 irreducible in $\widehat{\mathcal{M}}_n(k)$. Since both μ' and μ_0 have lifts in $\mathcal{M}_n(k)$, the former by (IH.2) and the latter by Lemma 4.18, it follows that the morphism μ has such a lift. Therefore the functor Λ_k^n is *surjective* on morphisms.

Next let us consider two morphisms $f, g : A \longrightarrow B$ in $\mathcal{M}_n(k)$, with A $\Box_r$–irreducible and B $\Box_r$–reducible. Obviously f and g can be factored, respectively, as

$$A \xrightarrow{f_0} A_1'\Box_i A_2' \xrightarrow{f'} B$$

$$A \xrightarrow{g_0} A_1''\Box_j A_2'' \xrightarrow{g''} B$$

with f' and g'' irreducible. But in this way we have obtained in fact the following diagram in $\mathcal{M}_n(k)$:

$$\begin{tikzcd} A \arrow[r, "f_0"] \arrow[d, "g_0"] & A_1'\Box_i A_2' \arrow[d, "f'"] \\ A_1''\Box_j A_2'' \arrow[r, "g''"] & B \end{tikzcd}$$

which, according to Lemmas 4.22 (if $i \neq j$) or 4.25 (if $i = j$), is commutative and therefore yields the equality $f = g$.

Thus the functor Λ_k^n is also *injective* on morphisms and the coherence theorem is completely proved. $\Box$

5 The Milgram Construction

This section is devoted to a detailed discussion of the relation between the Milgram construction and the premonad construction with respect to the Milgram subpreoperad $\overline{\mathcal{J}}$ of the n-fold monoidal operad $\mathcal{M}_n$, introduced in Section 3.

Definition 5.1. Let X be an object in the category $\overline{\mathcal{J}}_n(k)$. We denote by $\mathcal{S}(X)$ the full subcategory of $\overline{\mathcal{J}}_n(k)$ consisting of all the objects Y in $\mathcal{J}_n(k)$ which map into X (including X itself). As usual abusively we also use the same notation $\mathcal{S}(X)$ to denote the nerve of this category.

For $n = 2$ the natural homeomorphism of Theorem 3.12 between the Milgram construction and the premonad construction $\overline{J}_n(X)$ is a direct consequence of the following result:

Theorem 5.2 *$\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$ is homeomorphic to the permutohedron P_k. More precisely:*

1. *The simplicial triangulation of $\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$ arising from its definition as a nerve is isomorphic to the barycentric subdivision of the natural cell structure on P_k.*

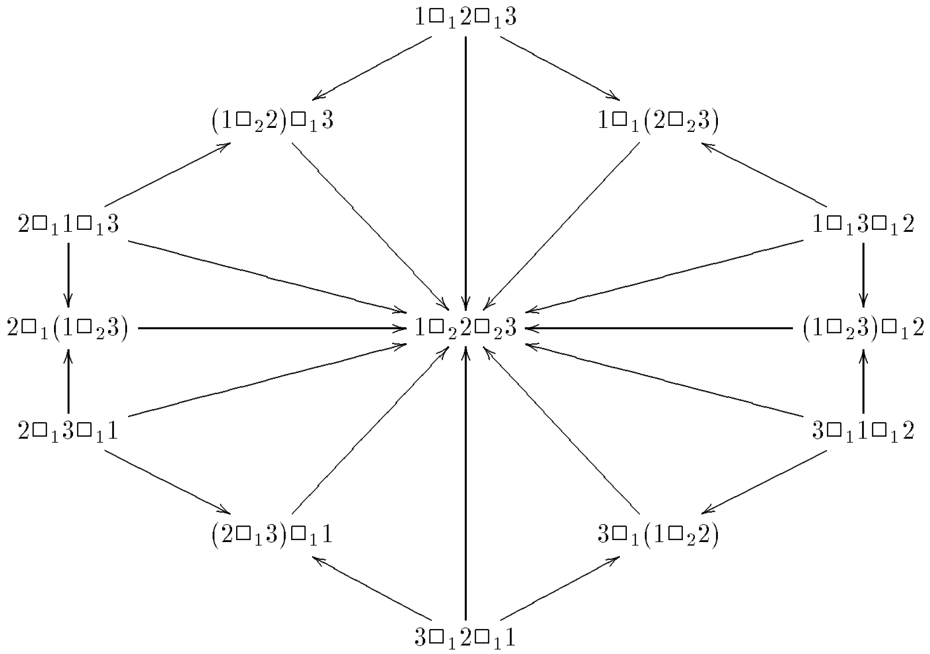
2. *There is a functorial action of the symmetric group Σ_k on the category $\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$ inducing an action on its nerve which corresponds under this isomorphism to the natural action of Σ_k on P_k.*

3. For each $i = 1, 2, \ldots, k$ the functor $\mathcal{S}(1\Box_2 2\Box_2 \ldots \Box_2 k) \longrightarrow \mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 (k-1)\right)$ induced by the map of generating elements:

$$1 \mapsto 1,\ 2 \mapsto 2,\ \ldots,\ (i-1) \mapsto (i-1),\ i \mapsto 0,\ (i+1) \mapsto i,\ (i+2) \mapsto (i+1),\ \ldots,\ k \mapsto (k-1)$$

corresponds to the i-degeneracy map $D_i : P_k \to P_{k-1}$.

Before we go on to the proof of this theorem, we illustrate this for the case $k = 3$. Recall that P_3 is a hexagon. Here is a picture of the nerve of $\mathcal{S}(1\Box_2 2\Box_2 3)$:



(Here, as elsewhere throughout this section, we rely heavily on the coherence theorem for n-fold monoidal categories. Thus we do not have to worry about labelling the arrows in our diagram, since there can be at most one between any pair of objects, and the existence of the arrows shown can be easily checked.)

Proof of Theorem 5.2. The coherence theorem implies that the objects of $\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$ have the form $A_1\Box_1 A_2\Box_1 \ldots \Box_1 A_s$, with

$$A_r = i_{r1}\Box_2 i_{r2}\Box_2 \ldots \Box_2 i_{rj_r} \qquad 1 \le i_{r1} < i_{r2} < \cdots < i_{rj_r} \le k,$$

ie. $\left((i_{rt})_{t=1}^{j_r}\right)_{r=1}^{s}$ forms a $(j_1, j_2, \ldots, j_s)$-shuffle in Σ_k.

We begin by defining a functorial action of the symmetric group Σ_k on $\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$. Given an element of $\sigma \in \Sigma_k$, there is a functor

$$\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k) \longrightarrow \mathcal{S}\left(\sigma(1)\Box_2\sigma(2)\Box_2\sigma(3)\Box_2 \ldots \Box_2\sigma(k)\right)$$

given by permuting the generating elements $\{1, 2, \ldots, k\}$ according to σ. We compose this with the functor

$$\mathcal{S}\left(\sigma(1)\Box_2\sigma(2)\Box_2\sigma(3)\Box_2 \ldots \Box_2\sigma(k)\right) \longrightarrow \mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$$

which reorders the generating elements within the inner parentheses in their natural order when read from left to right. (To see that this defines a functor one must use the coherence theorem.)

To illustrate this action consider the totally order reversing permutation $[6, 5, 4, 3, 2, 1]$ acting on the object $(2\Box_2 4)\Box_1(3\Box_2 5\Box_2 6)\Box_1 1 \in \mathcal{S}(1\Box_2 2\Box_2 3\Box_2 3\Box_2 4\Box_2 5\Box_2 6)$. We have

$$(2\Box_2 4)\Box_1(3\Box_2 5\Box_2 6)\Box_1 1 \mapsto (5\Box_2 3)\Box_1(4\Box_2 2\Box_2 1)\Box_1 6 \mapsto (3\Box_2 5)\Box_1(1\Box_2 2\Box_2 4)\Box_1 6$$

We now proceed by induction on k to prove part (1) of the theorem. For $k = 1$ this is trivially true, since both $\mathcal{S}(1)$ and P_1 are consist of a single point. We then note that by the coherence theorem, $\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$ is the cone, with respect to the vertex $1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k$, of the union

$$\cup_{p=1}^{k-1} \cup_{\alpha \in Sh_{p,k-p}} \alpha \left(\mathcal{S}\left((1\Box_2 2\Box_2 \ldots \Box_2 p)\Box_1 \left((p+1)\Box_2 (p+2)\Box_2 \ldots \Box_2 k\right)\right)\right),$$

where $Sh_{p,k-p}$ denotes the set of $(p, k-k)$ shuffles acting via the symmetric group action defined above.

Moreover by the coherence theorem, any object in

$$\mathcal{S}\left((1\Box_2 2\Box_2 \ldots \Box_2 p)\Box_1 \left((p+1)\Box_2 (p+2)\Box_2 \ldots \Box_2 k\right)\right)$$

must have a canonical splitting $X_1 \Box_1 X_2$, with X_1 in $\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 p\right)$. Thus there is a canonical isomorphism

$$\begin{aligned} &\mathcal{S}\left((1\Box_2 2\Box_2 \ldots \Box_2 p)\Box_1 \left((p+1)\Box_2 (p+2)\Box_2 \ldots \Box_2 k\right)\right) \\ \cong\ &\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 p\right) \times \mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 (k-p)\right) \end{aligned}$$

Hence $\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$ can be identified with the cone on

$$\cup_{p=1}^{k-1} \cup_{\alpha \in Sh_{p,k-p}} \alpha \left(\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 p\right) \times \mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 (k-p)\right)\right).$$

Now according to [18], the boundary of the permutohedron P_k has a similar decomposition as a union:

$$\cup_{p=1}^{k-1} \cup_{\alpha \in Sh_{p,k-p}} \alpha \left(P_k \times P_{k-p}\right).$$

Thus we construct our simplicial isomorphism by sending the vertex $1\Box_2 2\Box_2 \ldots \Box_2 k$ to the barycenter of P_k and then extending to the boundary by sending

$$\alpha \left(\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 p\right) \times \mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 (k-p)\right)\right)$$

to $\alpha\left(P_p \times P_{k-p}\right)$ via the inductively defined isomorphisms $\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 p\right) \cong P_p$ and $\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 (k-p)\right) \cong P_{k-p}$.

