# Modules of Solvable Infinitesimal Groups and the Structure of Representation-Finite Cocommutative Hopf Algebras

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**Abstract**

Let $k$ be an algebraically closed field $k$ of characteristic $p \geq 3$. In this paper we extend the representation and block theory of solvable infinitesimal groups to determine the structure of those cocommutative Hopf algebras over $k$ that have finite representation type.

## 0. Introduction

In continuation of earlier work [7], we study in this paper the structure of cocommutative Hopf algebras of finite representation type. Since every such algebra is the group algebra $H(\mathcal{G})$ of a finite algebraic group $\mathcal{G}$, this problem has to interrelated aspects. On the one hand, one has to understand the structure of the group scheme $\mathcal{G}$, while on the other the Morita equivalence class of $H(\mathcal{G})$ has to be determined. Although we shall here be mainly concerned with the latter, structural information on the underlying group schemes enters crucially at several points.

Experience shows that the theory naturally divides into two parts that tend to exhibit different features and require different methods. The classical theory deals with group algebras of constant groups. Here the representation theory of each block is entirely governed by the structure of its defect group. Our main focus are therefore the group algebras of infinitesimal groups, where support varieties have proven to be an effective tool. If $\mathcal{G}$ has height $\leq 1$, methods related to this notion lead to a complete description of the structure of the representation-finite blocks of $H(\mathcal{G})$ (cf. [4]). If one passess to groups of arbitrary height, one still has enough information to pin down the structure of the blocks and the stable Auslander-Reiten quiver of the infinitesimal representation-finite groups. In fact, one

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basic feature from the theory of restricted Lie algebras retains its validity: the simple modules of such a block can be obtained from one representative by repeated twisting with the Nakayama automorphism $\nu$ of $H(\mathcal{G})$, with each block having either 1 or $ord(\nu)$ simple modules.

We begin by collecting notation and general conventions that will be in force throughout the paper. With regard to algebraic groups we will follow [3] with the exception of quotients. These will be denoted $\mathcal{G}/\mathcal{H}$ rather than $\mathcal{G}\tilde{/}\mathcal{H}$. For a Hopf algebra $H$ we will employ the Heyneman-Sweedler sigma notation for the coproduct:

$$\Delta : H \longrightarrow H \otimes_k H \quad ; \quad h \mapsto \sum_{(h)} h_{(1)} \otimes h_{(2)}.$$

The counit and the antipode of $H$ will be denoted by $\varepsilon$ and $\eta$, respectively. Moreover, $H^{\dagger} := ker\,\varepsilon$ is the augmentation ideal of $H$. Aside from the regular representation, $H$ also acts on itself via the (left) *adjoint representation* that is given by

$$h \cdot x := \sum_{(h)} h_{(1)} x \eta(h_{(2)}) \qquad \forall\ h, x \in H.$$

The corresponding $H$-module will occasionally be written as $H_{ad}$.

By an $H$-module, we will always understand a left $H$-module, and all $k$-vector spaces are assumed to be of finite dimension. Given an $H$-module $M$ and a $k$-algebra automorphism $\alpha \in Aut_k(H)$ of $H$, we let $M^{(\alpha)}$ be the $H$-module with underlying $k$-space $M$ and twisted action

$$h \cdot m := \alpha^{-1}(h)m \qquad \forall\ h \in H,\ m \in M.$$

Since $M \mapsto M^{(\alpha)}$ is an equivalence on the category $mod_H$ of all $H$-modules, the group $Aut_k(H)$ operates on the set of isoclasses of $H$-modules such that the sets of indecomposables, simples, and projectives are stable. Note that the group of inner automorphisms fixes every isoclass of $mod_H$.

Let $G(H)$ be the group of group-like elements of $H$. Then $G(H^*)$, the set of group-likes of the dual Hopf algebra, coincides with the group of algebra homomorphism $H \longrightarrow k$. The operation of $G(H^*)$ is the convolution

$$(\lambda * \mu)(h) := \sum_{(h)} \lambda(h_{(1)})\mu(h_{(2)}) \qquad \forall\ \lambda,\ \mu \in G(H^*).$$

Note that $G(H^*)$ is commutative, whenever $H$ is cocommutative. If $H = H(\mathcal{G})$ is the Hopf algebra of a finite algebraic $k$-group, then $G(H(\mathcal{G})^*) = G(\mathcal{O}(\mathcal{G}))$ is the set of group-like elements of the function algebra $\mathcal{O}(\mathcal{G})$ of $\mathcal{G}$.

Given $\lambda \in G(H^*)$ we observe that the map

$$\lambda * id_H : H \longrightarrow H \quad ; \quad h \mapsto \sum_{(h)} \lambda(h_{(1)})h_{(2)}$$

is an automorphism, whose inverse is $(\lambda \circ \eta) * id_H$. If $M$ is an $H$-module, we will abuse notation and write $M^{(\lambda)}$ rather than $M^{(\lambda * id_H)}$. In the special case, where $M = k$ is the trivial $H$-module, we put $k_\lambda := k^{(\lambda\circ\eta)}$, so that

$$h \cdot 1 = \lambda(h)1 \qquad \forall\ h \in H.$$

Given a simple $H$-module $S$, we let $P(S)$ be its projective cover, and write $P(\lambda)$ instead of $P(k_\lambda)$. If $K \subset H$ is a subalgebra, we occasionally denote the restriction to $K$ of a given $H$-module $M$ by $M|_K$.

Finally, when dealing with algebraic groups, we will always consider a given group $\mathcal{G}$ a subfunctor of the functor $H(\mathcal{G})_a$ that sends a commutative $k$-algebra $R$ to the additive group $H(\mathcal{G}) \otimes_k R$. Accordingly, we will not distinguish between characters $\lambda : \mathcal{G} \longrightarrow \mu_k$ and the corresponding algebra homomorphism $\lambda : H(\mathcal{G}) \longrightarrow k$.

The paper is organized as follows. In section 1 we generalize a result of [5] to obtain information on the structure of projective covers of simple, induced modules. As is shown in §2, this result applies in particular to simple modules of solvable infinitesimal groups of characteristic $p \geq 3$. According to Theorem 2.1, which extends and refines earlier results (cf. [14, 17]), these modules are of the form $H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$, where $(\mathcal{K}, \lambda)$ is an "admissible pair", obtained by iterated application of the "Blattner process". In case $\mathcal{G}$ is supersolvable, the algebra $H(\mathcal{G})$ satisfies a nice linkage principle. When combined with the structural features of V-uniserial groups established in §3, this yields complete information on cocommutative Hopf algebras of finite representation type with local dual algebra. The relevant results are summarized in Theorem 4.2. The fusion of the theories concerning infinitesimal and constant groups often presents problems. In our context, finite algebraic groups generally either behave like constant groups, or like infinitesimal groups. There are, however, representation-finite algebraic groups of dimension zero that are not Morita equivalent to constant or infinitesimal groups.

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# 1. Projective Covers of Induced Modules

Throughout this section we fix an infinitesimal algebraic $k$-group $\mathcal{G}$ with group algebra $H(\mathcal{G})$. The unique largest multiplicative normal subgroup of $\mathcal{G}$ will be denoted $\mathcal{M}(\mathcal{G})$. We begin with the following generalization of [5, (2.3)].

**Proposition 1.1** *Let $\mathcal{G}' \subset \mathcal{G}$ be a supersolvable subgroup of $\mathcal{G}$ such that $\mathcal{M}(\mathcal{G}') \subset \mathcal{M}(\mathcal{G})$. Suppose that $\lambda \in G(\mathcal{O}(\mathcal{G}'))$ is an algebra homomorphism such that $S := H(\mathcal{G}) \otimes_{H(\mathcal{G}')} k_\lambda$ is simple. Let $\mathcal{M}' \subset \mathcal{G}'$ be a maximal multiplicative subgroup of $\mathcal{G}'$. Then there is an isomorphism*

$$P(S)|_{H(\mathcal{G}')} \cong H(\mathcal{G}') \otimes_{H(\mathcal{M}')} (\bigoplus_{\gamma \in A} k_\gamma)$$

*with the subset $A \subset G(\mathcal{O}(\mathcal{G}'))$ having cardinality $r = dim_k Soc_{H(\mathcal{G}')}(S)$.*

*Proof.* Since $\mathcal{M}(\mathcal{G}')$ is contained in $\mathcal{M}(\mathcal{G})$, the adjoint representation

$$H(\mathcal{G}') \longrightarrow End_k(H(\mathcal{G})_{ad})$$

factors through to $H(\mathcal{G}'/\mathcal{M}(\mathcal{G}'))$. Thanks to [7, (2.3)] the group $\mathcal{G}'/\mathcal{M}(\mathcal{G}')$ is trigonalizable, so that the $H(\mathcal{G}')$-module $H(\mathcal{G})_{ad}$ possesses a composition series with one-dimensional composition factors. We consider the canonical surjection

$$\psi : H(\mathcal{G})_{ad} \otimes_k k_\lambda \longrightarrow S \quad ; \quad u \otimes 1 \mapsto u \otimes 1,$$

and observe that for $h \in H(\mathcal{G}')$ and $u \in H(\mathcal{G})_{ad}$ we have

$$\begin{aligned} \psi(h \cdot (u \otimes 1)) &= \sum_{(h)} \psi((h_{(1)} \cdot u) \otimes (h_{(2)} \cdot 1)) = \sum_{(h)} \psi((h_{(1)} u\, \eta(h_{(2)})) \otimes (h_{(3)} \cdot 1) \\ &= \sum_{(h)} h_{(1)} u \otimes \lambda(\eta(h_{(2)}) h_{(3)}) 1 = \sum_{(h)} h_{(1)} u \otimes \varepsilon(h_{(2)}) 1 = hu \otimes 1. \end{aligned}$$

Consequently, $\psi$ is $H(\mathcal{G}')$-linear, and $S|_{H(\mathcal{G}')}$ also admits a composition series with one-dimensional composition factors.

