ON THE DISTRIBUTION OF AR-COMPONENTS OF RESTRICTED LIE ALGEBRAS

Rolf Farnsteiner*

Abstract

In this paper we study the distribution of indecomposable modules of the reduced enveloping algebra $u(L,\chi)$ associated to a finite dimensional restricted Lie algebra $(L,[p])$. Each component of the stable Auslander-Reiten quiver possesses only finitely modules of a given dimension. We combine this fact with geometric techniques in order to produce families of components of type $\mathbb{Z}[A_\infty]$.

0. Introduction and Preliminaries

Let F be an algebraically closed field, Λ a finite dimensional F-algebra of infinite representation type. If Λ is tame, it follows from Brauer Thrall II and the work of Crawley-Boevey [5, Corollary E] that there exists a number $d \in \mathbb{N}$, such that for each multiple ℓd there are infinitely many non-isomorphic indecomposable Λ-modules of dimension ℓd. In case Λ is wild, this conclusion continues to hold (with d even) since there is a representation embedding from the module category of the Kronecker algebra $F[X,Y]/(X^2,Y^2)$ into the module category of Λ. A related question concerning the distribution of the dimensions within a given Auslander-Reiten component was raised by Ringel [30]: Given an AR-component Θ and $d > 0$, is the set

$$\Theta(d) := \{[M] \in \Theta \ ; \ dim_F M \leq d\}$$

finite? While the answer is affirmative for some classes of algebras (cf. for instance [5, Corollary F]), there exist symmetric algebras which admit infinite $\Theta(d)$ (cf. [24]).

In this note we shall study questions related to the distribution of AR-components for reduced enveloping algebras of restricted Lie algebras. Our results, some of which will be

*Supported by N.S.A. Grant MDA904-96-1-0040 and by the D.F.G.

[0]1991 Mathematics Subject Classification: 16G70, 17B50

applied in a forthcoming paper [12], illustrate the utility of the geometric methods that are based on support varieties.

In the sequel we shall consider finite dimensional restricted Lie algebras $(L, [p])$, defined over an algebraically closed field F of characteristic $p > 0$. The universal enveloping algebra of L will be denoted $\mathcal{U}(L)$. Given a linear form $\chi \in L^*$, the *χ-reduced enveloping algebra* of L is defined by

$$u(L, \chi) := \mathcal{U}(L)/(\{x^p - x^{[p]} - \chi(x)^p 1 \ \ ; \ \ x \in L\}).$$

We refer the interested reader to [32] concerning basic facts of restricted Lie algebras and their enveloping algebras. A few properties of $u(L, \chi)$ that are relevant for our purposes are summarized in the following list (cf. [6, 8, 13]).

- The algebra $u(L, \chi)$ is finite dimensional, more precisely $dim_F u(L, \chi) = p^{dim_F L}$.
- $u(L, \chi)$ is a Frobenius algebra with Nakayama automorphism μ uniquely determined by
$$\mu(x) = x - tr(ad\, x)1 \quad \forall\, x \in L.$$
- The restricted enveloping algebra $u(L) := u(L, 0)$ is a Hopf algebra.
- Let Θ be a component of the stable Auslander-Reiten quiver $\Gamma_s(L, \chi)$ of $u(L, \chi)$. Then Θ is either a finite or infinite tube of rank $k \in \{1, p\}$, or
$$\Theta \cong \mathbb{Z}[A_\infty],\ \mathbb{Z}[A_\infty^\infty],\ \mathbb{Z}[D_\infty],\ \mathbb{Z}[\tilde{A}_n],\ \mathbb{Z}[\tilde{A}_{12}],\ \mathbb{Z}[\tilde{D}_n],\ \mathbb{Z}[\tilde{E}_\ell] \quad (6 \leq \ell \leq 8).$$

In the following all $u(L, \chi)$-modules are assumed to be finite dimensional. We also set $\Gamma_s(L) := \Gamma_s(L, 0)$.

Let M be a $u(L, \chi)$-module. In view of the work of Friedlander and Parshall (cf. [15, 16, 17]) and Jantzen [22] we define the *cohomological support variety* of M via

$$\mathcal{V}_L(M) := \{x \in L\ ;\ x^{[p]} = 0 \textit{ and } M|_{\langle x \rangle} \textit{ is not free}\}.$$

Here $M|_{\langle x \rangle}$ denotes the restriction of M to the subalgebra of $u(L, \chi)$ that is generated by x.

It is well-known (cf. [15, 16, 17]) that $\mathcal{V}_L(M)$ is a conical variety whose dimension coincides with the complexity $c_{u(L,\chi)}(M)$. In particular, M is projective if and only if $\mathcal{V}_L(M) = \{0\}$. Moreover, if $\Theta \subset \Gamma_s(L, \chi)$ is a component, then we have $\mathcal{V}_L(M) = \mathcal{V}_L(N) \quad \forall\, [M], [N] \in \Theta$. Accordingly, we may define

$$\mathcal{V}_L(\Theta) := \mathcal{V}_L(M) \qquad [M] \in \Theta.$$

After providing an affirmative answer to Ringel's question (Theorem 1.2), we turn in §2 to algebraic Lie algebras, i.e., those where $L = Lie(G)$ is the Lie algebra of an algebraic group G. In this setting the stabilizer G_χ of χ (with respect to the coadjoint action) operates on $\Gamma_s(L, \chi)$ via automorphisms. If G_χ is reductive (e.g., if G is reductive and χ is semisimple), then the components invariant relative to this action are shown to consist of modules that are stable under the twist by the adjoint representation. In the same vein, we employ support varieties to see that representation infinite enveloping algebras possess infinitely many indecomposables for each multiple of a suitable p-power. This result, as well as those of the final section extend recent work of [26] in various directions. In particular, we show that each restricted simple Lie algebra $L \not\cong s\ell(2)$ admits infinitely many stable AR-components of type $\mathbb{Z}[A_\infty]$ containing modules of certain p-power dimensions.

1. The Distribution of Indecomposable Modules

Throughout this section we fix a finite dimensional restricted Lie algebra $(L, [p])$ as well as a linear form $\chi \in L^*$. The following subsidiary result is the key step in proving the finiteness of $\Theta(d)$.

Lemma 1.1 *Let $\Theta \subset \Gamma_s(L, \chi)$ be a component, $\mathcal{T} \subset \Theta$ a τ-orbit. Then $\mathcal{T} \cap \Theta(d)$ is finite for every $d > 0$.*

Proof. Assume there is $d > 0$ such that $\mathcal{T} \cap \Theta(d)$ is infinite. Then Θ is a nonperiodic component. By assumption there exists a vertex $[M] \in \Theta(d)$ such that

$$dim_F \Omega^{2j}(M) \leq d$$

for j belonging to some infinite set $J \subset \mathbb{Z}$. Without loss of generality we may assume that $0 \in J$. Consequently,

$$dim_F Ext^{2j}_{u(L,\chi)}(M, M) \leq dim_F Hom_{u(L,\chi)}(\Omega^{2j}(M), M) \leq d^2 \quad \forall\, j \in J \cap \mathbb{N},$$

while

$$dim_F Ext^{-2j}_{u(L,\chi)}(M, M) \leq dim_F Hom_{u(L,\chi)}(M, \Omega^{2j}(M)) \leq d^2 \quad \forall\, j \in J \cap -\mathbb{N}.$$

Hence there are infinitely many $n \in \mathbb{N}$ such that

$$dim_F Ext^{2n}_{u(L,\chi)}(M, M) \leq d^2.$$

Since Θ is not periodic, it follows from [2, (3.4)] that $dim \mathcal{V}_L(M) = c_{u(L,\chi)}(M) \geq 2$. By work of Jantzen [22] and Friedlander-Parshall [15], the variety $\mathcal{V}_L(M)$ is the zero locus of the ideal $ker\, \Phi_M$, where

$$\Phi_M : S(L^*) \longrightarrow Ext^{ev}_{u(L,\chi)}(M, M)^{(-1)}$$

is a graded algebra homomorphism from the symmetric algebra of L^* into the even part of the Yoneda algebra of M. The superscript (-1) indicates the twisting of the underlying F-space by the Frobenius automorphism of F (cf. [16, (1.4)] for details). The Noether Normalization Theorem provides $X, Y \in L^*$ such that

$$dim_F Ext^{2n}_{u(L,\chi)}(M, M) \geq dim_F S^n[FX \oplus FY] \geq n + 1 \quad \forall\, n \geq 0,$$

a contradiction. $\square$

Theorem 1.2 *Let $\Theta \subset \Gamma_s(L,\chi)$ be a component. Then $\Theta(d)$ is finite for every $d > 0$.*

Proof. Suppose first that Θ is periodic. Then Θ is either finite or an infinite tube. From the structure of the latter (cf. [8, (5.3)]) we readily obtain $|\Theta(d)| \leq pd$.