To check that this is well-defined, we note that if two codimension 1 faces

$$\alpha \left(\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 p\right) \times \mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 (k-p)\right)\right)$$

and

$$\alpha' \left(\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 q\right) \times \mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 (k-q)\right)\right)$$

have a nonempty intersection, then we must have $p \neq q$ and the intersection must have the form

$$\beta \left(\mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 u\right) \times \mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 v\right) \times \mathcal{S}\left(1\Box_2 2\Box_2 \ldots \Box_2 w\right)\right),$$

where $u = \min(p, q)$, $w = \min(k-p, k-q)$, $v = k-u-w$, and β is a (u, v, w)-shuffle. Moreover β is determined as the only shuffle such that $\beta(1\Box_1 2\Box_1 \ldots \Box_1 k)$ is contained in both codimension 1 faces. We then note that the analogs of these facts are also true in P_k.

The rest of the proof is straightforward and is left as an exercise.

Proof of Theorem 3.12 for n = 2. The Milgram construction for $n = 2$ can be rearranged as the premonad construction on the preoperad whose k-th space is the quotient space $P_k \times \Sigma_k / \approx$. The equivalence relation $\approx$ identifies the codimension 1 face $\alpha\left(P_p \times P_{k-p}\right)$ in $P_k \times \{\sigma\}$ with the codimension 1 face $P_p \times P_{k-p}$ in $P_k \times \{\alpha^{-1}\sigma\}$, for any $(p, k-p)$-shuffle α.

The preoperad space $\overline{\mathcal{J}}_2(k)$ can be similarly expressed as a similar quotient space $\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k) \times \Sigma_k / \approx$, where we identify $\mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k) \times \{\sigma\}$ with $\mathcal{S}\left(\sigma(1)\Box_2 \sigma(2)\Box_2 \sigma(3)\Box_2 \ldots \Box_2 \sigma(k)\right)$. The result now follows directly from Theorem 5.2.

The following is left as an exercise for the interested reader. It gives an intrinsic description in terms of generators and relations of the categories $\overline{\mathcal{J}}_2(k)$ and $\mathcal{J}_2(k)/\Sigma_k$. (It is not difficult to do the exercise with the help of the coherence theorem.)

Exercise 5.3. A J_2 functor is a functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ between monoidal categories which is strongly monoidal in the following sense. Denote by $\square_2$, $\square_1$ the monoid operations in $\mathcal{A}$, $\mathcal{B}$ respectively and by 0 the unit object in either category. Then we require that $F(0) = 0$ and that for each (p,q)-shuffle $\sigma \in \Sigma_k$ there is given a natural transformation

$$\begin{aligned}\zeta^{\sigma}_{A_1,A_2,\ldots,A_p;A_{p+1},A_{p+2},\ldots,A_k} : F(A_1\square_2\ldots\square_2 A_p)\square_1 F(A_{p+1}\square_2\ldots\square_2 A_k) \\ \longrightarrow \quad F(A_{\sigma^{-1}1}\square_2 A_{\sigma^{-1}2}\square_2\ldots\square_2 A_{\sigma^{-1}k})\end{aligned}$$

satisfying the following properties:

1. (Unit condition) $\zeta^{\sigma}_{A_1,A_2,\ldots,A_p;0,0,\ldots 0} = id_{F(A_1\square_2 A_2\square_2\ldots\square_2 A_p)}$ and $\zeta^{\sigma}_{0,0\ldots,0;A_{p+1},A_{p+2},\ldots,A_k} = id_{F(A_{p+1}\square_2 A_{p+2}\square_2\ldots\square_2 A_k)}$.

2. (Substitution Property) If $A_i = B_{ij_1}\square_2 B_{ij_2}\square_2\ldots\square_2 B_{ij_i}$, then

$$\zeta^{\sigma}_{A_1,A_2,\ldots,A_p;A_{p+1},A_{p+2},\ldots,A_k} = \zeta^{\sigma'}_{B_{11},\ldots,B_{1j_1},B_{21},\ldots,B_{pj_p};B_{j+1,1},\ldots,B_{kj_k}}$$

   where $\sigma' \in \Sigma_{j_1+j_2+\cdots+j_k}$ permutes the blocks $\{1,2,\ldots,j_1\}$, $\{j_1+1, j_1+2,\ldots,j_1+j_2\}$, $\ldots$, $\{j_1+\cdots+j_{k-1}+1, j_1+\cdots+j_{k-1}+2,\ldots,j_1+\cdots+j_{k-1}+j_k\}$ the same way that σ permutes 1, 2, $\ldots$ k. A special case of this is if any $A_i = 0$ (ie. a 0-fold $\square_2$-sum), in which case the resulting ζ is the same as one where the corresponding 0 entries have been deleted.

3. (Associativity) Given $p+q+r = k$, a (p,q)-shuffle σ, a $(p+q,r)$-shuffle τ, a (q,r)-shuffle κ and a $(p,q+r)$-shuffle λ, such that $\lambda(id\oplus\kappa) = \tau(\sigma\oplus id) = \gamma$ in Σ_k, then the following diagram commutes

$$\begin{tikzcd}[column sep=tiny]
& F(A_1\square_2\ldots\square_2 A_p)\square_1 F(A_{p+1}\square_2\ldots\square_2 A_{p+q})\square_1 F(A_{p+q+1}\square_2\ldots\square_2 A_k) \arrow[dl, "id\square_1\zeta^{\kappa}_{A_{p+1},\ldots A_{p+q};A_{p+q+1},\ldots,A_k}"'] \arrow[dr, "\zeta^{\sigma}_{A_1,\ldots,A_p;A_{p+1},\ldots A_{p+q}}\square_1 id"] & \\
F(A_1\square_2\ldots\square_2 A_p)\square_1 F(A_{\tilde\kappa^{-1}(p+1)}\square_2\ldots\square_2 A_{\tilde\kappa^{-1}n}) \arrow[dr, "\zeta^{\lambda}_{A_1,\ldots,A_p;A_{\tilde\kappa^{-1}(p+1)},\ldots,A_{\tilde\kappa^{-1}n}}"] & & F(A_{\sigma^{-1}1}\square_2\ldots\square_2 A_{\sigma^{-1}(p+q)})\square_1 F(A_{p+q+1}\square_2\ldots\square_2 A_k) \arrow[dl, "\zeta^{\tau}_{A_{\sigma^{-1}1},\ldots,A_{\sigma^{-1}(p+q)};A_{p+q+1},\ldots,A_k}"] \\
& F(A_{\gamma^{-1}1}\square_2 A_{\gamma^{-1}2}\square_2\ldots\square_2 A_{\gamma^{-1}n}) &
\end{tikzcd}$$

   where $\tilde\kappa$ is the translation of κ to the set $\{p+1,p+2,\ldots,k\}$.

Show that $\overline{\mathcal{J}}_2(k)/\Sigma_k$ is the target of the universal J_2 functor from the free monoidal category on one object. Similarly $\overline{\mathcal{J}}_2(k)$ can be described as a subcategory of the universal J_2 functor from the free monoidal category on $\{1,2,\ldots,k\}$.

The basic building block of the Milgram construction $J_n(X)$ for $n > 2$ is the product $(P_k)^{n-1}$. In order to relate $\overline{J}_n(X)$ to the Milgram construction, we have to relate $(P_k)^{n-1}$ to $\mathcal{S}(1\square_n 2\square_n\ldots\square_n k)$. Unfortunately the analog of Theorem 5.2 breaks down when $n > 2$: $\mathcal{S}(1\square_n 2\square_n\ldots\square_n k)$ is not isomorphic as a cell complex to $(P_k)^{n-1}$ but rather to a quotient of $(P_k)^{n-1}$. Nevertheless as we show below, $\mathcal{S}(1\square_n 2\square_n\ldots\square_n k)$ is homeomorphic to a disk of dimension $(n-1)(k-1)$, and thus also to $(P_k)^{n-1}$.

Definition 5.4. Let A and B be two objects of $P_k = \mathcal{S}(1\square_2 2\square_2 3\square_2\ldots\square_2 k)$ (using Theorem 5.2). Suppose that $A = A_1\square_1 A_2\square_1\ldots\square_1 A_p$ where each A_i is $\square_1$-irreducible. Define a new object $\pi_A(B)$ of P_k by

$$\pi_A(B) = (B\cap|A_1|)\square_1(B\cap|A_2|)\square_1(B\cap|A_3|)\square_1\ldots\square_1(B\cap|A_p|)$$

It is obvious that this induces a map of posets

$$\pi_A : P_k \longrightarrow P_k$$

retracting P_k onto the face $\mathcal{S}(A)$.