Note that the central subalgebra $H(\mathcal{M}(\mathcal{G}')) \subset H(\mathcal{G})$ operates on $S$ via $\lambda|_{H(\mathcal{M}(\mathcal{G}'))}$. It therefore acts in this fashion on every simple module $T$ belonging to the block $\mathcal{B}(S) \subset H(\mathcal{G})$ associated to $S$. Hence $\mathcal{M}(\mathcal{G}')$ operates trivially on the $H(\mathcal{G}')$-module $T \otimes_k k_{\lambda \circ \eta}$, implying that all composition factors of this modules are one-dimensional. Consequently, the composition factors of $T|_{H(\mathcal{G}')}$ also have this property. It follows that $P(S)|_{H(\mathcal{G}')}$ possesses a composition series with one-dimensional factors.

Let $\gamma : H(\mathcal{G}') \longrightarrow k$ be an algebra homomorphism. Direct computation shows that the automorphism $(\gamma \circ \eta) * id_{H(\mathcal{G}')}$ maps the principal block of $H(\mathcal{G}')$ onto the block containing $k_\gamma$. From the proof of [7, (2.4)] we obtain $P(\gamma) \cong H(\mathcal{G}') \otimes_{H(\mathcal{M}')} k_\gamma$. The above observations then show that there exists a subset $A \subset G(\mathcal{O}(\mathcal{G}'))$ and integers $n_\gamma \in \mathbb{N}_0$ such that

$$P(S)|_{H(\mathcal{G}')} \cong H(\mathcal{G}') \otimes_{H(\mathcal{M}')} (\bigoplus_{\gamma \in A} n_\gamma\, k_\gamma).$$

The extension $H(\mathcal{G}) : H(\mathcal{G}')$ is a Frobenius extension of second kind (cf. [9, (1.7)]) with Frobenius automorphism $\alpha \in Aut_k(H(\mathcal{G}'))$, say. Given $\omega \in G(\mathcal{O}(\mathcal{G}'))$ we put $S(\omega) := H(\mathcal{G}) \otimes_{H(\mathcal{G}')} k_\omega$. In view of [11, p.96f] we have

$$S(\omega) \cong Hom_{H(\mathcal{G}')}(H(\mathcal{G}), k_{\omega \circ \alpha}).$$

Frobenius reciprocity then yields

$$\begin{aligned} dim_k Hom_{H(\mathcal{G})}(P(S), S(\omega)) &= dim_k Hom_{H(\mathcal{G}')}(P(S)|_{H(\mathcal{G}')}, k_{\omega \circ \alpha}) \\ &= \sum_{\gamma \in A} n_\gamma dim_k Hom_{H(\mathcal{M}')}(k_\gamma, k_{\omega \circ \alpha}). \end{aligned}$$

Since the left-hand side counts the multiplicity of $S$ in $S(\omega)$, it is bounded by 1 for dimension reasons. Noting that every element of $G(\mathcal{O}(\mathcal{G}'))$ is uniquely determined by its restriction to the maximal multiplicative subgroup $\mathcal{M}'$, we obtain $n_\gamma = 1$ and $S \cong S(\gamma \circ \alpha^{-1})$ for every $\gamma \in A$.

By our observations above, there exists a subset $B \subset G(\mathcal{O}(\mathcal{G}'))$ and $m_\zeta \in \mathbb{N}$ such that

$$Soc_{H(\mathcal{G}')}(S) = \bigoplus_{\zeta \in B} m_\zeta\, k_\zeta.$$

Since $S \cong S(\zeta)$ for each $\zeta \in B$ we have

$$1 = dim_k Hom_{H(\mathcal{G})}(S,S) = dim_k Hom_{H(\mathcal{G}')}(k_\zeta, S) = m_\zeta.$$

Given $\zeta \in B$ we have $S \cong S(\zeta) \cong Hom_{H(\mathcal{G}')}(H(\mathcal{G}), k_{\zeta\circ\alpha})$, whence

$$\begin{aligned} 1 = dim_k Hom_{H(\mathcal{G})}(P(S), S) &= dim_k Hom_{H(\mathcal{G}')}(P(S)|_{H(\mathcal{G}')}, k_{\zeta\circ\alpha}) \\ &= \sum_{\gamma \in A} dim_k Hom_{H(\mathcal{M}')}(k_\gamma, k_{\zeta\circ\alpha}). \end{aligned}$$

Consequently, $\zeta \circ \alpha \in A$. If $\gamma \in A$, then $S \cong S(\gamma \circ \alpha^{-1})$ and $\gamma \circ \alpha^{-1} \in B$. As a result, we have $card(A) = card(B) = dim_k Soc_{H(\mathcal{G}')}(S)$. $\square$

**Corollary 1.2** *Suppose that $S = H(\mathcal{G}) \otimes_{H(\mathcal{G}')} k_\lambda$ is a simple $H(\mathcal{G})$-module, that is induced from a nilpotent subgroup $\mathcal{G}' \subset \mathcal{G}$ satisfying $\mathcal{M}(\mathcal{G}') \subset \mathcal{M}(\mathcal{G})$. Then the block $\mathcal{B}(S) \subset H(\mathcal{G})$ corresponding to $S$ is primary and has dimension $dim_k H(\mathcal{G}/\mathcal{M}(\mathcal{G}'))$.*

*Proof.* We adopt the notation of (1.1), and observe that, owing to [3, (IV.§4 (1.10)], $\mathcal{M}(\mathcal{G}')$ is the unique maximal multiplicative subgroup of the nilpotent group $\mathcal{G}'$. According to (1.1) we thus have

$$P(S)|_{H(\mathcal{G}')} \cong H(\mathcal{G}') \otimes_{H(\mathcal{M}(\mathcal{G}'))} (\bigoplus_{\gamma \in A} k_\gamma)$$

for a suitable subset $A \subset G(\mathcal{O}(\mathcal{G}'))$. Let $\gamma$ be an element of $A$. Since $\lambda|_{H(\mathcal{M}(\mathcal{G}'))} = \gamma|_{H(\mathcal{M}(\mathcal{G}'))}$, the algebra homomorphism $\lambda * (\gamma \circ \eta)$ factors through to the unipotent group $\mathcal{G}'/\mathcal{M}(\mathcal{G}')$ (cf. [3, (IV.§4 (1.10)]). This readily yields $\lambda = \gamma$, so that $A = \{\lambda\}$.

It follows from [7, (2.4)] that $P(S)|_{H(\mathcal{G}')}$ is indecomposable. According to [16, (I.2.14)] this implies that $k_\lambda$ is the only composition factor of $P(S)|_{H(\mathcal{G}')}$. Consequently, every composition factor $T$ of $P(S)$ contains a copy of $k_\lambda$. Hence there is a surjection $S \longrightarrow T$, so that $S \cong T$. As a result, $\mathcal{B}(S)$ is primary and of dimension $dim_k H(\mathcal{G}/\mathcal{M}(\mathcal{G}'))$. $\square$

*Remark.* If $\mathcal{G}$ is nilpotent, then the inclusion $\mathcal{M}(\mathcal{G}') \subset \mathcal{M}(\mathcal{G})$ holds for any subgroup $\mathcal{G}' \subset \mathcal{G}$. Thanks to [16, (I.2.4)] the simple $H(\mathcal{G})$-modules are induced from one-dimensional modules. Hence (1.2) shows that the blocks of $H(\mathcal{G})$ are primary. This result was first obtained by the second author (cf. [16, (I.2.41)]).

We conclude this section with a subsidiary result that shows that the technical conditions of (1.1) and (1.2) obtain whenever the simple module $S$ is built up via an iterated application of Blattner's criterion.

**Lemma 1.3** *Let $M$ be a $H(\mathcal{G})$-module, and suppose that $N$ is an isotypical semisimple component of $M$ relative to a normal subgroup $\mathcal{G}'' \subset \mathcal{G}$. If $\mathcal{G}' = \{g \in \mathcal{G} \ ; \ gN = N\}$ denotes the stabilizer of $N$, then $\mathcal{M}(\mathcal{G}') \subset \mathcal{M}(\mathcal{G})$.*

*Proof.* The centralizer $\mathcal{C} = Cent_{\mathcal{G}}(\mathcal{G}'')$ of the normal subgroup $\mathcal{G}''$ is normal in $\mathcal{G}$. According to [3, (IV.§3(1.1))] the multiplicative center $\mathcal{M}(\mathcal{C})$ is invariant under $\mathcal{AUT}(\mathcal{C})$. This implies that $\mathcal{M}(\mathcal{C})$ is normal in $\mathcal{G}$. Thanks to [3, (IV.§1 (4.6))] $\mathcal{M}(\mathcal{C})$ is therefore contained in $\mathcal{M}(\mathcal{G})$.

As $N$ is an isotypical component of $M|_{H(\mathcal{G}'')}$, it readily follows that $\mathcal{C} \subset \mathcal{G}'$. Consequently, the group $\mathcal{M}(\mathcal{G}')$ is contained in the center of $\mathcal{C}$, so that $\mathcal{M}(\mathcal{G}') \subset \mathcal{M}(\mathcal{C})$. $\square$

## 2. Simple Modules of Solvable Groups

We keep the general conventions of the foregoing section and assume henceforth that $p = char(k) \geq 3$. Let $\mathcal{N} \subset \mathcal{G}$ be a subgroup. Given an $\mathcal{G}$-module $M$, and a character $\lambda : \mathcal{N} \longrightarrow \mu_k$, we put

$$M_\lambda := \{m \in M \ ; \ g \cdot m = \lambda(g)m \quad \forall \ g \in \mathcal{N}\}.$$

Moreover, if $\mathcal{N}$ is normal in $\mathcal{G}$, we let

$$Stab_{\mathcal{G}}(\lambda) := \{g \in \mathcal{G} \ ; \ \lambda(g^{-1}ng) = \lambda(n) \ \ \forall \ n \in \mathcal{N}\}$$

be the *stabilizer* of $\lambda$ in $\mathcal{G}$. Note that $Stab_{\mathcal{G}}(\lambda) = \{g \in \mathcal{G} \ ; \ gM_\lambda = M_\lambda\}$ whenever $M_\lambda \neq (0)$.

The following notion is motivated by the concept of a polarization that was employed in [14] in the study of representations of solvable restricted Lie algebras.