Let Θ be nonperiodic. According to (1.1) the set $\Theta(d)$ is finite whenever Θ has only finitely many τ-orbits. If Θ has infinitely many τ-orbits, then our introductory remarks leave us with $\Theta \cong \mathbb{Z}[A_\infty]$, $\mathbb{Z}[D_\infty]$, or $\Theta \cong \mathbb{Z}[A_\infty^\infty]$.

Suppose that $\Theta \cong \mathbb{Z}[A_\infty]$, so that $\mathbb{Z} \times \mathbb{N}$ is its set of vertices. By definition the arrows and the Auslander-Reiten translation of Θ are given by

$$(i,a) \to (i,a+1) \quad ; \quad (i,a) \to (i+1,a-1)$$

and

$$\tau(i,a) = (i-1,a),$$

respectively. If Θ is not regular, then [11, (4.1)] shows that there exists precisely one principal indecomposable $u(L,\chi)$-module P that is attached to Θ. By the same token, $[Rad(P)]$ is located at an end of Θ, and we may set

$$[Rad(P)] = (0,1).$$

Induction on n now shows that

$$dim_F(j,n) = \sum_{k=j}^{n+j-1} dim_F(k,1) - \sum_{k=j}^{n-2+j} \delta_{k,0} dim_F P \qquad \forall\, j \in \mathbb{Z}, \tag{1}$$

where we set $P = (0)$ in case Θ is regular.

Now consider the set

$$\Upsilon := \{n \in \mathbb{N}\ ;\ \exists\, j \in \mathbb{Z}\ :\ (j,n) \in \Theta(d)\}.$$

Suppose Υ is infinite and choose for each $n \in \Upsilon$ an element $j(n) \in \mathbb{Z}$ such that $(j(n),n) \in \Theta(d)$. If $\{j(n)\ ;\ n \in \Upsilon\}$ is infinite, then (1) implies that the τ-orbit of $(0,1)$ contains infinitely many elements of $\Theta(d + dim_F P)$. This contradicts (1.1). Alternatively, the set $\{n + j(n) - 1\ ;\ n \in \Upsilon\}$ is infinite, and we arrive at the same contradiction.

Consequently, Υ is finite and there exist τ-orbits $\mathcal{T}_1, \ldots, \mathcal{T}_r$ such that $\Theta(d) = \bigcup_{i=1}^r (\mathcal{T}_i \cap \Theta(d))$. Thanks to (1.1), the set $\Theta(d)$ is finite.

We next consider $\Theta \cong \mathbb{Z}[A_\infty^\infty]$ and label its vertices by $\mathbb{Z} \times \mathbb{Z}$. By definition there are arrows

$$(i,a) \to (i,a+1) \quad ; \quad (i,a) \to (i+1,a-1).$$

Let $\{P_1, \ldots, P_r\}$ be the set of principal indecomposable modules that are attached to Θ and put $t := \sum_{i=1}^r dim_F P_i$. Then we have

$$dim_F(i,a) + dim_F(0,0) = dim_F(i,-i) + dim_F(0,a+i) + q(i,a),$$

where $-t \leq q(i,a) \leq t$ depends on the number of projective meshes contained in the "parallelogram" $\{(0,0),\ (0,a+i),\ (i,-i),\ (i,a)\}$. Consequently,

$$(i,-i) \in \Theta(d + dim_F(0,0) + t)$$

whenever $(i,a) \in \Theta(d)$. The arguments of [25, p.131] now show that, for any $\ell > 0$ the set $\Theta(\ell) \cap \{(i,-i)\ ;\ i \in \mathbb{Z}\}$ is finite. Thus, $\Theta(d)$ is contained in a union of finitely many τ -orbits and we may apply (1.1) to see that $\Theta(d)$ is finite.

Finally, let $\Theta \cong \mathbb{Z}[D_\infty]$. The set of vertices is $\mathbb{Z} \times (\mathbb{N}_0 \cup \{-1\})$. The subset $\mathbb{Z} \times \mathbb{N}_0$ is a subquiver isomorphic to $\mathbb{Z}[A_\infty]$. To obtain Θ we add arrows

$$(i,-1) \to (i,1) \quad ; \quad (i,1) \to (i+1,-1) \quad \forall\ i \in \mathbb{Z}.$$

Induction on n shows that

$$dim_F(j,n) = \begin{cases} dim_F(j,0) + dim_F(j+n,0) + q(j,n) & n \equiv 1 \ \ mod(2) \\ dim_F(j,-1) + dim_F(j+n,0) + q(j,n) & n \equiv 0 \ \ mod(2) \end{cases} \qquad \forall\ n \geq 1.$$

As before $-t \leq q(j,n) \leq t$ depends on the number of projective meshes of the wing $\{(j,n),(j,-1),(j+n+1,-1)\}$. The proof may now be completed by using the arguments of the case $\Theta = \mathbb{Z}[A_\infty]$. $\square$

Corollary 1.3 *Let $\mathcal{B} \subset u(L,\chi)$ be a block. Then $\mathcal{B}$ has finite representation type if and only if its Auslander-Reiten quiver $\Gamma(\mathcal{B})$ is connected.*

Proof. Necessity follows from a well-known theorem of Auslander (cf. [1, (VI.1.4)]). The reverse direction is a direct consequence of Brauer-Thrall II and (1.2). By way of illustration, we sketch a second argument that is based on the geometric approach.

Suppose that $\Gamma(\mathcal{B})$ is connected. If $\Gamma_s(\mathcal{B})$ is not connected, then $\mathcal{B}$ is readily seen to be a Nakayama algebra. Alternatively, let S be a simple $\mathcal{B}$-module. Given $x \in \mathcal{V}_L(S) \setminus \{0\}$, a suitable modification of [15, (2.2)] implies that there exists an indecomposable $\mathcal{B}$-module M such that $\mathcal{V}_L(M) = Fx$. Since $[M]$ and $[S]$ belong to the same stable component, we obtain $\mathcal{V}_L(S) = Fx$. Thanks to [8, (3.2)] this implies that $\mathcal{B}$ is a Nakayama algebra. $\square$

2. Indecomposables of Algebraic Lie Algebras

In the following we shall be primarily interested in the situation where L is the Lie algebra of a (connected) algebraic group G. In that case a natural action of G on the set of stable Auslander-Reiten components provides an additional tool.

We begin with a subsidiary result that is valid for arbitrary restricted Lie algebras $(L,[p])$ admitting a *triangular decomposition*

$$L = L^- \oplus L_0 \oplus L^+.$$

By definition, the constituents are p-subalgebras satisfying $[L_0, L^{\pm}] \subset L^{\pm}$. We endow a $u(L_0)$-module M with the structure of a $u(L_0 \oplus L^+)$-module by letting L^+ act trivially and define

$$M\!\uparrow^L := u(L) \otimes_{u(L_0 \oplus L^+)} M.$$

A restricted Lie algebra $(L, [p])$ is said to be *p-nilpotent* if some power of its p-map is trivial. The Lie algebras of unipotent algebraic groups are p-nilpotent.

Lemma 2.1 *Suppose that L^- is p-nilpotent and let M and N be $u(L_0)$-modules.*

(1) If M is indecomposable, then $M\!\uparrow^L$ is indecomposable.

(2) If $M\!\uparrow^L \cong N\!\uparrow^L$, then $M \cong N$.

(3) $\mathcal{V}_{L_0}(M) = \mathcal{V}_{L_0}(M\!\uparrow^L) \subset \mathcal{V}_L(M\!\uparrow^L) \subset \mathcal{V}_{L_0 \oplus L^+}(F)$.

Proof. (1). This follows directly from [7, (4.1)].