We collect here, for future reference, the following basic properties of the retractions π_A:

Lemma 5.5 *If A and B are objects of $P_k = \mathcal{S}(1\Box_2 2\Box_2 3\Box_2 \ldots \Box_2 k)$, then*

(i) $\pi_A \pi_B = \pi_{\pi_A(B)}$

(ii) *If $\mathcal{S}(A) \cap \mathcal{S}(B) = \mathcal{S}(C)$, then*

$$\pi_A(B) = \pi_B(A) = C,$$

and consequently

$$\pi_A \pi_B = \pi_B \pi_A = \pi_C.$$

(iii) *If $\sigma \in \Sigma_k$, then*

$$\sigma \pi_A(B) = \pi_{\sigma A}(\sigma B).$$

Definition 5.6. Given a based space X we define the *thick Milgram construction* $\widetilde{\mathcal{J}}_n(X)$ to be the quotient of the disjoint union $\coprod_{k \geq 0} (P_k)^{n-1} \times X^k$ by the equivalence relation generated by the relations

(i) $(c_1, c_2, \ldots, c_{n-1}; x_1, \ldots, x_{i-1}, *, x_i, \ldots, x_{k-1}) \approx (s_i(c_1), s_i(c_2), \ldots, s_i(c_{n-1}); x_1, \ldots, x_{i-1}, x_i, \ldots, x_{k-1})$

(ii) If some c_i is in a boundary face $\alpha(P_p \times P_q)$, where α is a (p, q) shuffle in Σ_k, then

$$(c_1, c_2, \ldots, c_{n-1}; x_1, x_2, \ldots, x_k) \approx \big(\alpha^{-1}(c_1), \alpha^{-1}(c_2), \ldots, \alpha^{-1}(c_{n-1}); \alpha(x_1, x_2, \ldots, x_k)\big)$$

We define the *Milgram construction* $J_n(X)$ to be the quotient of the thick Milgram construction by the following additional equivalence relations:

(iii) If c_i is in a boundary face $\mathcal{S}(A)$, then

$$(c_1, c_2, \ldots, c_i, c_{i+1}, \ldots, c_{n-1}; x_1, x_2, \ldots, x_k) \approx (c_1, c_2, \ldots, c_i, \pi_A(c_{i+1}), \ldots, \pi_A(c_{n-1}); x_1, x_2, \ldots, x_k)$$

Finally we define the *thin Milgram construction* $\widehat{\mathcal{J}}_n(X)$ by conically extending the relations (iii) to the interior of P_k^{n-1}:

(iii') If $(c_1, c_2, \ldots, c_{n-1})$ is in the cone (with respect to $(1\Box_2 2\Box_2 \ldots \Box_2 k, 1\Box_2 2\Box_2 \ldots \Box_2 k, \ldots, 1\Box_2 2\Box_2 \ldots \Box_2 k)$) of $P_k \times \cdots \times P_k \times \mathcal{S}(A) \times P_k \times \cdots \times P_k$, then

$$(c_1, c_2, \ldots, c_{n-1}; x_1, x_2, \ldots, x_k) \approx (c'_1, c'_2, \ldots, c'_{n-1}; x_1, x_2, \ldots, x_k),$$

where $(c'_1, c'_2, \ldots, c'_{n-1})$ is the image of $(c_1, c_2, \ldots, c_{n-1})$ under the conical extension of the map

$$(d_1.d_2, \ldots, d_i, d_{i+1}, \ldots, d_{n-1}) \mapsto (d_1.d_2, \ldots, d_i, \pi_A(d_{i+1}), \ldots, \pi_A(d_{n-1}))$$

on the boundary face.

Remark 5.7. In Milgram's own description of his construction $J_n(X)$, the relations (ii) and (iii) are combined into a single relation, cf. [19, p. 24].

It is clear that each of the above variants of Milgram's construction arises as the premonad construction on a preoperad. The preoperad $\widetilde{\mathcal{J}}_n$ associated with the thick Milgram construction has the form

$$\widetilde{\mathcal{J}}_n(k) = P_k^{n-1} \times \Sigma_k / \approx,$$

where the equivalence relation identifies a point $(c_1, c_2, \dots, c_{n-1}; \sigma)$, if some c_i is in $\alpha(P_p \times P_q)$, with the point $(\alpha^{-1}(c_1), \alpha^{-1}(c_2), \dots, \alpha^{-1}(c_{n-1}); \alpha\sigma)$. The unit maps $s_i : \widetilde{\mathcal{J}}_n(k) \to \widetilde{\mathcal{J}}_n(k-1)$ are applied coordinatewise:

$$s_i(c_1, c_2, \dots, c_{n-1}; \sigma) = (s_i(c_1), s_i(c_2) \dots, s_i(c_{n-1}); s_i(\sigma))$$

The preoperad $\mathcal{J}_n$, corresponding to the Milgram construction, is obtained from $\widetilde{\mathcal{J}}_n$ by taking the quotient of P_k^{n-1} by the relations

(*) If c_i is in a boundary face $\mathcal{S}(A)$, then

$$(c_1, c_2, \dots, c_i, c_{i+1}, \dots, c_{n-1}) \approx (c_1, c_2, \dots, c_i, \pi_A(c_{i+1}), \dots, \pi_A(c_{n-1})).$$

The preoperad $\widehat{\mathcal{J}}_n$, corresponding to the thin Milgram construction, is obtained from $\widetilde{\mathcal{J}}_n$ by taking the quotient of P_k^{n-1} by the conical extension of the relations (*):

(**) If $(c_1, c_2, \dots, c_{n-1})$ is in the cone (with respect to $(1\Box_2 2\Box_2 \dots \Box_2 k, 1\Box_2 2\Box_2 \dots \Box_2 k, \dots, 1\Box_2 2\Box_2 \dots \Box_2 k)$) of $P_k \times \cdots \times P_k \times \mathcal{S}(A) \times P_k \times \cdots \times P_k$, then

$$(c_1, c_2, \dots, c_{n-1}) \approx (c'_1, c'_2, \dots, c'_{n-1}),$$

where $(c'_1, c'_2, \dots, c'_{n-1})$ is the image of $(c_1, c_2, \dots, c_{n-1})$ under the conical extension of the map

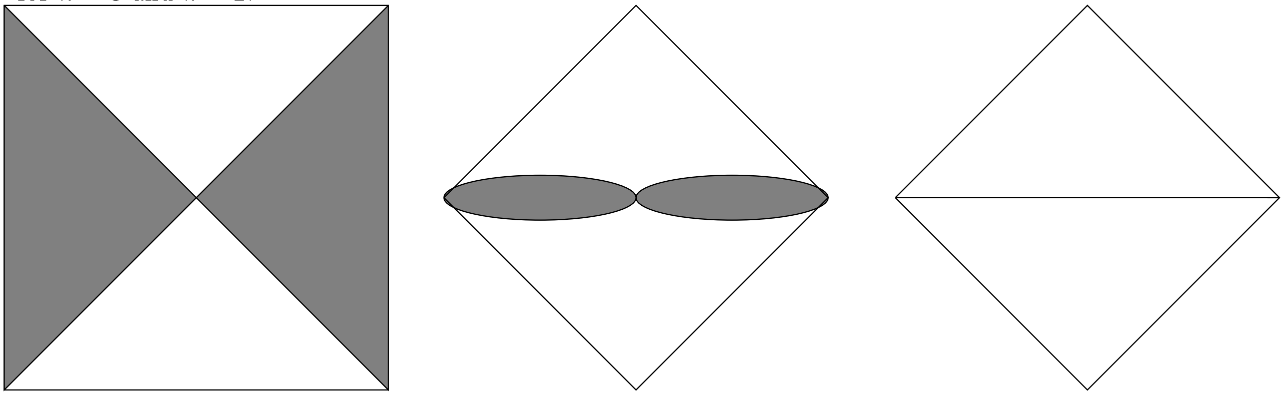
$$(d_1.d_2, \dots, d_i, d_{i+1}, \dots, d_{n-1}) \mapsto (d_1.d_2, \dots, d_i, \pi_A(d_{i+1}), \dots, \pi_A(d_{n-1}))$$

on the boundary face.

Thus all of these preoperads take the generic form

$$\begin{aligned}
\widetilde{\mathcal{J}}_n(k) &= \widetilde{D}_n(k) \times \Sigma_k / \approx \\
\mathcal{J}_n(k) &= D_n(k) \times \Sigma_k / \approx \\
\widehat{\mathcal{J}}_n(k) &= \widehat{D}_n(k) \times \Sigma_k / \approx
\end{aligned}$$

where $\widetilde{D}_n(k) = P_k^{n-1}$ and $D_n(k)$, resp. $\widehat{D}_n(k)$, are the quotients of P_k^{n-1} by the relations (*), resp. (**). (One needs Lemma 5.5 to check that the relations (*) and (**) commute with the equivariancy relations used to glue together the $k!$ copies of these quotients.) The following pictures illustrate these constructions for $n = 3$ and $k = 2$:



The first picture shows $\widetilde{D}_3(2) = P_2^2 = I \times I$. The second picture shows $D_3(2)$, which is obtained from the first picture by collapsing the vertical sides of the shaded triangles to points. The collapsed triangles become "polygons" with two sides. The third picture shows $\widehat{D}_3(2)$, which is obtained from the first picture by collapsing the shaded triangles to horizontal lines.

To complete the proof of Theorem 3.13 we will need the following elementary result from PL topology:

Lemma 5.8 *Let D^i denote the i-dimensional disk.*

(a) *If $D^m \subset \partial D^n$ is a PL imbedded disk and $\phi : D^m \to D^k$ is an elementary collapse to a boundary face, then $D^n \cup_{D^m} D^k \cong D^n$*

(b) *If D^n, $\phi : D^m \to D^k$ are as in (a) and C_pX denotes the cone with respect to an interior point $p \in D^n$, then $D^n \cup_{C_pD^m} C_pD^k \cong D^n$.*

Proof: We first prove part (a) for the case $m = n-1$. We take as a model for D^n the prism $\Delta^{n-1} \times I$ and we take the boundary disk we are collapsing to be the top face $\Delta^{n-1} \times \{1\}$. (That we can arrange this follows from the Disk Theorem of PL topology, cf. [21, p. 56].) Let K denote the convex hull in $\Delta^{n-1} \times I$ of $\Delta^{n-1} \times \{0\}$ and $\Delta^k \times \{1\}$. Then K is obviously an n-dimensional topological disk. Now consider the map of pairs $\lambda : (\Delta^{n-1} \times I, \Delta^{n-1} \times \{1\} \longrightarrow (K, \Delta^k \times \{1\})$ given by the formula

$$(\mathbf{x}, t) \mapsto ((1-t)\mathbf{x} + t\phi(\mathbf{x}), t)\,.$$

This map is a relative homeomorphism, since if $(\mathbf{x_1}, t)$ and $(\mathbf{x_2}, t)$ both mapped to the same point for some $t < 1$, then the vectors $\mathbf{x_1} - \mathbf{x_2}$ and $\phi(\mathbf{x_1}) - \phi(\mathbf{x_2})$ would have to point in opposite directions, which can't happen for a linear retraction ϕ. Since the restriction of λ to $\Delta^{n-1} \times \{1\} \to \Delta^k \times \{1\}$ is just ϕ, λ induces a homeomorphism

$$D^n \cup_{D^m} D^k = (\Delta^{n-1} \times I) \cup_{\Delta^{n-1} \times \{1\}} \Delta^k \times \{1\} \cong K \cong D^n$$

We can reduce the general case of part (a) to the special case proved above as follows:

$$D^n \cup_{D^m} D^k \cong (D^n \cup_{D^{n-1}} D^m) \cup_{D^m} D^k \cong D^n \cup_{D^{n-1}} D^k \cong D^n\,.$$

Finally we can reduce part (b) to part (a) as follows. Cut apart the given disk D^n along a suitable codimension 1 subdisk passing through the point p. (If $m = n-1$, excise the interior of the cone C_pD^m first.) Then the we can realize $D^n \cup_{C_pD^m} C_pD^k$ as the result of a two step process. In the first step we are attaching D^{k+1} to each of the two n-dimensional disks we obtained after the cut, by an attachment of the form given in part (a). By part (a) we know that the resulting spaces are homeomorphic to n-disks. In the second step we glue together these disks along the parts of the boundaries of the two pieces which were originally $(n-1)$-disks (where we originally made the cut), but where we attached D^{k+1}'s. That the resulting part of the boundaries are still $(n-1)$-disks follows by noting that the complementary parts of the boundaries are PL imbedded $(n-1)$-disks.

