THE FLUCTUATIONS OF THE OVERLAP IN THE HOPFIELD MODEL WITH FINITELY MANY PATTERNS AT THE CRITICAL TEMPERATURE

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ABSTRACT. We investigate the limiting fluctuations of the order parameter in the Hopfield model of spin glasses and neural networks with finitely many patterns at the critical temperature $1/\beta_c = 1$. At the critical temperature, the measure-valued random variables given by the distribution of the appropriately scaled order parameter under the Gibbs measure converge weakly towards a random measure which is non-Gaussian in the sense that it is not given by a Dirac measure concentrated in a Gaussian distribution. Moreover, we analyze the limiting behaviour of the fluctuations as the temperature $1/\beta_N$ converges to one and obtain a threshold for the speed of this convergence to one: If β_N converges to $\beta_c = 1$ at speed $o(1/\sqrt{N})$, then the non-Gaussian behaviour of the fluctuations persists, whereas for β_N converging to one at a speed which is slower than $\mathcal{O}(1/\sqrt{N})$, the order parameter obeys a weak central limit theorem.

1. Introduction and Description of the Model

The Hopfield model, together with the Sherrington–Kirkpatrick model of spin glasses, is one of the most popular models of disordered spin systems. The origin of the interest in it is at least twofold. On the one hand, it can be regarded as a very simple model of the brain, hence as a neural network. This point of view has been taken in the original paper by Hopfield [15] for instance, as well as in the papers [21], [22], [17], [19], [20], and many others. On the other hand, the Hopfield model may also be regarded as a spin glass with the additional feature that the complexity of the disorder can be adjusted by an additional parameter, the number of patterns. This has been the motivation for a variety of papers including [23], where the Hopfield model has been first introduced, and [24], [5], [6], [7], [1], [11], [12], [2], [3], and [26] to list some of the papers in this area (of course, this list is far from being complete). For an overview of recent results on the Hopfield model and related models, see [8] and, in particular, [4] therein.

To be more specific, let us define the Hopfield model. First of all we fix a number $M \in \mathbb{N}$ which will denote the number of so-called patterns, which in contrast to the most general set-up of the model will not depend on the number $N \in \mathbb{N}$ of spins or "neurons". By

$$H_{N,\xi}(\sigma) = -\frac{1}{2N} \sum_{\mu=1}^{M} \sum_{i,j=1}^{N} \sigma_i \sigma_j \xi_i^\mu \xi_j^\mu, \qquad \sigma \in \{-1,+1\}^N, \tag{1.1}$$

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we will denote the (random) Hamiltonian of the Hopfield model, which we will regard as a function of the spin configuration $\sigma \in \{-1,+1\}^N$. The variable $\xi_i^\mu \in \{-1,+1\}$ stands for the ith component of the μth pattern. We will assume that the ξ_i^μ are i.i.d. unbiased random variables, i. e., that

$$\mathbb{P}(\xi_i^\mu = +1) = \mathbb{P}(\xi_i^\mu = -1) = \frac{1}{2}.$$

Expectations with respect to $\mathbb{P}$ will be denoted by $\mathbb{E}$. For convenience, we denote by ξ the $(N \times M)$-matrix consisting of the $(\xi_i^\mu)_{i,\mu}$, while $\xi_i = (\xi_i^1, \dots, \xi_i^M)$ and $\xi^\mu = (\xi_1^\mu, \dots, \xi_N^\mu)$, respectively, stand for the ith row and the μth column of this matrix, respectively.

The spin variables' a priori distribution P is assumed to be unbiased as well, i. e.,

$$\mathrm{P}(\sigma_i = +1) = \mathrm{P}(\sigma_i = -1) = \frac{1}{2}$$

for all $i \in \mathbb{N}$. Expectations with respect to the spin variables' a priori distribution P will be denoted by E. We shall assume throughout this note that the family $\{\, \xi_i^\mu : i \in \{1, \dots, N\}, \mu \in \{1, \dots, M\} \,\}$ is independent of the family of the spin variables $\{\, \sigma_i : i \in \{1, \dots, N\} \,\}$.

The Hopfield model at temperature $1/\beta \in (0, \infty)$ may now either be identified with the Hamiltonian (1.1) or, equivalently, with the Gibbs measure

$$\varrho_{N,\beta,\xi}(\sigma) = 2^{-N} \exp\{-\beta H_{N,\xi}(\sigma)\}/Z_{N,\beta}(\xi), \qquad \sigma \in \{-1,+1\}^N,$$

where the so-called partition function

$$Z_{N,\beta}(\xi) = \frac{1}{2^N} \sum_{\sigma \in \{-1,+1\}^N} \exp\{-\beta H_{N,\xi}(\sigma)\}$$

is the appropriate normalization to make $\varrho_{N,\beta,\xi}$ a probability measure.

Note that the Hamiltonian (1.1) may be rewritten in the following convenient form as a quadratic functional of the overlap m_N:

$$H_{N,\xi}(\sigma) = -\frac{N}{2} \|m_N(\sigma)\|_2^2,$$

where

$$m_N(\sigma) = (m_N^\mu(\sigma))_{\mu=1,\dots,M} \qquad \text{with} \qquad m_N^\mu(\sigma) = \sum_{i=1}^N \xi_i^\mu \sigma_i.$$

Here and below, $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^M$. The μth component of the overlap compares the spin configuration to the μth pattern ξ^μ. Therefore, the overlap m_N is an important quantity for the investigation of the Hopfield model, a so-called order parameter. Its distribution under $\varrho_{N,\beta,\xi}$ has been of major interest in recent studies of the model. In [5], the asymptotic behaviour of the overlap under the Gibbs measure $\varrho_{N,\beta,\xi}$ was first investigated. The authors established a law of large numbers which holds for $\mathbb{P}$-almost all realizations of the random patterns ξ. They show that for $\mathbb{P}$-almost all ξ, the distribution of the overlap m_N under the Gibbs measure with external magnetic field of strength $h \neq 0$ in the direction of the first unit vector e_1 of the canonical basis in $\mathbb{R}^M$ converges weakly towards the Dirac measure $\delta_{\pm z(\beta) e_1}$ concentrated in $\pm z(\beta) e_1$ as first the system size $N \to \infty$ and then the strength $h \to 0\pm$. Here $z(\beta)$ denotes the largest root $z \in [0, 1)$ of the Curie–Weiss equation

$$z = \tanh(\beta z).$$

Note that $z(\beta) = 0$ for $\beta \leq \beta_c = 1$ so that δ_0 is the unique limiting measure in the high-temperature region $\beta \leq \beta_c = 1$, whereas $z(\beta) > 0$ for $\beta > \beta_c$. This law of large numbers holds in the more general case where M depends on N in such a way that $M(N)/N \to 0$ as $N \to \infty$. Note that it strongly resembles the law of large numbers for the magnetization in the Curie–Weiss model, see [9, Theorem IV.4.1(a)], for example. This is, of course, not accidental. Indeed, for $M = 1$ the Hopfield model and the Curie–Weiss model agree by a simple change of variable, and generally, the law of large numbers for the overlap in the Hopfield model may be summarized by saying that for M growing slowly enough (as a function of N) the Hopfield model almost surely behaves like M orthogonal Curie–Weiss models.

However, on the scale of fluctuations, i.e., when analyzing the distribution of $\sqrt{N}(m_N(\sigma) - z(\beta)e_1)$, the character of the disorder becomes visible even when the number of patterns remains bounded (in N). Indeed, for M increasing not too fast ($M/N \to 0$ suffices) and $(\beta, h) \neq (1, 0)$, the overlap satisfies $\mathbb{P}$-almost surely a central limit theorem with the covariance matrix which could be expected from the analogy with the Curie–Weiss model and a centring which differs in the cases $\beta > 1$ or $h \neq 0$ from the naïvely expected one by a ξ-dependent adjustment, see [11], [12], [14] and Bovier and Gayrard [2].

In this paper we will investigate the fluctuations of the overlap at the critical temperature $\beta = \beta_c = 1$. Recall that in the Curie–Weiss model the criticality at temperature $1/\beta = 1$ can be seen in various ways, one of them being the breakdown of the central limit theorem. As a matter of fact at the critical temperature the magnetization in the Curie–Weiss model—scaled by a factor $N^{1/4}$—converges weakly towards a random variable given by its density with respect to Lebesgue measure which is proportional to $\exp(-x^4/12)$, cf. [9, Theorem V.9.5]. One might expect a similar behaviour in the Hopfield model. On the other hand observe that, other than the magnetization in the Curie–Weiss model, the quantities $m_N(\sigma)$ depend on an extra random parameter and that therefore limit theorems can only be stated with respect to this additional randomness. While the above-mentioned central limit theorem holds for $\mathbb{P}$-almost all realizations of the random patterns, at criticality $\mathbb{P}$-almost sure convergence is too strong to ask for. As it turns out, the right point of view is to consider the random distributions of the overlap under the Gibbs measures at temperature one. We consider the random variables

$$\xi \mapsto \varrho_{N,1,\xi}(N^{1/4} m_N)^{-1} = \varrho_{N,1,\xi}(\{\sigma \in \{-1,+1\}^N \mid N^{1/4} m_N(\sigma) \in \cdot\})$$

taking values in the set $\mathcal{M}_1(\mathbb{R}^M)$ of probability measures on $\mathbb{R}^M$ and show their weak convergence (with respect to $\mathbb{P}$) to a limiting random measure. This limiting measure is characterized by its random density with respect to the M-dimensional Lebesgue measure which is proportional to

$$\exp\left(-\frac{1}{12}\sum_{\mu=1}^{M} x_\mu^4 - \frac{1}{2}\sum_{1\leq\mu<\nu\leq M} x_\mu^2 x_\nu^2 + \sum_{1\leq\mu<\nu\leq M} \eta_{\mu,\nu} x_\mu x_\nu\right),$$

where η is an $M(M-1)/2$-dimensional Gaussian random variable with mean zero and the covariance matrix being the identity matrix, namely,

$$\Sigma = \big(\Sigma_{(\mu,\nu),(\mu',\nu')}\big)_{(\mu,\nu),(\mu',\nu')}$$

and

$$\Sigma_{(\mu,\nu),(\mu',\nu')} = \begin{cases} 1, & \text{if } (\mu,\nu) = (\mu',\nu'), \\ 0, & \text{otherwise,} \end{cases}$$

for $1 \le \mu < \nu \le M$ and $1 \le \mu' < \nu' \le M$.

This shows that at the critical temperature $1/\beta = 1$, the fluctuations of the overlap depend strongly on the random disorder as even the distribution of the limiting fluctuations is random. Of course, a result valid for almost all realizations of ξ would be more in the spirit of previous results on the fluctuations in the Hopfield model. On the other hand, some recent discussion for the Hopfield model (see e. g. [4, Section 8]) might indicate that an almost sure result is a stroke of luck and that usually all one can hope for is convergence in distribution.

Moreover, it is interesting to ask what happens, if β depends on N in such a way that $\beta = \beta_N$ converges to 1 as the system size N tends to infinity. Depending on the speed of convergence either the above limiting behaviour at criticality persists or the limiting fluctuations are Gaussian. On the borderline, a mixture of the two different regimes occurs.

This paper has three more sections. Section 2 contains the explicit statements of the results described above, while Sections 3 and 4 are devoted to the proofs which are based on the Hubbard–Stratonovich transform of the measures of interest together with a Taylor expansion of the resulting density. In Section 3, we prove the results for all cases except for $\beta = \beta_N$ tending to one slowly from above. This case, which is slightly more involved, is treated separately in Section 4. In the appendix, we prove a convergence criterion needed in the proofs.

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2. Statement of Results

In this section we shall give the precise statements of the announced results on the limiting fluctuations of the overlap at the critical temperature. The case which has always been in the center of our interest is the case where $\beta = \beta_c = 1$ is fixed. Note that the measures of interest

$$Q_{N,1} = \varrho_{N,1,\xi}(N^{1/4} m_N(\sigma))^{-1}$$

are random variables in ξ taking values in the Polish space $\mathcal{M}_1(\mathbb{R}^M)$ of probability measures on $\mathbb{R}^M$, endowed with the Prohorov metric.

Theorem 2.1. *Let $\beta = \beta_c = 1$. Then*

$$\mathbb{P}Q_{N,1}^{-1} = \mathbb{P}(\varrho_{N,1,\xi}(N^{1/4} m_N(\sigma))^{-1})^{-1} \Longrightarrow \mathbb{P}Q^{-1}, \qquad \text{as } N \to \infty.$$

Here '$\Rightarrow$' denotes weak convergence and Q is the random probability measure on $\mathbb{R}^M$ which has the (random) density φ with respect to the M-dimensional Lebesgue measure given by

$$\varphi(x) = \exp\biggl(-\frac{1}{12}\sum_{\mu=1}^{M} x_\mu^4 - \frac{1}{2}\sum_{1\le\mu<\nu\le M} x_\mu^2 x_\nu^2 + \sum_{1\le\mu<\nu\le M} \eta_{\mu,\nu} x_\mu x_\nu\biggr)\Big/ \Xi, \qquad x \in \mathbb{R}^M,$$

where

$$\Xi = \int_{\mathbb{R}^M} \varphi(x)\, dx,$$

and $\eta = (\eta_{\mu,\nu})_{1\le\mu<\nu\le M}$ *is a random vector with distribution* $\mathbb{P}\eta^{-1}$ *being the* $M(M-1)/2$*-dimensional standard Gaussian distribution* $\mathcal{N}(0,\Sigma)$.

Remark 2.2. *Note that the scaling factor* $N^{1/4}$ *for the overlap vector is the same as the one for the mean magnetization in the Curie–Weiss model at the critical temperature, see* [9, Theorem V.9.5]. *Similar to that case we have a non-Gaussian limiting distribution in the Hopfield model.*

As it turns out, Theorem 2.1 above can be regarded as a special case of the following theorem treating the case of size-dependent temperature $1/\beta_N \to 1$. There are two different regimes: for β_N tending to one fast enough, the limiting behaviour is the same as at criticality, whereas for β_N tending to one slowly, the limiting fluctuations are Gaussian. Recall from [5, Theorem 1], that for (fixed) $\beta > 1$ and fixed M, there are $2M$ possible limiting points in the law of large numbers for the distribution of the overlap under the Gibbs measure. Therefore, to obtain a central limit theorem, we need either to introduce an external magnetic field or to condition on the event that the overlap is already close to one of the $2M$ possible limiting points as in [11, Theorem 1.1]. Similarly, to obtain a weak central limit theorem for β_N tending to one slowly from above, we shall condition on the event that the overlap m_N is close to the random centring $z_N(\xi)$ related to $z(\beta_N)e_1$, see Part 3 of the theorem below. Finally, the theorem treats the "borderline case" where the limiting fluctuations differ from the ones at criticality by an additional term.

To ease the notation, we shall employ the following abbreviation.

Notation 2.3. *For two sequences* $(a_N)_{N\in\mathbb{N}}$ *and* $(b_N)_{N\in\mathbb{N}}$ *with* $a_N, b_N \ge 0$, *let us write* $a_N \ll b_N$ *if* $a_N \le b_N$ *and* $a_N/b_N \to 0$ *and, similarly, we shall write* $a_N \gg b_N$ *if* $b_N \ll a_N$.