(2). According to [7, (3.2)] there exists an automorphism ω of $u(L_0)$ such that the $u(L_0)$-module $(M\!\uparrow^L)^{L^-}$ of L^--invariants of $M\!\uparrow^L$ is isomorphic to the $u(L_0)$-module $M^{(\omega)}$ obtained from M by twisting the action of $u(L_0)$. This readily yields the assertion.

(3). We apply [9, (3.4)] to see that the support variety of $M\!\uparrow^L$ is contained in $\mathcal{V}_{L_0 \oplus L^+}(F)$. Let us denote by $u(L)_{ad}$ the L-module with action $x \cdot u := xu - ux \;\; \forall \, x \in L, u \in u(L)$. Since $[L_0, L^-] \subset L^-$, it follows that $u(L^-)$ and its augmentation ideal $u(L^-)^{\dagger}$ are L_0-submodules of $u(L)_{ad}$. If $u(L^-)$ is endowed with this structure, then the map

$$u(L^-) \otimes_F M \longrightarrow M\!\uparrow^L \;\; ; \;\; u \otimes m \mapsto u \otimes m$$

is an isomorphism of L_0-modules. From the decomposition $u(L^-) = F \oplus u(L^-)^{\dagger}$, we infer that the L_0-module M is a direct summand of $M\!\uparrow^L$. Consequently, $\mathcal{V}_{L_0}(M) \subset \mathcal{V}_{L_0}(M\!\uparrow^L)$. For the reverse inclusion we combine the above isomorphism with [15, (2.1)] to obtain

$$\mathcal{V}_{L_0}(M\!\uparrow^L) = \mathcal{V}_{L_0}(u(L^-) \otimes_F M) = \mathcal{V}_{L_0}(u(L^-)) \cap \mathcal{V}_{L_0}(M) \subset \mathcal{V}_{L_0}(M). \quad \square$$

Remark. Properties (1) and (2) can be rephrased by saying that $M \mapsto M\!\uparrow^L$ is a representation embedding from the module category of $u(L_0)$ to the module category of $u(L)$.

Let $(L, [p])$ be a restricted Lie algebra with group $Aut_p(L)$ of automorphisms (of the restricted Lie algebra). Given a homomorphism $\pi : G \longrightarrow Aut_p(L)$ of linear algebraic groups, and a linear form $\chi \in L^*$ we consider the stabilizer

$$G_\chi := \{g \in G \; ; \; \chi \circ \pi(g^{-1}) = \chi\}$$

as well as its connected component G_χ°. For a $u(L, \chi)$-module M and $g \in G_\chi$, we let $M^{(g)}$ denote the $u(L, \chi)$-module with underlying F-space M and twisted action

$$x \cdot m := \pi(g^{-1})(x)\, m \quad \forall \, x \in L, \; m \in M.$$

There results an operation

$$G_\chi \times \Gamma_s(L, \chi) \longrightarrow \Gamma_s(L, \chi) \;\; ; \;\; (g, [M]) \mapsto [M^{(g)}]$$

of G_χ on $\Gamma_s(L,\chi)$. As noted in [22, (2.2)], we have

$$\mathcal{V}_L(M^{(g)}) = \pi(g)(\mathcal{V}_L(M)) \quad \forall\, g \in G_\chi. \tag{2}$$

Let $g \in G_\chi$. The equivalence $M \mapsto M^{(g)}$ induces an automorphism $[M] \mapsto [M^{(g)}]$ of the quiver $\Gamma_s(L,\chi)$. Accordingly, G_χ operates on the set $\mathcal{C}_s(L,\chi)$ of components of $\Gamma_s(L,\chi)$, and for every component $\Theta \subset \Gamma_s(L,\chi)$ and $g \in G_\chi$ we either have

$$\Theta^{(g)} = \Theta$$

or

$$\Theta^{(g)} \cap \Theta = \emptyset.$$

Owing to (2) the stabilizer

$$G_\Theta := \{g \in G_\chi \ ; \ \Theta^{(g)} = \Theta\}$$

of a component $\Theta \subset \Gamma_s(L,\chi)$ is contained in the closed subgroup

$$G_{\mathcal{V}_L(\Theta)} := \{g \in G_\chi \ ; \ \pi(g)(\mathcal{V}_L(\Theta)) = \mathcal{V}_L(\Theta)\}.$$

In particular, if the orbit $\{\Theta^{(g)} \ ; \ g \in G_\chi\}$ is finite, then $G_{\mathcal{V}_L(\Theta)}$ is a closed subgroup of finite index. Consequently, $G_\chi^\circ \subset G_{\mathcal{V}_L(\Theta)}$, so that $\mathcal{V}(\Theta)$ is G_χ°-invariant.

Let G be a connected algebraic group with Lie algebra L, $\chi \in L^*$ a linear form. Considering the adjoint representation $Ad : G \longrightarrow Aut_p(L)$, we call a component $\Theta \subset \Gamma_s(L,\chi)$ *invariant* if $\Theta^{(g)} = \Theta \quad \forall\, g \in G_\chi^\circ$. A $u(L,\chi)$-module M is referred to as *Ad-stable* if $M \cong M^{(g)}$ for every $g \in G_\chi^\circ$. If M is a rational G-module, then the corresponding $u(L)$-module is Ad-stable. Moreover, since G_χ° is connected, every simple $u(L,\chi)$-module is Ad-stable.

Evidently, every component containing an Ad-stable vertex is stable. The converse is the content of our first application of (1.2), which has turned out to be useful in the study of invariant components of reductive groups (cf. [12]).

Proposition 2.2 *Let G be a connected algebraic group with Lie algebra L. Suppose that $\chi \in L^*$ is given such that G_χ is reductive. If $\Theta \subset \Gamma_s(L,\chi)$ an invariant component, then every vertex of Θ is defined by an Ad-stable module.*

Proof. Let $\Theta \subset \Gamma_s(L,\chi)$ be an invariant component and suppose that $[M] \in \Theta$. By assumption we have

$$\mathcal{O} := \{[M^{(g)}] \ ; \ g \in G_\chi^\circ\} \subset \Theta(dim_F M),$$

and (1.2) implies that $\mathcal{O}$ is finite. There results a homomorphism

$$\varrho : G_\chi^\circ \longrightarrow Sym(\mathcal{O})$$

of abstract groups sending G_χ° to the symmetric group of $\mathcal{O}$. Let $G_i \subset G_\chi^\circ$ be an almost simple constituent. Then $ker\, \varrho|_{G_i}$ is a normal subgroup of G_i of finite index that contains the center of G_i. Thanks to [20, Corollary 29.5] we obtain $ker\, \varrho|_{G_i} = G_i$. Since the reductive group G_χ° is the product of its almost simple constituents and its center, ϱ is the trivial homomorphism and $M^{(g)} \cong M \quad \forall\, g \in G_\chi^\circ$. $\square$

Remark. Let G be a connected, reductive group of characteristic $p > 3$, $\chi \in L^*$ a linear form. We can decompose

$$\chi = \chi_s + \chi_n$$

into its semisimple and nilpotent parts (cf. [23, Thm.4] and [29, (3.1)]). Since the stabilizer G_{χ_s} of the semisimple part χ_s is reductive (cf. [23, (3.1)]), (2.2) shows that every vertex of an invariant component $\Theta \subset \Gamma_s(L, \chi_s)$ is Ad-stable.

We continue by considering the distribution of indecomposable modules within $\Gamma_s(L, \chi)$. For a natural number d, we let $\mathcal{C}_s(L, \chi, d)$ and $\mathcal{C}_s^{per}(L, \chi, d)$ be the subsets of $\mathcal{C}_s(L, \chi)$ containing a module of dimension d, and a periodic module of dimension d, respectively.

Let $\Theta \subset \Gamma_s(L, \chi)$ be a periodic component, $[M_1] \in \Theta$ a vertex of minimal dimension d. According to the remark of [8, p.62] there exists, for any $\ell > 0$, a vertex $[M_\ell] \in \Theta$ with $dim_F M_\ell = \ell\, d$. In case $L = s\ell(2)$ and $p \geq 3$, the indecomposable modules of length 2 are periodic and of dimension p. For $p = 2$ the principal block of $u(s\ell(2))$ is isomorphic to the Kronecker algebra $F[X, Y]/(X^2, Y^2)$. Hence there exist nonprojective indecomposables of dimension ℓp in all cases. Since the principal indecomposable $u(s\ell(2))$-modules are known to have dimension p or $2p$, it readily follows from Pollack's work [28, Thm.4] and the representation theory of the Kronecker algebra that $u(s\ell(2))$ admits nonprojective indecomposable modules in each dimension. We shall use these observations repeatedly in the sequel.