**Definition.** Let $\mathcal{K} \subset \mathcal{G}$ be a subgroup, $\lambda : \mathcal{K} \longrightarrow \mu_k$ a character. An *inducing sequence for* $(\mathcal{K}, \lambda)$ is a filtration $e_k \subset \mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots \subset \mathcal{N}_r = \mathcal{K} = \mathcal{G}_r \subset \mathcal{G}_{r-1} \subset \cdots \subset \mathcal{G}_1 \subset \mathcal{G}_0 = \mathcal{G}$ of $\mathcal{G}$ by subgroups such that

(a) $\mathcal{N}_i$ is a normal subgroup of $\mathcal{G}_{i-1}$ for $1 \leq i \leq r$, and

(b) $\mathcal{G}_i = Stab_{\mathcal{G}_{i-1}}(\lambda|_{\mathcal{N}_i}) \quad 1 \leq i \leq r$.

A pair $(\mathcal{K}, \lambda)$ consisting of a subgroup $\mathcal{K} \subset \mathcal{G}$, and a character $\lambda : \mathcal{K} \longrightarrow \mu_k$ is called *admissible* if there exists an inducing sequence for $(\mathcal{K}, \lambda)$.

Following [3] we denote by $\mathcal{F} = \mathcal{F}_{\mathcal{G}}$ and ${}_{\mathcal{F}}\mathcal{G} = {}_{\mathcal{F}_{\mathcal{G}}} \mathcal{G}$ the Frobenius homomorphism and the first Frobenius kernel of $\mathcal{G}$, respectively.

**Theorem 2.1** *The following statements hold:*

*(1) Let $(\mathcal{K}, \lambda)$ be an admissible pair. Then $H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$ is a simple $H(\mathcal{G})$-module.*

*(2) Let $S$ be a simple module for the solvable group $\mathcal{G}$.*

*(a) There exists an admissible pair $(\mathcal{K}, \lambda)$ such that $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$.*

*(b) Let $\mathcal{N} \subset \mathcal{G}$ be a commutative normal subgroup, $\gamma : \mathcal{N} \longrightarrow \mu_k$ a character such that $S_\gamma \neq (0)$ and $Stab_{\mathcal{G}}(\gamma) \neq \mathcal{G}$. Then there exists an admissible pair $(\mathcal{K}, \lambda)$ such that $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$, $\mathcal{N} \subset \mathcal{K}$, and $\lambda|_{\mathcal{N}} = \gamma$.*

*Proof.* (1). We put $S := H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$ and proceed by induction on $dim_k S$. Suppose that $dim_k S > 1$, and let

$$e_k \subset \mathcal{N}_1 \subset \mathcal{N}_2 \subset \cdots \subset \mathcal{N}_r = \mathcal{K} = \mathcal{G}_r \subset \mathcal{G}_{r-1} \subset \cdots \subset \mathcal{G}_1 \subset \mathcal{G}_0 = \mathcal{G}$$

be an inducing sequence for the pair $(\mathcal{K}, \lambda)$. By inductive hypothesis

$$S_1 := H(\mathcal{G}_1) \otimes_{H(\mathcal{K})} k_\lambda$$

is a simple submodule of $S|_{H(\mathcal{G}_1)}$ such that $H(\mathcal{G})S_1 = S$. Since the $H(\mathcal{G}_1)$-submodule

$$S_\lambda := \{s \in S \ ; \ gs = \lambda(g)s \ \ \forall \ g \in \mathcal{N}_1\}$$

intersects $S_1$ non-trivially, it follows that $S_1 \subset S_\lambda$. Consequently, the image of the natural map

$$H(\mathcal{G}) \otimes_{H(\mathcal{G}_1)} S_\lambda \longrightarrow S$$

contains $H(\mathcal{G})S_1 = S$. According to [16, (I.1.4)] the left-hand module above is simple, implying that $S$ has the same property.

(2)(a). We proceed by induction on $n := dim_k S$. If $n = 1$, we set $\mathcal{N}_1 = \mathcal{G}_1 = \mathcal{G}$. Alternatively, we let $\mathcal{Q} \subset \mathcal{G}$ be the kernel of the representation of $S$, and denote by $\pi$ the canonical projection onto $\mathcal{G}/\mathcal{Q}$. Since $dim_k S > 1$, the group $\mathcal{G}/\mathcal{Q}$ is not commutative, and [17, (2.1)] provides a commutative, normal subgroup $\mathcal{L} \subset \mathcal{G}/\mathcal{Q}$ of height $\leq 1$ such that $\mathcal{M}({}_{\mathcal{F}}(\mathcal{G}/\mathcal{Q}))$ is properly contained in $\mathcal{L}$.

We set $\mathcal{N}_1 := \pi^{-1}(\mathcal{L})$. Since $\mathcal{N}_1/\mathcal{Q}$ is commutative, [3, (IV.§3 (1.1))] ensures the existence of subgroups $\mathcal{M}, \mathcal{U} \subset \mathcal{N}_1$ containing $\mathcal{Q}$ such that $\mathcal{M}/\mathcal{Q}$ and $\mathcal{U}/\mathcal{Q}$ are the multiplicative and unipotent components of $\mathcal{L}$, respectively.

Owing to [16, (II.9.8)] all composition factors of $S|_{H(\mathcal{N}_1)}$ are isomorphic. Since $\mathcal{N}_1/\mathcal{Q}$ is commutative, there thus exists a character $\gamma : \mathcal{N}_1 \longrightarrow \mu_k$ such that $k_\gamma$ is the only composition factor of $S|_{H(\mathcal{N}_1)}$. Observing the semisimpliciy of $H(\mathcal{M}/\mathcal{Q})$ we therefore obtain

$$S_\gamma = Soc_{H(\mathcal{N}_1)}(S) = S^{\mathcal{U}},$$

where the latter space is the set of $\mathcal{U}$-invariants.

We put $\mathcal{G}_1 := Stab_{\mathcal{G}}(\gamma)$. Since $\mathcal{U} = ker\,\gamma$, application of [17, (4.1)] yields

$$\mathcal{G}_1 = Nor_{\mathcal{G}}(\mathcal{U}).$$

The assumption $\mathcal{G}_1 = \mathcal{G}$ implies $S^{\mathcal{U}} = S$, so that $\mathcal{U} \subset \mathcal{Q}$. This, however, contradicts the choice of $\mathcal{N}_1$. According to [16, (II.9.14)] the canonical map

$$H(\mathcal{G}) \otimes_{H(\mathcal{G}_1)} S_\gamma \longrightarrow S$$

is injective. As $S$ is simple, this map is an isomorphism, and $S_\gamma \subset S$ is a simple $H(\mathcal{G}_1)$-module with $dim_k S_\gamma < dim_k S$.

Application of the inductive hypothesis to the simple $\mathcal{G}_1/\mathcal{U}$-module $S_\gamma$ yields an admissible pair $(\mathcal{K}, \lambda)$ and an inducing sequence

$$e_k \subset \mathcal{N}_2 \subset \mathcal{N}_3 \subset \cdots \subset \mathcal{N}_r = \mathcal{K} = \mathcal{G}_r \subset \mathcal{G}_{r-1} \subset \cdots \subset \mathcal{G}_2 \subset \mathcal{G}_1$$

for $(\mathcal{K}, \lambda)$ such that $\mathcal{U} \subset \mathcal{N}_2 \cap ker\,\lambda$ and

$$S_\gamma \cong H(\mathcal{G}_1) \otimes_{H(\mathcal{K})} k_\lambda.$$

Since $\mathcal{N}_1/\mathcal{U} = \mathcal{M}\mathcal{U}/\mathcal{U}$ is contained in the multiplicative center of $\mathcal{G}_1/\mathcal{U}$, it follows inductively that $\mathcal{N}_1 \subset Stab_{\mathcal{G}_{i-1}}(\lambda|_{\mathcal{N}_i}) = \mathcal{G}_i$ for $2 \leq i \leq r$. Consequently, $\mathcal{N}_1 \subset \mathcal{K}$ and $\lambda|_{\mathcal{N}_1} = \gamma$. Hence

$$e_k \subset \mathcal{N}_1 \subset \mathcal{N}_1\mathcal{N}_2 \subset \mathcal{N}_1\mathcal{N}_3 \subset \cdots \subset \mathcal{N}_1\mathcal{N}_r = \mathcal{K} = \mathcal{G}_r \subset \mathcal{G}_{r-1} \subset \cdots \subset \mathcal{G}_2 \subset \mathcal{G}_1 \subset \mathcal{G}$$

is an inducing sequence for $(\mathcal{K}, \lambda)$, and the canonical isomorphisms

$$S \cong H(\mathcal{G}) \otimes_{H(\mathcal{G}_1)} S_\gamma \cong H(\mathcal{G}) \otimes_{H(\mathcal{G}_1)} (H(\mathcal{G}_1) \otimes_{H(\mathcal{K})} k_\lambda) \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$$

yield the desired result.

(2)(b). This follows from the arguments of part (a). □

We briefly digress from our main theme in order to present a first application, which constitutes a refinement of [18, Thm.3]. The reader is referred to [2] for the definition of tameness.

**Theorem 2.2** *Let $\mathcal{G}$ be a nilpotent, infinitesimal $k$-group of characteristic $p \geq 3$. Then $H(\mathcal{G})$ does not possess any tame blocks.*

*Proof.* Let $\mathcal{B} \subset H(\mathcal{G})$ be a block, $S$ a simple $\mathcal{B}$-module. According to (2.1) and (1.3) there exists a subgroup $\mathcal{K} \subset \mathcal{G}$ containing $\mathcal{M}(\mathcal{G})$, and $\lambda \in G(\mathcal{O}(\mathcal{K}))$ such that $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$. In virtue of (1.2) $\mathcal{B}$ is primary with $dim_k \mathcal{B} = dim_k H(\mathcal{G}/\mathcal{M}(\mathcal{G}))$. This, however, implies that [6, (1.1)] continues to hold for $H(\mathcal{G})$. The arguments of [6, (1.2)] now yield the desired result. □

Given a subgroup $\mathcal{K} \subset \mathcal{G}$, we recall that $H(\mathcal{G}) : H(\mathcal{K})$ is a Frobenius extension (cf. [9, (1.7)]). This implies in particular that there is an automorphism $\nu \in Aut_k(H(\mathcal{K}))$, the so-called *Frobenius automorphism*, such that for a character $\lambda : \mathcal{K} \longrightarrow \mu_k$ we have an isomorphism

$$H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda \cong Hom_{H(\mathcal{K})}(H(\mathcal{G}), k_{\lambda \circ \nu})$$

of $H(\mathcal{G})$-modules (cf. [11, p.96f] for details).