Theorem 2.4. *Let* $0 < \beta_N < \infty$ *depend on* N *in such a way that* $\beta_N \to 1$ *monotonically as* $N \to \infty$. *Then the following assertions hold:*

1. *If* $|\beta_N - 1| \ll 1/\sqrt{N}$, *then the random measures*
$$Q_{N,\beta_N} = \varrho_{N,\beta_N,\xi}(N^{1/4} m_N(\sigma))^{-1}$$
again satisfy
$$\mathbb{P}Q_{N,\beta_N}^{-1} \Longrightarrow \mathbb{P}Q^{-1}, \qquad \text{as } N \to \infty,$$
where Q *is defined as in Theorem 2.1.*
2. *If* $1 - \beta_N \gg 1/\sqrt{N}$, *then the random measures*
$$\tilde{Q}_{N,\beta_N} = \varrho_{N,\beta_N,\xi}(\sqrt{N(1-\beta_N)}\, m_N(\sigma))^{-1}$$
satisfy a weak central limit theorem: As $N \to \infty$, *the measures* $\mathbb{P}\tilde{Q}_{N,\beta_N}^{-1}$ *converge weakly towards the Dirac measure concentrated in the* M*-dimensional standard Gaussian distribution.*
3. *If* $\beta_N - 1 \gg 1/\sqrt{N}$, *then there exists a random centring* $z_N(\xi)$ *depending on* N, β_N *and* ξ *such that for arbitrary* $r \in (0, 1/\sqrt{2})$, *the random measures*
$$\tilde{Q}_{N,\beta_N} = \varrho_{N,\beta_N,\xi}\Bigg(\sqrt{N}\sqrt{\frac{1-\beta_N(1-z(\beta_N)^2)}{1-z(\beta_N)^2}}[m_N - z_N(\xi)] \in \cdot \;\Bigg|\; \|m_N - z_N(\xi)\|_2 < r z(\beta_N)\Bigg)$$

satisfy a weak central limit theorem: As $N \to \infty$, the measures $\mathbb{P}\tilde{Q}^{-1}_{N,\beta_N}$ converge weakly towards the Dirac measure concentrated in the M-dimensional standard Gaussian distribution.

4. *If $\beta_N - 1 = \gamma/\sqrt{N}$ for some $\gamma \neq 0$, then*

$$\mathbb{P}Q^{-1}_{N,\beta_N} \Longrightarrow \mathbb{P}\hat{Q}^{-1}$$

where $\hat{Q}$ is the random measure which has a (random) density $\hat{\varphi}$ with respect to the M-dimensional Lebesgue measure given by

$$\hat{\varphi}(x) = \exp\biggl(-\frac{1}{12}\sum_{\mu=1}^{M} x_\mu^4 - \frac{1}{2}\sum_{1\le\mu<\nu\le M} x_\mu^2 x_\nu^2 + \frac{\gamma}{2}\sum_{\mu=1}^{M} x_\mu^2 + \sum_{1\le\mu<\nu\le M} \eta_{\mu,\nu} x_\mu x_\nu\biggr)\Big/\hat{\Xi},$$

where $\hat{\Xi}$ is the corresponding normalizing factor

$$\hat{\Xi} = \int_{\mathbb{R}^M} \hat{\varphi}(x)\,dx,$$

and $(\eta_{\mu,\nu})_{1\le\mu<\nu\le M}$ is the same Gaussian random vector as in Theorem 2.1.

Remark 2.5. 1. *The random centring $z_N(\xi)$ in Theorem 2.4, Part 3, is the one also necessary for the central limit theorem in the case of fixed $\beta > 1$, cf.* [11, Theorem 1.1]. *One way to define this centring is to consider a linear adjustment of the deterministic centring $z(\beta_N)e_1$, namely,*

$$z_N(\xi) = \biggl[\mathrm{Id} + \frac{1}{1-\beta_N(1-z(\beta_N)^2)}\biggl(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\biggr)\biggr]\, z(\beta_N)e_1. \tag{2.1}$$

Recall that ξ is the $(N \times M)$-matrix with the M columns $\xi^1, \dots, \xi^M$, so that $\xi^{\mathrm{T}}\xi$ is an $(M \times M)$-matrix. This matrix approaches the identity matrix as $N \to \infty$ in the following sense: Fix an arbitrary $\varepsilon > 0$. Then, for $\mathbb{P}$-almost all realizations of the random patterns ξ, there exists an $N_0(\xi) \ge 3$ such that for all $N \ge N_0(\xi)$

$$\biggl\|\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\biggr\|_{\mathrm{Op}} \le \varepsilon_N \searrow 0, \tag{2.2}$$

where

$$\varepsilon_N = (1+\varepsilon)\sqrt{2\frac{M}{N}\log\log N},$$

cf. [11, Lemma 2.2]. *A similar bound holds in the general case $M \to \infty$, see* [4, Section 4] *for a particularly elegant proof.*

2. *Note that*

$$z(\beta) = 3(\beta-1)[1 + \mathcal{O}(\beta-1)], \qquad \text{as } \beta \searrow 1, \tag{2.3}$$

and, therefore,

$$\frac{1-\beta(1-z(\beta)^2)}{1-z(\beta)^2} = 2(\beta-1)[1+\mathcal{O}(\beta-1)], \qquad \text{as } \beta \searrow 1. \tag{2.4}$$

3. Proofs

This section and the following one contain the proofs of the results presented in the previous section. First we give a criterion for weak convergence in the Polish space $\mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^M))$ of probability measures on the probability measures on $\mathbb{R}^M$, which is what we need to prove Theorem 2.4. Then we calculate the Hubbard–Stratonovich transform ([16], [25]) of the measures in question. The core of the proofs is the investigation of the limiting behaviour of these Hubbard–Stratonovich transforms as $N \to \infty$ by an adaptation of Laplace's method as the limiting behaviour of the measures is determined by the limiting behaviour of their Hubbard–Stratonovich transforms.

To prove weak convergence in $\mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^M))$, we shall apply the following criterion to the Hubbard–Stratonovich transforms calculated in Lemma 3.4.

Lemma 3.1. *Let $M \in \mathbb{N}$ be any finite number and let $(Q_N)_{N\in\mathbb{N}}$ be a sequence of random probability measures on $\mathbb{R}^M$ and Q be another random probability measure on $\mathbb{R}^M$. Denote by*

$$\mu_N = \mathbb{P}Q_N^{-1} \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^M))$$

the distribution of Q_N and similarly by

$$\mu = \mathbb{P}Q^{-1} \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^M)).$$

the distribution of Q. Then μ_N converges weakly towards μ if and only if

$$\int g\Big(\int f\,dQ_N\Big)\,d\mathbb{P} \to \int g\Big(\int f\,dQ\Big)\,d\mathbb{P}, \qquad \text{as } N \to \infty, \tag{3.1}$$

holds for all continuous functions $f : \mathbb{R}^M \to \mathbb{R}^k$ with compact support, all continuous functions $g : \mathbb{R}^k \to \mathbb{R}$, and all k.

The proof of this technical lemma is given in the appendix. The next lemma shows that the weak convergence of a sequence of random measures can be established by proving the weak convergence of their Hubbard–Stratonovich transforms.

Notation 3.2. *We denote by $\mu * \nu$ the convolution of two measures μ and ν.*

Lemma 3.3. *Let* w-lim *denote the weak limit of a sequence of probability measures in the Polish space $\mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^M))$ and let $(Q_N)_{N\in\mathbb{N}}$ be a sequence of random probability measures on $\mathbb{R}^M$. Assume $c_N \to 0$ as $N \to \infty$. Then* $\text{w-lim}_{N\to\infty} \mathbb{P}Q_N^{-1}$ *exists if and only if* $\text{w-lim}_{N\to\infty} \mathbb{P}(Q_N * \mathcal{N}(0, c_N\mathrm{Id}))^{-1}$ *exists, and if one of the limits exists, then*

$$\operatorname*{w-lim}_{N\to\infty} \mathbb{P}Q_N^{-1} = \operatorname*{w-lim}_{N\to\infty} \mathbb{P}(Q_N * \mathcal{N}(0, c_N\mathrm{Id}))^{-1}.$$

Proof. By Lemma 3.1 it is sufficient to show that

$$\Big|\int g\Big(\int f\,dQ_N\Big)\,d\mathbb{P} - \int g\Big(\int f\,d(Q_N * \mathcal{N}(0, c_N\mathrm{Id}))\Big)\,d\mathbb{P}\Big| \to 0 \tag{3.2}$$

as $N \to \infty$ for all continuous $f : \mathbb{R}^M \to \mathbb{R}^k$ with compact support, all continuous $g : \mathbb{R}^k \to \mathbb{R}$, and all $k \in \mathbb{N}$. Fix an arbitrary $\varepsilon > 0$. The continuous function g is uniformly continuous on the compact set $K = f(\mathbb{R}^M)$. (Recall that f is assumed to be continuous with compact support.) Therefore, there exists a $\delta = \delta(\varepsilon) > 0$ such that $|g(s) - g(t)| < \varepsilon$ for all $s, t \in K$ satisfying $\|s - t\|_2 < \delta$. Furthermore,

the uniform continuity of f implies that there exists a $\delta' = \delta'(\delta) > 0$ such that $\|f(x) - f(y)\|_2 < \delta/2$ for all $x, y \in \mathbb{R}^M$ satisfying $\|x - y\|_2 < \delta'$. Now,

$$\int f\, d(Q_N * \mathcal{N}(0, c_N\mathrm{Id})) = \iint f(x+y)\, Q_N(dy)\, \mathcal{N}(0, c_N\mathrm{Id})(dx)$$

implies that

$$\begin{aligned}\Big\| \int f\, d(Q_N * \mathcal{N}(0, c_N\mathrm{Id})) - \int f\, dQ_N \Big\|_2 \\ &\le \iint \|f(x+y) - f(y)\|_2\, Q_N(dy)\, \mathcal{N}(0, c_N\mathrm{Id})(dx) \\ &\le \frac{\delta}{2} + 2\|f\|_\infty \Big(\frac{1}{2\pi c_N}\Big)^{M/2} \int_{B(0,\delta')^{\mathrm{c}}} \exp\Big\{-\frac{1}{2c_N}\|x\|_2^2\Big\}\, dx \le \delta,\end{aligned}$$

for large enough $N \in \mathbb{N}$. This estimate holds uniformly for all possible choices of the random measures Q_N and, therefore, implies (3.2). □

So let us now calculate the Hubbard–Stratonovich transforms of the measures we are interested in. Lemma 3.4 will be used in all cases except for $\beta_N - 1 \gg 1/\sqrt{N}$, which will be treated with the help of Lemma 4.1 below.

Lemma 3.4. *Let $0 < \beta < \infty$ and $a > 0$. Then the convolution*

$$\chi_{N,\beta,a} = \varrho_{N,\beta,\xi}(\sqrt{a}m_N)^{-1} * \mathcal{N}\big(0, \frac{a}{N\beta}\mathrm{Id}\big)$$

of $\varrho_{N,\beta,\xi}(\sqrt{a}m_N)^{-1}$ with the M-dimensional Gaussian distribution with mean zero and covariance matrix $\frac{a}{N\beta}\mathrm{Id}$ is the random measure on $\mathbb{R}^M$ which is given by its (random) density

$$f_{N,\beta,a}(x) = \frac{\exp\{-N\beta\Phi_{N,\beta}(x/\sqrt{a})\}}{\int_{\mathbb{R}^M} \exp\{-N\beta\Phi_{N,\beta}(x/\sqrt{a})\}\, dx}, \qquad x \in \mathbb{R}^M,$$

with respect to the M-dimensional Lebesgue measure, where

$$\Phi_{N,\beta}(x) = \frac{1}{2}\|x\|_2^2 - \frac{1}{\beta N}\sum_{i=1}^{N} \log\cosh(\beta\langle x, \xi_i\rangle), \qquad x \in \mathbb{R}^M, \tag{3.3}$$

depends on the random patterns. Here $\langle\cdot,\cdot\rangle$ stands for the inner product in $\mathbb{R}^M$ and $\|\cdot\|_2$ for the corresponding Euclidean norm as before.

We omit the proof of Lemma 3.4 here as the proof follows from a straight-forward calculation similar to the ones given in [5, Lemma 2.2] or [10, Lemma 3.3.].

Now we are ready to prove Theorem 2.4 with the help of the preceding lemmas.

Proof of Theorem 2.4. Let $\gamma_N = \sqrt{N}(\beta_N - 1)$. As in the statement of the theorem, we have to distinguish the following four cases:

1. $\gamma_N \to 0$ corresponds to the case $|\beta_N - 1| \ll 1/\sqrt{N}$, where we expect the same asymptotic behaviour as for $\beta_N \equiv \beta_{\mathrm{c}} = 1$, whereas
2. $\gamma_N \to -\infty$ corresponds to the case $1 - \beta_N \gg 1/\sqrt{N}$, where we expect the weak central limit theorem to hold with centring zero.
3. $\gamma_N \to \infty$ corresponds to the case $\beta_N - 1 \gg 1/\sqrt{N}$, where we expect the weak central limit theorem to hold with the random centring $z_N(\xi)$, and, finally,
4. $\gamma_N \equiv \gamma \neq 0$ corresponds to the borderline case $\beta_N - 1 = \gamma/\sqrt{N}$.

According to Lemma 3.1 and Lemma 3.3, we may prove the claimed convergence results by showing that the following assertions hold for all $k \in \mathbb{N}$, all continuous $f : \mathbb{R}^M \to \mathbb{R}^k$ with compact support and all continuous $g : \mathbb{R}^k \to \mathbb{R}$:

1. In the case $\gamma_N \to 0$:
$$\int g\Big(\int f\, d\chi_{N,\beta_N,a_N}\Big)\, d\mathbb{P} \to \int g\Big(\int f\, dQ\Big)\, d\mathbb{P}$$
with $a_N = \sqrt{N}$,
2. in the case $\gamma_N \to -\infty$:
$$\int g\Big(\int f\, d\chi_{N,\beta_N,a_N}\Big)\, d\mathbb{P} \to g\Big(\int f\, d\mathcal{N}(0, \mathrm{Id})\Big)$$
with $a_N = N(1-\beta_N)$,
3. in the case $\gamma_N \to \infty$:
$$\int g\Big(\int f\, d\tilde{\chi}_{N,\beta_N,a_N}\Big)\, d\mathbb{P} \to g\Big(\int f\, d\mathcal{N}(0, \mathrm{Id})\Big)$$
with $a_N = N(1-\beta_N(1-z(\beta_N)^2))/(1-z(\beta_N)^2)$, and
4. in the case $\gamma_N \equiv \gamma \neq 0$:
$$\int g\Big(\int f\, d\chi_{N,\beta_N,a_N}\Big)\, d\mathbb{P} \to \int g\Big(\int f\, d\hat{Q}\Big)\, d\mathbb{P}$$
with $a_N = \sqrt{N}$.

The case $\gamma_N \to \infty$ is treated in Section 4. In the other three cases, we apply Lemma 3.4, thereby obtaining

$$\int f\, d\chi_{N,\beta_N,a_N} = \frac{\int f(x) \exp\{-N\beta_N \Phi_{N,\beta_N}(x/\sqrt{a_N})\}\, dx}{\int \exp\{-N\beta_N \Phi_{N,\beta_N}(x/\sqrt{a_N})\}\, dx},$$

with Φ_{N,β_N} given by (3.3). To prove the theorem, we need to investigate the asymptotic behaviour as $N \to \infty$ of the integral

$$\int f(x) \exp\{-N\beta_N \Phi_{N,\beta_N}(x/\sqrt{a_N})\}\, dx, \tag{3.4}$$

and, in addition, we have to show that the contribution arising from

$$\int_{B(0,R)^c} \exp\{-N\beta_N \Phi_{N,\beta_N}(x/\sqrt{a_N})\}\, dx$$

is negligible. Here $B(0,R)$ denotes the Euclidean ball $\{x \in \mathbb{R}^M : \|x\|_2 < R\}$ of radius $R > 0$, centred at the origin.

In the sequel, we give a detailed proof in the cases $\gamma_N \to 0$, $\gamma_N \to -\infty$ and $\gamma_N \equiv \gamma \neq 0$. The most interesting case is $\gamma_N \to 0$ as it includes the case $\beta_N \equiv \beta_c = 1$. In Section 4, we show how to treat the remaining case $\gamma_N \to \infty$, which is more involved, with the same method.