Example. Let G be connected, reductive with maximal torus $T \neq G$ and Lie algebra L. Let $B \supset T$ be a Borel subgroup, Δ the corresponding set of simple roots. We choose a proper subset $\emptyset \neq I \subset \Delta$ as well as a standard parabolic subgroup $P_I \supset B$. The Levi decomposition of $Lie(P_I)$ induces a triangular decomposition

$$L = L^- \oplus L_0 \oplus L^+,$$

where L^+ and L^- are p-nilpotent subalgebras and L_0 is a Levi factor of $K := Lie(P_I) = L_0 \oplus L^+$.

If I is a singleton, contained in a connected component of Δ, then $L_0 \cong Q \oplus s\ell(2)$, $Q \subset C(L_0)$, and $\mathcal{V}_K(F)$ does not contain any G-invariant subsets $\neq \{0\}$. Hence (1) and (3) of (2.1) imply that a nonprojective indecomposable $u(s\ell(2))$-module M defines infinitely many components containing a module $(M{\uparrow}^L)^{(g)}$ of dimension $p^{dim_F L^-}(dim_F M)$. In particular, there are infinitely many nonisomorphic indecomposables of dimension $\ell\, p^r$. (More precisely, we can choose $r := max\{|\Phi^+| - 1, 1\}$, where Φ^+ is the set of positive roots.) This result, which readily extends to Lie algebras of Cartan type (cf. §3 below), is the main theorem of [26] (cf. [26, (4.8)]).

The following subsidiary result deals with the complementary case of a solvable Lie algebra. We fix a toral Lie algebra T and linear forms $\alpha \neq \beta \in T^*$ such that $\gamma(t^{[p]}) = \gamma(t)^p \ \ \forall\ t \in T, \gamma \in \{\alpha, \beta\}$. Let $L(\alpha, \beta)$ denote the restricted Lie algebra with underlying F-vector space $T \oplus Fx_\alpha \oplus Fx_\beta$, Lie bracket

$$[t, x_\gamma] = \gamma(t)x_\gamma \quad \gamma \in \{\alpha, \beta\}, \quad [x_\alpha, x_\beta] = 0,$$

and p-map given by $x_\gamma^{[p]} = 0 \ \ \gamma \in \{\alpha, \beta\}$.

Lemma 2.3 *Let M be a $u(L(\alpha,\beta))$-module such that*
(a) $x_\alpha,\, x_\beta \notin \mathcal{V}_{L(\alpha,\beta)}(M)$, and
(b) $dim_F Top(M) = 1$.
Then we have $dim_F M = p^2$.

Proof. Since the top of M is simple, M is generated by a weight vector m_λ. Accordingly, we have

$$(*) \qquad\qquad M = \sum_{0\le r,s\le p-1} F x_\alpha^r x_\beta^s m_\lambda,$$

so that $dim_F M \le p^2$. Let $\varrho : u(L(\alpha,\beta)) \longrightarrow End_F(M)$ be the representation of M. From $(*)$ we obtain $\varrho(x_\alpha)^{p-1}(M) = \sum_{s=0}^{p-1} F x_\alpha^{p-1} x_\beta^s m_\lambda$. Thus, if $d = dim_F \varrho(x_\alpha)^{p-1}(M)$, then

$$\varrho(x_\alpha)^{p-1}(M) = \bigoplus_{s=0}^{d-1} F x_\alpha^{p-1} x_\beta^s m_\lambda.$$

Condition (a) implies

$$dim_F M = p\, dim_F im \varrho(x_\gamma)^{p-1}(M) \qquad\qquad \gamma = \alpha, \beta,$$

so that

$$(**) \qquad\qquad M = \bigoplus_{r=0}^{p-1}\bigoplus_{s=0}^{d-1} F x_\alpha^r x_\beta^s m_\lambda.$$

By interchanging the rôles of x_α and x_β we obtain

$$(***) \qquad\qquad M = \bigoplus_{r=0}^{d-1}\bigoplus_{s=0}^{p-1} F x_\alpha^r x_\beta^s m_\lambda.$$

Let $M = \bigoplus_{\zeta\in\Lambda} M_\zeta$ denote the weight space decomposition of M relative to the torus T. From $(**)$ and $(***)$ it follows that

$$\{\lambda + r\alpha + s\beta \;;\; (r,s) \in \mathbb{N}_{p-1} \times \mathbb{N}_{d-1}\} = \Lambda = \{\lambda + r\alpha + s\beta \;;\; (r,s) \in \mathbb{N}_{d-1} \times \mathbb{N}_{p-1}\}.$$

Here we have set $\mathbb{N}_q := \{0, 1, \ldots, q\}$.

If $\{\alpha, \beta\}$ is linearly independent over the Galois field $GF(p)$, then we readily obtain $d = p$ and $dim_F M = p^2$. Alternatively, we may assume that $\beta = k\alpha$ for $k \in \{0, 2, 3, \ldots, p-1\}$. Since $dim \mathcal{V}_{L(\alpha,\beta)}(M) \le 1$, it follows from the result dual to [10, (4.4)] that $Soc(M)$ is simple. Writing $Soc(M) = F_\zeta$ we obtain from $(**)$ and $(***)$

$$((d-1)k - 1)\alpha + \lambda = \zeta = ((d-1) - k)\alpha + \lambda,$$

whence $kd = d$. Since $k \ne 1$, we obtain $d = p$, as desired. $\square$

Theorem 2.4 *Let G be a connected algebraic group with Lie algebra L. If $u(L)$ has infinite representation type, then there exists $r > 0$ such that $\mathcal{C}_s(L, \ell p^r)$ is infinite for all $\ell > 0$. Moreover, if G is reductive, then there exists $d > 0$ such that $\mathcal{C}_s^{per}(L, \ell d)$ is infinite for all $\ell > 0$.*

Proof. In view of (1.2) it suffices to show the existence of infinitely many indecomposable modules of the desired dimension.

Let $R_u(G) \subset G$ be the unipotent radical of G. According to [19, (11.11)], $Lie(R_u(G)) = rad_p(L)$ is the p-nilpotent radical of L. We consider $\hat{G} := G/R_u(G)$ as well as its Lie algebra $\hat{L} = L/rad_p(L)$. If $\hat{G}$ is not a torus, then the example preceding (2.3) yields our result. Alternatively, G is solvable, so that G is a semidirect product of a torus and $R_u(G)$ (cf. [20, (19.3)]). Thus,

$$L = T \oplus rad_p(L),$$

where $T \subset L$ is a maximal torus. Note that $K := [rad_p(L), rad_p(L)] + \langle rad_p(L)^{[p]} \rangle$ is a p-ideal of L.

If $dim_F H^1(u(rad_p(L)), F) = 1$, then the local algebra $u(rad_p(L))$ is a Nakayama algebra and $rad_p(L) = (Fx)_p$ is cyclic (cf. [28, §4] or [8, p.58]). Thus, by a result due to Pfautsch and Voigt [27, Thm.3] (cf. also [8, (4.3)] and [14, (2.4)]), the algebra $u(L)$ has finite representation type, a contradiction. An application of [18, (2.1)] now shows that $dim_F rad_p(L)/K = dim_F H^1(u(rad_p(L)), F) \geq 2$. We consider the restricted Lie algebra $\mathcal{L} := L/K$

$$\mathcal{L} = T \oplus \tilde{V} \;\; ; \;\; \tilde{V}^{[p]} = \{0\} \; , \; [\tilde{V}, \tilde{V}] = (0), \; dim_F \tilde{V} = n \geq 2$$

and decompose $\tilde{V}$ into its root spaces relative to T:

$$\tilde{V} = \bigoplus_{\alpha \in R} \tilde{V}_\alpha.$$

If there is $\alpha_0 \in R$ with $dim_F \tilde{V}_{\alpha_0} \geq 2$, then we consider the factor algebra

$$\mathcal{L}_1 := \mathcal{L}/(\sum_{\alpha \neq \alpha_0} \tilde{V}_\alpha) \cong T \oplus \tilde{V}_{\alpha_0}.$$

Given $v \in \tilde{V}_{\alpha_0} \setminus \{0\}$, the module

$$M_v := u(\mathcal{L}_1) \otimes_{u(T \oplus Fv)} F$$

is indecomposable with support variety Fv. The assertion now follows from the structure of periodic components of $\Gamma_s(\mathcal{L}_1)$.