Proof of Theorem 3.13: $\widetilde{\mathcal{J}}_n(k) = P_k^{n-1}$ is evidently a $(k-1)(n-1)$-dimensional disk. By repeated applications of part (a) of Lemma 5.8, so is $\mathcal{J}_n(k)$. By repeated applications of part (b) of Lemma 5.8, $\widehat{\mathcal{J}}_n(k)$ is also a $(k-1)(n-1)$-dimensional disk.

We now construct a map of posets $q : \widetilde{\mathcal{J}}_n(k) = P_k^{n-1} \to \overline{\mathcal{J}}_n(k) = \mathcal{S}(1\Box_n 2\Box_n \ldots \Box_n k)$ as follows. Given $(A_1, A_2, \ldots, A_{n-1}) \in P_k^{n-1} = (\mathcal{S}(1\Box_n 2\Box_n \ldots \Box_n k))^{n-1}$, first replace it by

$$(B_1, B_2, \ldots, B_{n-1}) = \big(A_1, \pi_{A_1}(A_2), \pi_{A_1}\pi_{A_2}(A_3), \ldots, \pi_{A_1}\pi_{A_2}\ldots\pi_{A_{n-2}}(A_{n-1})\big)$$

We then have

$$B_{n-1} \le B_{n-2} \le \cdots \le B_2 \le B_1,$$

from which it follows that the parenthesization of any object B_i induces a (usually redundant) parenthesization of the object B_{i+1}. From this it follows that we can endow B_{n-1} with $n-1$ levels of parentheses: the innermost coming from the original parenthesization of B_{n-1}, the next level coming from the parenthesization of B_{n-1}, ... , the outermost level coming from the parenthesization of B_1. Now define $q(A_1, A_2, \ldots, A_{n-1})$ to be the object constructed from this heavily parenthesized version of B_{n-1} as follows. Replace each $\Box_2$ in the innermost level of parentheses by $\Box_n$. Then replace each $\Box_1$ in the next level of parentheses by $\Box_{n-1}$, etc. At the penultimate step replace each $\Box_1$ in the next to outermost level of parentheses by $\Box_2$. At the final step leave the outermost $\Box_1$'s alone.

The following example (with $n = 4$, $k = 5$) illustrates this process. Let

$$(A_1, A_2, A_3) = ((1\Box_2 3)\Box_1(2\Box_2 4\Box_2 5), (1\Box_2 3\Box_2 4)\Box_1(2\Box_2 5), (1\Box_2 2\Box_2 4\Box_2 5)\Box_1 3)\,.$$

Then

$$(B_1, B_2, B_3) = ((1\Box_2 3)\Box_1(2\Box_2 4\Box_2 5), (1\Box_2 3)\Box_1 4\Box_1(2\Box_2 5), 3\Box_1 1\Box_1 4\Box_1(2\Box_2 5)),$$

and the resulting redundant parenthesization of B_3 is

$$B_3 = (((3)\Box_1(1)))\Box_1((4)\Box_1((2\Box_2 5))$$

Thus

$$q(A_1, A_2, A_3) = (3\Box_3 1)\Box_1(4\Box_2(2\Box_4 5)).$$

It is easy to see that this map of posets extends to a map of preoperads $q : \widetilde{\mathcal{J}}_n \to \overline{\mathcal{J}}_n$, which factors through a map of preoperads $q' : \widehat{\mathcal{J}}_n \to \overline{\mathcal{J}}_n$. To check that q' is a simplicial isomorphism, it is only necessary to note that $X \leq Y$ in $\mathcal{S}(1\Box_n 2\Box_n \ldots \Box_n k)$ if and only if we can find A, B in $(P_k)^{n-1}$ so that $q(A) = X$, $q(B) = Y$ and $A \leq B$.

Thus we have quotient maps of preoperads

$$\widetilde{\mathcal{J}}_n \xrightarrow{q_1} \mathcal{J}_n \xrightarrow{q_2} \widehat{\mathcal{J}}_n \cong \overline{\mathcal{J}}_n$$

We have that q_2 is an equivalence, since $D_n(k) \to \widehat{D}_n(k)$ are both given by elementary collapses of $\widetilde{D}_n(k)$, and since the gluings of the $k!$ copies of $D_n(k)$, resp. $\widehat{D}_n(k)$ are along the boundaries where $D_n(k)$ and $\widehat{D}_n(k)$ are isomorphic.

So it remains to show that

$$q = q_2 q_1 : \widetilde{\mathcal{J}}_n \longrightarrow \overline{\mathcal{J}}_n$$

is an equivalence. Since this map is given by a map of posets, we use Quillen's Theorem A: we show that for any object in the poset $\overline{\mathcal{J}}_n$ the overcategory of objects in $\widetilde{\mathcal{J}}_n$ is contractible. But this is easy: elementary collapses of this overcategory given by relations (**) above gives the cone over that object in $\overline{\mathcal{J}}_n$. Since the cone is obviously contractible, and elementary collapse do not change the homotopy type, the overcategory must be contractible too. This completes the proof.

Remark 5.9. In [12, p. 55], Getzler and Jones consider a poset closely related to $\overline{\mathcal{J}}_n(k)$. More precisely their poset is isomorphic to the "dual" $\overline{\mathcal{J}}_n(k)^*$. By this we mean the full subcategory of $\mathcal{M}_n(k)$ consisting of objects whose nesting of operations is opposite to those in $\overline{\mathcal{J}}_n(k)$: the $\Box_n$ operations are nested on the outermost level, the $\Box_{n-1}$ operations are nested at the next outermost level, ..., the $\Box_1$ operations are nested at the innermost level. Getzler and Jones denote the objects of their poset as "multiple bar codes": permutations $\sigma \in \Sigma_k$ with their elements separated by multiple bars

$$\sigma(1)|_{i_1}\sigma(2)|_{i_2}\ldots|_{i_{k-1}}\sigma(k)$$

where the subscript on each bar is $\leq n$ and denotes the number of times the bar is supposed to be repeated. The poset isomorphism with $\overline{\mathcal{J}}_n(k)^*$ is given by the replacement $|_i \mapsto \Box_i$, with the resulting object parenthesized according to the operation precedence rules: $\Box_1$ has the highest precedence, $\Box_2$ has the next highest precedence, ..., $\Box_n$ has the lowest precedence.

There is a duality anti-automorphism of $\mathcal{M}_n(k)$ given by $\Box_i \mapsto \Box_{n-i+1}$, which is easy to verify using the coherence theorem. This anti-automorphism takes $\overline{\mathcal{J}}_n(k)$ to $\overline{\mathcal{J}}_n(k)^*$. Thus $\overline{\mathcal{J}}_n(k)$ is anti-isomorphic to $\overline{\mathcal{J}}_n(k)^*$ and hence also to the Getzler-Jones poset. It follows that the nerve of $\overline{\mathcal{J}}_n(k)$ is isomorphic to the nerve of the Getzler-Jones poset.

Getzler and Jones also consider an operad freely generated by these posets. This operad obviously maps into our operad $\mathcal{M}_n(k)$. It will be shown in a forthcoming paper that this map of operads is an equivalence.

There is also an extensive discussion of the Getzler-Jones posets and their relation to various other constructions in [2, p. 46].

6 Relation to Little n-Cubes

Boardman and Vogt [4] introduced the little n-cubes operad to parametrize multiplications on an n-fold loop space. Later May [15] used these operads to construct small models of $\Omega^n S^n X$, an alternative to Milgram's models. This section is devoted to the proof of our main result Theorem 3.14, relating the n-fold monoidal operad $\mathcal{M}_n$ to the little n-cubes operad $\mathcal{C}_n$, and then derive some consequences relating n-fold monoidal categories to n-fold loop spaces.

We begin by associating to each object of $\mathcal{M}_n(k)$ a contractible space of k-fold configurations of little n-cubes.