From now on, we shall use the following conventions: We shall write β instead of β_N, γ instead of γ_N, a instead of a_N, and Φ instead of Φ_{N,β_N}. Furthermore, we shall write z instead of $z(\beta) = z(\beta_N)$ and z_N instead of $z_N(\xi)$. Keep in mind that $z = z(\beta_N) \neq z_N(\xi) = z_N$ in general.

The inner region. As we shall see, the main contribution to the integral

$$\int \exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx \tag{3.5}$$

arises from small x. So let us investigate the asymptotic behaviour in the *inner region* $B(0,R)$ first.

We start by calculating the Taylor expansion of Φ around zero. Although for $\beta > 1$, Φ does not posses a minimum in zero, it suffices to consider the Taylor expansion of Φ around zero, as the minimizing point, which is ξ-dependent (cf. [3, Theorem 3.1] or [12, Proposition 2.3]), tends to zero rather fast when β tends to one fast enough. Recall from (3.3) that

$$\Phi(x) = \frac{1}{2}\|x\|_2^2 - \frac{1}{N\beta}\sum_{i=1}^{N} \log\cosh(\beta\langle x,\xi_i\rangle).$$

By the Taylor expansion of Φ, we know that there exists a $\theta \in (0,1)$ such that

$$\Phi(x) = \frac{1}{2}\|x\|_2^2 - \frac{1}{N\beta}\sum_{i=1}^{N}\left[\frac{\beta^2}{2}\langle x,\xi_i\rangle^2 - \frac{\beta^4}{12}\langle x,\xi_i\rangle^4\right] + R_{N,\beta}(x,\xi),$$

where

$$R_{N,\beta}(x,\xi) = -\frac{1}{N\beta}\sum_{i=1}^{N}\frac{\beta^5}{15}\,h(\beta\langle\theta x,\xi_i\rangle)\langle x,\xi_i\rangle^5$$

and

$$h(t) = \frac{\tanh(t)}{\cosh^4(t)}[2-\sinh^2(t)], \qquad t\in\mathbb{R}.$$

Let us assume for the moment that $\|x\|_2 \le R$ for some $R > 0$. Then the remainder of the Taylor expansion is negligible when investigating the asymptotic behaviour of (3.5), as $|h(t)| \le 2|t|$ and $0 < \theta < 1$ imply

$$\begin{aligned} |R_{N,\beta}(x/\sqrt{a},\xi)| &\le \frac{1}{N\beta}\sum_{i=1}^{N}\frac{2}{15}\,\beta^6\left\langle\frac{x}{\sqrt{a}},\xi_i\right\rangle^6 \\ &\le \frac{1}{N\beta}\mathcal{O}(1)\frac{NM^3}{a^3}\|x\|_2^6 \le \frac{1}{N\beta}\mathcal{O}_M(NR^6/a^3) \end{aligned} \tag{3.6}$$

by the coarse estimate $|\langle x,\xi_i\rangle| \le \sqrt{M}\|x\|_2$. Here we use the following notation:

Notation 3.5. *By $\mathcal{O}(\cdot)$ we denote numerical bounds only, whereas $\mathcal{O}_M(\cdot)$ denotes a bound which depends on M.*

As we consider M independent of N only, for the purpose of this paper we generally do not need the precise form of the dependence on M.

Now, still assuming that $\|x\|_2 \le R$, we find that

$$\begin{aligned} &-N\beta\Phi(x/\sqrt{a}) \\ &\quad = -\frac{N\beta}{2a}\|x\|_2^2 + \frac{\beta^2}{2a}\sum_{i=1}^{N}\langle x,\xi_i\rangle^2 - \frac{\beta^4}{12a^2}\sum_{i=1}^{N}\langle x,\xi_i\rangle^4 + \mathcal{O}_M(NR^6/a^3). \end{aligned} \tag{3.7}$$

Rewriting (3.7),

$$\begin{aligned}
&-N\beta\Phi(x/\sqrt{a}) \\
&\quad = -\frac{N\beta}{2a}\|x\|_2^2 + \frac{\sqrt{N}\beta^2}{2a}\sum_{\mu_1,\mu_2=1}^{M} x_{\mu_1}x_{\mu_2}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2} \\
&\quad = --\frac{N\beta^4}{12a^2}\sum_{\mu_1,\mu_2,\mu_3,\mu_4=1}^{M} x_{\mu_1}x_{\mu_2}x_{\mu_3}x_{\mu_4}\frac{1}{N}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4} + \mathcal{O}_M(NR^6/a^3)
\end{aligned}$$

follows. Let us introduce the notation $\sum^*_{\mu_1,\dots,\mu_k}$ with arbitrary $k \geq 2$ to denote the summation over all k-tuples $(\mu_1,\dots,\mu_k) \in \{1,\dots,M\}^k$ with pairwise disjoint components. Recalling $\gamma = \sqrt{N}(\beta-1)$, we write

$$\begin{aligned}
&-N\beta\Phi(x/\sqrt{a}) \\
&\quad = \gamma\frac{\sqrt{N}\beta}{2a}\|x\|_2^2 + \frac{\sqrt{N}\beta^2}{2a}\sum_{\mu_1,\mu_2}^{*} x_{\mu_1}x_{\mu_2}\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2} \\
&\qquad - \frac{N\beta^4}{12a^2}\|x\|_4^4 - \frac{N\beta^4}{4a^2}\sum_{\mu_1,\mu_2}^{*} x_{\mu_1}^2x_{\mu_2}^2 - \frac{N\beta^4}{3a^2}\sum_{\mu_1,\mu_2}^{*} x_{\mu_1}x_{\mu_2}^3\frac{1}{N}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2} \\
&\qquad - \frac{N\beta^4}{2a^2}\sum_{\mu_1,\mu_2,\mu_3}^{*} x_{\mu_1}x_{\mu_2}x_{\mu_3}^2\frac{1}{N}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2} \\
&\qquad - \frac{N\beta^4}{12a^2}\sum_{\mu_1,\mu_2,\mu_3,\mu_4}^{*} x_{\mu_1}x_{\mu_2}x_{\mu_3}x_{\mu_4}\frac{1}{N}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4} + \mathcal{O}_M(NR^6/a^3), \qquad (3.8)
\end{aligned}$$

where $\|x\|_4^4 = \sum_{\mu=1}^{M} x_\mu^4$.

The proof will finally follow from the fact that, for $\mu_1 \neq \mu_2$,

$$\eta_N^{\mu_1,\mu_2}(\xi) := \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2} \Longrightarrow \mathcal{N}(0,1),$$

as $N \to \infty$, and that, for $\mu_1,\dots,\mu_4$ pairwise disjoint,

$$\frac{1}{N}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2} \to 0 \qquad \text{as well as} \qquad \frac{1}{N}\sum_{i=1}^{N}\xi_i^{\mu_1}\xi_i^{\mu_2}\xi_i^{\mu_3}\xi_i^{\mu_4} \to 0$$

$\mathbb{P}$-almost surely, as $N \to \infty$. Because of the integration with respect to x in (3.4) and (3.5), we need to be careful when using these facts. So let us define the set "where everything is fine", namely,

$$D_{N,L,\delta} = \left(\bigcup_{\mu_1,\mu_2}^{*} A_{N,L}^{\mu_1,\mu_2} \cup \bigcup_{\mu_1,\mu_2}^{*} B_{N,\delta}^{\mu_1,\mu_2} \cup \bigcup_{\mu_1,\mu_2,\mu_3,\mu_4}^{*} C_{N,\delta}^{\mu_1,\mu_2,\mu_3,\mu_4}\right)^{\mathrm{c}},$$

where $\bigcup^*_{\mu_1,\dots,\mu_k}$ denotes the union over all k-tuples $(\mu_1,\dots,\mu_k) \in \{1,\dots,M\}^k$ with pairwise disjoint components, and

$$\begin{aligned} A^{\mu_1,\mu_2}_{N,L} &= \left\{ \left| \frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i^{\mu_1} \xi_i^{\mu_2} \right| > L \right\}, \\ B^{\mu_1,\mu_2}_{N,\delta} &= \left\{ \left| \frac{1}{N} \sum_{i=1}^N \xi_i^{\mu_1} \xi_i^{\mu_2} \right| > \delta \right\}, \\ C^{\mu_1,\mu_2,\mu_3,\mu_4}_{N,\delta} &= \left\{ \left| \frac{1}{N} \sum_{i=1}^N \xi_i^{\mu_1} \xi_i^{\mu_2} \xi_i^{\mu_3} \xi_i^{\mu_4} \right| > \delta \right\}. \end{aligned}$$

We shall choose $L > 0$ large and $\delta > 0$ small later. For the moment just note that by Chebyshev's inequality and the central limit theorem with the Berry–Esseen correction,

$$\begin{aligned} \mathbb{P}(D^{\mathrm{c}}_{N,L,\delta}) &\le M^2 \mathbb{P}(A^{1,2}_{N,L}) + M^2 \mathbb{P}(B^{1,2}_{N,\delta}) + M^4 \mathbb{P}(C^{1,2,3,4}_{N,\delta}) \\ &\le 2M^2 [1 - \Phi_{\mathcal{N}(0,1)}(L) + C/\sqrt{N}] + \frac{M^2}{N\delta^2} + \frac{M^4}{N\delta^2}, \end{aligned} \tag{3.9}$$

where $\Phi_{\mathcal{N}(0,1)}$ denotes the standard normal distribution function and $C > 0$ is the (universal) Berry–Esseen constant. So, for arbitrary $\delta > 0$, choosing N large, and choosing L large independently makes $\mathbb{P}(D^{\mathrm{c}}_{N,L,\delta})$ arbitrarily small.

Recall that we were considering a function $f : \mathbb{R}^M \to \mathbb{R}^k$ with compact support. This allows choosing $R > 1$ so large that $\operatorname{supp} f \subset B(0,R)$. Then, for all $\xi \in D_{N,L,\delta}$,

$$\begin{aligned} &\int f(x) \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \\ &\quad = \int f(x) \exp\left\{ \frac{N}{a^2} \left[-\frac{\beta^4}{12} \|x\|_4^4 - \frac{\beta^4}{2} \sum_{\mu<\nu} x_\mu^2 x_\nu^2 \right] + \gamma \frac{\sqrt{N}\beta}{2a} \|x\|_2^2 \right\} \\ &\qquad \times \exp\left\{ \frac{\sqrt{N}\beta^2}{a} \sum_{\mu<\nu} \eta_N^{\mu,\nu} x_\mu x_\nu + \frac{N}{a^2} \mathcal{O}(M^4 R^4 \delta) + \mathcal{O}_M(NR^6/a^3) \right\} dx. \end{aligned}$$

Let us now have a look at the different cases.

1. In the case $\gamma \to 0$, we have $a = \sqrt{N}$. Therefore, for all $\xi \in D_{N,L,\delta}$,

$$\begin{aligned} &\int f(x) \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \\ &\quad = \exp\{\mathcal{O}_M(R^4\delta + R^6/\sqrt{N} + |\beta - 1| R^4 (L \vee 1))\} \\ &\qquad \times \int f(x) \exp\left\{ -\frac{1}{12} \|x\|_4^4 - \frac{1}{2} \sum_{\mu<\nu} x_\mu^2 x_\nu^2 + \sum_{\mu<\nu} \eta_N^{\mu,\nu} x_\mu x_\nu \right\} dx. \end{aligned} \tag{3.10}$$

2. In the case $\gamma \to -\infty$, we have $a = N(1-\beta)$, which implies $N/a^2 \to 0$ and $\gamma\sqrt{N}/a \equiv -1$. Therefore, for all $\xi \in D_{N,L,\delta}$,

$$\begin{aligned} &\int f(x) \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \\ &\quad = \exp\{|\gamma|^{-1} \mathcal{O}_M(R^6 + R^4\delta + R^2 L) + \mathcal{O}_M(|\beta - 1| R^2)\} \\ &\qquad \times \int f(x) \exp\left\{ -\frac{1}{2} \|x\|_2^2 \right\} dx. \end{aligned} \tag{3.11}$$

4. In the case $\gamma_N \equiv \gamma \neq 0$, again $a = \sqrt{N}$, and

$$\begin{aligned}\int f(x) \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \\ = \exp\{\mathcal{O}_M(R^4\delta + R^6/\sqrt{N} + |\beta-1|R^4(L\vee 1))\} \\ \times \int f(x) \exp\biggl\{-\frac{1}{12}\|x\|_4^4 - \frac{1}{2}\sum_{\mu<\nu} x_\mu^2 x_\nu^2 + \frac{\gamma}{2}\|x\|_2^2 + \sum_{\mu<\nu}\eta_N^{\mu,\nu} x_\mu x_\nu\biggr\}\, dx\end{aligned} \tag{3.12}$$

holds for all $\xi \in D_{N,L,\delta}$.

Note that we have not used the continuity of f so far, and that, therefore, in (3.10), (3.11) and (3.12), f can be replaced by the indicator function on any bounded set. In particular, we shall use $f = 1_{B(0,R)}$ later on.

Before we make use of the fact that

$$\eta_N = (\eta_N^{\mu,\nu})_{1\le\mu<\nu\le M} \Longrightarrow \mathcal{N}(0,\Sigma),$$

as $N\to\infty$, where Σ denotes the identity matrix on $\mathbb{R}^{M(M-1)/2}$, we shall investigate the integral $\int \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx$ and show that the main contribution to the asymptotic behaviour actually arises from those x satisfying $\|x\| \le R$.

So let us now show that

$$\int_{B(0,R)^{\mathrm{c}}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \tag{3.13}$$

is negligible. More precisely, we shall show that for ξ from a set of sufficiently large $\mathbb{P}$-measure, the integral (3.13) tends to zero as $R\to\infty$. For this purpose, we split the domain of integration into two parts, namely, the *outer region* $B(0,\sqrt{a}r)^{\mathrm{c}}$ and the *intermediate region* $W_{R,r}^N = B(0,\sqrt{a}r)\setminus B(0,R)$, where we still may choose $r>0$ small.

The outer region. Let us first treat the outer region. We follow along the lines of the analogous proof of the $\mathbb{P}$-almost sure central limit theorem at fixed temperature $1/\beta \neq 1$, cf. [12, Section 4] or [13, Section 4.1].

First we show that there exists an $r_0 > 0$ such that

$$\int_{B(0,\sqrt{a}r_0)^{\mathrm{c}}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \le \left(\frac{36a\pi}{N}\right)^{M/2} \exp\{-Nr_0^2/36\}$$

for large enough N and all $\xi \in \tilde{D}_N$ for some set $\tilde{D}_N$ satisfying $\mathbb{P}(\tilde{D}_N)\to 0$ as $N\to\infty$. Then we show that we can replace r_0 by an arbitrarily small $r>0$.