Hence we may assume that $dim_F \tilde{V}_\alpha = 1$ for every $\alpha \in R$. In that case $\mathcal{L}$ has an algebra $L(\alpha, \beta)$ as a quotient, and it suffices to establish the result for such an algebra. Letting $P(\gamma) := u(L(\alpha, \beta)) \otimes_{u(T)} F_\gamma$ denote the principal indecomposable $u(L(\alpha, \beta))$-module with top F_γ (cf. [7, (4.3)]), we see that

$$(0) \longrightarrow \Omega^2(F) \longrightarrow P(\alpha) \oplus P(\beta) \longrightarrow P(0) \longrightarrow F \longrightarrow (0)$$

are the initial terms of a minimal projective resolution of F. In particular, we have

$$dim_F \Omega^2(F) = p^2 + 1.$$

Now let $v \neq 0$ be an element of $V := Fx_\alpha \oplus Fx_\beta$. Choose a linear form $f \in L(\alpha, \beta)^*$ such that $(ker\, f) \cap V = Fv$. According to [16, (4.2)], there exists a linear map $\zeta : \Omega^2(F) \longrightarrow F$

corresponding to f such that such that the support variety of $\mathcal{K}_\zeta := ker\zeta$ equals Fv. Note that $dim_F \mathcal{K}_\zeta = p^2$. From the above resolution we obtain

$$\Omega^2(F) \subset P(\alpha) \oplus P(\beta),$$

so that $dim_F Soc(\mathcal{K}_\zeta) \leq 2$. If the socle had dimension 1, then $\mathcal{K}_\zeta$ would embed into a principal indecomposable module $P(\lambda)$. Since $dim_F \mathcal{K}_\zeta = p^2 = dim_F P(\lambda)$, this contradicts the fact that $\mathcal{K}_\zeta$ is not projective. As $\mathcal{K}_\zeta$ is periodic, the arguments of [10, (4.4)] show that

$$dim_F Top(\mathcal{K}_\zeta) = dim_F Soc(\mathcal{K}_\zeta) = 2.$$

Thus, if $\mathcal{K}_\zeta$ is decomposable, then

$$\mathcal{K}_\zeta = M_1 \oplus M_2,$$

where each M_i has a simple top and support variety Fv. It now follows from (2.3) that $\mathcal{K}_\zeta$ is indecomposable, whenever $v \notin Fx_\alpha \cup Fx_\beta$. Thus, there are infinitely many periodic components containing an indecomposable module of dimension p^2, so that our assertion follows.

Finally, we assume G to be reductive. Since L is not a torus, there is $x \in \mathcal{V}_L(F) \setminus \{0\}$. Owing to [15, (2.2)] there exists an Auslander-Reiten component $\Theta \subset \Gamma_s(L)$ with support variety $\mathcal{V}_L(\Theta) = F\,x$. If Fx is stable under $Ad(G)$, then Fx is a p-nilpotent ideal of L. Since Lie algebras of reductive groups do not possess non-trivial p-nilpotent ideals, $\mathcal{V}_L(\Theta)$ is not invariant and $\{\Theta^{(g)}\ ;\ g \in G\}$ is infinite. □

Remark. Theorem 2.4 continues to hold if L is a completely solvable Lie algebra. In that case the derived algebra $[L, L]$ of L is nilpotent. If $T(L)$ denotes the largest toral ideal of L, then the factor algebra $\tilde{L} := L/T(L)$ admits a decomposition $\tilde{L} = \tilde{T} \oplus rad_p(\tilde{L})$. Moreover, $u(\tilde{L})$ is representation infinite whenever $u(L)$ has this property. Consequently, the above arguments apply.

3. AR-Components for Lie Algebras of Classical and Cartan Type

In this section we apply geometric methods in order to see that Auslander-Reiten components of type $\mathbb{Z}[A_\infty]$ usually occur most often. The exception is given by the algebra $u(s\ell(2))$: Here the Auslander-Reiten quiver has infinitely many periodic components of type $\mathbb{Z}[A_\infty]/(\tau)$, containing all modules of even length, and $p-1$ components of type $\mathbb{Z}[\tilde{A}_{12}]$ containing all modules of odd length.

We denote by $A_\infty(L, \chi, d)$ the set of those components of $\mathcal{C}_s(L, \chi, d)$ that are isomorphic to $\mathbb{Z}[A_\infty]$. As usual, we will suppress $\chi = 0$ in our notation. Throughout this section we assume that $p \geq 5$.

Theorem 3.1 *Let G be a connected, reductive algebraic group of semisimple rank ≥ 2. Then there exists $r > 0$ such that $A_\infty(L, \ell p^r)$ is infinite for every $\ell \geq 1$.*

Proof. Let $T \subset G$ be a maximal torus, Φ the root system corresponding to T, $B \supset T$ a Borel subgroup with corresponding simple system Δ. We proceed in several steps and first assume G to be almost simple. By assumption Δ contains at least two elements. If Δ is not of type A_2, there exists a singleton $I := \{\alpha\} \subset \Delta$ such that there either are $\beta_1 \neq \beta_2 \in \Delta$ such that $\alpha + \beta_i$ is a root $1 \leq i \leq 2$, or there is $\beta_1 \in \Delta$ such that $\alpha + i\beta_1$ is a root $1 \leq i \leq 2$. Consider the standard parabolic subgroup P_I, denote its opposite group by P_I^-, and let U_I^+ and U_I^- be the unipotent radicals of P_I and P_I^-, respectively. There results a triangular decomposition

$$L = L^- \oplus L_0 \oplus L^+,$$

where $L_0 = C(L_0) \oplus s\ell(2)$ is the Levi subalgebra of $Lie(P_I)$ and $L^\pm = Lie(U_I^\pm)$. We consider the root space decomposition

$$L = Lie(T) \oplus \bigoplus_{\gamma \in \Phi} L_\gamma$$

of L relative to T. We have $L_\gamma = Fe_\gamma \ \ \forall\ \gamma \in \Phi$ and $L_0 = Lie(T) \oplus Fe_\alpha \oplus Fe_{-\alpha}$ (cf. [20, (26.2)]). Let Φ^+ be the set of positive roots. The group U_I^+ is generated by the root groups U_β, $\beta \in \Phi^+ \setminus \{\alpha\}$. Given $x_\beta \in U_\beta$, general theory (cf. [20, §27]) shows that $Ad(x_\beta)(e_\gamma) \in \sum_{k \in \mathbb{N}_0} L_{\gamma + k\beta}$. Since $Ad(x_\beta)$ is unipotent, this implies

$$Ad(x_\beta)(e_\gamma) \equiv e_\gamma \quad mod(L^+)$$

for $\gamma \in \{\alpha, -\alpha\}$. By the same token, we have

$$Ad(x_\beta)(t) \equiv t \quad mod(L^+) \quad \forall\ t \in Lie(T).$$

Now let M be a $u(L_0)$-module. It follows from the above that the $u(L_0 \oplus L^+)$-module M with trivial L^+-structure is $Ad(U_I^+)$-stable. Direct computation then shows that $M\!\uparrow^L$ also enjoys this property. In view of (2.1), we therefore have

$$U_I^+ \cdot \mathcal{V}_{L_0}(M) \subset \mathcal{V}_L(M\!\uparrow^L).$$

Let H be a closed subgroup of G, x and element of L. We consider the centralizers $C_H(x) := \{h \in H \ ;\ Ad(h)(x) = x\}$ and $C_{Lie(H)}(x) := \{y \in Lie(H) \ ;\ [y, x] = 0\}$. General theory yields

$$\begin{aligned} dim(H \cdot x) &= dimH - dimC_H(x) = dim_F Lie(H) - dim_F Lie(C_H(x)) \\ &\geq dim_F Lie(H) - dim_F C_{Lie(H)}(x). \end{aligned}$$