Definition 6.1. We think of a little n-cube c as a product of closed subintervals of the unit interval. Thus the elements of the k-th space of little n-cubes $\mathcal{C}_n(k)$ have the form $(c_1, c_2, \ldots, c_k)$ where

$$c_j = [u_{j1}, v_{j1}] \times [u_{j2}, v_{j2}] \times \cdots \times [u_{jn}, v_{jn}]$$

where the interiors of the little n-cubes c_j are required to be pairwise disjoint. If

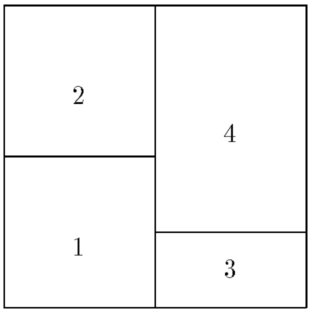
$$c = [u_1, v_1] \times [u_2, v_2] \times \cdots \times [u_n, v_n] \qquad d = [z_1, w_1] \times [z_2, w_2] \times \cdots \times [z_n, w_n]$$

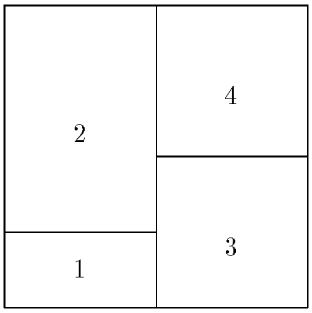
are little n-cubes we write $c <_i d$ to mean that $v_i \leq z_i$. Equivalently $c <_i d$ if there is a hyperplane perpendicular to the i-coordinate axis such that the interior of c lies on the negative side of the hyperplane and the interior of d on the positive side of the hyperplane. (Note that this condition forces the interiors of c and d to be disjoint.) If A is an object of $\mathcal{M}_n(k)$ let $G(A)$ denote the space of all k-fold configurations of little n-cubes satisfying the following conditions:

$$G(A) = \{(c_1, c_2, \ldots, c_k) \mid c_a <_i c_b \text{ if } a \Box_i b \text{ in } A\}$$

(cf. Definition 3.5). Note that we do not have to explicitly require that $G(A)$ be a subspace of $\mathcal{C}_n(k)$ – the ordering relations defining $G(A)$ force the little n-cubes in a configuration in $G(A)$ to have pairwise disjoint interiors thus forcing the configuration to be in $\mathcal{C}_n(k)$. Because of this, $G(A)$ may be identified with a convex, hence contractible, subspace of $\mathbb{R}^{2k}$ given by a set of inequalities between the coordinates. For the same reason $G(A)$ is a closed subspace of $\mathcal{C}_n(k)$ (but not of $\mathbb{R}^{2k}$ and hence not compact, since the requirement that each little n-cube have a nonvacuous interior is an open condition given by strict inequalities).

Example 6.2. The following two figures represent configurations belonging to $G((1\Box_2 2)\Box_1(3\Box_2 4))$:





More generally such configurations could have the subcube i properly contained in the region marked i $1 \leq i \leq 4$.

Remark 6.3. If $A = B\Box_i C$ then for any configuration $(d_1, d_2, \ldots, d_k)$ in $G(B\Box_i C)$, we can find a hyperplane perpendicular to the i-th coordinate axis such that all little cubes in the configuration having labels coming from the generating objects in B have their interiors on the negative side of the hyperplane and all little cubes in the configuration whose labels come from the generating objects in C have their interiors on the negative side of the hyperplane. For if

$$d_j = [u_{j1}, v_{j1}] \times [u_{j2}, v_{j2}] \times \cdots \times [u_{jn}, v_{jn}]$$

and if we let

$$M = \max\{v_{bi} \mid b \text{ in } B\} \qquad m = \min\{u_{ci} \mid c \text{ in } C\},$$

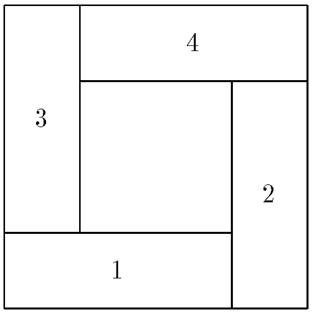
then the conditions that $(d_1, d_2, \ldots, d_k)$ must satisfy in order to be in $G(B\Box_i C)$ imply that $M \leq m$. Thus we can take $x_i = M$ as a separating hyperplane with the required properties. ($x_i = m$ would also work, as would any hyperplane in between those two.) It follows from this observation that

$$\cup_{A \in obj(\mathcal{M}_n(k))} G(A) = \mathcal{D}_n(k)$$

where $\mathcal{D}_n(k) \subset \mathcal{C}_n(k)$ is the subspace of *decomposable configurations* of little cubes. Decomposability is defined recursively as follows. First of all a configuration consisting of a single n-cube, ie. an element of $\mathcal{C}_n(1)$, is declared to be decomposable. For a k-fold configuration to be decomposable, we require that there be a hyperplane perpendicular to one of the coordinate axes which does not pass through the interior of any little n-cube in the configuration and which divides the configuration into two proper subconfigurations. We further require that the subconfigurations on both sides of the separating hyperplane to be themselves decomposable. It is trivially true that all $\mathcal{C}_1(k)$, $\mathcal{C}_n(1)$, and $\mathcal{C}_n(2)$ consist entirely of decomposable configurations. The same is also true for $\mathcal{C}_2(3)$, but all other spaces in the little n-cubes operads contain nondecomposable configurations. For instance (c_1, c_2, c_3) where

$$c_1 = [0, \frac{1}{2}] \times [0,1] \times [0, \frac{1}{2}] \quad c_2 = [0,1] \times [0, \frac{1}{2}] \times [\frac{1}{2}, 1] \quad c_3 = [\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \times [0,1]$$

is a nondecomposable configuration of little 3-cubes in $\mathcal{C}_3(3)$ and the following figure shows a 4-fold configuration of little 2-cubes which is nondecomposable.



The decomposable little n-cubes form a suboperad $\mathcal{D}_n$ of $\mathcal{C}_n$. By sufficiently shrinking every little n-cube in a configuration towards its barycenter, we can convert any configuration into a decomposable one. This shows that the inclusion $\mathcal{D}_n \subset \mathcal{C}_n$ is an equivalence of operads. The operad $\mathcal{D}_n$ was studied by Dunn [7] who showed it is homeomorphic to the n-fold tensor product $\mathcal{C}_1 \otimes \mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_1$ of the little 1-cubes operad.

The assignment $A \mapsto G(A)$ is only defined on objects, not on morphisms. In order to construct a functor on $\mathcal{M}_n(k)$ we proceed as follows:

Definition 6.4. For any object $A \in \mathcal{M}_n(k)$ define

$$F(A) = \cup_{X \to A} G(X),$$

where the union is indexed over all objects $X \in \mathcal{M}_n(k)$ which map into A. Then by definition given a morphism $B \longrightarrow A$ in $\mathcal{M}_n(k)$, there is an induced inclusion of subspaces $F(B) \subset F(A)$. Thus we have constructed a functor $F : \mathcal{M}_n(k) \longrightarrow$ **Top**

Remark 6.5. This construction and proof of Theorem 3.14, based on the analysis of the resulting colimits, was inspired by the work of Clemens Berger on cellular operads. Our original proposed line of proof was to form similar colimits over the barycentric subdivision of $M_n(k)$, associating to the barycenter the intersection of the spaces $G(X)$ over all the vertices in the simplex. This caused a great number of technical difficulties due to the fact that some of these intersections are empty.

Lemma 6.6 *For any object $A \in \mathcal{M}_n(k)$ the inclusion*

$$G(A) \subset F(A)$$

is a strong deformation retract. Thus $F(A)$ is contractible.

Proof. The deformation retraction is constructed in a number of stages. If $A = B\Box_i C$, we first show that the subspace

$$\cup_{X_1\Box_i X_2 \to A} G(X_1\Box_i X_2) \subset F(A)$$

is a strong deformation retract, where the union is taken over all objects of the form $X_1\Box_i X_2$ which map into A, with X_1, X_2 having the same underlying sets of generating objects as B and C respectively.

Suppose X is an arbitrary object of $\mathcal{M}_n(k)$ which maps into A. Then define

$$X_1 = X - |C| \qquad X_2 = X - |B|,$$

(cf. Definition 4.5). By the coherence theorem $X_1\Box_i X_2$ maps into X. Now let $(d_1, d_2, \ldots, d_k)$ be a configuration of little n-cubes contained in $G(X)$, with

$$d_j = [u_{j1}, v_{j1}] \times [u_{j2}, v_{j2}] \times \cdots \times [u_{jn}, v_{jn}],$$

and let

$$M = \max\{v_{bi} \mid b \text{ in } B\} \qquad m = \min\{u_{ci} \mid c \text{ in } C\}.$$

If $M \leq m$ then $(d_1, d_2, \ldots, d_k)$ is contained in $G(X_1\Box_i X_2)$ and we leave the configuration alone. Otherwise if $M > m$, let D_1 denote the linear deformation which takes the closed interval $[0, M]$ onto the closed interval $[0, \frac{M+m}{2}]$ and let D_2 denote the linear deformation which takes the closed interval $[m, 1]$ to the closed interval $[\frac{M+m}{2}, 1]$. Now apply the deformation D_1 (resp. D_2) simultaneously to the i-th coordinates of all little cubes d_b (resp. d_c) whose labels correspond to generators b in B (resp. c in C). We claim that this defines a strong deformation retraction of $G(X)$ onto $G(X) \cap G(X_1\Box_i X_2)$. The only nonobvious point is that the retraction stays within $G(X)$. This follows from the coherence theorem. For the relative position of any two little cubes in the configuration can change only if the label of one, say d_b is in B and the label of the other d_c is in C. Moreover this only happens in the i-th coordinate direction and only if $d_b \not<_i d_c$. So the only trouble which could arise is if $(d_1, d_2, \ldots, d_k) \in G(X)$ required that $d_c <_i d_b$. But this could only happen if $c\Box_i b$ in X. But if that were the case, by the coherence theorem, there couldn't be a morphism $X \to A = B\Box_i C$ since $b\Box_i c$ in A.

By gluing together the deformations of $G(X)$ onto $G(X) \cap G(X_1\Box_i X_2)$ over all objects X in $\mathcal{M}_n(k)$ mapping into A one obtains that

$$\cup_{X_1\Box_i X_2 \to A} G(X_1\Box_i X_2) \subset F(A)$$

is a strong deformation retract. In the next stage of the deformation one decomposes $B = B'\Box_r B''$ and $C = C'\Box_s C''$ obtaining a decomposition

$$A = (B'\Box_r B'')\Box_i(C'\Box_s C''),$$

and then one shows by a similar argument that

$$\cup_{(X_1'\Box_r X_2'')\Box_i(X_2'\Box_s X_2'')} G\left((X_1'\Box_r X_2'')\Box_i(X_2'\Box_s X_2'')\right) \subset \cup_{X_1\Box_i X_2 \to A} G(X_1\Box_i X_2)$$

is a strong deformation retract. Composing the two retractions, one obtains that

$$\cup_{(X_1'\Box_r X_2'')\Box_i(X_2'\Box_s X_2'')} G\left((X_1'\Box_r X_2'')\Box_i(X_2'\Box_s X_2'')\right) \subset F(A)$$

is a strong deformation retract. One continues this refinement process, restricting to objects X which map into A and which resemble A to an ever deeper level of parentheses and operations, showing at each stage that the resulting union of $G(X)$ is a strong deformation retract of the union of $G(X)$ at the preceding stage and hence is also a strong deformation retract of $F(A)$. After finitely many stages the only object X left is A itself. Thus we obtain that $G(A) \subset F(A)$ is a strong deformation retract. Now as we noted in Definition 6.1, $G(A)$ can be identified with a convex subspace of Euclidean space and hence is contractible. Therefore $F(A)$ is also contractible.