**Proposition 2.3** *Let $S$ be a simple $\mathcal{G}$-module with admissible pair $(\mathcal{K}, \lambda)$ satisfying $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$. If $\mathcal{K}$ is supersolvable, then the following statements hold:*

*(1) $Soc_{H(\mathcal{K})}(S) = k_\lambda$.*

*(2) There exists a maximal multiplicative subgroup $\mathcal{M} \subset \mathcal{K}$, such that $P(S)|_{H(\mathcal{K})} \cong H(\mathcal{K}) \otimes_{H(\mathcal{M})} k_{\lambda \circ \nu}$.*

*Proof.* (1). Using induction on the length $r$ of the inducing filtration, we shall also verify $\mathcal{M}(\mathcal{K}) \subset \mathcal{M}(\mathcal{G})$. Consider the normal subgroup $\mathcal{N}_1 \subset \mathcal{G}$. For this group the space $S_\lambda$ is non-trivial (as it contains $1 \otimes 1$) and [16, (II.9.14)] implies $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{G}_1)} S_\lambda$. Since $S_\lambda \cong H(\mathcal{G}_1) \otimes_{H(\mathcal{K})} k_\lambda$, the inductive hypothesis implies $\mathcal{M}(\mathcal{K}) \subset \mathcal{M}(\mathcal{G}_1)$. On the other hand, (1.3) shows that $\mathcal{M}(\mathcal{G}_1) \subset \mathcal{M}(\mathcal{G})$.

We propose to show that $Soc_{H(\mathcal{K})}(S) \subset S_\lambda$. Let $\gamma : H(\mathcal{K}) \longrightarrow k$ be an algebra homomorphism such that $k_\gamma \subset S|_{H(\mathcal{K})}$. Since $S$ is simple, an application of [16, (II.9.8)] ensures $k_\gamma|_{H(\mathcal{N}_1)} \cong k_\lambda|_{H(\mathcal{N}_1)}$, so that $\gamma|_{H(\mathcal{N}_1)} = \lambda|_{H(\mathcal{N}_1)}$. Consequently, $k_\gamma \subset S_\lambda$. Since $\mathcal{M}(\mathcal{K}) \subset \mathcal{M}(\mathcal{G})$ (1.1) implies that $Soc_{H(\mathcal{K})}(S)$ is a direct sum of one-dimensional modules. Accordingly, we have $Soc_{H(\mathcal{K})}(S) \subset S_\lambda$. As $H(\mathcal{G}_1) \otimes_{H(\mathcal{K})} k_\lambda \cong S_\lambda$, the inductive hypothesis now yields

$$Soc_{H(\mathcal{K})}(S) \subset Soc_{H(\mathcal{K})}(S_\lambda) \subset k_\lambda,$$

which readily implies the assertion.

(2). By combining (1) with (1.1) we obtain a character $\omega : \mathcal{K} \longrightarrow \mu_k$, and a maximal multiplicative $\mathcal{M} \subset \mathcal{K}$ such that

$$P(S)|_{H(\mathcal{K})} \cong H(\mathcal{K}) \otimes_{H(\mathcal{M})} k_\omega.$$

In particular, $P(S)|_{H(\mathcal{K})}$ has top $k_\omega$, so that its image $S|_{H(\mathcal{K})}$ has the same property. On the other hand, the isomorphism

$$S \cong Hom_{H(\mathcal{K})}(H(\mathcal{G}), k_{\lambda\circ\nu})$$

readily shows that $S|_{H(\mathcal{K})}$ surjects onto $k_{\lambda\circ\nu}$. Consequently, $\omega = \lambda\circ\nu$, as desired. $\square$

We conclude this section by providing a linkage principle for simple modules of supersolvable infinitesimal groups. For groups of height $\leq 1$, the first part of the succeeding result can be found in [8, Thm.3].

**Corollary 2.4** *Let $\mathcal{G}$ be supersolvable, $\mathcal{B} \subset H(\mathcal{G})$ a block.*

*(1) If $S, T$ are a simple modules belonging to $\mathcal{B}$, then there exists a character $\gamma : \mathcal{G} \longrightarrow \mu_k$ such that $\mathcal{M}(\mathcal{G}) \subset ker\,\gamma$ and $T \cong S \otimes_k k_\gamma$.*

*(2) There exists an admissible pair $(\mathcal{K}, \lambda)$ such that every simple $\mathcal{B}$-module $S$ is of the form $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\gamma$, where $k_\gamma$ belongs to the block $\mathcal{B}_\lambda \subset H(\mathcal{K})$ associated to $k_\lambda$.*

*(3) Let $S$ be a simple $\mathcal{B}$-module, $\omega : \mathcal{G} \longrightarrow \mu_k$ a character. Then $S \cong S \otimes_k k_\omega$ if and only if $\mathcal{K} \subset ker\,\omega$.*

*Proof.* (1). Consider the $H(\mathcal{G})$-module $V := Hom_k(S, T)$. Since $\mathcal{M}(\mathcal{G})$ operates on $S$ and $T$ via the same character, $V$ is a module for the trigonalizable group $\mathcal{G}/\mathcal{M}(\mathcal{G})$. Hence $V$ contains a one-dimensional submodule $k_\gamma$, defined by a character $\gamma$ of $\mathcal{G}$ whose kernel comprises $\mathcal{M}(\mathcal{G})$. Let $\psi \in k_\gamma$ be a non-zero element and consider

$$\hat{\psi} : S \otimes_k k_\gamma \longrightarrow T \quad ; \quad x \otimes 1 \mapsto \psi(x).$$

Since $h \cdot \psi = \gamma(h)\psi \;\; \forall\, h \in H(\mathcal{G})$ we have, observing the cocommutativity of $H(\mathcal{G})$,

$$\begin{aligned}\hat{\psi}(h(x \otimes 1)) &= \sum_{(h)} \gamma(h_{(1)})\psi(h_{(2)}x) = \sum_{(h)} (h_{(1)} \cdot \psi)(h_{(2)}x) = \sum_{(h)} h_{(1)}\psi(\eta(h_{(2)})h_{(3)}x) \\ &= \sum_{(h)} h_{(1)}\psi(\varepsilon(h_{(2)})x) = h\psi(x) = h\hat{\psi}(x \otimes 1).\end{aligned}$$

Consequently, $\hat{\psi}$ is, as a non-zero $H(\mathcal{G})$-linear map between two simple modules, an isomorphism.

(2). Let $S$ be a simple $\mathcal{B}$-module. According to (2.1) there exists an admissible pair $(\mathcal{K}, \lambda)$ such that $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$. If $T$ is another simple $\mathcal{B}$-module, then (1) provides a character $\omega : \mathcal{G} \longrightarrow \mu_k$ such that $T \cong S \otimes_k k_\omega$, whence

$$T \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_{\lambda*(\omega|_{H(\mathcal{K})})}.$$

From (1.3) we obtain $\mathcal{M}(\mathcal{K}) \subset \mathcal{M}(\mathcal{G})$. Thus, $\mathcal{M}(\mathcal{K})$ operates on $S$ and $T$ via the same character, and it follows from [7, (2.4)] that $k_{\lambda*(\omega|_{H(\mathcal{K})})}$ belongs to the block $\mathcal{B}_\lambda$.

(3). We write $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$ so that $S \otimes_k k_\omega \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_{\lambda*(\omega|_{H(\mathcal{K})})}$. This readily shows the condition on $\omega$ is sufficient. If $S$ and $S \otimes_k k_\omega$ are isomorphic, then (2.3) implies $\omega|_{H(\mathcal{K})} = \varepsilon$, as desired. $\square$

**Example.** We illustrate the results obtained so far by considering an infinitesimal group of height $\leq 1$ that is given by the so-called "diamond algebra". By definition, this is the restricted Lie algebra $(L, [p])$ with basis

$$L = k\,t \oplus k\,x \oplus k\,y \oplus k\,z,$$

multiplication table

$$[t, x] = x \ \ ; \ \ [t, y] = -y \ \ ; \ \ [t, z] = 0 \ \ ; \ \ [x, y] = z \ \ ; \ \ [x, z] = 0 \ \ ; \ \ [y, z] = 0,$$

and $p$-map defined by

$$t^{[p]} = t \ \ ; \ \ x^{[p]} = 0 = y^{[p]} \ \ ; \ \ z^{[p]} = z.$$

On the $p$-subalgebra $K := k\,t \oplus k\,y \oplus k\,z$, we consider a linear form $\lambda \in K^*$ satisfying

$$\lambda(t) \in GF(p) \ \ ; \ \ \lambda(y) = 0 \ \ ; \ \ \lambda(z) \in GF(p) \setminus \{0\}.$$

Letting $N_1 := k\,y \oplus k\,z$, it follows that $(K, \lambda)$ is an induced pair or a "V-polarization" in the language of [14]. By (2.1) or [14, Satz3] the module $S(\lambda) := u(L) \otimes_{u(K)} k_\lambda$ is simple. Here $u(L)$ denotes the restricted universal enveloping algebra of $L$. Since $K$ has a maximal torus of dimension 2, it readily follows from (2.3) that $dim_k P(S(\lambda)) = p$. Thus, the block $\mathcal{B}(S(\lambda))$ corresponding to $S(\lambda)$ is simple. Accordingly, primary blocks also occur in situations not covered by (1.2). Moreover, (2.4) shows that $S(\lambda) \cong S(\lambda) \otimes_k k_\omega$ if and only if $\omega = \varepsilon$. Thus, the character group of $u(L/kz)$ does not operate on the simples of $\mathcal{B}(S(\lambda))$. Finally, the block $\mathcal{B}_\lambda \subset u(K)$ associated to $k_\lambda$ is isomorphic to $u(K/kz)$, so there is no structural relationship between $\mathcal{B}_\lambda$ and $\mathcal{B}(S(\lambda))$.

# 3. Properties of V-Uniserial Groups

In this section we collect a few structural results involving V-uniserial groups. These will be employed in following section, where we will be concerned with the structure of the group algebras of representation-finite infinitesimal groups.

Recall that a unipotent, commutative infinitesimal group $\mathcal{U}$ is referred to as *V-unserial* if the Verschiebung $V_{\mathcal{U}}$ induces an exact sequence

$$\mathcal{U}^{(p)} \xrightarrow{V_{\mathcal{U}}} \mathcal{U} \longrightarrow {}_p\alpha_k \longrightarrow e_k.$$

Owing to [7, (2.6)] the V-uniserial groups are precisely the unipotent infinitesimal groups of finite representation type.