Denote by $f_{\mathrm{CW}}(\beta)$ the free energy in the Curie–Weiss model at temperature $1/\beta$, i. e.,

$$f_{\mathrm{CW}}(\beta) = -\frac{\beta}{2}z(\beta)^2 + \log\cosh(\beta z(\beta)). \tag{3.14}$$

Then,

$$\log\cosh x \le \frac{1}{4\beta}x^2 + \max_{t\in\mathbb{R}}\biggl\{-\frac{1}{4\beta}t^2 + \log\cosh t\biggr\} = \frac{1}{4\beta}x^2 + f_{\mathrm{CW}}(2\beta)$$

implies that

$$\begin{aligned}\Phi(x/\sqrt{a}) &= \frac{1}{2a}\|x\|_2^2 - \frac{1}{N\beta}\sum_{i=1}^N \log\cosh(\beta\langle x/\sqrt{a},\xi_i\rangle)\\ &\geq \frac{1}{2a}\|x\|_2^2 - \frac{1}{4aN}\sum_{i=1}^N\langle x,\xi_i\rangle^2 - \frac{1}{\beta}\,f_{\mathrm{CW}}(2\beta).\end{aligned}$$

According to (2.2), for $\mathbb{P}$-almost all ξ, there exists an $N_0(\xi)$ such that for all $N \geq N_0(\xi)$,

$$\left\|\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right\|_{\mathrm{Op}} \leq \varepsilon_N = (1+\varepsilon)\sqrt{2\frac{M}{N}\log\log N}.$$

Therefore, for $N \geq N_0(\xi)$,

$$\Phi(x/\sqrt{a}) \geq \frac{1}{2a}\|x\|_2^2 - \frac{1}{4a}(1+\varepsilon_N)\|x\|_2^2 - \frac{1}{\beta}\,f_{\mathrm{CW}}(2\beta).$$

Set $r_0 = 3\sqrt{f_{\mathrm{CW}}(3)}$ and assume that $N \geq N_0(\xi)$ is large enough to satisfy $\varepsilon_N \leq 1/5$ and $5/6 \leq \beta_N \leq 3/2$. Then, we find for $\mathbb{P}$-almost all ξ and all $x \in B(0,\sqrt{a}r_0)^{\mathrm{c}}$ that

$$-N\beta\Phi(x/\sqrt{a}) \leq -\frac{N}{6a}\|x\|_2^2 + N\,f_{\mathrm{CW}}(3) \leq -\frac{N}{18a}\|x\|_2^2.$$

Therefore, for $N \geq N_0(\xi)$ large enough,

$$\begin{aligned}\int_{B(0,\sqrt{a}r_0)^{\mathrm{c}}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx &\leq \int_{B(0,\sqrt{a}r_0)^{\mathrm{c}}} \exp\left\{-\frac{N}{18a}\|x\|_2^2\right\}dx\\ &\leq \exp\{-Nr_0^2/36\}\int_{\mathbb{R}^M}\exp\left\{-\frac{N}{36a}\|x\|_2^2\right\}dx\\ &= \left(\frac{36a\pi}{N}\right)^{M/2}\exp\{-Nr_0^2/36\}. \qquad (3.15)\end{aligned}$$

We need a bound similar to (3.15) holding with r_0 replaced by an arbitrary $r \in (0,r_0)$. Such a bound will be an immediate consequence of the following lemma.

Lemma 3.6. *For all $0 < r < r_0$,*

$$\liminf_{N\to\infty}\inf_{x\in B(0,r_0)\setminus B(0,r)}\Phi_{N,\beta_N}(x) > 0$$

$\mathbb{P}$-almost surely.

Once we have proved the lemma, we know that for $\mathbb{P}$-almost all ξ, there exist a $\delta(\xi,r,r_0) > 0$ and an $\bar{N}_1(\xi,r,r_0)$ such that $\Phi(x) > \delta(\xi,r,r_0)$ holds for all $x \in B(0,r_0)\setminus B(0,r)$ and for all $N \geq \bar{N}_1(\xi,r,r_0)$, which implies

$$\begin{aligned}&\int_{B(0,\sqrt{a}r_0)\setminus B(0,\sqrt{a}r)}\exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx\\ &\qquad\leq \exp\{-N\beta\delta(\xi,r,r_0)\}\,\mathrm{Vol}(B(0,\sqrt{a}r_0))\\ &\qquad= (ar_0^2)^{M/2}\,\mathrm{Vol}(B(0,1))\exp\{-N\beta\delta(\xi,r,r_0)\}\end{aligned}$$

for $\mathbb{P}$-almost all ξ and all $N \geq \bar{N}_1(\xi, r, r_0)$. So, for $\mathbb{P}$-almost all ξ, there exists an $N_1(\xi, r, r_0)$ such that for all $N \geq N_1(\xi, r, r_0)$,

$$\int_{B(0,\sqrt{a}r_0)\setminus B(0,\sqrt{a}r)} \exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx \leq \exp\{-N\beta\delta(\xi, r, r_0)/2\}. \tag{3.16}$$

By (3.15) and (3.16), we know that for arbitrary $r > 0$,

$$\begin{aligned}\int_{B(0,\sqrt{a}r)^c} &\exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx \\ &\leq \left(\frac{36a\pi}{N}\right)^{M/2} \exp\{-Nr_0^2/36\} + \exp\{-N\beta\delta(\xi, r, r_0)/2\}\end{aligned} \tag{3.17}$$

for $\mathbb{P}$-almost all ξ and all $N \geq N_1(\xi, r, r_0) \vee N_0(\xi)$.

The proof of Lemma 3.6 is a slight modification of the proof of [12, Lemma 4.1], where the case of fixed temperature is considered.

Proof of Lemma 3.6. First of all recall the self-averaging property [12, Lemma 4.2]: For all $\tau > 0$ and all Lipschitz continuous functions $f : \mathbb{R} \to \mathbb{R}$,

$$\lim_{N\to\infty} \sup_{\|x\|_2 \leq \tau} \left| \frac{1}{N} \sum_{i=1}^{N} f(\langle x, \xi_i\rangle) - \mathbb{E} f(\langle x, \xi_1\rangle) \right| = 0$$

holds $\mathbb{P}$-almost surely.

Applied to the Lipschitz continuous function $\log\cosh$, this self-averaging property yields

$$\lim_{N\to\infty} \sup_{\|x\|_2 \leq r_0} \left| \Phi(x) - \frac{1}{2}\|x\|_2^2 + \frac{1}{\beta}\mathbb{E}\log\cosh(\beta\langle x, \xi_1\rangle) \right| = 0$$

$\mathbb{P}$-almost surely. Therefore, it suffices to show that

$$\liminf_{N\to\infty} \inf_{x \in B(0,r_0)\setminus B(0,r)} \mathbb{E}\left\{ \frac{1}{2}\langle x, \xi_1\rangle^2 - \frac{1}{\beta}\log\cosh(\beta\langle x, \xi_1\rangle) \right\} > 0. \tag{3.18}$$

Because $\beta \to 1$, (3.18) is satisfied, whenever

$$\begin{aligned}\liminf_{N\to\infty} &\inf_{y \in B(0,\beta r_0)\setminus B(0,\beta r)} \mathbb{E}\left\{ \frac{1}{2\beta^2}\langle y, \xi_1\rangle^2 - \frac{1}{\beta}\log\cosh(\langle y, \xi_1\rangle) \right\} \\ &\geq \liminf_{N\to\infty} \inf_{y \in B(0,2r_0)\setminus B(0,r/2)} \left(\frac{1}{\beta}\mathbb{E}\left\{ \frac{1}{2}\langle y, \xi_1\rangle^2 - \log\cosh(\langle y, \xi_1\rangle) \right\} - \frac{\beta - 1}{2\beta^2}\|y\|_2^2 \right) \\ &= \inf_{y \in B(0,2r_0)\setminus B(0,r/2)} \mathbb{E}\left\{ \frac{1}{2}\langle y, \xi_1\rangle^2 - \log\cosh(\langle y, \xi_1\rangle) \right\} > 0.\end{aligned}$$

Now, the continuous function

$$\phi(t) = t^2/2 - \log\cosh t, \qquad t \in \mathbb{R},$$

attains its unique minimum at $t = 0$. The fact that $\langle y, \xi_i\rangle$ is a (finite) Rademacher average (see [18, Chapter I.4], for instance), implies

$$\mathbb{P}(|\langle y, \xi_i\rangle| \geq \tfrac{1}{8}\|y\|_2) > 1/3$$

(cf. [12, Lemma 4.3]), so that

$$\inf_{y \in B(0,2r_0)\setminus B(0,r/2)} \mathbb{E}\,\phi(\langle y, \xi_i\rangle) > 0,$$

because there is a set of positive $\mathbb{P}$-measure, on which ϕ is bounded away from its unique minimum at zero. □

The intermediate region. It remains to investigate the intermediate region. For $x \in W^N_{R,r} = B(0,\sqrt{a}r) \setminus B(0,R)$, we consider again the Taylor expansion of Φ, cf. (3.8). But note that the bound on the remainder (3.6) has to be replaced by

$$|R_{N,\beta}(x/\sqrt{a},\xi)| \le \frac{1}{N\beta}\mathcal{O}(1)NM^3\left\|\frac{x}{\sqrt{a}}\right\|_2^6 \le \frac{1}{N\beta}\mathcal{O}_M(r^2)\frac{N}{a^2}\|x\|_2^4.$$

On $D_{N,L,\delta}$,

$$\begin{aligned}-N\beta\Phi(x/\sqrt{a}) \;\le\; & \gamma\frac{\sqrt{N}\beta}{2a}\|x\|_2^2 + \frac{\sqrt{N}\beta^2}{2a}M^2L\|x\|_2^2 - \frac{N\beta^4}{12a^2}\|x\|_4^4 \\ & + \frac{N\beta^4\delta}{a^2}\|x\|_2^4\left[\frac{M^2}{3}+\frac{M^3}{2}+\frac{M^4}{12}\right] + \frac{N}{a^2}\mathcal{O}_M(r^2)\|x\|_2^4.\end{aligned}$$

By now, we need to distinguish two cases: either $\gamma \to -\infty$, or not. In the cases $\gamma \to 0$ and $\gamma_N \equiv \gamma \neq 0$, we have $a = \sqrt{N}$, and therefore, by the estimate $\|x\|_2^2 \le \sqrt{M}\|x\|_4^2$,

$$-N\beta\Phi(x/\sqrt{a}) \le \gamma\frac{\beta}{2}\sqrt{M}\|x\|_4^2 - \frac{\beta^4}{12}\|x\|_4^4 + \mathcal{O}_M(L)\|x\|_4^2 + \mathcal{O}_M(\delta)\|x\|_4^4 + \mathcal{O}_M(r^2)\|x\|_4^2.$$

Choose now N large enough to satisfy $\beta^4 \ge 2/3$ and choose $\delta > 0$ and $r > 0$ small enough, so that

$$-N\beta\Phi(x/\sqrt{a}) \le -\frac{1}{20}\|x\|_4^4 + (\mathcal{O}_M(\gamma) + \mathcal{O}_M(L))\|x\|_4^2.$$

Then, the integral

$$\begin{aligned}&\int_{W^N_{R,r}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx \\ &\qquad \le \int_{B(0,R)^c} \exp\left\{-\frac{1}{20}\|x\|_4^4 + (\mathcal{O}_M(\gamma)+\mathcal{O}_M(L))\|x\|_4^2\right\}dx \qquad (3.19)\end{aligned}$$

exists for every choice of γ and L, and, therefore,

$$\int_{W^N_{R,r}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx \longrightarrow 0 \qquad \text{as } R \to \infty.$$

It remains to consider the case $\gamma \to -\infty$, where the quadratic (Gaussian) term determines the asymptotic behaviour. In this case, $a = N(1-\beta) = -\sqrt{N}\gamma$, so that $\sqrt{N}\gamma/a \equiv -1$ and $\sqrt{N}/a = -1/\gamma \to 0$ as $N \to \infty$. Therefore, again using $\|x\|_2^2 \le \sqrt{M}\|x\|_4^2$, we obtain

$$-N\beta\Phi(x/\sqrt{a}) \le -\frac{\beta}{2}\|x\|_2^2 + \frac{\beta^2}{2|\gamma|}M^2L\|x\|_2^2 - \frac{1}{\gamma^2}\left[\frac{\beta^4}{12} + \mathcal{O}_M(\delta) + \mathcal{O}_M(r^2)\right]\|x\|_4^4$$

on $D_{N,L,\delta}$. Choosing δ and r small, we can achieve $\beta^4/12 + \mathcal{O}(\delta) + \mathcal{O}(r^2) \ge 0$, so that

$$-N\beta\Phi(x/\sqrt{a}) \le \left[-\frac{\beta}{2} + \frac{1}{|\gamma|}\mathcal{O}_M(L)\right]\|x\|_2^2.$$

For every fixed L, choosing N large enough, yields

$$-N\beta\Phi(x/\sqrt{a}) \le -\frac{1}{4}\|x\|_2^2,$$

which implies that

$$\int_{W_{R,r}^N} \exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx \le \int_{B(0,R)^c} \exp\{-\tfrac{1}{4}\|x\|_2^2\}\,dx \to 0 \quad \text{as } R\to\infty \tag{3.20}$$

for all $\xi \in D_{N,L,\delta}$.

So we have seen that in the intermediate region, for every L, we can choose $r>0$ and $\delta>0$ so small and N so large that on $D_{N,L,\delta}$,

$$\int_{W_{R,r}^N} \exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx \longrightarrow 0 \qquad \text{as } R\to\infty.$$

Convergence. To conclude the proof of Theorem 2.4 in the cases $\gamma\to 0$, $\gamma\to-\infty$ and $\gamma_N\equiv\gamma\neq 0$, we use the preceding results on the asymptotic behaviour of $\int f(x)\exp\{-N\beta\Phi(x/\sqrt{a})\}\,dx$ to investigate $\lim_{N\to\infty}\int g(\int f\,d\chi_{N,\beta,a})\,d\mathbb{P}$.

Recall that we are considering a continuous function $f:\mathbb{R}^M\to\mathbb{R}^k$ with compact support and a continuous function $g:\mathbb{R}^k\to\mathbb{R}$, where $k\in\mathbb{N}$ is arbitrary. On the closure of the ball $B(0,\|f\|_\infty)$, the function g is bounded and uniformly continuous. Therefore, given $\varepsilon>0$, there exists a $\kappa>0$ such that $|g(x)-g(y)|<\varepsilon$ for all $x,y\in B(0,\|f\|_\infty)$ satisfying $\|x-y\|_2<\kappa$. We shall show that we can choose the parameters L, δ and R in such a way that for all large enough N the following assertions (3.21), (3.22) and (3.23) hold:

We need a set $H_{N,L,\delta}$ of sufficiently large $\mathbb{P}$-measure on which the ξ-variables behave well. Therefore, we request that

$$\mathbb{P}(H_{N,L,\delta}^c)<\varepsilon, \tag{3.21}$$

where

$$H_{N,L,\delta} = D_{N,L,\delta}\cap F_N\cap G_L$$

with

$$G_L = \Big(\bigcup_{\mu<\nu}\{|\eta^{\mu,\nu}|>L\}\Big)^c.$$

The set F_N will be defined via (3.27) below.

Then we want to neglect the remainder terms in the Taylor expansion of Φ and we want to restrict to integrals over compact subsets of $\mathbb{R}^M$. This is summarized by requesting that

$$\Big\|\int f\,d\chi_{N,\beta,a} - \frac{\int f(x)\exp\{\Psi(x,\eta_N)\}\,dx}{\int 1_{B(0,R)}(x)\exp\{\Psi(x,\eta_N)\}\,dx}\Big\|_2 < \kappa, \tag{3.22}$$

for all $\xi\in H_{N,L,\delta}$, where the definition of Ψ depends on the case we are considering:

1. In the case $\gamma\to 0$,

$$\Psi(x,y) = -\frac{1}{12}\|x\|_4^4 - \frac{1}{2}\sum_{\mu<\nu}x_\mu^2x_\nu^2 + \sum_{\mu<\nu}y^{\mu,\nu}x_\mu x_\nu,$$

2. in the case $\gamma\to-\infty$,

$$\Psi(x,y) = -\frac{1}{2}\|x\|_2^2,$$

4. in the case $\gamma_N\equiv\gamma\neq 0$,

$$\Psi(x,y) = -\frac{1}{12}\|x\|_4^4 - \frac{1}{2}\sum_{\mu<\nu}x_\mu^2x_\nu^2 + \frac{\gamma}{2}\|x\|_2^2 + \sum_{\mu<\nu}y^{\mu,\nu}x_\mu x_\nu,$$

for $(x,y) \in \mathbb{R}^M \times \mathbb{R}^{M(M-1)/2}$.