Let M_ℓ be an ℓp-dimensional periodic $u(s\ell(2))$-module with support variety $\mathcal{V}_{L_0}(M_\ell) = Fe_\alpha$. (Take for instance a vertex of quasilength ℓ of the periodic component containing a non-projective Verma module.) We give M_ℓ the structure of an L_0-module by letting $C(L_0)$ act trivially and consider $M_\ell\!\uparrow^L$. According to the above, we have

$$B \cdot e_\alpha \subset \mathcal{V}_L(M_\ell\!\uparrow^L).$$

Let $\gamma := \beta_2$ for $|\Delta| \geq 3$ and $\gamma := \alpha + \beta_1$, otherwise. Owing to [19, (3.2)] there is an element $t_0 \in Lie(T)$ such that the differential $d(\alpha)$ satisfies $d(\alpha)(t_0) \neq 0$. Note that $C_{Lie(B)}(e_\alpha)$ is

T-invariant. Using standard properties of classical Lie algebras (cf. [31, (II.3)]), we thus obtain

$$(Ft_0 \oplus Fe_{\beta_1} \oplus Fe_\gamma) \cap C_{Lie(B)}(e_\alpha) = (0),$$

so that

$$dim \mathcal{V}_L(M_\ell{\uparrow}^L) \geq dim(B \cdot e_\alpha) \geq 3.$$

We proceed by considering the case where $\Delta = \{\alpha, \beta\}$ is of type A_2. Then $L \cong s\ell(3)$ and we may assume that $G \cong SL(3)$. Consider the element $x_0 := e_\alpha + e_\beta \in \mathcal{V}_L(F)$. We propose to show that $x_0 \in \mathcal{V}_L(M_\ell{\uparrow}^L)$ and note that this is equivalent to $Ext^1_{u(Fx_0)}(F, M_\ell{\uparrow}^L) \neq (0)$. Consider the induced module $A_0 := u(L) \otimes_{u(Fx_0)} F$. Since $u(L) : u(Fx_0)$ is a Frobenius extension of first kind, [13, (1.6)] provides isomorphisms

$$A_0^* \cong Hom_{u(Fx_0)}(u(L), F) \cong A_0$$

of $u(L)$-modules. Let $\omega : u(Lie(P_I)) \longrightarrow u(Lie(P_I))$ be the automorphism that is given by $\omega(x) = x - tr((ad_{L/Lie(P_I)}(x))1 \quad \forall\, x \in Lie(P_I)$. Then $u(L) : u(Lie(P_I))$ is an ω-Frobenius extension (cf. [13, p.158]), and there result isomorphisms

$$\begin{aligned} Ext^1_{u(Fx_0)}(F, M_\ell{\uparrow}^L) &\cong Ext^1_{u(L)}(A_0, M_\ell{\uparrow}^L) \cong Ext^1_{u(Lie(P_I))}(A_0, M_\ell^{(\omega)}) \\ &\cong H^1(u(Lie(P_I)), A_0^* \otimes_F M_\ell^{(\omega)}) \cong H^1(u(Lie(P_I)), A_0 \otimes_F M_\ell^{(\omega)}). \end{aligned}$$

Here the superscript "(ω)" indicates the twisting of the module structure by ω. Since $\mathcal{V}_{L^+}(A_0) \subset L^+ \cap Fx_0 = \{0\}$, the module $A_0 \otimes_F M_\ell^{(\omega)}$ is $u(L^+)$-projective. Consequently, the Grothendieck spectral sequence

$$H^p(u(L_0), H^q(u(L^+), A_0 \otimes_F M_\ell^{(\omega)})) \Rightarrow H^n(u(Lie(P_I)), A_0 \otimes_F M_\ell^{(\omega)})$$

collapses, and we obtain in particular an isomorphism

$$H^1(u(Lie(P_I)), A_0 \otimes_F M_\ell^{(\omega)}) \cong H^1(u(L_0), (A_0 \otimes_F M_\ell^{(\omega)})^{L^+}).$$

Since L^+ operates trivially on $M_\ell^{(\omega)}$, the foregoing isomorphisms combine to

$$Ext^1_{u(Fx_0)}(F, M_\ell{\uparrow}^L) \cong H^1(u(L_0), A_0^{L^+} \otimes_F M_\ell^{(\omega)}).$$

The $u(L^+)$-projectivity of A_0 readily yields $dim_F A_0^{L^+} = p^5$. Direct computation then shows that

$$A_0^{L^+} = \{\epsilon_0 u \otimes 1 \ ; \ u \in u(L_0 \oplus L^-)\},$$

where $\epsilon_0 = e_\beta^{p-1} e_{\alpha+\beta}^{p-1}$ is an integral of the Hopf algebra $u(L^+)$. Let $\nu : u(L_0) \longrightarrow u(L_0)$ be the automorphism of $u(L_0)$ given by

$$\nu(x) = x - tr(ad_{L^+}(x))1 \quad \forall\, x \in L_0.$$

Using [13, (1.2)] it is not hard to see that

$$a\,\epsilon_0 = \epsilon_0\,\nu(a) \qquad \forall\, a \in u(L_0).$$

Since L^- is an ideal of $L_0 \oplus L^-$, we have thus have a decomposition

$$A_0^{L^+} = V \oplus W$$

of $u(L_0)$-modules, with $V := \{\epsilon_0 u \otimes 1\ ;\ u \in u(L_0)\}$ and $W := \{\epsilon_0 u \otimes 1\ ;\ u \in u(L_0 \oplus L^-)L^-\}$. The canonical map

$$u(L_0) \times F \longrightarrow V \quad ; \quad (u, r) \mapsto \epsilon_0 \nu(u) \otimes r$$

induces an isomorphism

$$u(L_0) \otimes_{u(Fe_\alpha)} F \longrightarrow V$$

of $u(L_0)$-modules. Observing the self-duality of $u(L_0) \otimes_{u(Fe_\alpha)} F$ we therefore obtain

$$\begin{aligned} dim_F H^1(u(L_0), A_0^{L^+} \otimes_F M_\ell^{(\omega)}) &\geq dim_F H^1(u(L_0), V \otimes_F M_\ell^{(\omega)}) \\ &= dim_F H^1(u(L_0), V^* \otimes_F M_\ell^{(\omega)}) \\ &= dim_F Ext^1_{u(L_0)}(V, M_\ell^{(\omega)}) = dim_F Ext^1_{u(Fe_\alpha)}(F, M_\ell) \neq (0), \end{aligned}$$

where the last inequality follows from $e_\alpha \in \mathcal{V}_{L_0}(M_\ell)$. Thus, $Ext^1_{u(Fx_0)}(F, M_\ell{\uparrow}^L) \neq (0)$ and $x_0 \in \mathcal{V}_L(M_\ell{\uparrow}^L)$. Consequently, we have

$$dim \mathcal{V}_L(M_\ell{\uparrow}^L) \geq dim(B \cdot x_0) = 3$$

in this case as well.

It now follows from [11, (2.1)] that $M_\ell{\uparrow}^L$ belongs to a component of type $\mathbb{Z}[A_\infty]$. Since $\mathcal{V}_{Lie(P_I)}(F)$ does not contain any G-invariant subsets we have established our assertion in case G is almost simple.

Now let G be semisimple with almost simple components $G_1, \ldots, G_r$. Then we have

$$L = \bigoplus_{i=1}^{r} Lie(G_i) \qquad (r \geq 2),$$

and the first part of the proof yields the assertion provided there exists a component of rank ≥ 2. Alternatively, $s\ell(2)$ is a direct summand of L. Writing $L = s\ell(2) \oplus K$, we give every $s\ell(2)$-module M the structure of an L-module by letting K act trivially. Thus,

$$dim \mathcal{V}_L(M) = dim \mathcal{V}_{s\ell(2)}(M) + dim \mathcal{V}_K(F) \geq dim \mathcal{V}_{s\ell(2)}(M) + 2,$$

and the above arguments show that the periodic $s\ell(2)$-modules give rise to $u(L)$-modules belonging to components of type $\mathbb{Z}[A_\infty]$.