**Theorem 3.1** *Let $\mathcal{U}$ be a V-uniserial group of height $\geq 2$. Then the connected component $\mathcal{AUT}(\mathcal{U})^\circ$ of the automorphism group of $\mathcal{U}$ is unipotent.*

*Proof.* We proceed in several steps.

(a). *Let $\mathcal{U}$ be a unipotent group, and suppose that ${}_p\mu_k$ operates on $\mathcal{U}$ via automorphisms. Then ${}_p\mu_k$ operates trivially on $\mathcal{U}/_{\mathcal{F}}\mathcal{U}$.*

Let $m : {}_p\mu_k \times \mathcal{U} \longrightarrow \mathcal{U}$ be the structure map, $m^{(p)} : {}_p\mu_k^{(p)} \times \mathcal{U}^{(p)} \longrightarrow \mathcal{U}^{(p)}$ the induced operation. Then we have, observing that ${}_p\mu_k$ has height $\leq 1$,

$$\mathcal{F}_{\mathcal{U}} \circ m = m^{(p)} \circ (\mathcal{F}_{{}_p\mu_k} \times \mathcal{F}_{\mathcal{U}}) = m^{(p)} \circ (\varepsilon_{{}_p\mu_k} \times \mathcal{F}_{\mathcal{U}}).$$

Consequently, $\mathcal{F}_{\mathcal{U}}(g \cdot u) = \mathcal{F}_{\mathcal{U}}(u) \;\; \forall \, g \in {}_p\mu_k, u \in \mathcal{U}$, so that ${}_p\mu_k$ operates trivially on $\mathcal{U}/_{\mathcal{F}_{\mathcal{U}}}\mathcal{U}$.

(b). *Let $\mathcal{U}$ be a commutative unipotent group, and suppose that ${}_p\mu_k$ operates on $\mathcal{U}$ via automorphisms. Then ${}_p\mu_k$ operates trivially on $V_{\mathcal{U}}(\mathcal{U}^{(p)})$.*

Let $\mathcal{W}$ be the Cartier dual $\mathcal{D}(\mathcal{U})$ of $\mathcal{U}$. Then ${}_p\mu_k$ operates on $\mathcal{W}$ via automorphisms, and (a) shows that the induced operation on $\mathcal{W}/_{\mathcal{F}_{\mathcal{W}}}\mathcal{W}$ is trivial. Consider the factorization

$$\mathcal{W} \xrightarrow{\alpha} \mathcal{W}/_{\mathcal{F}_{\mathcal{W}}}\mathcal{W} \xrightarrow{\beta} \mathcal{W}^{(p)}$$

of the Frobenius homomorphism $\mathcal{F}_{\mathcal{W}}$. In view of [3, (IV.§3(4.9))] we obtain

$$V_{\mathcal{U}} = \mathcal{D}(\mathcal{F}_{\mathcal{W}}) = \mathcal{D}(\alpha) \circ \mathcal{D}(\beta).$$

Since $\alpha$ is injective and $\beta$ is surjective, $\mathcal{D}(\beta)$ is injective and $\mathcal{D}(\alpha)$ is surjective. Consequently, $V_{\mathcal{U}}(\mathcal{U}^{(p)}) \cong \mathcal{D}(\mathcal{W}/_{\mathcal{F}_{\mathcal{W}}}\mathcal{W})$, and we obtain (b).

Now suppose $\mathcal{U}$ to be V-uniserial of height $\geq 2$. Then $V_{\mathcal{U}}(\mathcal{U}^{(p)})$ is the unique maximal subgroup of $\mathcal{U}$. Since the height of $\mathcal{U}$ is $\geq 2$, the Frobenius kernel ${}_{\mathcal{F}}\mathcal{U}$ is contained in $V_{\mathcal{U}}(\mathcal{U}^{(p)})$. Suppose that ${}_p\mu_k \subset \mathcal{AUT}(\mathcal{U})^{\circ}$. It follows from (b) that ${}_p\mu_k$ operates trivially on ${}_{\mathcal{F}}\mathcal{U}$. Consequently, it also operates trivially on $Lie(\mathcal{U})$. Consider the induced action of ${}_p\mu_k$ on the function algebra $\mathcal{O}(\mathcal{U})$, whose augmentation ideal we denote by $I$. By what we have just seen, ${}_p\mu_k$ operates trivially on $I/I^2$ and thereby on all spaces $I^n/I^{n+1}$. Since $H({}_p\mu_k)$ is semisimple, ${}_p\mu_k$ acts trivially on $\mathcal{O}(\mathcal{U})$, and thus on $H(\mathcal{U})$, a contradiction.

By our foregoing discussion, the group $\mathcal{AUT}(\mathcal{U})^{\circ}$ does not contain any multiplicative groups. It therefore remains to be shown that any such infinitesimal group $\mathcal{G}$ is unipotent. We proceed by induction on the height $h(\mathcal{G})$ of $\mathcal{G}$, the case $h(\mathcal{G}) \leq 1$ being a consequence of the Jordan-Chevalley decomposition [15, (3.5)]. If $h(\mathcal{G}) > 1$, then ${}_{\mathcal{F}}\mathcal{G}$ is unipotent, and an application of [3, (IV.§2(3.5))] shows that $\mathcal{G}/_{\mathcal{F}}\mathcal{G}$ contains no multiplicative subgroups. By inductive hypothesis this group is unipotent, and our assertion now follows from [3, (IV.§2 (2.2))]. □

In view of (3.1), V-uniserial groups of height $\leq 1$ are of particular interest for the structure theory of representation-finite infinitesimal groups. We turn to the classification of the former groups. In the sequel, we let ${}_p\mathcal{W}(n)_k$ be the infinitesimal group corresponding to the restricted enveloping algebra $u(L_n)$ of the $n$-dimensional $p$-nilpotent abelian restricted Lie algebra $L_n := \bigoplus_{i=0}^{n-1} kx^{[p]^i}$ with $x^{[p]^n} = 0$. It turns out that ${}_p\mathcal{W}(n)_k$ is the first Frobenius kernel of the *n-th Witt group* $\mathcal{W}(n)_k$ (cf. [3, (V.§1(1.6))]).

**Proposition 3.2** *Let $\mathcal{U}$ be an infinitesimal, unipotent group of height $\leq 1$ and length $n$. Then $\mathcal{U}$ is V-uniserial if and only if $\mathcal{U} \cong {}_p\mathcal{W}(n)_k$.*

*Proof.* If $\mathcal{U}$ is V-uniserial, then $\mathcal{U}$ is commutative, and [3, (IV.§2 (2.14)] shows that $\mathcal{U} \cong \prod_{i=1}^{r} {}_p\mathcal{W}(n_i)_k$. Since $\mathcal{U}$ is indecomposable of length $n$, it follows that $\mathcal{U} \cong {}_p\mathcal{W}(n)_k$. For the converse, we first invoke [3, (II.§7 (3.9))] to see that $H({}_p\mathcal{W}(n)_k) \cong u(L_n)$. It follows that $H({}_p\mathcal{W}(n)_k)$ has finite representation type (cf. [4, (4.3)]). Owing to [7, (2.7)] this implies that ${}_p\mathcal{W}(n)_k$ is V-uniserial. $\square$

**Theorem 3.3** *Let $\mathcal{G}$ be an infinitesimal group such that $\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong {}_p\mathcal{W}(n)_k$. Then $\mathcal{G}$ is commutative, and $\mathcal{G} \cong {}_p\mathcal{W}(n)_k \times \mathcal{M}(\mathcal{G})$.*

*Proof.* Let $\mathcal{M}$ be a multiplicative group. Following [3, (III.§6 no.1)] we shall prove that

$$E\tilde{x}^1({}_p\mathcal{W}(n)_k, \mathcal{M}) = (0)$$

by induction on the length $m$ of $\mathcal{M}$. If $m = 1$, then $\mathcal{M} \cong {}_p\mu_k$.

Consider an extension

$$e_k \longrightarrow {}_p\mu_k \longrightarrow \mathcal{G} \xrightarrow{\pi} {}_p\mathcal{W}(n)_k \longrightarrow e_k.$$

We shall show by induction on $n$ that $\mathcal{G}$ is commutative. The commutator on $\mathcal{G}$ induces a bilinear form

$$\chi : {}_p\mathcal{W}(n)_k \times {}_p\mathcal{W}(n)_k \longrightarrow {}_p\mu_k \;\; ; \;\; (\pi(a), \pi(b)) \mapsto aba^{-1}b^{-1}.$$

Since $\mathcal{HOM}({}_p\mathcal{W}(n)_k, {}_p\mu_k) \cong \mathcal{HOM}({}_p\mathcal{W}(n)_k, \mu_k) \cong \mathcal{D}({}_p\mathcal{W}(n)_k) \cong {}_{p^n}\alpha_k$ [3, (V.§4 (4.7))], our bilinear form $\chi$ induces a homomorphism

$$\varphi_\chi : {}_p\mathcal{W}(n)_k \longrightarrow {}_{p^n}\alpha_k.$$

As the height of ${}_p\mathcal{W}(n)_k$ is one, $im\,\varphi_\chi \subset {}_{\mathcal{F}}({}_{p^n}\alpha_k) \cong {}_p\alpha_k$. If $n = 1$, then ${}_p\mathcal{W}(n)_k = {}_p\alpha_k$. If $\mathcal{G}$ does not contain a copy of ${}_p\alpha_k$, then [3, (IV.§3(3.7))] implies that $\mathcal{G}$ is multiplicative, a contradiction. Hence the exact sequence above splits, and $\mathcal{G}$ is commutative. Now assume that $n > 1$. Then $\varphi_\chi$ has a non-trivial kernel. Accordingly, the center $\mathcal{C}$ of $\mathcal{G}$ properly contains ${}_p\mu_k = \mathcal{M}(\mathcal{G})$. Let $\mathcal{U}$ be the unipotent part of $\mathcal{C}$. There results an exact sequence

$$e_k \longrightarrow {}_p\mu_k \longrightarrow \mathcal{G}/\mathcal{U} \longrightarrow {}_p\mathcal{W}(m)_k \longrightarrow e_k$$

for some $m < n$. By inductive hypothesis the group $\mathcal{G}/\mathcal{U}$ is commutative, so that

$$(\mathcal{G}, \mathcal{G}) \subset \mathcal{U} \cap \mathcal{M}(\mathcal{G}) = e_k.$$

As a result, $\mathcal{G}$ is commutative.