Once we have taken the limit $N \to \infty$, we want to remove the restriction to compact subsets of $\mathbb{R}^M$ again. Therefore, we request in addition that

$$\left\| \frac{\int f(x) \exp\{\Psi(x,\eta)\}\,dx}{\int 1_{B(0,R)}(x) \exp\{\Psi(x,\eta)\}\,dx} - \frac{\int f(x) \exp\{\Psi(x,\eta)\}\,dx}{\int \exp\{\Psi(x,\eta)\}\,dx} \right\| < \kappa, \tag{3.23}$$

for all $\xi \in H_{N,L,\delta}$.

Having established (3.21), (3.22) and (3.23), we proceed as follows: Let

$$F(y) = \frac{\int f(x) \exp\{\Psi(x,y)\}\,dx}{\int 1_{B(0,R)}(x) \exp\{\Psi(x,y)\}\,dx}, \qquad y \in \mathbb{R}^{M(M-1)/2}.$$

Then, by (3.21), (3.22) and (3.23),

$$\begin{aligned}
&\left| \int g\biggl(\int f\,d\chi_{N,\beta,a}\biggr)\,d\mathbb{P} - \int g\biggl(\frac{\int f(x) \exp\{\Psi(x,\eta)\}\,dx}{\int \exp\{\Psi(x,\eta)\}\,dx}\biggr)\,d\mathbb{P} \right| \\
&\qquad \le 2\|g\|_\infty \mathbb{P}(H^{\mathrm{c}}_{N,L,\delta}) + 2\varepsilon + \left| \int g(F(\eta_N))\,d\mathbb{P} - \int g(F(\eta))\,d\mathbb{P} \right| \\
&\qquad \le 2(\|g\|_\infty + 1)\varepsilon + \left| \int g(F(\eta_N))\,d\mathbb{P} - \int g(F(\eta))\,d\mathbb{P} \right|
\end{aligned}$$

Thus it suffices to show that

$$\int g(F(\eta_N))\,d\mathbb{P} \to \int g(F(\eta))\,d\mathbb{P} \qquad \text{as } N \to \infty.$$

In the case $\gamma \to -\infty$, F is constant because $\Psi(x,y)$ does not depend on y. Therefore, there is nothing left to show in this case. In the other cases, let R be large enough to satisfy $\operatorname{supp} f \subset B(0,R)$. Note that

$$\mathbb{R}^{M(M-1)/2} \ni y \mapsto F(y) \in \mathbb{R}^k$$

is a bounded and continuous function, so that the weak convergence

$$\eta_N = \biggl(\frac{1}{\sqrt{N}} \sum_{i=1}^N \xi_i^\mu \xi_i^\nu\biggr)_{\mu<\nu} \Longrightarrow \mathcal{N}(0,\Sigma) \qquad \text{as } N \to \infty$$

implies

$$F(\eta_N) \Longrightarrow F(\eta) \qquad \text{as } N \to \infty.$$

By the definition of weak convergence,

$$\int g(F(\eta_N))\,d\mathbb{P} \to \int g(F(\eta))\,d\mathbb{P} \qquad \text{as } N \to \infty$$

follows.

It only remains to choose the parameters L, δ and R properly. Note that by (3.9),

$$\begin{aligned}
&\mathbb{P}(H^{\mathrm{c}}_{N,L,\delta}) \\
&\qquad = \mathbb{P}(D^{\mathrm{c}}_{N,L,\delta}) + \mathbb{P}(F^{\mathrm{c}}_N) + \mathbb{P}(G^{\mathrm{c}}_L) \\
&\qquad \le 2M^2[1 - \Phi_{\mathcal{N}(0,1)}(L) + C/\sqrt{N}] + \frac{2M^4}{N\delta^2} + \mathbb{P}(F^{\mathrm{c}}_N) + 2M^2[1 - \Phi_{\mathcal{N}(0,1)}(L)].
\end{aligned}$$

In a first step, we choose $L > 0$ so large that

$$4M^2[1 - \Phi_{\mathcal{N}(0,1)}(L)] < \varepsilon/2$$

and fix this L from now on.

In our investigation of the intermediate region we have seen (cf. (3.19) or (3.20), respectively) that for all small enough $\delta > 0$ and $r > 0$ and all large enough N, there exists an upper bound for the integral

$$\int_{W^N_{R,r}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx,$$

which depends only on R and L and tends to zero as $R \to \infty$. Given an arbitrary small $\bar\kappa > 0$, we fix a small $r > 0$ and choose $R > 0$ so large that

$$\int_{W^N_{R,r}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \le \bar\kappa \qquad \text{on } D_{N,L,\delta} \tag{3.24}$$

and

$$\int_{B(0,R)^{\mathrm{c}}} \exp\{\Psi(x,\eta)\}\, dx \le \bar\kappa \qquad \text{on } G_L.$$

The latter is possible because $x \mapsto \exp\{\Psi(x,\eta)\}$ is uniformly integrable for $\eta \in G_L$. Let us fix such an R.

Next, we return to our study of the inner region (3.10), (3.11) and (3.12). Choosing $\delta > 0$ small and N large enough, we can achieve

$$\begin{aligned} \mathrm{e}^{-\bar\kappa} \int f(x) \exp\{\Psi(x,\eta_N)\}\, dx &\le \int f(x) \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \\ &\le \mathrm{e}^{\bar\kappa} \int f(x) \exp\{\Psi(x,\eta_N)\}\, dx \end{aligned} \tag{3.25}$$

and

$$\begin{aligned} \mathrm{e}^{-\bar\kappa} \int 1_{B(0,R)}(x) \exp\{\Psi(x,\eta_N)\}\, dx &\le \int 1_{B(0,R)}(x) \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \\ &\le \mathrm{e}^{\bar\kappa} \int 1_{B(0,R)}(x) \exp\{\Psi(x,\eta_N)\}\, dx \end{aligned} \tag{3.26}$$

simultaneously on $D_{N,L,\delta}$.

Let F_N be the set of all ξ such that

$$\int_{B(0,\sqrt{a}r)^{\mathrm{c}}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \le \bar\kappa \tag{3.27}$$

holds. By (3.17),

$$\mathbb{P}(F_N) \nearrow 1, \qquad \text{as } N \to \infty.$$

Now, that our parameters are chosen, we can verify (3.21):

$$\mathbb{P}(H^{\mathrm{c}}_{N,L,\delta}) \le \frac{\varepsilon}{2} + 2M^2 \frac{C}{\sqrt{N}} + \frac{2M^4}{N\delta^2} + \mathbb{P}(F^{\mathrm{c}}_N) < \varepsilon$$

holds for all large enough N.

Let us now prove (3.22). Recall that

$$\int f\, d\chi_{N,\beta,a} = \frac{\int f(x) \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx}{\int \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx}.$$

By (3.24) and (3.27),

$$\int_{B(0,R)^{\mathrm{c}}} \exp\{-N\beta\Phi(x/\sqrt{a})\}\, dx \le 2\bar\kappa,$$

which implies by (3.25) and (3.26) that

$$\int f\,d\chi_{N,\beta,a} = \frac{\mathrm{e}^{\mathcal{O}(\bar\kappa)}\int f(x)\exp\{\Psi(x,\eta_N)\}\,dx}{\mathrm{e}^{\mathcal{O}(\bar\kappa)}\int 1_{B(0,R)}(x)\exp\{\Psi(x,\eta_N)\}\,dx + 2\bar\kappa}$$

for all $\xi \in H_{N,L,\delta}$. Now (3.22) follows as soon as we know that the integral in the denominator is bounded away from zero on $D_{N,L,\delta}$. But this is obvious because the integrand is bounded away from zero—uniformly for $\xi \in D_{N,L,\delta}$.

Similarly, (3.23) can be established: Just note that

$$\int 1_{B(0,R)}(x)\exp\{\Psi(x,\eta)\}\,dx > 0,$$

$\int f(x)\exp\{\Psi(x,\eta)\}\,dx$ is bounded, and by our choice of R,

$$\int_{B(0,R)^c}(x)\exp\{\Psi(x,\eta)\}\,dx \le \bar\kappa \qquad \text{on } G_L.$$

This finishes the proof of Theorem 2.4 in the cases $\gamma \to 0$ and $\gamma \to -\infty$ as well as $\gamma_N \equiv \gamma \neq 0$. □

4. The case $\beta_N - 1 \gg 1/\sqrt{N}$

In this section, we shall complete the proof of the main result which is Theorem 2.4. The remaining case is the one of β tending to one slowly from above, where we expect the weak central limit theorem to hold with a random centring $z_N(\xi)$ and scaling

$$\sqrt{a} = \sqrt{N}\sqrt{\frac{1-\beta(1-z^2)}{1-z^2}}.$$

The following lemma is the substitute for Lemma 3.4. It is a first step in showing that the convolution of the conditional probabilities $\tilde Q_{N,\beta}$ with the appropriate Gaussian distribution behaves asymptotically like a (random) measure with density proportional to

$$1_{B(0,R)}(x)\exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}.$$

Lemma 4.1. *Assume $\beta - 1 \gg 1/\sqrt{N}$ and let a denote the appropriate scaling of the centred overlap as above. Then the convolution*

$$\tilde\chi_{N,\beta,a} = \tilde Q_{N,\beta} * \mathcal{N}\Big(0, \frac{a}{N\beta}\mathrm{Id}\Big)$$

of the conditional probability $\tilde Q_{N,\beta}$ with the M-dimensional Gaussian distribution with mean zero and covariance matrix $\frac{a}{N\beta}\mathrm{Id}$ satisfies

$$\begin{aligned}&\int f\,d\tilde\chi_{N,\beta,a}\\ &\quad= \frac{\int f(x)1_{B(0,R)}(x)\exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\,dx + \mathcal{O}(\|f\|_\infty F_{N,\beta,R}(\xi))}{\int 1_{B(0,R)}(x)\exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\,dx + \mathcal{O}(F_{N,\beta,R}(\xi))}\end{aligned}$$

for all ξ, all Borel measurable and bounded $f : \mathbb{R}^M \to \mathbb{R}$ and all $R > 0$. The remainder $F_{N,\beta,R}(\xi)$ equals

$$\begin{aligned} F_{N,\beta,R}(\xi) &= \int_{\mathbb{R}^M} 1_{V^N_{R,\tilde{r}}}(x) \exp\{-N\beta\Phi(x/\sqrt{a} + z_N)\}\, dx \\ &\quad + \left(\frac{4\pi a}{N\beta}\right)^{M/2} Z_{N,\beta}(\xi) \exp\{-c_1 N(\beta - 1)\} \end{aligned}$$

with

$$V^N_{R,\tilde{r}} = B(0, \sqrt{a}\tilde{r}z) \setminus B(0, R),$$

with an arbitrary $\tilde{r} \in (r, 1/\sqrt{2})$ and some constant $c_1 = c_1(r, \tilde{r}) > 0$ which depends only on r and $\tilde{r}$.

Remark 4.2. *The $\mathcal{O}$-term in the preceding lemma is deterministic in the following sense: There exists an $\bar{N} \in \mathbb{N}$ depending only on r, R and the speed at which β tends to one and there exists a constant $C > 0$ such that for all $N \geq \bar{N}$ and all ξ*

$$\mathcal{O}(F_{N,\beta,R}(\xi)) \leq C F_{N,\beta,R}(\xi).$$

We shall show later on, that $F_{N,\beta,R}(\xi)$ becomes small compared to the main term on a set of large $\mathbb{P}$-measure.

Proof of Lemma 4.1. Let $f : \mathbb{R}^M \to \mathbb{R}$ be Borel measurable (not necessarily continuous at the moment) and bounded. Then, by the definitions of $\tilde{\chi}_{N,\beta,a}$ and $\tilde{Q}_{N,\beta}$,

$$\begin{aligned} \int f\, d\tilde{\chi}_{N,\beta,a} &= \iint f(u+v)\tilde{Q}_{N,\beta}(du)\mathcal{N}\big(0, \frac{a}{N\beta}\mathrm{Id}\big)(dv) \\ &= \left(\frac{N\beta}{2\pi a}\right)^{M/2} [Z_{N,\beta}(\xi)\varrho_{N,\beta,\xi}(\|m_N - z_N\|_2 < rz)]^{-1} \\ &\quad \times \iint f(\sqrt{a}[m_N(\sigma) - z_N] + v) 1_{\{\|m_N - z_N\|_2 < rz\}}(\sigma) \\ &\qquad \times \exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\} \exp\left\{-\frac{N\beta}{2a}\|v\|_2^2\right\} \mathrm{P}(d\sigma)\, dv. \end{aligned}$$

Replacing $\sqrt{a}[m_N - z_N] + v$ by x yields

$$\begin{aligned} \int f\, d\tilde{\chi}_{N,\beta,a} &= \left(\frac{N\beta}{2\pi a}\right)^{M/2} [Z_{N,\beta}(\xi)\varrho_{N,\beta,\xi}(\|m_N - z_N\|_2 < rz)]^{-1} \\ &\quad \times \iint f(x) 1_{\{\|m_N - z_N\|_2 < rz\}}(\sigma) \exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\} \\ &\qquad \times \exp\left\{-\frac{N\beta}{2a}\|x - \sqrt{a}[m_N(\sigma) - z_N]\|_2^2\right\} \mathrm{P}(d\sigma)\, dx. \end{aligned}$$

$\tilde{\chi}_{N,\beta,a}$ being a probability measure,

$$\begin{aligned}\int f\,d\tilde{\chi}_{N,\beta,a} &= \iint f(x)1_{\{\|m_N-z_N\|_2<rz\}}(\sigma)\exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\}\\ &\qquad\times\exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}}-[m_N(\sigma)-z_N]\right\|_2^2\right\}\mathrm{P}(d\sigma)\,dx\ \Big/\\ &\quad\iint 1_{\{\|m_N-z_N\|_2<rz\}}(\sigma)\exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\}\\ &\qquad\times\exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}}-[m_N(\sigma)-z_N]\right\|_2^2\right\}\mathrm{P}(d\sigma)\,dx\end{aligned}$$

is immediate. Next note, that the main contribution to the Gaussian integral in the nominator as well as in the denominator arises from those x which are close to $\sqrt{a}[m_N - z_N]$. As we shall see, the condition $\|m_N - z_N\|_2 < rz$ can be replaced by a similar condition on $x/\sqrt{a}$, namely $\|x/\sqrt{a}\|_2 < \tilde{r}z$ with $r < \tilde{r} < 1/\sqrt{2}$. Then we shall strengthen this condition to $\|x\|_2 < R$ with an arbitrary $R > 0$. To do so, we consider

$$\begin{aligned}&\iint f(x)1_{\{\|m_N-z_N\|_2<rz\}}(\sigma)\exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\}\\ &\qquad\times\exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}}-[m_N(\sigma)-z_N]\right\|_2^2\right\}\mathrm{P}(d\sigma)\,dx = \mathrm{(I)}+\mathrm{(II)}+\mathrm{(III)}-\mathrm{(IV)},\end{aligned}$$

where

$$\begin{aligned}\mathrm{(I)} &= \iint f(x)1_{B(0,R)}(x)\exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\}\\ &\qquad\times\exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}}-[m_N(\sigma)-z_N]\right\|_2^2\right\}\mathrm{P}(d\sigma)\,dx\end{aligned}$$

with $R > 0$ is the main part, whereas all the other terms

$$\begin{aligned}\mathrm{(II)} &= \iint f(x)1_{V_{R,\tilde{r}}^N}(x)1_{\{\|m_N-z_N\|_2<rz\}}(\sigma)\exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\}\\ &\qquad\times\exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}}-[m_N(\sigma)-z_N]\right\|_2^2\right\}\mathrm{P}(d\sigma)\,dx\end{aligned}$$

with $\tilde{r} \in (r, 1/\sqrt{2})$,

$$\begin{aligned}\mathrm{(III)} &= \iint f(x)1_{B(0,\sqrt{a}\tilde{r}z)^c}(x)1_{\{\|m_N-z_N\|_2<rz\}}(\sigma)\exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\}\\ &\qquad\times\exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}}-[m_N(\sigma)-z_N]\right\|_2^2\right\}\mathrm{P}(d\sigma)\,dx\end{aligned}$$

and, finally,

$$\begin{aligned}\mathrm{(IV)} &= \iint f(x)1_{B(0,R)}(x)1_{\{\|m_N-z_N\|_2\geq rz\}}(\sigma)\exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\}\\ &\qquad\times\exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}}-[m_N(\sigma)-z_N]\right\|_2^2\right\}\mathrm{P}(d\sigma)\,dx\end{aligned}$$

become small compared to (I) as $N \to \infty$. In fact,

$$\begin{aligned}
&\text{(III)}\\
&\le \|f\|_\infty \exp\left\{-\frac{N\beta}{4}[\tilde r - r]^2 z^2\right\}\\
&\quad \times \iint \exp\left\{\frac{N\beta}{2}\|m_N(\sigma)\|_2^2\right\} \exp\left\{-\frac{N\beta}{4}\left\|\frac{x}{\sqrt{a}} - [m_N(\sigma) - z_N]\right\|_2^2\right\} \mathrm{P}(d\sigma)\, dx\\
&= \|f\|_\infty \left(\frac{4\pi a}{N\beta}\right)^{M/2} Z_{N,\beta}(\xi) \exp\left\{-\frac{N\beta}{4}[\tilde r - r]^2 z^2\right\}.
\end{aligned}$$

Now, let N be large enough to satisfy $z^2 \ge 2(\beta - 1)$, which is possible by the bound (2.3). Then,

$$\text{(III)} \le \|f\|_\infty \left(\frac{4\pi a}{N\beta}\right)^{M/2} Z_{N,\beta}(\xi) \exp\{-\tilde c_1 N(\beta - 1)\}$$

follows with $\tilde c_1 = [\tilde r - r]^2/2$.