Lemma 6.7 *For any two objects A, B of $\mathcal{M}_n(k)$,*

$$F(A) \cap F(B) = \bigcup_{\substack{X \to A \\ X \to B}} F(X),$$

where the union is indexed over all objects X in $\mathcal{M}_n(k)$ which map into both A and B.

Proof. First note that the inclusion

$$\bigcup_{\substack{X \to A \\ X \to B}} F(X) \subseteq F(A) \cap F(B)$$

is immediate from definition. To prove equality, we proceed by double induction. Our primary induction is on k, the number of generating objects, starting with the observation that the lemma holds trivially if $k = 1$. Building on this induction we first prove the following:

Claim. If there are nontrivial decompositions $A = A_1 \Box_i A_2$ and $B = B_1 \Box_i B_2$ with A_1 and B_1 (and hence also A_2 and B_2) having the same underlying set of generating objects, then the intersection $F(A) \cap F(B)$ satisfies the lemma.

By our primary induction:

$$F(A_1) \cap F(B_1) = \bigcup_{\substack{X_1 \to A_1 \\ X_1 \to B_1}} F(X_1) \qquad F(A_2) \cap F(B_2) = \bigcup_{\substack{X_2 \to A_2 \\ X_2 \to B_2}} F(X_2)$$

It follows immediately that

$$F(A) \cap F(B) = \bigcup_{\substack{X_1 \to A_1 \\ X_1 \to B_1 \\ X_2 \to A_2 \\ X_2 \to B_2}} F(X_1 \Box_i X_2)$$

and thus implies the lemma in this case, proving the claim.

Our secondary induction is on objects A and B with respect to the ordering in the poset $\mathcal{M}_n(k)$. If A or B, say A, is minimal in the poset $\mathcal{M}_n(k)$, then A has the form

$$A = j_1 \Box_1 j_2 \Box_1 j_3 \Box_1 \ldots \Box_1 j_k.$$

Now there are two possibilities. First of all if $m_1 \Box_1 m_2$ in B implies that $m_1 \Box_1 m_2$ in A, then by the coherence theorem A maps into B and we have

$$F(A) \cap F(B) = F(A)$$

and we are done. Conversely if there is a pair of generating objects m_1, m_2 such that $m_1 \Box_1 m_2$ in A, whereas $m_2 \Box_1 m_1$ in B, then by the coherence theorem $m_2 \Box_1 m_1$ in X for any object X mapping into B. Hence $G(A) \cap G(X) = \emptyset$ for all such X, since a requirement for a configuration $(c_1, c_2, \ldots, c_k)$ to lie in $G(A)$ is that $c_{m_1} <_1 c_{m_2}$ whereas a requirement for that configuration to lie in $G(X)$ is that $c_{m_2} <_1 c_{m_1}$, and no configuration can simultaneously satisfy both requirements. It follows that

$$F(A) \cap F(B) = \emptyset,$$

and the lemma again holds in this case. This starts the secondary induction.

Now suppose we have shown that the lemma holds for all intersections $F(C) \cap F(D)$ where C maps into A, D maps into B, and at least one of C, D is not equal to A, resp. B. Let us suppose that the outermost operation in A is $\Box_i$ and the outermost operation in B is $\Box_j$. Thus $A = A_1 \Box_i A_2$ and $B = B_1 \Box_j B_2$. Without loss of generality, we may suppose that $i \leq j$.

Consider the partial objects

$$A'_1 = B \cap |A_1| \qquad A'_2 = B \cap |A_2|$$

(cf. Definition 4.5). We clearly have

$$F(A) \cap F(B) = F(A) \cap F(A'_1 \Box_i A'_2) \cap F(B)$$

We can apply the claim above to the intersection $F(A) \cap F(A'_1 \Box_i A'_2)$. We can then distribute the intersection with $F(B)$ over the resulting union. If $A = A'_1 \Box_i A'_2$, we get no reduction, since then $F(A) \cap F(A'_1 \Box_i A'_2) = F(A)$. Otherwise we can apply our secondary induction to the resulting union of intersections. Similarly we consider

$$B'_1 = A \cap |B_1| \qquad B'_2 = A \cap |B_2|,$$

note that

$$F(A) \cap F(B) = F(A) \cap F(B'_1 \Box_j B'_2) \cap F(B)$$

and apply the claim to the intersection $F(B'_1 \Box_j B'_2) \cap F(B)$. Again using our secondary induction we obtain that the lemma applies unless $B = B'_1 \Box_j B'_2$.

Thus we are left with the case when both $A = A'_1 \Box_i A'_2$ and $B = B'_1 \Box_j B'_2$ hold. But in this case we must have decompositions

$$A = (C \Box_j D) \Box_i (U \Box_j V) \qquad B = (C \Box_i U) \Box_j (D \Box_i V)$$

for some objects C, D, U and V. Now if $i < j$, then there is a morphism $\eta^{ij}_{C,D,U,V} : A \longrightarrow B$. Hence $F(A) \cap F(B) = F(A)$ and the lemma holds. If $i = j$ and either $D = 0$ or $U = 0$, then $A = B$ and again we are done. Finally if both $D \neq 0$ and $U \neq 0$, then $G(A) \cap G(B) = \emptyset$ for a configuration of little n-cubes in the intersection would have to satisfy contradictory specifications on the relative positions of little n-cubes with labels in D and U. This then means that

$$F(A) \cap F(B) = \bigcup_{\substack{X \to A \\ X \neq A}} F(X) \cap F(B) \bigcup \bigcup_{\substack{Y \to B \\ Y \neq B}} F(A) \cap F(Y)$$

and we can apply our secondary induction. This concludes the induction and proof.

In view of Remark 6.3 and the obvious fact that the inclusion of a finite union of closed convex spaces of $\mathbb{R}^N$ into a bigger such finite union is a closed cofibration, a direct consequence of the preceding lemma is:

Corollary 6.8 *The natural map induced by inclusions*

$$colim_{A \in \mathcal{M}_n(k)} F(A) \longrightarrow \cup_{A \in Obj(\mathcal{M}_n(k))} F(A) = \mathcal{D}_n(k)$$

is a homeomorphism. Moreover for each object A in $\mathcal{M}_n(k)$ the induced map

$$colim_{X \xrightarrow{\neq} A} F(X) \longrightarrow F(A)$$

is a closed cofibration.

The main technical ingredient in the proof of Theorem 3.14 is the following:

Proposition 6.9 *Let $\mathcal{P}$ be a finite poset and let $F : \mathcal{P} \longrightarrow$ **Top** be a functor satisfying the property that for each object i in $\mathcal{P}$ the induced map*

$$colim_{j<i} F(j) \longrightarrow F(i)$$

is a closed cofibration. Then the natural map $hocolim_{\mathcal{P}} F \longrightarrow colim_{\mathcal{P}} F$ is an equivalence.

Proof: We first observe that

$$\mathrm{hocolim}_{\mathcal{P}} F = \mathrm{colim}_{\mathcal{P}} G$$

where $G : \mathcal{P} \longrightarrow$ **Top** is given by

$$G(i) = \mathrm{hocolim}_{j \leq i} F$$

and that G satisfies the cofibration condition also. We note that $G(i) \longrightarrow F(i)$ is an equivalence for all objects $i \in \mathcal{P}$.

Then we filter the objects of $\mathcal{P}$ according to the length of the largest increasing chain of objects which terminates in the given object. Thus the objects of filtration 0 are the minimal objects. We denote by $\mathcal{P}_k$ the full subcategory of $\mathcal{P}$ whose objects have filtration $\leq k$. We proceed by induction on k to show that

$$\mathrm{colim}_{\mathcal{Q}_k} G \longrightarrow \mathrm{colim}_{\mathcal{Q}_k} F$$

is an equivalence, for any subposet $\mathcal{Q}_k \subseteq \mathcal{P}_k$ satisfying the condition that if $j < i$ and $i \in \mathcal{Q}_k$, then $j \in \mathcal{Q}_k$. This is true for $k = 0$ since in that case the colimits are just disjoint unions of the values of G and F over minimal objects.

The induction step from $k-1$ to k is based upon the pushout lemma for equivalences: suppose given a commutative cubical diagram of spaces and maps as shown

$$\begin{tikzcd}
& \bullet \arrow[rr, hook] \arrow[dl, "\alpha"'] \arrow[dd] & & \bullet \arrow[dl, "\beta"'] \arrow[dd] \\
\bullet \arrow[rr, hook] \arrow[dd] & & \bullet \arrow[dd] & \\
& \bullet \arrow[rr, hook] \arrow[dl, "\gamma"'] & & \bullet \arrow[dl, "\delta"] \\
\bullet \arrow[rr, hook] & & \bullet &
\end{tikzcd}$$

Assume that the front and back faces are pushout squares with the map across the top being a closed cofibration in each case. (In the sequel we will refer to such pushout squares as cofibration squares. It will also be useful to note that in such a cofibration square the map across the bottom is also a cofibration.) If the maps marked α, β, and γ are equivalences, then so is the map marked δ.

We note that we have a pushout square

$$\begin{tikzcd}
\coprod_{i \in \mathcal{Q}_k \,\&\, \mathrm{filt}(i)=k} \mathrm{colim}_{j<i} F(j) \arrow[r, hook] \arrow[d] & \coprod_{i \in \mathcal{Q}_k \,\&\, \mathrm{filt}(i)=k} F(i) \arrow[d] \\
\mathrm{colim}_{\mathcal{Q}_{k-1}} F \arrow[r, hook] & \mathrm{colim}_{\mathcal{Q}_k} F
\end{tikzcd}$$

with $\mathcal{Q}_{k-1} = \mathcal{Q}_k \cap \mathcal{P}_{k-1}$. The map across the top is a closed cofibration by hypothesis.