It now follows from [3, (IV.§3 (1.1))] that $E\tilde{x}^1({}_p\mathcal{W}(n)_k, {}_p\mu_k) = (0)$. For $m > 1$ we consider a subgroup $\mathcal{M}_1 \subset \mathcal{M}$ of length $m-1$, and observe that the exact sequence

$$e_k \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M} \longrightarrow {}_p\mu_k \longrightarrow e_k$$

induces an exact sequence

$$E\tilde{x}^1({}_p\mathcal{W}(n)_k, \mathcal{M}_1) \longrightarrow E\tilde{x}^1({}_p\mathcal{W}(n)_k, \mathcal{M}) \longrightarrow E\tilde{x}^1({}_p\mathcal{W}(n)_k, {}_p\mu_k).$$

By inductive hypothesis the extreme terms vanish, so that the middle terms also equals zero.

Since $E\tilde{x}^1({}_p\mathcal{W}(n)_k, \mathcal{M}(\mathcal{G}))$ is trivial, we now obtain the statements of our theorem. $\square$

# 4. Representation-Finite Infinitesimal Groups

In this section we will determine the block structure and the stable Auslander-Reiten quiver $\Gamma_s(\mathcal{G})$ for the group algebra $H(\mathcal{G})$ of an infinitesimal group $\mathcal{G}$ of finite representation type. According to [7, (2.2)] such a group is necessarily supersolvable, so that the results of the preceding sections can be brought to bear. More precisely, given an infinitesimal group $\mathcal{G}$ of finite representation type, it was shown in [7, (2.7)] that

$$\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \times {}_{p^m}\mu_k$$

is a semidirect product with a V-uniserial normal subgroup $\mathcal{U}$.

In the following subsidiary result we collect a few technical properties that are interesting in their own right. Given a block $\mathcal{B} \subset H(\mathcal{G})$, the set of isoclasses of simple $\mathcal{B}$-modules will be denoted $\mathcal{S}(\mathcal{B})$. If $M$ is a $H(\mathcal{G})$-module with projective cover $P_M \xrightarrow{f} M$, then the kernel $\Omega(M)$ of $f$ is unique up to isomorphism. The resulting operator $\Omega$ is called the *Heller operator*. We refer the reader to [2] concerning its basic properties.

**Lemma 4.1** *Suppose that $\mathcal{G}/\mathcal{M}(\mathcal{G}) = \mathcal{U} \times {}_{p^m}\mu_k$ is a semidirect product with a V-uniserial normal subgroup $\mathcal{U}$ of length $n$. Then there exists a character $\alpha : \mathcal{G} \longrightarrow \mu_k$ with the following properties:*

*(1) The character $\alpha \in G(\mathcal{O}(\mathcal{G}))$ has order $p^m$, and $\alpha^{p^n-1} * id_{H(\mathcal{G})}$ is a Nakayama automorphism of $H(\mathcal{G})$.*

*(2) For any non-projective, simple $H(\mathcal{G})$-module $S$ we have $\Omega^2(S) \cong S \otimes_k k_{\alpha^{p^n}}$.*

*(3) Let $\mathcal{B} \subset H(\mathcal{G})$ be a block. Then there exists an admissible pair $(\mathcal{K}, \lambda)$ such that $dim_k Ext^1_{H(\mathcal{G})}(S, S \otimes_k k_\gamma) = \delta_{\gamma|_\mathcal{K}, \alpha|_\mathcal{K}} \quad \forall\ \gamma \in G(\mathcal{O}(\mathcal{G})),\ [S] \in \mathcal{S}(\mathcal{B})$.*

*Proof.* (1). We put $\mathcal{G}' := \mathcal{G}/\mathcal{M}(\mathcal{G})$ and let $\pi : \mathcal{G} \longrightarrow \mathcal{G}'$ denote the canonical projection. The algebra $H({}_{p^m}\mu_k)$ operates on $H(\mathcal{U})$ via the restriction of the adjoint representation. Since $H({}_{p^m}\mu_k)$ is semisimple, there results a weight space decomposition

$$H(\mathcal{U}) = \bigoplus_{\beta \in R} H(\mathcal{U})_\beta,$$

where $R$ is a finite subset of $G(\mathcal{O}({}_{p^m}\mu_k)) \cong \mathbb{Z}/(p^m)$. Note that $H(\mathcal{U})_\beta H(\mathcal{U})_\gamma \subset H(\mathcal{U})_{\beta * \gamma}$. Moreover, since $\mathcal{U}$ is V-uniserial, we have $dim_k H(\mathcal{U})^\dagger/(H(\mathcal{U})^\dagger)^2 = 1$, so that there exists a root vector $x \in H(\mathcal{U})_\beta$ such that $\{1, x, x^2, \ldots, x^{p^n-1}\}$ is a basis of $H(\mathcal{U})$. Thus,

$$R = \{\beta^i\ ;\ 0 \le i \le p^n - 1\}.$$

Consider $\beta : {}_{p^m}\mu_k \longrightarrow \mu_k$. By the above observation, $ker\,\beta$ operates trivially on $H(\mathcal{U})$. Thus, $\beta$ is injective, so that $ord(\beta) = p^m$. Consider $\alpha := \beta \circ \pi$. Since $\pi$ induces an injection $G(\mathcal{O}(\mathcal{G}')) \hookrightarrow G(\mathcal{O}(\mathcal{G}))$, it follows that $\alpha$ also has order $p^m$.

Let $\mathcal{M}$ be the inverse image of ${}_{p^m}\mu_k$ under $\pi$. The proof of [7, (2.4)] shows that the projective cover $P(0)$ of the trivial $H(\mathcal{G})$-module is isomorphic to $H(\mathcal{G}) \otimes_{H(\mathcal{M})} k$. By the same token we have $Soc_{H(\mathcal{G})}(P(0)) = k_{\alpha^{p^n-1}}$.

According to [9, (1.5)] a Nakayama automorphism $\nu$ of $H(\mathcal{G})$ is given by

$$\nu = \zeta * id_{H(\mathcal{G})},$$

where $\zeta : H(\mathcal{G}) \longrightarrow k$ is the modular function of $H(\mathcal{G})$. Observing [4, (3.1)] we thus obtain

$$k_{\alpha^{p^n-1}} \cong Soc_{H(\mathcal{G})}(P(0)) \cong k_\zeta,$$

implying the desired identity $\nu = \alpha^{p^n-1} * id_{H(\mathcal{G})}$.

(2). Let $J := H(\mathcal{G}')H(\mathcal{U})^\dagger$ be the Jacobson radical of $H(\mathcal{G}')$. Writing

$$H(\mathcal{U})_{ad} = \bigoplus_{i=0}^{p^n-1} H(\mathcal{U})_{\alpha^n},$$

we have $JP(0)/J^2P(0) \cong k_\alpha$. Since $P(0)$ and $P(\alpha)$ have the same dimension, it follows that $\Omega^2(k)$ is one-dimensional. There results an exact sequence

$$(0) \longrightarrow Soc(P(\alpha)) \longrightarrow P(\alpha) \longrightarrow P(0) \longrightarrow k \longrightarrow (0),$$

so that

$$\Omega^2(k) \cong Soc(P(\alpha)) \cong Soc(P(0) \otimes_k k_\alpha) \cong k_{\alpha^{p^n}}.$$

By general principles (cf. [2, (3.1.6)]) any non-projective simple $H(\mathcal{G})$-module $S$ therefore satisfies

$$\Omega^2(S) \cong S \otimes_k k_{\alpha^{p^n}}.$$

(3). Let $S$ be a simple $\mathcal{B}$-module. Since $\Omega^2(S) \cong S \otimes_k k_{\alpha^{p^n}}$ there is an exact sequence

$$(0) \longrightarrow S \otimes_k k_{\alpha^{p^n}} \longrightarrow P(S \otimes_k k_\alpha) \longrightarrow P(S) \longrightarrow S \longrightarrow (0).$$

By (2.1) there exists an admissible pair $(\mathcal{K}, \lambda)$ such that $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$. If $T = S \otimes_k k_\gamma$ for some $\gamma \in G(\mathcal{O}(\mathcal{G}))$, then, letting $\nu$ be the Frobenius automorphism of the Frobenius extension $H(\mathcal{G}) : H(\mathcal{K})$, we obtain

$$T \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_{\lambda*\gamma} \cong Hom_{H(\mathcal{K})}(H(\mathcal{G}), k_{(\lambda*\gamma)\circ\nu}).$$

Let $\mathcal{M} \subset \mathcal{K}$ be a maximal multiplicative subgroup. Thanks to [2, (2.5.4)] and (2.3) there are isomorphisms

$$\begin{aligned} Ext^1_{H(\mathcal{G})}(S,T) &\cong Hom_{H(\mathcal{G})}(P(S \otimes_k k_\alpha), T) \cong Hom_{H(\mathcal{K})}(P(S \otimes_k k_\alpha), k_{(\lambda*\gamma)\circ\nu}) \\ &\cong Hom_{H(\mathcal{M})}(k_{(\lambda*\alpha)\circ\nu}, k_{(\lambda*\gamma)\circ\nu}). \end{aligned}$$

Accordingly, this space is not trivial if and only if

$$[(\lambda * \alpha) \circ \nu]|_{H(\mathcal{M})} = [(\lambda * \gamma) \circ \nu]|_{H(\mathcal{M})}.$$

A consecutive application of [9, (1.5)] and [12, Satz7] shows that $\nu(H(\mathcal{M})) = H(\mathcal{M})$. Since $\mathcal{K}$ is supersolvable, we therefore obtain

$$(\lambda * \alpha)|_{H(\mathcal{K})} = (\lambda * \gamma)|_{H(\mathcal{K})},$$

whence

$$\alpha|_{H(\mathcal{K})} = \gamma|_{H(\mathcal{K})}.$$

Consequently, $dim_k Ext^1_{H(\mathcal{G})}(S,T) = \delta_{\alpha|_{\mathcal{K}},\gamma|_{\mathcal{K}}}$. $\square$

*Remark.* Let $t := max\{0, m-n\}$. From (2) of (4.1) we see that the order of $\Omega$ is a divisor of $2p^t$. Note that for infinitesimal groups of height $\leq 1$ the period of a periodic module is at most 2 (cf. [4, (2.5)]).