To estimate (IV), we use that for $\|x\|_2 < R$, $\|m_N - z_N\|_2 \ge rz$, and large enough N (depending on our choice of R and r),

$$\|x/\sqrt{a} - [m_N - z_N]\|_2 \ge rz - R/\sqrt{a} \ge \sqrt{\beta - 1}\, r$$

by the bounds $z^2 \ge 2(\beta - 1)$ and $a \ge N(\beta - 1)$, cf. (2.3) and (2.4). So for large enough N, we find

$$\begin{aligned}
\text{(IV)} \;&\le\; \|f\|_\infty \exp\left\{-\frac{N\beta}{4}(\beta - 1) r^2\right\}\\
&\qquad \times \iint \exp\left\{-\frac{N\beta}{4}\left\|\frac{x}{\sqrt{a}} - [m_N(\sigma) - z_N]\right\|_2^2\right\} \exp\left\{\frac{N\beta}{2}\|m_N\|_2^2\right\} \mathrm{P}(d\sigma)\, dx\\
&\le\; \|f\|_\infty \left(\frac{4\pi a}{N\beta}\right)^{M/2} Z_{N,\beta}(\xi) \exp\left\{-\bar c_1 N(\beta - 1)\right\}
\end{aligned}$$

with $\bar c_1 = r^2/4$.

Obviously,

$$\begin{aligned}
\text{(II)} \;&\le\; \|f\|_\infty \iint 1_{V_{R,\tilde r}^N}(x) \exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}} + z_N\right\|_2^2\right\}\\
&\qquad \times \exp\left\{N\beta\left\langle \frac{x}{\sqrt{a}} + z_N, m_N\right\rangle\right\} \mathrm{P}(d\sigma)\, dx.
\end{aligned}$$

Performing the integration with respect to P, we find that the right-hand side equals

$$\|f\|_\infty \int 1_{V_{R,\tilde r}^N}(x) \exp\{-N\beta \Phi_{N,\beta}(x/\sqrt{a} + z_N)\}\, dx,$$

where $\Phi_{N,\beta}$ is given by (3.3).

The main contribution arises from

$$\begin{aligned}
\text{(I)} \;&=\; \iint f(x) 1_{B(0,R)}(x) \exp\left\{-\frac{N\beta}{2}\left\|\frac{x}{\sqrt{a}} + z_N\right\|_2^2\right\}\\
&\qquad \times \exp\left\{N\beta\left\langle \frac{x}{\sqrt{a}} + z_N, m_N\right\rangle\right\} \mathrm{P}(d\sigma)\, dx,
\end{aligned}$$

and, again, performing the integration with respect to P yields the desired expression, namely,

$$(\mathrm{I}) = \int f(x) 1_{B(0,R)}(x) \exp\{-N\beta\Phi_{N,\beta}(x/\sqrt{a} + z_N)\}\, dx.$$

□

Proof of Theorem 2.4 (continued). Recall from the beginning of Section 3 that we want to show that the Hubbard–Stratonovich transform $\tilde{\chi}_{N,\beta,a}$ of the conditional probability $\tilde{Q}_{N,\beta}$ converges, as N tends to infinity, in $\mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^M))$ towards the Dirac measure concentrated in the standard Gaussian distribution on $\mathbb{R}^M$. We have seen that it suffices to show

$$\int g\left(\int f\, d\tilde{\chi}_{N,\beta,a}\right) d\mathbb{P} \to g\left(\int f\, d\mathcal{N}(0, \mathrm{Id})\right), \qquad \text{as } N \to \infty,$$

for all $k \in \mathbb{N}$, all continuous $f : \mathbb{R}^M \to \mathbb{R}^k$ with compact support and all continuous $g : \mathbb{R}^k \to \mathbb{R}$. By the preceding lemma, we need to investigate the asymptotic behaviour of the integral

$$\int f(x) 1_{B(0,R)}(x) \exp\{-N\beta\Phi(x/\sqrt{a} + z_N)\}\, dx. \tag{4.1}$$

Furthermore, we need to show that the contribution arising from the integral

$$\int_{B(0,R)^c} \exp\{-N\beta\Phi(x/\sqrt{a} + z_N)\}\, dx$$

is negligible compared to (4.1).

The treatment of the case β tending to one slowly from above is more involved than the other cases because it does not suffice to consider the Taylor expansion of Φ around zero in this case. Because the minimizing points of Φ tend to zero only slowly as $\gamma = \sqrt{N}(\beta - 1) \to \infty$, we had to condition on the set where the overlap is close to one of the possible limiting points in the statement of Theorem 2.4, and, similarly, in the proof, we need to consider the Taylor expansion around the corresponding minimizing point—or at least around a point which is close enough to the minimizing point. We choose the latter possibility and use the second-order Taylor expansion of Φ around ze_1.

The main contribution. The second-order Taylor expansion of Φ around ze_1 is given by

$$\begin{aligned}\Phi(x) \;=\; & -\frac{1}{\beta} f_{\mathrm{CW}}(\beta) - \left\langle \left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right) ze_1, x - ze_1 \right\rangle \\ & + \frac{1}{2}\left\langle x - ze_1, \left([1 - \beta(1 - z^2)]\mathrm{Id} - \beta(1 - z^2)\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)\right)[x - ze_1] \right\rangle \\ & + R_{N,\beta}(x, \xi),\end{aligned} \tag{4.2}$$

where we used

$$z = \tanh(\beta z) \qquad \text{as well as} \qquad \cosh^2(\beta z) = \frac{1}{1 - z^2},$$

and $f_{\mathrm{CW}}(\beta)$ denotes the free energy in the Curie–Weiss model at temperature $1/\beta$ as defined in (3.14). The remainder $R_{N,\beta}$ equals

$$R_{N,\beta}(x,\xi) = \frac{\beta^2}{3N}\sum_{i=1}^{N}\frac{\tanh}{\cosh^2}(\beta\langle ze_1+\theta(x-ze_1),\xi_i\rangle)\langle x-ze_1,\xi_i\rangle^3. \tag{4.3}$$

Recall from (2.1) the definition of the centring

$$z_N = z_N(\xi) = \left[\mathrm{Id} + \frac{1}{1-\beta(1-z^2)}\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)\right]ze_1.$$

The trivial estimate $|\tanh(x)|/\cosh(x)^2 \le |x|$ yields the following bound on the remainder:

$$\begin{aligned}
&N\beta|R_{N,\beta}(x/\sqrt{a}+z_N,\xi)| \\
&\quad\le \mathcal{O}_M(1)\frac{N}{a^{3/2}}\|x+\sqrt{a}[z_N-ze_1]\|_2^3\left(z+\frac{1}{\sqrt{a}}\|x+\sqrt{a}[z_N-ze_1]\|_2\right) \\
&\quad= \mathcal{O}_M(1)\left[\frac{Nz}{a^{3/2}}\|x+\sqrt{a}[z_N-ze_1]\|_2^3+\frac{N}{a^2}\|x+\sqrt{a}[z_N-ze_1]\|_2^4\right].
\end{aligned}$$

Now recall from (2.3) and (2.4) that

$$z\sim\sqrt{3(\beta-1)} \qquad \text{and} \qquad a = N\frac{1-\beta(1-z^2)}{1-z^2}\sim 2N(\beta-1) \qquad \text{as } N\to\infty,$$

which yields

$$\frac{Nz}{a^{3/2}}\sim\frac{\sqrt{3/8}}{\sqrt{N}(\beta-1)} = \frac{\sqrt{3/8}}{\gamma}\to 0$$

and

$$\frac{N}{a^2}\sim\frac{1}{4N(\beta-1)^2} = \frac{1}{4\gamma^2}\to 0.$$

In addition,

$$\begin{aligned}
\sqrt{a}[z_N-ze_1] &= \sqrt{N}\sqrt{\frac{1-\beta(1-z^2)}{1-z^2}}\frac{1}{1-\beta(1-z^2)}\left(\frac{1}{N}\xi^{\mathrm{T}}\xi-\mathrm{Id}\right)ze_1 \\
&= \frac{z}{\sqrt{(1-z^2)(1-\beta(1-z^2))}}\sqrt{N}\left(\frac{1}{N}\xi^{\mathrm{T}}\xi-\mathrm{Id}\right)e_1,
\end{aligned}$$

where

$$\frac{z}{\sqrt{(1-z^2)(1-\beta(1-z^2))}}\sim\sqrt{3/2}$$

and $\sqrt{N}(\frac{1}{N}\xi^{\mathrm{T}}\xi-\mathrm{Id})e_1$ is the vector in $\mathbb{R}^M$ with its first component being zero and its μth component given by $\eta_N^{1,\mu} = (1/\sqrt{N})\sum_{i=1}^N\xi_i^\mu\xi_i^1$ for $\mu\in\{2,\dots,M\}$. On the set

$$A_{N,L} = \left(\bigcup_{\mu\neq\nu}A_{N,L}^{\mu,\nu}\right)^{\mathrm{c}} = \left(\bigcup_{\mu\neq\nu}\left\{\left|\frac{1}{\sqrt{N}}\sum_{i=1}^N\xi_i^\mu\xi_i^\nu\right|>L\right\}\right)^{\mathrm{c}},$$

$\sqrt{N}(\frac{1}{N}\xi^{\mathrm{T}}\xi-\mathrm{Id})e_1$ is bounded by $\sqrt{M-1}L$. So we have seen that, for $\|x\|_2<R$ with some $R>1$ and $\xi\in A_{N,L}$,

$$N\beta|R_{N,\beta}(x/\sqrt{a}+z_N,\xi)|\le\frac{1}{\gamma}\mathcal{O}_M(R^4+L^4)\to 0, \qquad \text{as } N\to\infty, \tag{4.4}$$

which means that we may neglect the remainder on $A_{N,L}$ when discussing the inner region $\|x\|_2 < R$.

Now, on $A_{N,L}$,

$$\begin{aligned}
-N\beta\Phi(x/\sqrt{a}+z_N) &\\
&= N f_{\mathrm{CW}}(\beta) + \frac{\sqrt{N}\beta z}{\sqrt{a}}\left\langle \sqrt{N}\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)e_1, \sqrt{a}[z_N - ze_1]\right\rangle \\
&\quad - \frac{1}{2}\langle\sqrt{a}[z_N - ze_1], \mathbb{M}\sqrt{a}[z_N - ze_1]\rangle \\
&\quad + \frac{\beta^2 z}{\sqrt{N}}\left(\frac{1-z^2}{1-\beta(1-z^2)}\right)^{3/2}\left\langle N\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)^2 e_1, x\right\rangle \\
&\quad - \frac{1}{2}\langle x, \mathbb{M}\, x\rangle - N\beta R_{N,\beta}(x/\sqrt{a}+z_N, \xi), \qquad (4.5)
\end{aligned}$$

where

$$\mathbb{M} = \frac{N\beta}{a}\left([1-\beta(1-z^2)]\mathrm{Id} - \beta(1-z^2)\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)\right).$$

For $\xi \in A_{N,L}$, the matrix $\mathbb{M}$ approaches the identity matrix in operator norm as $N \to \infty$. To see this, just note that

$$\frac{N\beta}{a}[1-\beta(1-z^2)] = \beta(1-z^2) \to 1, \qquad \text{as } N \to \infty,$$

and that, for all $\mu < \nu$ and all $\xi \in A_{N,L}$,

$$\left|\left\langle e_\mu, \frac{N\beta^2}{a}(1-z^2)\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)e_\nu\right\rangle\right| \sim \frac{|\eta_N^{\mu,\nu}|}{2\gamma} \le \frac{L}{2\gamma} \to 0, \qquad \text{as } N \to \infty. \quad (4.6)$$

Let

$$\begin{aligned}
x_N^\mu &= x_N^\mu(\xi) \\
&= N f_{\mathrm{CW}}(\beta) + \frac{\sqrt{N}\beta z}{\sqrt{a}}\left\langle \sqrt{N}\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)e_\mu, \sqrt{a}[z_N^\mu - ze_\mu]\right\rangle \\
&\quad - \frac{1}{2}\langle\sqrt{a}[z_N^\mu - ze_\mu], \mathbb{M}\sqrt{a}[z_N^\mu - ze_\mu]\rangle,
\end{aligned}$$

where z_N^μ is given by

$$z_N^\mu = z_N^\mu(\xi) = \left[\mathrm{Id} + \frac{1}{1-\beta_N(1-z(\beta_N)^2)}\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)\right] z(\beta_N)e_\mu,$$

cf. the definition (2.1) of $z_N = z_N^1$. Then $x_N = x_N^1$ denotes those terms in (4.5), which do not depend on x. For $\xi \in A_{N,L}$ and $\|x\|_2 < R$, we find by our bound (4.4) on the remainder that

$$\begin{aligned}
-N\beta\Phi(x/\sqrt{a}+z_N) &\\
&= x_N + \frac{\beta^2 z}{\sqrt{N}}\left(\frac{1-z^2}{1-\beta(1-z^2)}\right)^{3/2}\mathcal{O}\left(\left\|\sqrt{N}\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)\right\|_{\mathrm{Op}}^2\right)\|x\|_2 \\
&\quad - \frac{1}{2}\|x\|_2^2 + \mathcal{O}(\|\mathbb{M} - \mathrm{Id}\|_{\mathrm{Op}})\|x\|_2^2 + \frac{1}{\gamma}\mathcal{O}_M(R^4 + L^4). \qquad (4.7)
\end{aligned}$$