The same considerations apply to the functor G and we get an analogous cofibration square. We thus obtain a commutative cube as in the pushout lemma, with the front face being the cofibration square for F and the back face being the cofibration square for G, and the maps from the back face to the front face being induced by the natural transformation $G \to F$. It is immediate that the map corresponding to β is an equivalence, while the maps corresponding to γ and β are equivalences by the induction hypothesis. This completes the induction and proof.

Remark 6.10. Proposition 6.9 is true for any cofinite strongly directed set $\mathcal{P}$ (ie. $\mathcal{P}$ is a directed set such that $a \leq b$ and $b \leq a$ implies $a = b$, and each $a \in \mathcal{P}$ has only a finite number of predecessors). This statement is a fairly immediate consequence of the closed model category structure on the category of $\mathcal{P}$-diagrams in **Top** dual to the one constructed by Edwards and Hastings in [8, §(3.2)]

Lemma 6.11 *Let $\{\mathcal{M}(n)\}_{n\geq 0}$ be an operad in the category of small categories. Let $\{F_n : \mathcal{M}(n) \longrightarrow$ **Top**$\}_{n\geq 0}$ be a collection of functors satisfying the following conditions:*

1. *There is an operad $\mathcal{C}$ such that for each object A of $\mathcal{M}(n)$ $F_n(A) \subseteq \mathcal{C}(n)$, and for each morphism $f : A \to B$ in $\mathcal{M}(n)$ $F_n(f) : F_n(A) \longrightarrow F_n(B)$ is an inclusion.*

2. *For each permutation $\sigma \in \Sigma_n$, action by σ on $\mathcal{C}(n)$ sends the subspace $F_n(A)$ to the subspace $F_n(A\sigma)$.*

3. *Given objects $A \in \mathcal{M}(n)$, $B_i \in \mathcal{M}(j_i)$ $1 \leq i \leq n$, the structure map*

$$\mathcal{C}(n) \times \mathcal{C}(j_1) \times \mathcal{C}(j_2) \times \cdots \times \mathcal{C}(j_n) \longrightarrow \mathcal{C}(j_1 + j_2 + \cdots + j_n)$$

sends the subspace $F_n(A) \times F_{j_1}(B_1) \times \cdots \times F_{j_n}(B_n)$ to the subspace $F_{j_1+\cdots+j_n}(\gamma(A; B_1, B_2, \ldots, B_n))$, where γ denotes the structure map of $\mathcal{M}$.

4. *The unit element in $\mathcal{C}(1)$ is contained in $F_1(1)$, where 1 denotes the unit element of $\mathcal{M}(1)$.*

Then $\{hocolim_{\mathcal{M}(n)} F_n\}_{n\geq 0}$ is an operad and the natural map $\{hocolim_{\mathcal{M}(n)} F_n\}_{n\geq 0} \longrightarrow \mathcal{C}$ is a map of operads.

The proof of this lemma is completely straightforward and will be left as an exercise for the reader. Moreover we also note that in case the action of Σ_n on both $\mathcal{M}(n)$ and $\mathcal{C}(n)$ is free, then the same is the case with the action on $\mathrm{hocolim}_{\mathcal{M}(n)} F_n$.

Proof of Theorem 3.14. Corollary 6.8, Proposition 6.9, Lemma 6.11 and Remark 6.3 imply that the chain

$$\left\{\mathrm{hocolim}_{\mathcal{M}_n(k)} F \longrightarrow \mathrm{colim}_{\mathcal{M}_n(k)} F \cong \mathcal{D}_n(k) \subset \mathcal{C}_n(k)\right\}_{k\geq 0}$$

is a chain of operad maps which are also equivalences. Similarly by Lemma 6.6 the natural map of the diagram $F : \mathcal{M}_n(k) \longrightarrow \mathbf{Top}$ to the trivial diagram $* : \mathcal{M}_n(k) \longrightarrow \mathbf{Top}$ induces a map of operads which is also an equivalence:

$$\left\{\mathrm{hocolim}_{\mathcal{M}_n(k)} F \longrightarrow \mathrm{hocolim}_{\mathcal{M}_n(k)} * = \mathcal{N}\mathcal{M}_n(k) = \mathcal{M}_n(k)\right\}_{k\geq 0}$$

where the last equality is our usual notational abuse of using the same symbol for a category and its nerve.

It remains to show that the inclusion of the Milgram preoperad $\overline{\mathcal{J}}_n(k)$ in the operad $\mathcal{M}_n(k)$ is an equivalence. To do this requires defining a subdiagram of subspaces of the diagram F, indexed over the Milgram subcategory $\overline{\mathcal{J}}_n(k)$. Specifically given an object A in $\overline{\mathcal{J}}_n(k)$ we define

$$\overline{F}(A) = \cup_{X\to A} G(X)$$

where the union is indexed over all objects X in $\overline{\mathcal{J}}_n(k)$ (*not* $\mathcal{M}_n(k)$) mapping into A. The inclusion of diagrams then induces a commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{J}}_n(k) \arrow[d] & \mathrm{hocolim}_{\overline{\mathcal{J}}_n(k)} \overline{F} \arrow[l] \arrow[r, hook] \arrow[d] & \mathrm{colim}_{\overline{\mathcal{J}}_n(k)} \overline{F} \arrow[d] \\
\mathcal{M}_n(k) & \mathrm{hocolim}_{\mathcal{M}_n(k)} F \arrow[l, "\simeq"'] \arrow[r, hook, "\simeq"] & \mathcal{C}_n(k)
\end{tikzcd}$$

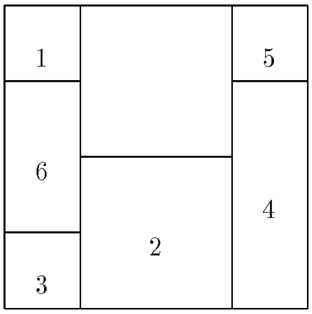
We have already shown that the maps across the bottom row are equivalences. Using similar arguments, first proving the analogs of Lemmas 6.6 and 6.7 and Corollary 6.8 hold for the diagram $\overline{F}$, we can show the maps across the top row are also equivalences. Thus it suffices to show that the right hand vertical arrow is an equivalence.

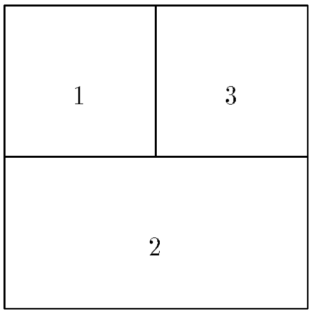
By the analog of Corollary 6.8 we can identify $\mathrm{colim}_{\overline{\mathcal{J}}_n(k)} \overline{F}$ with the union

$$\bigcup_{A\in\overline{\mathcal{J}}_n(k)} G(A) \subset \mathcal{D}_n(k) \subset \mathcal{C}_n(k)$$

This in turn is the subspace of Milgram decomposable configurations of little n-cubes. A configuration in $\mathcal{C}_n(k)$ is said to be *Milgram decomposable* if one can cut through the configuration with a finite set of hyperplanes perpendicular to the first coordinate axis which miss the interiors of all the little n-cubes and each of the resulting strips individually can then be cut through by a finite number of hyperplanes perpendicular to the second coordinate axis (again missing the interiors of all the little cubes in the strip), and each of those resulting strips can then be cut by hyperplanes perpendicular to the the third coordinate axis, etc. with the final cuts being done by hyperplanes perpendicular to the last coordinate axis, so that at the end of this process there is exactly one little cube in each compartment.

The following two figures in $\mathcal{C}_2(k)$ illustrate the concept of Milgram decomposability





The configuration on the left is Milgram decomposable, whereas the one on the right is not (although it is decomposable).

We now show that that the inclusion of the space of Milgram decomposable configurations of little n-cubes into $\mathcal{C}_n(k)$ is an equivalence. Given any configuration of little n-cubes in $\mathcal{C}_n(k)$ let m be the minimum distance between barycenters of different subcubes in the ℓ_∞ norm. Define a map $\mathcal{C}_n(k) \to \mathcal{C}_n(k)$ which linearly shrinks (towards their barycenters) those the little cubes in a configuration whose dimensions are bigger than $\frac{m}{2k}$ by $\frac{m}{2k}$ to subcubes of this size (leaving alone dimensions of cubes which are smaller). This map is clearly homotopic to the identity.

It also takes any configuration to a Milgram decomposable one by the following argument. We say that two little cubes in a configuration overlap in the first coordinate direction if there is a hyperplane perpendicular to the first coordinate direction which passes through the interiors of both. We say that two little cubes are in the same 1-clump if there is chain of little cubes from one to the other such that any two adjacent ones in the chain overlap in the first coordinate direction. Clearly the 1-clumps of little cubes can be separated from each other by hyperplanes perpendicular to the first coordinate direction. The barycenters of any two little cubes in the same 1-clump are separated in the first coordinate direction by a distance at most $\frac{m}{2}$. (There at most k elements in the chain connecting the little cubes, with the barycenters of adjacent subcubes in the chain having separation in the first coordinate direction at most $\frac{m}{2k}$.) Thus the separation in at least one of the other coordinate directions between the barycenters of any two little cubes in the 1-clump must be at least m. Now for the little cubes within a given 1-clump define an analogous notion of 2-clump and repeat. At the final stage of this process we will have an $(n-1)$-clump of cubes which overlap in all the coordinate directions except the last. It will follow that all the little cubes in this $(n-1)$-clump must have barycenters separated in the last coordinate direction by distances of at least m. Since the little cubes have dimensions $\frac{m}{2k}$, they can then be separated from one another by hyperplanes perpendicular to the last coordinate direction, proving the configuration is Milgram decomposable.

Moreover the homotopy from the shrinking map to the identity restricts to the subspace of Milgram decomposable configurations. It follows that the inclusion of the space of Milgram decomposable configurations in $\mathcal{C}_n(k)$ is an equivalence, completing the proof of Theorem 3.14.

Before we proceed to the proof of Theorem 3.16 we recall the basic definitions, due to Berger [3].