Let $\mathcal{G}$ be an infinitesimal $k$-group. The largest nilpotent normal subgroup of $\mathcal{G}$ is referred to as the *nilpotent radical* and will be denoted $\mathcal{N}(\mathcal{G})$. Note that $\mathcal{M}(\mathcal{G}) \subset \mathcal{N}(\mathcal{G})$. Moreover, $\mathcal{N}(\mathcal{G})/\mathcal{M}(\mathcal{G})$ is the unipotent radical of $\mathcal{G}/\mathcal{M}(\mathcal{G})$. We let $n(\mathcal{G})$ and $m(\mathcal{G})$ be the length of $\mathcal{N}(\mathcal{G})/\mathcal{M}(\mathcal{G})$ and $\mathcal{G}/\mathcal{N}(\mathcal{G})$, respectively. We will write $\ell(M)$ for the length of a $H(\mathcal{G})$-module $M$.

Recall that $Aut_k(H(\mathcal{G}))$ operates on the set $\mathcal{S}(H(\mathcal{G}))$ of all isoclassess of simple $H(\mathcal{G})$-modules via

$$\lambda \cdot [S] := [S^{(\lambda)}].$$

Consider the subgroup $C_\nu \subset Aut_k(H(\mathcal{G}))$ that is generated by a Nakayama automorphism of $\nu$ of $H(\mathcal{G})$. The general theory of Frobenius algebras shows that two Nakyama automorphisms of $H(\mathcal{G})$ differ only by an inner automorphism [12, Satz1]. Consequently, the operation of $C_\nu$ on $\mathcal{S}(H(\mathcal{G}))$ does not depend on the choice of $\nu$.

If $a$, $b$ are non-negative integers, we consider the algebra $\Lambda(a,b) := k[Q]/I$ given by the following quiver $Q$ and relations $I$: $Q := \mathbb{Z}/(a)$ with arrows $x_i := i \to i+1$. The ideal $I$ is generated by all paths of length $b$. For more information on path algebras and the definition of the stable Auslander-Reiten quiver $\Gamma_s(\Lambda)$ of a self-injective algebra $\Lambda$ the reader may consult [1].

We are now in a position to state the structure theorem for group algebras of infinitesimal groups of finite representation type. For the reader's convenience, parts of its proof will be given by a series of propositions, each of which contains additional information.

**Theorem 4.2** *Let $\mathcal{G}$ be an infinitesimal group of finite representation type, $\mathcal{B} \subset H(\mathcal{G})$ a block. Then the following statements hold:*

*(1) $ord(\nu) = p^{m(\mathcal{G})}$ and $\mathcal{S}(\mathcal{B}) = C_\nu \cdot [S] \quad \forall\, [S] \in \mathcal{S}(\mathcal{B})$.*

*(2) The block $\mathcal{B}$ has dimension $p^{m(\mathcal{G})+n(\mathcal{G})}$ and is either primary or basic. In the latter case $|\mathcal{S}(\mathcal{B})| = p^{m(\mathcal{G})}$.*

*(3) There exists $s \in \{1, p^{m(\mathcal{G})}\}$ and $n \in \{0, \ldots, n(\mathcal{G})\}$ such that $\mathcal{B}$ is Morita equivalent to $\Lambda(s, p^n)$ and $\Gamma_s(\mathcal{B}) \cong \mathbb{Z}[A_{p^n-1}]/(\tau^s)$.*

*Remarks.* Part (1) of (4.2) does not hold for constant groups. For $p = 3$ the group algebra of the symmetric group $S_3$ on 3 letters has finite representation type. Its Sylow-3-subgroup is normal and not central, so that the inertial index $e$ of the principal block $\mathcal{B}_0$ of the group algebra $k[S_3]$ is 2. According to [2, (6.5.4)] $\mathcal{B}_0$ is a symmetric Nakayama algebra with 2 simple modules.

Let $\mathcal{G}$ be an arbirtrary infinitesimal group of height $\leq 1$. Then [4, (3.2)] implies that a representation-finite block $\mathcal{B} \subset H(\mathcal{G})$ is Morita equivalent to $\Lambda(m, \frac{\ell}{m})$, where $\ell > 0$ and $m \in \{1, p\}$ divides $\ell$.

**Proposition 4.3** *Let $\mathcal{G}$ be an infinitesimal group of finite representation type such that $\mathcal{N}(\mathcal{G})/\mathcal{M}(\mathcal{G})$ has height $\geq 2$.*

*(1) We have $(\mathcal{G},\mathcal{G}) \subset \mathcal{M}(\mathcal{G})$. In particular $\mathcal{G} = \mathcal{N}(\mathcal{G})$, and each block of $H(\mathcal{G})$ is isomorphic to a full matrix ring over a truncated polynomial ring $k[X]/(X^{p^n})$.*

*(2) If $S$ is a simple $H(\mathcal{G})$-module, such that $\mathcal{M}(\mathcal{G})$ operates on $S$ via the character $\lambda : \mathcal{M}(\mathcal{G}) \longrightarrow \mu_k$, then $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\gamma$, where $\mathcal{K}/ker\,\lambda$ is the unique maximal abelian subgroup of $\mathcal{G}/ker\,\lambda$, and $\gamma$ is the unique extension of $\lambda$ to $\mathcal{K}$.*

*Proof.* (1). According to [7, (2.7)] we have

$$\mathcal{G}/\mathcal{M}(\mathcal{G}) \cong \mathcal{U} \times {}_{p^m}\mu_k,$$

where $\mathcal{U}$ is a V-uniserial normal subgroup and $m = m(\mathcal{G})$. In view of (3.1) the group ${}_{p^m}\mu_k$ operates trivially on $\mathcal{U}$ by conjugation. Thus, $\mathcal{G}/\mathcal{M}(\mathcal{G})$ is abelian and we obtain the first claim.

In particular, $\mathcal{G}$ is nilpotent, and a consecutive application of (2.1) and (1.2) implies that every block $\mathcal{B}$ of $H(\mathcal{G})$ is primary. Thus, $\mathcal{B} \cong Mat_n(k[X]/(X^\ell))$, where $\ell$ is the Cartan invariant of $\mathcal{B}$. This number is the length of the unique principal indecomposable $\mathcal{B}$-module, which, owing to (2.1) and (1.1), is a $p$-power.

(2). Let $\mathcal{Q}$ be the kernel of $\lambda$. We begin by observing that a maximal commutative subgroup of $\mathcal{G}' := \mathcal{G}/\mathcal{Q}$ contains $\mathcal{M}(\mathcal{G}')$. It now follows from (1) that $\mathcal{G}'/\mathcal{M}(\mathcal{G}')$ is V-uniserial. Hence $\mathcal{G}'$ has a unique maximal commutative subgroup $\mathcal{K}'$.

We decompose $\mathcal{K}' = \mathcal{M}(\mathcal{G}') \times \mathcal{U}'$ into its multiplicative and unipotent part, and denote by $\mathcal{N}'$ the normalizer of $\mathcal{U}'$ in $\mathcal{G}'$. Then $\mathcal{K}' \subset \mathcal{N}'$. If the two groups do not coincide, then there exists a subgroup $\mathcal{K}' \subset \mathcal{L}' \subset \mathcal{N}'$ such that $\mathcal{L}'/\mathcal{K}' \cong {}_p\alpha_k$. It readily follows that the group $\mathcal{L}'/\mathcal{U}'$ is trigonalizable. According to [3, (IV.§4 (1.12))] $\mathcal{L}'$ is, as a trigonalizable and nilpotent group, a direct product of its unipotent radical and its multiplicative center $\mathcal{M}(\mathcal{G}')$. Since $\mathcal{G}'/\mathcal{M}(\mathcal{G}')$ is commutative, so is $\mathcal{L}'$. This contradicts the maximality of $\mathcal{K}'$, and we conclude that $\mathcal{K}'$ is the normalizer of $\mathcal{U}'$ in $\mathcal{G}'$.

Note that $S$ is a module for the group $\mathcal{G}'$ such that $\mathcal{M}(\mathcal{G}')$ operates faithfully on $S$. Let $\lambda' : \mathcal{M}(\mathcal{G}') \longrightarrow \mu_k$ be the character induced by $\lambda$. Let $\mathcal{K}'$ be the maximal commutative subgroup of $\mathcal{G}'$. We extend $\lambda'$ to a character $\gamma'$ of $\mathcal{K}'$ such that $\gamma'|_{\mathcal{U}'} = \varepsilon$. Consequently, $ker\gamma' = \mathcal{U}'$. According to [17, (4.1)] the group $Stab_{\mathcal{G}'}(\gamma')$ coincides with $\mathcal{N}' = \mathcal{K}'$. Consequently, $(\mathcal{K}', \gamma')$ is an admissible pair for $S$, and [16, (I.1.4)] yields

$$S \cong H(\mathcal{G}') \otimes_{H(\mathcal{K}')} k_{\gamma'}.$$

We consider the quotient map $\pi : \mathcal{G} \longrightarrow \mathcal{G}'$ and put $\gamma := \gamma' \circ \pi$. Then $\gamma|_{\mathcal{M}(\mathcal{G})} = \lambda$, and setting $\mathcal{K} := \pi^{-1}(\mathcal{K}')$ we have

$$S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\gamma,$$

as well as $Stab_{\mathcal{G}}(\gamma) = \mathcal{K}$. □

**Proposition 4.4** *Let $\mathcal{G}$ be a supersolvable infinitesimal $k$-group such that $\mathcal{N}(\mathcal{G})/\mathcal{M}(\mathcal{G}) \cong {}_p\mathcal{W}(n)_k$. Then the following statements hold:*

*(1) Let $S$ be a simple $H(\mathcal{G})$-module of dimension $> 1$.*

*(a) There exists a character $\lambda : \mathcal{N}(\mathcal{G}) \longrightarrow \mu_k$ such that $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{N}(\mathcal{G}))} k_\lambda$.*

*(b) The block $\mathcal{B}(S)$ associated to $S$ is primary and $\ell(P(S)) = p^{n(\mathcal{G})-m(\mathcal{G})}$.*

*(2) Each block of $H(\mathcal{G})$ has dimension $p^{m(\mathcal{G})+n(\mathcal{G})}$.*

*(3) The group $\mathcal{G}$ is trigonalizable if and only if there exists a multiplicative subgroup $\mathcal{M} \subset \mathcal{G}$ such that $\mathcal{G} \cong \mathcal{U} \rtimes \mathcal{M}$ is a semidirect product with a normal subgroup $\mathcal{U} \cong {}_p\mathcal{W}(n)_k$.*

*Proof.* Owing to (3.3) the nilpotent radical $\mathcal{N}(\mathcal{G})$ is commutative, $\mathcal{N}(\mathcal{G}) = \mathcal{M}(\mathcal{G}) \cdot \mathcal{U}$, where $\mathcal{U} \cong {}_p\mathcal{W}(n)_k$.