Because each component of $\sqrt{N}(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id})$ is bounded by L on $A_{N,L}$ and because

$$\frac{\beta^2 z}{\sqrt{N}}\left(\frac{1-z^2}{1-\beta(1-z^2)}\right)^{3/2} \sim \sqrt{\frac{3(\beta-1)}{N}}\left(\frac{1}{2(\beta-1)}\right)^{3/2} \sim \frac{\sqrt{3/8}}{\gamma},$$

we know that the second term on the right-hand side of (4.7) tends to zero as $N \to \infty$ for $\|x\|_2 < R$ and $\xi \in A_{N,L}$. Furthermore, by (4.6), on $A_{N,L}$,

$$\begin{aligned} \|\mathbb{M} - \mathrm{Id}\|_{\mathrm{Op}} &\leq \left|\frac{N\beta}{a}[1-\beta(1-z^2)] - 1\right| + \mathcal{O}_M(L/\gamma) \\ &= 1 - \beta(1-z^2) + \mathcal{O}_M(L/\gamma) \\ &\sim 2(\beta-1) + \mathcal{O}_M(L/\gamma) \to 0, \qquad \text{as } N \to \infty. \end{aligned}$$

Therefore, for all $\xi \in A_{N,L}$ and all $\|x\|_2 < R$,

$$-N\beta\Phi(x/\sqrt{a} + z_N) = x_N - \frac{1}{2}\|x\|_2^2 + \mathcal{O}_M\left(\frac{L^2R}{\gamma} + (\beta-1)R^2 + \frac{LR^2}{\gamma} + \frac{R^4}{\gamma} + \frac{L^4}{\gamma}\right).$$

Once $R > 0$ and $L > 0$ are fixed, we obtain for continuous $f : \mathbb{R}^M \to \mathbb{R}^k$ with compact support contained in $B(0,R)$

$$\begin{aligned} &\int f(x)\exp\{-N\beta\Phi(x/\sqrt{a} + z_N)\}\,dx \\ &\qquad = \mathrm{e}^{x_N}\exp\{\mathcal{O}_M((\beta-1) + 1/\gamma)\}\int f(x)\exp\{-\tfrac{1}{2}\|x\|_2^2\}\,dx \end{aligned}$$

and

$$\begin{aligned} &\int_{B(0,R)} \exp\{-N\beta\Phi(x/\sqrt{a} + z_N)\}\,dx \\ &\qquad = \mathrm{e}^{x_N}\exp\{\mathcal{O}_M((\beta-1) + 1/\gamma)\}\int_{B(0,R)} \exp\{-\tfrac{1}{2}\|x\|_2^2\}\,dx \end{aligned} \tag{4.8}$$

for all $\xi \in A_{N,L}$. This concludes the investigation of the integral over $\{\|x\|_2 < R\}$ which determines the asymptotic behaviour of $\int f\,d\tilde{\chi}_{N,\beta,a}$.

An upper bound for the partition function. Next, we shall show how to obtain an upper bound on the partition function $Z_{N,\beta}(\xi)$, valid for all $\xi \in A_{N,L}$. We follow closely the presentation in [3] and [4], where a concentration-of-measure result in the case $M/N \to \alpha > 0$ is proved. First note that, by the usual Gaussian integration,

$$\begin{aligned} Z_{N,\beta}(\xi) &= 2^{-N}\sum_{\sigma\in\{-1,+1\}^N} \exp\{-\beta H_{N,\xi}(\sigma)\} \\ &= \left(\frac{N\beta}{2\pi}\right)^{M/2}\int_{\mathbb{R}^M} \exp\{-N\beta\Phi(x)\}\,dx. \end{aligned}$$

To investigate the above integral, we split the domain of integration into four parts as in [3], namely the *region far out*

$$U_d^1 = \{\, x \in \mathbb{R}^M : |\|x\|_2 - z| \geq dz \,\}$$

with some arbitrarily fixed $d \in (0,1)$, the *outer region*

$$U_{d,\tau}^2 = \{\, x \in \mathbb{R}^M : |\|x\|_2 - z| < dz \,\} \cap \Bigg(\bigcup_{\substack{1\leq\mu\leq M,\\ s\in\{-1,+1\}}} B(sze_\mu, \tau z)\Bigg)^{\mathrm{c}}$$

with some small $\tau \in (0, 3/10)$ to be chosen later, the *intermediate region*

$$U^3_{d,\tau,R} = \{\, x \in \mathbb{R}^M : |\|x\|_2 - z| < dz \,\} \cap \bigcup_{\substack{1\le\mu\le M,\\ s\in\{-1,+1\}}} \big(B(sze_\mu, \tau z) \setminus B(sze_\mu, R/\sqrt{a})\big)$$

with $R > 0$ to be chosen large, and, finally, the *inner region*

$$U^4_{d,\tau,R} = \{\, x \in \mathbb{R}^M : |\|x\|_2 - z| < dz \,\} \cap \bigcup_{\substack{1\le\mu\le M,\\ s\in\{-1,+1\}}} B(sze_\mu, R/\sqrt{a}).$$

The region far out. For x from the region far out, we use the fact that $\Phi(x)$ is close to $\frac{1}{N}\sum_{i=1}^N \phi(\langle x, \xi_i\rangle)$, where

$$\phi(t) = \frac{t^2}{2} - \frac{1}{\beta}\log\cosh(\beta t), \qquad t \in \mathbb{R},$$

satisfies the lower bound

$$\phi(t) - \phi(z) \ge \frac{z^2}{12}(|t| - z)^2 \tag{4.9}$$

for all $t \in \mathbb{R}$. Doing so, we obtain for all $\xi \in A_{N,L}$

$$\begin{aligned}
\Phi(x) &\ge \phi(z) + \frac{z^2}{12N}\sum_{i=1}^N (|\langle x, \xi_i\rangle| - z)^2 - \frac{1}{2}\left\langle x, \left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)x\right\rangle \\
&\ge -\frac{1}{\beta} f_{\mathrm{CW}}(\beta) + \frac{z^2}{12}\left(\sqrt{\frac{1}{N}\sum_{i=1}^N \langle x, \xi_i\rangle^2} - z\right)^2 + \mathcal{O}_M\left(\frac{L}{\sqrt{N}}\right)\|x\|_2^2,
\end{aligned}$$

where we used the definition (3.14) of the free energy in the Curie–Weiss model, Schwarz' inequality and the fact that, for $\xi \in A_{N,L}$, each component of the matrix $\sqrt{N}(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id})$ is bounded by L. For $x \in U^1_d$, we know that

$$\|x\|_2 \le |\|x\|_2 - z| + z \le \left(1 + \frac{1}{d}\right)|\|x\|_2 - z|,$$

and, therefore, for $x \in U^1_d$ and $\xi \in A_{N,L}$,

$$\begin{aligned}
\Phi(x) \ge\ & -\frac{1}{\beta} f_{\mathrm{CW}}(\beta) \\
&+ \left[\frac{z^2}{12}\left(1 + \left(1 + \frac{1}{d}\right)\mathcal{O}_M\left(\frac{L}{\sqrt{N}}\right)\right)^2 + \mathcal{O}_M\left(\frac{L}{\sqrt{N}}\right)\left(1 + \frac{1}{d}\right)^2\right]|\|x\|_2 - z|^2.
\end{aligned}$$

As $z^2 \sim 3(\beta - 1) \gg 1/\sqrt{N}$,

$$\Phi(x) \ge -\frac{1}{\beta} f_{\mathrm{CW}}(\beta) + \frac{z^2}{20}|\|x\|_2 - z|^2$$

follows for all $x \in U_d^1$, all $\xi \in A_{N,L}$ and all large enough N, which in turn implies

$$\begin{aligned}
&\left(\frac{N\beta}{2\pi}\right)^{M/2} \int_{U_d^1} \exp\{-N\beta\Phi(x)\}\, dx \\
&\quad \le \left(\frac{N\beta}{2\pi}\right)^{M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\} \int_{U_d^1} \exp\left\{-\frac{N\beta z^2}{20} |\|x\|_2 - z|^2\right\} dx \\
&\quad \le \left(\frac{N\beta}{2\pi}\right)^{M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\} \exp\left\{-\frac{N\beta z^2}{40} d^2 z^2\right\} \\
&\qquad \times \int \exp\left\{-\frac{N\beta z^2}{40} |\|x\|_2 - z|^2\right\} dx \\
&\quad \le \left(\frac{N\beta}{2\pi}\right)^{M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\} \exp\{-d^2\gamma^2/10\} \\
&\qquad \times \int \exp\left\{-\frac{N\beta z^2}{40} \left(\frac{d}{d+1}\right)^2 \|x\|_2^2\right\} dx \\
&\quad \le \mathcal{O}_M(1)(\beta-1)^{-M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\} \exp\{-d^2\gamma^2/10\}
\end{aligned}$$

for all $\xi \in A_{N,L}$ and all large enough N. This concludes our treatment of the region far out.

The outer region. For the investigation of the outer region, we take advantage from the facts that x is bounded and that the number of patterns M remains finite which allows us to apply coarser estimates than those used in [4] or [12], for example. Note that

$$\Phi(x) = \mathbb{E}\phi(\langle x, \xi_1\rangle) - \frac{1}{N\beta} \sum_{i=1}^{N} \big[\log\cosh(\beta\langle x, \xi_i\rangle) - \mathbb{E}\log\cosh(\beta\langle x, \xi_1\rangle)\big].$$

For $\|x\|_2 \le (1+d)z$, $\xi \in A_{N,L}$ and large enough N, the second term is bounded by

$$\begin{aligned}
&\left|\frac{1}{N\beta} \sum_{i=1}^{N} \big[\log\cosh(\beta\langle x, \xi_i\rangle) - \mathbb{E}\log\cosh(\beta\langle x, \xi_1\rangle)\big]\right| \\
&\qquad \le \frac{1}{N\beta} \sum_{i=1}^{N} \frac{\beta^2}{2} \big(\langle x, \xi_i\rangle^2 - \mathbb{E}(\langle x, \xi_1\rangle^2)\big)(1 + \mathcal{O}_M(z^2)) \le \mathcal{O}\left(\frac{L}{\sqrt{N}}\right)\|x\|_2^2,
\end{aligned}$$

where we used that $\|\sqrt{N}(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id})\|_{\mathrm{Op}} \le \mathcal{O}_M(L)$ for $\xi \in A_{N,L}$. By the estimate (4.9) and [4, Lemma 6.8], we obtain for all $\|x\|_2 \le (1+d)z$ the estimate

$$\begin{aligned}
\mathbb{E}\phi(\langle x, \xi_1\rangle) &\ge -\frac{1}{\beta}\, f_{\mathrm{CW}}(\beta) + \frac{z^2}{12}\mathbb{E}\{(|\langle x, \xi_1\rangle| - z)^2\} \\
&\ge -\frac{1}{\beta}\, f_{\mathrm{CW}}(\beta) + c_2 z^2 \inf_{\substack{1\le\mu\le M,\\ s\in\{-1,+1\}}} \|x - sze_\mu\|_2^2
\end{aligned}$$

with some small constant $c_2 > 0$. Once τ is fixed, this estimate immediately implies the desired bound for the outer region:

$$\begin{aligned}
&\left(\frac{N\beta}{2\pi}\right)^{M/2} \int_{U^2_{d,\tau}} \exp\{-N\beta\Phi(x)\}\,dx \\
&\quad\le \left(\frac{N\beta}{2\pi}\right)^{M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\} \\
&\qquad\times \int_{U^2_{d,\tau}} \exp\{-(N\beta c_2 z^2 - \sqrt{N}\beta\mathcal{O}_M(L)(1+d)^2/\tau^2) \inf_{\substack{1\le\mu\le M,\\ s\in\{-1,+1\}}} \|x - sze_\mu\|_2^2\}\,dx \\
&\quad\le \left(\frac{N\beta}{2\pi}\right)^{M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\} \exp\{-N\beta c_2\tau^2 z^4/2\} \\
&\qquad\times 2M \int_{\mathbb{R}^M} \exp\{-\tfrac{1}{2}N\beta c_2 z^2\|x\|_2^2\}\,dx \\
&\quad\le \mathcal{O}_M(1)(\beta-1)^{-M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\} \exp\{-c_2\tau^2\gamma^2\}
\end{aligned}$$

for all $\xi \in A_{N,L}$ and all large enough N.

The intermediate region. For x from the intermediate region $U^3_{d,\tau,R}$, we may assume that $x \in B(ze_1, \tau z) \setminus B(ze_1, R/\sqrt{a})$. By the Taylor expansion (4.2) of Φ around ze_1 with remainder (4.3), we find

$$\begin{aligned}
&-N\beta\Phi(x) \\
&\quad= N\, f_{\mathrm{CW}}(\beta) + N\beta\left\langle \left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)ze_1, x - ze_1\right\rangle \\
&\qquad - \frac{N\beta}{2}\left\langle x - ze_1, \left([1-\beta(1-z^2)]\mathrm{Id} - \beta(1-z^2)\left(\frac{1}{N}\xi^{\mathrm{T}}\xi - \mathrm{Id}\right)\right)[x - ze_1]\right\rangle \\
&\qquad - N\beta R_{N,\beta}(x,\xi),
\end{aligned}$$

where the remainder satisfies

$$|R_{N,\beta}(x,\xi)| \le \frac{\beta^3 M^2}{3}(z + \|x - ze_1\|_2)\|x - ze_1\|_2^3.$$

Therefore, for $x \in B(ze_1, \tau z) \setminus B(ze_1, R/\sqrt{a})$ and all $\xi \in A_{N,L}$,

$$\begin{aligned}
&-N\beta\Phi(x) \\
&\quad\le N\, f_{\mathrm{CW}}(\beta) + \sqrt{N}\beta\mathcal{O}_M(L)z\|x - ze_1\|_2 \frac{\|x - ze_1\|_2}{R/\sqrt{a}} \\
&\qquad - \frac{N\beta}{2}\Big([1-\beta(1-z^2)] - \beta(1-z^2)\mathcal{O}_M(L/\sqrt{N})\Big)\|x - ze_1\|_2^2 \\
&\qquad + N\beta(1+\tau)\tau z^2 \frac{\beta^3 M^2}{3}\|x - ze_1\|_2^2 \\
&\quad\le N\, f_{\mathrm{CW}}(\beta) \\
&\qquad - \frac{N\beta}{2}\left([1-\beta(1-z^2)] + \mathcal{O}_M\left(\frac{L}{\sqrt{N}}\right) + \mathcal{O}_M(L)\frac{z\sqrt{a}}{\sqrt{N}R} - \frac{2}{3}\beta^3 M^2(1+\tau)\tau z^2\right) \\
&\qquad\quad\times \|x - ze_1\|_2^2
\end{aligned}$$

follows. As $1-\beta(1-z^2) \sim 2(\beta-1)$, τ can be chosen so small and R and N so large that

$$[1-\beta(1-z^2)] + \mathcal{O}_M(L/\sqrt{N}) + \mathcal{O}_M(L)\frac{z\sqrt{a}}{\sqrt{N}R} - \frac{2}{3}\beta^3 M^2(1+\tau)\tau z^2 \geq \beta - 1.$$

Then, for all large enough N,

$$-N\beta\Phi(x) \leq N\, f_{\mathrm{CW}}(\beta) - \frac{N\beta}{2}(\beta-1)\|x - ze_1\|_2^2.$$

Now, for all $\xi \in A_{N,L}$ and all large enough N,

$$\begin{aligned}
&\left(\frac{N\beta}{2\pi}\right)^{M/2} \int_{U^3_{d,\tau,R}} \exp\{-N\beta\Phi(x)\}\, dx \\
&\quad \leq \left(\frac{N\beta}{2\pi}\right)^{M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\}\, 2M \left(\frac{1}{N\beta(\beta-1)}\right)^{M/2} \int_{B(0,R/2)^{\mathrm{c}}} \exp\{-\tfrac{1}{2}\|x\|_2^2\}\, dx \\
&\quad \leq \mathcal{O}_M(1)(\beta-1)^{-M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\} \exp\{-R^2/8\}.
\end{aligned}$$