If G is reductive, then we consider the semisimple group $\mathcal{G} := G/Rad(G)$, whose Lie algebra we denote by $\mathcal{L}$. As $Rad(G) = Z(G)^\circ$ is a torus, it readily follows that the canonical projection

$$\pi : L \longrightarrow \mathcal{L}$$

induces an isomorphism $\pi : \mathcal{V}_L(F) \longrightarrow \mathcal{V}_{\mathcal{L}}(F)$. Thus, if M is a $u(\mathcal{L})$-module, then its pull-back $\tilde{M}$ has support variety

$$\mathcal{V}_L(\tilde{M}) \cong \mathcal{V}_{\mathcal{L}}(M).$$

As $\mathcal{G}$ has rank ≥ 2, our result follows. $\square$

Remark. Let $L = Lie(G)$, where G is connected and reductive. Suppose there is $d > 0$ such that $\mathcal{C}_s(L,d)$ is finite. Since $G/Z(G)$ operates on the finite set $\mathcal{C}_s(L,d)$, it follows from the structure of semisimple groups that G operates trivially on $C_s(L,d)$. Hence every component $\Theta \in \mathcal{C}_s(L,d)$ is invariant. It can be shown (cf. [12]) that such components are either isomorphic to $\mathbb{Z}[A_\infty]$ or $\mathbb{Z}[\tilde{A}_{12}]$, with the latter occurring only if Θ belongs to a tame block of $u(L)$.

Let G be reductive with maximal torus $T \subset G$. As before we consider a triangular decomposition

$$L = L^- \oplus L_0 \oplus L^+ \quad ; \quad L_0 = Lie(T)$$

of L. Following Kac and Weisfeiler [23] we refer to a linear form $\chi \in L^*$ as *semisimple* if there exists $g \in G$ such that $g \cdot \chi \in (L^- \oplus L^+)^\perp$. It is well-known (cf. [29, (3.1)]) that the stabilizer G_χ of a semisimple linear form $\chi \in L^*$ is reductive and that $Lie(G_\chi)$ coincides with the inifinitesimal stabilizer

$$L_\chi := \{x \in L \ ; \ \chi([x,L]) = (0)\}.$$

Corollary 3.2 *Suppose that $p > 5$ and let G be a semisimple, simply connected algebraic group. If $\chi \in L^*$ is semisimple such that G_χ has semisimple rank ≥ 2, then there exists $r > 0$ such that $A_\infty(L,\chi,\ell p^r)$ is infinite for every $\ell > 0$.*

Proof. Owing to [17, (3.2)] there exists a parabolic subalgebra $L_\chi \subset P \subset L$ such that the induction functor $X \mapsto u(L,\chi) \otimes_{u(P,\chi|_P)} X$ is an equivalence between the module categories of $u(L_\chi, \chi|_{L_\chi})$ and $u(L,\chi)$. Since $dim_F u(L,\chi) \otimes_{u(P,\chi|_P)} X = p^{dim_F L/P} dim_F X$, it suffices to verify the result for the former algebra.

We put $K := [L_\chi, L_\chi]$. Then K is a p-subalgebra of L_χ such that $\chi(K) = (0)$, and there exists a central torus $T \subset L_\chi$ such that $L_\chi = T \oplus K$. Consequently,

$$u(L_\chi, \chi|_{L_\chi}) \cong u(T, \chi|_T) \otimes_F u(K) \cong u(T) \otimes_F u(K) \cong u(L_\chi).$$

The desired result now follows directly from (3.1). $\square$

Remark. If G_χ is a torus, that is, if χ is regular semisimple, then $u(L,\chi)$ is semisimple. If G_χ has semisimple rank 1, then the arguments of the proof of (3.2) show that

$$u(L_\chi) \cong \bigoplus_{i=1}^{r} u(s\ell(2)).$$

As a result, $\Gamma_s(L,\chi)$ does not contain components of type $\mathbb{Z}[A_\infty]$ in that case.

Aside from the classical simple Lie algebras of types $A-G$ there exist four series of restricted simple Lie algebras, labelled $(W(n))_{n\geq 1}$, $(S(n))_{n\geq 3}$, $(H(2r))_{r\geq 1}$ and $K(2r+1)_{r\geq 1}$, that are not associated to algebraic groups. These are referred to as the restricted Lie algebras of *Cartan type*. The Block-Wilson classification theorem (cf. [3]) insures that every restricted simple Lie algebra of characteristic $p > 7$ is either classical or of Cartan type.

Corollary 3.3 *Let $L \ncong W(1)$ be a restricted Lie algebra of Cartan type. Then there exists $r > 0$ such that $A_\infty(L, \ell p^r)$ is infinite for every $\ell \geq 1$.*

Proof. Since L is of Cartan type, L possesses a $\mathbb{Z}$-grading $L = \bigoplus_{i=-r}^{s} L_i$ such that $L_i^{[p]} \subset L_{ip}$. This grading induces a triangular decomposition

$$L = L^- \oplus L_0 \oplus L^+$$

such that $L^+ := \sum_{i>0} L_i$ and $L^- := \sum_{i<0} L_i$ are p-nilpotent. Moreover, we have $W(n)_0 \cong g\ell(n)$, $S(n)_0 \cong s\ell(n)$, $H(2r)_0 \cong sp(2r)$, and $K(2r+1)_0 \cong sp(2r) \oplus F$. In particular, L_0 is the Lie algebra of a reductive group G_0 (cf. [32, Chapter IV] for details). If $L \ncong W(2), H(2), K(3)$, then G_0 has semisimple rank ≥ 2. According to (3.1) there exists $s > 0$ such that for every $\ell \geq 1$ there is an infinite set $\mathcal{X}_\ell$ of isoclasses of indecomposable $u(L_0)$-modules of dimension ℓp^s with $dim \mathcal{V}_{L_0}(X) \geq 3 \ \ \forall\, [X] \in \mathcal{X}_\ell$. We may now apply (2.1) to see that $\{[X\!\uparrow^L] \ ; \ [X] \in \mathcal{X}_\ell\}$ is an infinite set of isoclasses of indecomposable $u(L)$-modules of dimension $\ell p^{s+dim_F L^-}$. By the same token each consituent has a support variety of dimension ≥ 3. The result now follows from a consecutive application of [11, (2.1)] and (1.2).

It remains to consider the cases $L \cong W(2), H(2), K(3)$. We put $L_{(s-1)} := L_{s-1} \oplus L_s$ and note that $L_{(s-1)} \subset \mathcal{V}_L(F)$. Moreover, $dim_F H(2)_{(s-1)} = 5$, while $dim_F K(3)_{(s-1)} = 3 = dim_F W(2)_{(s-1)}$. Since $L_0 \cong s\ell(2) \oplus T$, where $T \subset L_0$ is a central torus, there exists for every $x \in \mathcal{V}_{L_0}(F) \setminus \{0\}$ and every $\ell \geq 1$ an indecomposable $u(L_0)$-module $M(x,\ell)$ of dimension ℓp and with support variety Fx. An application of the Cartan-Weyl formula shows that the annihilator of an element $y \in L_{(s-1)}$ in $M(x,\ell)\!\uparrow^L$ has dimension $> \frac{1}{p} dim_F M(x,\ell)\!\uparrow^L$. Consequently, $L_{(s-1)} \subset \mathcal{V}_L(M(x,\ell)\!\uparrow^L)$, so that $[M(x,\ell)\!\uparrow^L]$ belongs to a component $\Theta(x,\ell)$ of type $\mathbb{Z}[A_\infty]$. According to (2.1) we have $\mathcal{V}_L(\Theta(x,\ell)) \cap L_0 = Fx$, implying that $A_\infty(L, \ell p^{dim_F L^- + 1})$ is infinite. $\square$

Remark. Consider the p-dimensional Witt algebra $W(1) := \bigoplus_{i=-1}^{p-2} Fe_i$ whose multiplication and p-map are given by

$$[e_i, e_j] = (j-i)\, e_{i+j} \quad ; \quad e_i^{[p]} = \delta_{i,0}\, e_0,$$

respectively. (For $p = 3$ we have $W(1) \cong s\ell(2)$.) Note that $W(1)$ has a triangular decomposition with $W(1)^- := Fe_{-1}$, $W(1)_0 := Fe_0$, and $W(1)^+ := \sum_{i\geq 1} Fe_i$. The subalgebra $Fe_{-1} \oplus Fe_0 \oplus Fe_1$ is isomorphic to $s\ell(2)$. We set $B := W(1)_0 \oplus W(1)^-$.