Definition 6.12. An *acyclic orientation* of the complete graph on the set of vertices $\{1, 2, 3, \dots, k\}$ is an assignment of direction to each edge of the graph such that no directed cycles occur. Equivalently an acyclic orientation is determined uniquely by a total ordering (ie. a permutation) of the vertices. A *coloring* of the complete graph on k vertices is an assignment of colors to each edge of the graph from the countable set of colors $\{1, 2, 3, \dots\}$. The poset $\mathcal{K}(k)$ has as elements pairs (μ, σ), where μ is a coloring and σ is an acyclic orientation of the complete graph on k vertices. The order relation on $\mathcal{K}(k)$ is determined as follows: we say that $(\mu_1, \sigma_1) \le (\mu_2, \sigma_2)$ if for every edge $a \xrightarrow{i} b$ in (μ_1, σ_1) the corresponding edge in (μ_2, σ_2) has either orientation and coloring $a \xrightarrow{j} b$ with $j \ge i$ or $b \xrightarrow{j} a$ with $j > i$. Per our usual abuse we also denote by $\mathcal{K}(k)$ the nerve of this poset.

The action of the symmetric group Σ_k on $\mathcal{K}(k)$ is via permutation of the vertices. The structure map

$$\mathcal{K}(k) \times \mathcal{K}(m_1) \times \mathcal{K}(m_2) \times \cdots \times \mathcal{K}(m_k) \longrightarrow \mathcal{K}_(m_1 + m_2 + \cdots + m_k)$$

assigns to a tuple of orientations and colorings in $\mathcal{K}(k) \times \mathcal{K}(m_1) \times \mathcal{K}(m_2) \times \cdots \times \mathcal{K}(m_k)$ the orientation and coloring obtained by subdividing the set of $m_1 + m_2 + \cdots + m_k$ vertices into k blocks containing m_1, m_2, $\ldots$, m_k vertices respectively. The edges connecting vertices within the i-th block are oriented and colored according to the given element in $\mathcal{K}(m_i)$. The edges connecting vertices between blocks i and j are all oriented and colored according to the corresponding edge in the given element of $\mathcal{K}(k)$. It is easy to check that this specification gives $\mathcal{K}(k)_{k\geq 0}$ the structure of an E_∞ operad.

The n-th filtration $\mathcal{K}^{(n)}(k)$ is the subposet of $\mathcal{K}(k)$ where the colorings are restricted to take values in the subset $\{1, 2, 3, \ldots, n\}$. It is obvious that $\mathcal{K}^{(n)}(k)_{k\geq 0}$ is a suboperad of $\mathcal{K}$. There is an inclusion of posets $\mathcal{M}_n(k) \subset \mathcal{K}^{(n)}(k)$ which takes an object A to the complete graph on $\{1, 2, 3, \ldots, k\}$ with edges oriented and colored as follows:

$$a \xrightarrow{i} b \qquad \text{if } a \Box_i b \text{ in } A$$

These inclusions define a map of operads $\mathcal{M}_n \to \mathcal{K}^{(n)}$.

Remark 6.13. This definition departs slightly from that in [3] in that Berger takes colorings with values in the set $\{0, 1, 2, \ldots\}$ rather than $\{1, 2, 3, \ldots\}$.

Definition 6.14. Define $\Gamma(k)$ to be the category whose objects are permutations in Σ_k and with a unique morphism between any two objects (which is hence an isomorphism). The nerve of this category, also abusively denoted $\Gamma(k)$, can be identified with the standard simplicial model $E\Sigma_k$ of the total space of the universal principal Σ_k bundle. By rewriting the objects of $\Gamma(k)$ in the form

$$i_1 \Box i_2 \Box \ldots \Box i_k,$$

instead of $[i_1, i_2, \ldots, i_k]$, and appealing to MacLane's coherence theorem, we can identify $\Gamma(k)$ with a full subcategory of the free strict symmetric monoidal category on k generators. Thus the spaces $\{\Gamma(k)\}_{k\geq 0}$ can be naturally endowed with the structure of an operad which acts on the nerves of strict symmetric monoidal categories. The operad Γ was extensively studied by Barratt and Eccles [1] and May [16] (who denotes the operad $\mathcal{D}$ instead).

Smith [24] defined a filtration on Γ as follows. First of all he defined $\Gamma^{(n)}(2)$ to be the $n-1$ skeleton of $\Gamma(2)$, which is easily identified as the standard $\mathbb{Z}/2$-equivariant simplicial model of S^{n-1}. Then he defined a simplex in $\Gamma(k)$ to be in the n-th filtration $\Gamma^{(n)}(k)$ if its images under all restriction maps

$$R_{a,b} : \Gamma(k) \longrightarrow \Gamma(2)$$

lies in $\Gamma^{(n)}(2)$ (cf. Remark 3.8). Equivalently an r-simplex

$$\sigma_0 \longrightarrow \sigma_1 \longrightarrow \sigma_2 \longrightarrow \cdots \longrightarrow \sigma_r$$

lies in $\Gamma^{(n)}(k)$ if any pair of elements a, b in $\{1, 2, \ldots, k\}$ change their relative order in the given sequence of permutations at most $n-1$ times. For example the 3-simplex

$$[1, 2, 3] \longrightarrow [2, 1, 3] \longrightarrow [2, 3, 1] \longrightarrow [2, 1, 3]$$

lies in $\Gamma^{(3)}(3)$ since the pair $(1, 2)$ changes order once, the pair $(1, 3)$ changes order twice and the pair $(2, 3)$ doesn't change order at all. It is easy to see that $\Gamma^{(n)}(k)_{k\geq 0}$ forms a suboperad of Γ.

The forgetful map $(\mu, \sigma) \mapsto \sigma$, which forgets the coloring, defines a functor and hence a map of operads $\mathcal{K} \to \Gamma$. It also preserves filtrations. For given an r-simplex

$$(\mu_0, \sigma_0) \longrightarrow (\mu_1, \sigma_1) \longrightarrow (\mu_2, \sigma_2) \longrightarrow \cdots \longrightarrow (\mu_r, \sigma_r)$$

in $\mathcal{K}^{(n)}(k)$, any edge connecting two given vertices a and b can only change direction at most $n-1$ times. For every change in direction must correspond to an incrementation of the coloring of that edge.

The composite
$$\mathcal{M}_n \longrightarrow \mathcal{K}^{(n)} \longrightarrow \Gamma^{(n)}$$
can be identified with the map of operads arising from the fact that any symmetric monoidal category is n-fold monoidal (cf. Remark 1.9).

Smith [24] conjectured that $\Gamma^{(n)}$ has the same homotopy type as the little n-cubes operad $\mathcal{C}_n$, and thus could also be used to parametrize the structure of an n-fold loop space. This conjecture was proved by Berger [3]. Our proof of Theorem 3.16 below gives an alternative proof of this conjecture.

Proof Sketch of Theorem 3.16 The diagram F in the proof of Theorem 3.14 can be expanded in the evident way to a diagram on $\mathcal{K}^{(n)}(k)$ containing F as a subdiagram of subspaces, and the inclusion $\mathcal{M}_n \to \mathcal{K}^{(n)}$ can be shown to be an equivalence by an argument similar to the proof we used above to prove that $\overline{\mathcal{J}}_n \subset \mathcal{M}_n$ is an equivalence. See [3] for details.

To show that the map $p : \mathcal{K}^{(n)} \longrightarrow \Gamma^{(n)}$ is an equivalence we have to show that for any simplex $\mathfrak{S}$ in $\Gamma^{(n)}$ the inverse image $p^{-1}(\mathfrak{S})$ is contractible. We prove this by induction on the dimension of $\mathfrak{S}$. If $\mathfrak{S} = \sigma$ is a vertex, then $p^{-1}(\sigma)$ is a simplicial cone on the object (μ_0, σ), where μ_0 is the coloring which assigns to each edge the color 1.

Assume we have already shown the contractibility of inverse images for simplices of dimension less than that of
$$\mathfrak{S} \quad = \quad \sigma_0 \longrightarrow \sigma_1 \longrightarrow \sigma_2 \longrightarrow \cdots \longrightarrow \sigma_r$$
We note that
$$p^{-1}(\mathfrak{S}) = T(\mathfrak{S}) \cup \bigcup_{i=0}^{r} p^{-1}(\mathfrak{S}_i),$$
where $T(\mathfrak{S})$ is the union of all simplices in $\mathcal{K}^{(n)}$ which map surjectively onto $\mathfrak{S}$ and the $\mathfrak{S}_i$ are the codimension 1 faces of $\mathfrak{S}$. To show that this union is contractible, it suffices to show that all the intersections
$$\begin{aligned} \bigcap_{j \in J} p^{-1}(\mathfrak{S}_j) \quad &= \quad p^{-1}\left(\bigcap_{j \in J} \mathfrak{S}_j\right) \\ T(\mathfrak{S}) \cap \bigcap_{j \in J} p^{-1}(\mathfrak{S}_j) \quad &= \quad T(\mathfrak{S}) \cap p^{-1}\left(\bigcap_{j \in J} \mathfrak{S}_j\right) \end{aligned}$$
are contratible. Intersections of the first kind are contractible by induction hypothesis. To see that intersections of the second kind are contractible, we first consider the following distinguished simplex in $T(\mathfrak{S})$:
$$(\mu_0, \sigma_0) \longrightarrow (\mu_1, \sigma_1) \longrightarrow (\mu_2, \sigma_2) \longrightarrow \cdots \longrightarrow (\mu_r, \sigma_r)$$
where the coloring μ_i assigns to the edges joining a pair of vertices a, b the color which is 1 more than the number of times that this pair of elements changes relative position in the subsimplex
$$\sigma_0 \longrightarrow \sigma_1 \longrightarrow \sigma_2 \longrightarrow \cdots \longrightarrow \sigma_i$$
of $\mathfrak{S}$. Then it is easy to see that $T(\mathfrak{S}) \cap p^{-1}\left(\bigcap_{j \in J} \mathfrak{S}_j\right)$ is a cone on the vertex (μ_m, σ_m) where σ_m is the initial vertex of the face $\bigcap_{j \in J} \mathfrak{S}_j$ of $\mathfrak{S}$, and is thus contractible. This completes the induction and proof.

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