The inner region. In the inner region $U^4_{d,\tau,R}$, we need an upper bound for the integral $\int_{U^4_{d,\tau,R}} \exp\{-N\beta\Phi(x)\}\, dx$ which gives asymptotically the correct behaviour. First note that

$$U^4_{d,\tau,R} = \bigcup_{\substack{1\leq\mu\leq M,\\ s\in\{-1,+1\}}} B(sze_\mu, R/\sqrt{a})$$

for large enough N. Therefore, we may assume that $x \in B(ze_1, R/\sqrt{a})$. In the beginning of this section, we have already investigated the corresponding integral, cf. (4.8), the only difference being the centring ze_1 instead of z_N. But by choosing R so large that $\|\sqrt{a}[ze_1 - z_N]\|_2 \leq R/2$ on $A_{N,L}$, (4.8) becomes applicable, and we gain the following upper bound, valid for all $\xi \in A_{N,L}$:

$$\begin{aligned}
&\left(\frac{N\beta}{2\pi}\right)^{M/2} \int_{B(ze_1,R/\sqrt{a})} \exp\{-N\beta\Phi(x)\}\, dx \\
&\qquad = \left(\frac{N\beta}{2\pi a}\right)^{M/2} \int_{B(\sqrt{a}[ze_1-z_N],R)} \exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\, dx \\
&\qquad \leq \left(\frac{N\beta}{2\pi a}\right)^{M/2} \int_{B(0,2R)} \exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\, dx \\
&\qquad \leq \left(\frac{N\beta}{2\pi a}\right)^{M/2} \exp\{x_N^1 + \mathcal{O}_M((\beta-1)+1/\gamma)\} \int_{B(0,2R)} \exp\{-\tfrac{1}{2}\|x\|_2^2\}\, dx,
\end{aligned}$$

which implies

$$\begin{aligned}
&\left(\frac{N\beta}{2\pi}\right)^{M/2} \int_{U^4_{d,\tau,R}} \exp\{-N\beta\Phi(x)\}\,dx \\
&\quad\le \left(\frac{N\beta}{2\pi a}\right)^{M/2} \exp\{\mathcal{O}_M((\beta-1)+1/\gamma)\}\, 2\sum_{\mu=1}^{M} \exp\{x_N^\mu\} \int_{B(0,2R)} \exp\{-\tfrac{1}{2}\|x\|_2^2\}\,dx \\
&\quad\le \mathcal{O}_M(1)(\beta-1)^{-M/2} \sum_{\mu=1}^{M} \exp\{x_N^\mu\}.
\end{aligned}$$

On $A_{N,L}$, $\sup_{\mu\in\{1,\dots,M\}} |x_N^\mu - N\, f_{\mathrm{CW}}(\beta)|$ is bounded, so that

$$\begin{aligned}
Z_{N,\beta}(\xi) &= \left(\frac{N\beta}{2\pi}\right)^{M/2} \int_{\mathbb{R}^M} \exp\{-N\beta\Phi(x)\}\,dx \\
&\le \mathcal{O}_M(1)(\beta-1)^{-M/2} \exp\{N\, f_{\mathrm{CW}}(\beta)\}
\end{aligned}$$

on $A_{N,L}$, provided, τ is chosen small enough, R is chosen large enough and N is sufficiently large.

An upper bound for $\mathbf{F_{N,\beta,R}(\xi)}$. To obtain such an upper bound, it remains to show that

$$\int_{V^N_{R,\tilde r}} \exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\,dx$$

is not too large. This requires reconsidering the intermediate region:

$$\begin{aligned}
&\int_{V^N_{R,\tilde r}} \exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\,dx \\
&\qquad = a^{M/2} \int_{B(z_N,\tilde r z)\setminus B(z_N,R/\sqrt{a})} \exp\{-N\beta\Phi(x)\}\,dx \\
&\qquad \le a^{M/2} \int_{B(ze_1,z/\sqrt{2})\setminus B(ze_1,R/(2\sqrt{a}))} \exp\{-N\beta\Phi(x)\}\,dx
\end{aligned}$$

for large enough N, where we used $\tilde r < 1/\sqrt{2}$ and $\|ze_1 - z_N\|_2 \le R/(2\sqrt{a})$ on $A_{N,L}$ by our choice of R. As $\tau < 1/\sqrt{2}$,

$$\begin{aligned}
&\int_{V^N_{R,\tilde r}} \exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\,dx \\
&\qquad \le a^{M/2} \Bigg(\int_{U^1_d} \exp\{-N\beta\Phi(x)\}\,dx + \int_{U^2_{d,\tau}} \exp\{-N\beta\Phi(x)\}\,dx \\
&\qquad\qquad + \int_{U^3_{d,\tau,R/2}} \exp\{-N\beta\Phi(x)\}\,dx \Bigg)
\end{aligned}$$

follows. In the preceding steps, we already obtained bounds on these integrals, valid for $\xi \in A_{N,L}$ and large N. These bounds yield

$$\int_{V^N_{R,\tilde r}} \exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\,dx \le \mathcal{O}_M(1) \exp\{N\, f_{\mathrm{CW}}(\beta)\}[o(1) + \exp\{-R^2/32\}].$$

(Recall that we replaced R by $R/2$ in order to be able to apply the preceding result for the intermediate region.)

We have shown so far that for R chosen large enough, all $\xi \in A_{N,L}$ and all large enough N, the remainder $F_{N,\beta,R}$ obeys the following bound:

$$\begin{aligned} &F_{N,\beta,R}(\xi) \\ &\quad \le \mathcal{O}_M(1)\exp\{N\, f_{\mathrm{CW}}(\beta)\} \\ &\qquad \times \left[o(1) + \exp\{-R^2/32\} + \left(\frac{4\pi a}{N\beta}\right)^{M/2} (\beta-1)^{-M/2}\exp\{-c_1 N(\beta-1)\}\right] \\ &\quad \le \mathcal{O}_M(1)\exp\{N\, f_{\mathrm{CW}}(\beta)\}[o(1) + \exp\{-R^2/32\}]. \end{aligned}$$

Convergence. Now, we are ready to finish the proof of Theorem 2.4. In the beginning of the section, we investigated the integrals over the inner region by showing that

$$\begin{aligned} &\int f(x)\exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\,dx \\ &\qquad\qquad = \mathrm{e}^{x_N}\exp\{\mathcal{O}_M((\beta-1)+1/\gamma)\}\int f(x)\exp\{-\tfrac{1}{2}\|x\|_2^2\}\,dx \end{aligned}$$

and

$$\begin{aligned} &\int_{B(0,R)} \exp\{-N\beta\Phi(x/\sqrt{a}+z_N)\}\,dx \\ &\qquad\qquad = \mathrm{e}^{x_N}\exp\{\mathcal{O}_M((\beta-1)+1/\gamma)\}\int_{B(0,R)}\exp\{-\tfrac{1}{2}\|x\|_2^2\}\,dx \end{aligned}$$

for all $\xi \in A_{N,L}$. Therefore, recalling that $|x_N - N\, f_{\mathrm{CW}}(\beta)|$ is bounded on $A_{N,L}$,

$$\int f\, d\tilde{\chi}_{N,\beta,a} = \frac{\int f(x) 1_{B(0,R)}(x)\exp\{-\frac{1}{2}\|x\|_2^2\}\,dx}{\int 1_{B(0,R)}(x)\exp\{-\frac{1}{2}\|x\|_2^2\}\,dx} + \mathcal{O}_M(\|f\|_\infty[o(1)+\exp\{-R^2/32\}])$$

follows. So, choosing R large shows that on $A_{N,L}$, the integral $\int f\, d\tilde{\chi}_{N,\beta,a}$ is arbitrarily close to $\int f\, d\mathcal{N}(0,\mathrm{Id})$. Proceeding as in the Subsection "Convergence" of Section 3, the claimed weak convergence

$$\mathbb{P}\tilde{Q}_{N,\beta_N}^{-1} \Longrightarrow \delta_{\mathcal{N}(0,\mathrm{Id})}$$

can be established. This completes the proof of Theorem 2.4.

□

Appendix

In this appendix we give the proof of the criterion Lemma 3.1 for weak convergence in the Polish space $\mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^M))$ of the probability measures on the probability measures on $\mathbb{R}^M$. We applied this criterion to show the weak convergence of the Hubbard–Stratonovich transforms of the measures in question.

Proof of Lemma 3.1. First note that

$$\int g\left(\int f\, dQ_N\right) d\mathbb{P} = \int g\left(\int f\, d\nu\right)\mu_N(d\nu)$$

and, similarly,

$$\int g\left(\int f\, dQ\right) d\mathbb{P} = \int g\left(\int f\, d\nu\right)\mu(d\nu).$$

For $k \in \mathbb{N}$, $f : \mathbb{R}^M \to \mathbb{R}^k$ bounded and continuous and $g : \mathbb{R}^k \to \mathbb{R}$ continuous, the mapping

$$\mathcal{M}_1(\mathbb{R}^M) \ni \nu \mapsto g\left(\int f \, d\nu\right) \in \mathbb{R}$$

is bounded and continuous in the weak topology. Therefore, (3.1) is an immediate consequence of the definition of the weak convergence $\mu_N \Rightarrow \mu$.

To show the converse, namely that (3.1) implies the weak convergence $\mu_N \Rightarrow \mu$, we shall show that (3.1) implies

$$\liminf_{N\to\infty} \mu_N(G) \geq \mu(G) \tag{A.1}$$

for all sets $G \subset \mathcal{M}_1(\mathbb{R}^M)$ which are open in the weak topology.

Let us start with some topological considerations: Denote by $\mathcal{C}_c(\mathbb{R}^M, \mathbb{R})$ the set of all continuous functions $\mathbb{R}^M \to \mathbb{R}$ with compact support. Given a finite subset $F \subset \mathcal{C}_c(\mathbb{R}^M, \mathbb{R})$, a probability measure $\nu_0 \in \mathcal{M}_1(\mathbb{R}^M)$ and an $\varepsilon > 0$, we define the corresponding cylinder set $B(\nu_0, F, \varepsilon)$ by

$$B(\nu_0, F, \varepsilon) = \{\, \nu \in \mathcal{M}_1(\mathbb{R}^M) : h_{F,\nu_0}(\nu) < \varepsilon \,\},$$

where

$$h_{F,\nu_0}(\nu) = \max_{f\in F} \left| \int f \, d\nu - \int f \, d\nu_0 \right|.$$

The set

$$\mathcal{B} = \{\, B(\nu_0, F, \varepsilon) : \nu_0 \in \mathcal{M}_1(\mathbb{R}^M), F \subset \mathcal{C}_c(\mathbb{R}^M, \mathbb{R}) \text{ finite}, \ \varepsilon > 0 \,\},$$

consisting of all these cylinder sets, provides a basis of the vague topology on $\mathcal{M}_1(\mathbb{R}^M)$. Since we are considering probability measures only, the vague and the weak topology agree.

In order to show (A.1), let $G \subset \mathcal{M}_1(\mathbb{R}^M)$ be open in the weak topology. Since $\mathcal{B}$ is a basis of the topology, there exists an index set I such that

$$G = \bigcup_{i\in I} B_i$$

where $B_i = B(\nu_i, F_i, \varepsilon_i) \in \mathcal{B}$ with $\nu_i \in \mathcal{M}_1(\mathbb{R}^M)$, a finite subset $F_i \subset \mathcal{C}_c(\mathbb{R}^m, \mathbb{R})$ and $\varepsilon_i > 0$. As $\mathcal{M}_1(\mathbb{R}^M)$ is a Polish space, the measure $\mu \in \mathcal{M}_1(\mathcal{M}_1(\mathbb{R}^M))$ is regular. Therefore, for given $\delta > 0$, there exists a compact set $K \subset G$ satisfying

$$\mu(G) \leq \mu(K) + \delta.$$

By the compactness of K there are finitely many indices, say $i_1, \ldots, i_l \in I$ such that $K \subset \bigcup_{j=1}^{l} B_{i_j}$. Without loss of generality, we may assume that $\{i_1, \ldots, i_l\} = \{1, \ldots, l\}$. Now,

$$\mu(G) - \delta \leq \mu(K) \leq \mu\left(\bigcup_{i=1}^{l} B_i\right)$$

and

$$\mu_N(G) = \mu_N\left(\bigcup_{i\in I} B_i\right) \geq \mu_N\left(\bigcup_{i=1}^{l} B_i\right).$$

Thus it suffices to prove (A.1) for sets of the form $G = \bigcup_{i=1}^{l} B_i$.

So let $G = \bigcup_{i=1}^{l} B_i$ with $B_i = B(\nu_i, F_i, \varepsilon_i)$, where $F_i = \{f_1^i, \dots, f_{k_i}^i\}$. Put

$$c_i = \left(\int f_j^i \, d\nu_i \right)_{j=1,\dots,k_i} \in \mathbb{R}^{k_i}$$

and define for $L > 0$ the continuous function $g_L : \mathbb{R}^{\sum_{i=1}^{l} k_i} \to \mathbb{R}$ by

$$g_L(x_1, \dots, x_l) = \min\left\{1, \max\left\{0, L \max_{i=1,\dots,l}\left(1 - \frac{\|x_i - c_i\|_\infty}{\varepsilon_i}\right)\right\}\right\}, \qquad x_i \in \mathbb{R}^{k_i}.$$

Then

$$\begin{aligned}
\nu \in \bigcup_{i=1}^{l} B_i \quad &\Leftrightarrow \quad \exists\, i \in \{1, \dots, l\} \text{ with } h_{F_i,\nu_i}(\nu) < \varepsilon_i \\
&\Leftrightarrow \quad \exists\, i \in \{1, \dots, l\} \text{ with } 1 - \frac{\| \left(\int f_j^i \, d\nu \right)_{j=1,\dots,k_i} - c_i \|_\infty}{\varepsilon_i} > 0 \\
&\Leftrightarrow \quad \max_{i=1,\dots,l}\left(1 - \frac{\| \left(\int f_j^i \, d\nu \right)_{j=1,\dots,k_i} - c_i \|_\infty}{\varepsilon_i}\right) > 0 \\
&\Leftrightarrow \quad g_L\left(\left(\left(\int f_j^i \, d\nu\right)_{j=1,\dots,k_i}\right)_{i=1,\dots,l}\right) \nearrow 1 \qquad \text{as } L \to \infty.
\end{aligned}$$

Now suppose that (3.1) holds. Then

$$\begin{aligned}
&\liminf_{N\to\infty} \mu_N\left(\bigcup_{i=1}^{l} B_i\right) \\
&\qquad \geq \liminf_{N\to\infty} \int g_L\left(\left(\left(\int f_j^i \, d\nu\right)_{j=1,\dots,k_i}\right)_{i=1,\dots,l}\right) \mu_N(d\nu) \qquad \forall L > 0 \\
&\qquad \geq \int g_L\left(\left(\left(\int f_j^i \, d\nu\right)_{j=1,\dots,k_i}\right)_{i=1,\dots,l}\right) \mu(d\nu) \qquad \forall L > 0, \qquad \text{(A.2)}
\end{aligned}$$

where the first inequality follows from the fact that the indicator function on $\bigcup_{i=1}^{l} B_i$ majorizes the integrand on the right-hand side (by the above considerations) and the second inequality is just our assumption (3.1). Now, for $L \to \infty$, the integrand on the right-hand side of (A.2) converges monotonely to the indicator function on $\bigcup_{i=1}^{l} B_i$ and thus the right-hand side of (A.2) converges to $\mu(\bigcup_{i=1}^{l} B_i)$. □

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