According to [4] the algebra $u(W(1))$ has p simple modules $S_1, \ldots, S_p$. Aside from the trivial module there exists one module of dimension $p-1$ and $p-2$ modules of dimension p. For $i \in \{0, \ldots, p-1\}$ we let F_i denote the one-dimensional B-module with trivial $W(1)^-$-action and $e_0 \cdot 1 = i$. In virtue of (2.1) the modules $M_i := u(W(1)) \otimes_{u(B)} F_i$ are indecomposable and pairwise nonisomophic. Moreover, [9, (3.4)] implies $\mathcal{V}_{W(1)}(M_i) \subset Fe_{-1}$. If each of the M_i is projective, then they form a full set of representatives of the principal indecomposable $u(W(1))$-modules and we obtain

$$p^p = dim_F u(W(1)) = p^{p-2} + (p-1)p^{p-2} + (p-2)p^{p-1},$$

a contradiction. Hence there is an index i such that $\mathcal{V}_{W(1)}(M_i) = Fe_{-1}$. The Auslander-Reiten component of M_i contains modules of dimension ℓp^{p-2} for every $\ell \geq 1$. Since the space Fe_{-1} is not invariant under the action of the connected component of the automorphism group of $W(1)$ (cf. [21, Thm.6]), there are infinitely many indecomposables in each dimension ℓp^{p-2}. (Note that this result is also valid for $p = 3$).

Consider $\mathcal{V} := \{x + y \in W(1) \ ; \ x \in \mathcal{V}_{s\ell(2)}(F), \ y \in Fe_2\}$. Then $\mathcal{V} \cong \mathcal{V}_{s\ell(2)}(F) \times F$ is an irreducible, conical variety of dimension 3. According to [15, (2.2)] there exists a $u(W(1))$-module M with $\mathcal{V}_{W(1)}(M) = \mathcal{V}$. Since $\mathcal{V}$ is irreducible, at least one indecomposable summand N of M has the same property. By [11, (2.1)] the component Θ of N is isomorphic to $\mathbb{Z}[A_\infty]$. Since $\mathcal{V}$ is not invariant under $Aut_p(W(1))^\circ$, there are infinitely many such components. Let $[X] \in \Theta$ be a quasi-simple vertex of minimal dimension. As the dimension of every principal indecomposable $u(W(1))$-module is divisible by p^{p-2}, it readily follows from (1) that

$$dim_F Y \equiv q\ell(Y) dim_F X \quad mod(p^{p-2}) \quad \forall \ [Y] \in \Theta.$$

Here $q\ell(Y)$ denotes the quasilength of $[Y]$. Since the support variety $\mathcal{V}_L(\Theta) = \mathcal{V}$ intersects the p-nilpotent algebra $W(1)_{(3)} = \sum_{i \geq 3} Fe_i$ trivially, it follows that X is free over the local algebra $u(W(1)_{(3)})$. Consequently, $dim_F X = \ell_0 p^{p-4}$. For each $r > 0$, let $[Y_r]$ be a vertex of quasilength rp^2. The above discussion ensures the existence of $\ell_r \geq 0$ such that

$$dim_F Y_r = (r\ell_0 + \ell_r)p^{p-2}.$$

Owing to (1.2) the dimensions of the Y_r are not bounded. Consequently, $A_\infty(W(1), \ell p^{p-2})$ is infinite for infinitely many $\ell > 0$.

References

[1] M. Auslander, I. Reiten and S. Smalø. *Representation Theory of Artin Algebras.* Cambridge Studies in Advanced Math. **36.** Cambridge University Press, 1995

[2] D. Benson and J. Carlson. *Complexity and Multiple Complexes.* Math. Z. **195** (1987), 221-238

[3] R. Block and R. Wilson. *Classification of Restricted Simple Lie Algebras.* J. Algebra **114** (1988), 115-259

[4] H. Chang. *Über Wittsche Lie Ringe.* Abh. Math. Sem. Univ. Hamburg **14** (1941), 151-184

[5] W. Crawley-Boevey. *On Tame Algebras and Bocses.* Proc. London Math. Soc. **56** (1988), 451-483

[6] K. Erdmann. *The Auslander-Reiten Quiver of Restricted Enveloping Algebras.* CMS Conf. Proc. **18** (1996), 201-214

[7] R. Farnsteiner. *Representations of Lie algebras with Triangular Decomposition.* Symposia Gaussiana **2** (1995), 275-286

[8] R. Farnsteiner. *Periodicity and Representation Type of Modular Lie Algebras.* J. reine angew. Math. **464** (1995), 47-65

[9] R. Farnsteiner. *Representations of Blocks Associated to Induced Modules of Restricted Lie Algebras.* Math. Nachr. **179** (1996), 57-88

[10] R. Farnsteiner. *On Auslander-Reiten Quivers of Enveloping Algebras of Restricted Lie Algebras.* Math. Nachr. (to appear)

[11] R. Farnsteiner. *On Support Varieties of Auslander-Reiten Components.* Indag. Math. (to appear)

[12] R. Farnsteiner. *Auslander-Reiten Components for Lie Algebras of Reductive Groups.* Manuscript in Preparation.

[13] R. Farnsteiner and H. Strade. *Shapiro's Lemma and its Consequences in the Cohomology Theory of Modular Lie Algebras.* Math. Z. **206** (1991), 153-168

[14] J. Feldvoss and H. Strade. *Restricted Lie Algebras with Bounded Cohomology and Related Classes of Algebras.* Manuscripta math. **74** (1992), 47-67

[15] E. Friedlander and B. Parshall. *Support Varieties for Restricted Lie Algebras.* Invent. math. **86** (1986), 553-562

[16] E. Friedlander and B. Parshall. *Geometry of p-Unipotent Lie Algebras.* J. Algebra **109** (1987), 25-45

[17] E. Friedlander and B. Parshall. *Modular Representation Theory of Lie Algebras.* Amer. J. Math. **110** (1988), 1055-1098

[18] G. Hochschild. *Cohomology of Restricted Lie Algebras.* Amer. J. Math. **76** (1954), 555-580

[19] J. Humphreys. *Algebraic Groups and Modular Lie Algebras.* Mem. Amer. Math. Soc. **71** (1967)

[20] J. Humphreys. *Linear Algebraic Groups.* Graduate Texts in Mathematics **21.** Springer-Verlag, 1981

[21] N. Jacobson. *Classes of Restricted Lie Algebras of Characteristic p, II.* Duke Math. J. **10** (1943), 107-121

[22] J. Jantzen. *Kohomologie von p-Lie-Algebren und nilpotente Elemente.* Abh. Math. Sem. Univ. Hamburg **56** (1986), 191-219

[23] V. Kac and B. Weisfeiler. *Coadjoint Action of a Semisimple Algebraic Group and the Center of the Enveloping Algebra in Characteristic* p. Indag. Math. **38** (1976), 136-151

[24] S. Liu and R. Schulz. *The Existence of a Bounded Infinite DTr-Orbit.* Proc. Amer. Math. Soc. **122** (1994), 1003-1005

[25] E. Marmolejo and C. Ringel. *Modules of Bounded Length in Auslander-Reiten Components.* Arch. Math. **50** (1988), 128-133

[26] D. Nakano and R. Pollack. *On the Construction of Indecomposable Modules over Restricted Enveloping Algebras.* J. Pure Appl. Algebra **107** (1996), 61-73

[27] W. Pfautsch and D. Voigt. *The Representation-finite Algebraic Groups of Dimension zero.* C. R. Acad. Sci. Paris **306** (1988), 685-689

[28] R. Pollack. *Restricted Lie Algebras of Bounded Type.* Bull. Amer. Math. Soc. **74** (1968), 326-331

[29] A. Premet. *Irreducible Representations of Lie Algebras of Reductive Groups and the Kac-Weisfeiler Conjecture.* Invent. math. **121** (1995), 79-117

[30] C. Ringel. *Representation Theory of Finite Dimensional Algebras.* London Math. Soc. Lecture Notes **116** (1986), 7-79

[31] G. Seligman. *Modular Lie Algebras.* Ergebnisse der Mathematik **40** Springer-Verlag, 1967

[32] H. Strade and R. Farnsteiner. *Modular Lie Algebras and their Representations.* Marcel Dekker Monographs **116** New York, 1988

Department of Mathematics
University of Wisconsin
Milwaukee, WI 53201
U.S.A.

Fakultät für Mathematik
Universität Bielefeld
33501 Bielefeld
GERMANY