(1)(a). We begin by showing that there exists an admissible pair $(\mathcal{K}, \lambda)$ such that $\mathcal{N}(\mathcal{G}) \subset \mathcal{K}$ and $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$.

Since $\mathcal{N}(\mathcal{G})$ is a commutative subgroup, there exists a simple module $k_\gamma \subset S|_{H(\mathcal{N}(\mathcal{G}))}$. The assumption $Stab_{\mathcal{G}}(\gamma) = \mathcal{G}$ implies that $S = S_\gamma$ is an isotypical $H(\mathcal{N}(\mathcal{G}))$-module (cf. [16, (I.7.4)]). In particular, $\mathcal{U}$ operates trivially on $S$. Letting $\mathcal{Q}$ be the kernel of the representation of $S$, we observe that $\mathcal{Q} \supset \mathcal{U}$. Consequently, $\mathcal{G}/\mathcal{Q}$ is multiplicative, and $S$ has dimension 1, a contradiction. We may now apply (2.1(2b)) to obtain the desired result.

Since $\mathcal{G}/\mathcal{N}(\mathcal{G})$ is multiplicative, it follows in particular that $\mathcal{K}$ is normal in $\mathcal{G}$. Thanks to [16, (II.9.8)] every composition factor of $S|_{H(\mathcal{K})}$ is isomorphic to $k_\lambda$. Now consider the projective cover $P(S)$ of $S$. Owing to (2.3) there exists an automorphism $\nu$ of $H(\mathcal{K})$ such that $P(S)|_{H(\mathcal{K})} \cong P(\lambda \circ \nu)$ is the projective cover of the $H(\mathcal{K})$-module $k_{\lambda\circ\nu} = Top(S|_{H(\mathcal{K})})$. By the above observation we have $\lambda \circ \nu = \lambda$. According to [7, (2.7)] the algebra $H(\mathcal{K})$ also has finite representation type, and $P(\lambda)$ is uniserial. Let $J$ be the Jacobson radical of $H(\mathcal{K})$. Since $P(\lambda) \cong P(S)|_{H(\mathcal{K})}$ surjects onto $S$, and $dim_k S > 1$, it follows that $JP(\lambda)/J^2P(\lambda) \cong k_\lambda$. Observing that twisting by $\lambda$ is an auto-equivalence, we obtain

$$1 = dim_k Ext^1_{H(\mathcal{K})}(k_\lambda, k_\lambda) = dim_k Ext^1_{H(\mathcal{K})}(k, k).$$

Thus, the principal block $\mathcal{B}_0$ of $H(\mathcal{K})$, being a Nakayama algebra, is local, and [16, (I.2.29)] ensures that $\mathcal{K}$ is nilpotent. Consequently, $\mathcal{K} = \mathcal{N}(\mathcal{G})$.

(1)(b). We apply (1.2) to see that $\mathcal{B}(S)$ is primary. From (2.3) we obtain $\ell(P(S)) = p^{n(\mathcal{G})-m(\mathcal{G})}$, as desired.

(2). According to (1) every block of $H(\mathcal{G})$ is either basic or primary. It follows from [7, (2.4)] that each basic block is isomorphic to $H(\mathcal{G}/\mathcal{M}(\mathcal{G}))$. For the primary blocks, the desired formula is a direct consequence of (1).

(3). The condition is obviously sufficient. For the reverse conclusion, we let $\tilde{\mathcal{U}}$ be the unipotent radical of $\mathcal{G}$. As $\mathcal{G}$ is trigonalizable, an application of [3, (IV.§2 (3.5))] shows that $\mathcal{G} = \tilde{\mathcal{U}} \rtimes \mathcal{M}$ with a multiplicative subgroup $\mathcal{M} \supset \mathcal{M}(\mathcal{G})$. Consequently, $\mathcal{U} \subset \tilde{\mathcal{U}} \subset \mathcal{N}(\mathcal{G})$. Since $\tilde{\mathcal{U}}$ does not meet $\mathcal{M}(\mathcal{G})$, we have $\tilde{\mathcal{U}} = \mathcal{U}$. □

*Proof of (4.2).* According to [7, (2.7)] the group $\mathcal{G}$ is supersolvable and $\mathcal{G}' := \mathcal{G}/\mathcal{M}(\mathcal{G})$ has the structure given in (4.1).

(1). By (1) and (3) of (4.1) and (1) of (2.4) there exists a character $\alpha \in G(\mathcal{O}(\mathcal{G}))$ of order $p^{m(\mathcal{G})}$ such that the group $C_\alpha \subset Aut_k(H(\mathcal{G}))$ generated by $\alpha * id_{H(\mathcal{G})}$ operates transitively on $\mathcal{S}(\mathcal{B})$. Owing to (2) of (4.1) we have $\nu = \alpha^{p^n - 1} * id_{H(\mathcal{G})}$, implying that $C_\nu = C_\alpha$.

(2). According to [7, (2.7)] the group $\mathcal{U} := \mathcal{N}(\mathcal{G})/\mathcal{M}(\mathcal{G})$ is V-uniserial. If the height of $\mathcal{U}$ is $\geq 2$, then (4.3) shows that $\mathcal{G}$ is nilpotent and $\mathcal{B}$ is primary. Moreover, the arguments of (2.2) yield $dim_k\mathcal{B} = p^{m(\mathcal{G})+n(\mathcal{G})}$.

Alternatively, we invoke (3.2) to see that $\mathcal{U} \cong {}_p\mathcal{W}(n)_k$. According to (4.4) the block $\mathcal{B}$ is primary whenever it admits a simple module of dimension $> 1$. Since all simple modules of $\mathcal{B}$ have the same dimension (cf. (2.4)), $\mathcal{B}$ is basic otherwise. In view of (1) and (2.4(3)) we also have $|\mathcal{S}(\mathcal{B})| = p^{m(\mathcal{G})}$ in that case. Thanks to (4.4(3)) all blocks have dimension $p^{m(\mathcal{G})+n(\mathcal{G})}$.

(3). Let $S$ be a simple $\mathcal{B}$-module. According to (2.1) and (2.3) there exists an admissible pair $(\mathcal{K}, \lambda)$, a maximal multiplicative subgroup $\mathcal{M} \subset \mathcal{K}$ containing $\mathcal{M}(\mathcal{G})$, and an automorphism $\tilde{\nu} \in Aut_k(H(\mathcal{K}))$ such that

(a) $S \cong H(\mathcal{G}) \otimes_{H(\mathcal{K})} k_\lambda$, and

(b) $P(S)|_{H(\mathcal{K})} \cong H(\mathcal{K}) \otimes_{H(\mathcal{M})} k_{\lambda\circ\tilde{\nu}}$.

Consequently, $P(S)$ has length $p^n$, with $0 \leq n \leq n(\mathcal{G})$, and (2) implies that the Kupisch series of $\mathcal{B}$ is an $s$-tuple $(p^n, \ldots, p^n)$ for $s \in \{1, p^{n(\mathcal{G})}\}$. By combining [10, Satz 8] with [1, (IV.2.13)] we see that the basic algebra of $\mathcal{B}$ is isomorphic to $\Lambda(s, p^n)$.

General theory (cf. [1, p.253]) now shows that

$$\Gamma_s(\mathcal{B}) \cong \mathbb{Z}[A_{p^n-1}]/(\tau^s),$$

as desired. $\square$

# 5. Blocks of Finite Algebraic Groups

In this section we determine the block structure of arbitrary representation-finite cocommutative Hopf algebras by considering the group algebra $H(\mathcal{G})$ of a finite algebraic group $\mathcal{G}$. According to general theory (cf. [3, (II.§5(2.4))]) such a group decomposes into a semidirect product

$$\mathcal{G} = \mathcal{G}^0 \rtimes \mathcal{G}_{red},$$

where $\mathcal{G}_{red}$ is a constant group, and $\mathcal{G}^0$ is an infinitesimal, normal subgroup. Thanks to [7, (3.1)] the algebra $H(\mathcal{G})$ has finite representation type if and only if $H(\mathcal{G}^0)$ and $H(\mathcal{G}_{red})$ are representation-finite with at least one constituent being semisimple. Our first result provides a reduction to skew group algebras over blocks of $H(\mathcal{G}^0)$. The group algebra of a finite group $G$ will be denoted $k[G]$.

**Proposition 5.1** *There exist subgroups $G_1, \ldots, G_\ell$ of $\mathcal{G}(k)$ and blocks $\mathcal{B}_1, \ldots, \mathcal{B}_\ell$ of $H(\mathcal{G}^0)$ with the following properties:*

*(a) $\mathcal{G}(k) \in \{G_1, \ldots, G_\ell\}$.*

*(b) $H(\mathcal{G})$ is Morita equivalent to $\bigoplus_{i=1}^\ell \mathcal{B}_i[G_i]$, a direct sum of skew group algebras.*

*(c) If $\mathcal{G}^0$ is multiplicative, then $H(\mathcal{G})$ is Morita equivalent to $\bigoplus_{i=1}^\ell k[G_i]$.*

*Proof.* We put $\Lambda := H(\mathcal{G}^0)$ and note that

$$H(\mathcal{G}) = \Lambda[\mathcal{G}(k)]$$

is a skew group algebra. We decompose $\Lambda$ into its blocks and write

$$\Lambda = \Lambda e_1 \oplus \cdots \oplus \Lambda e_n,$$

with central, orthogonal, primitive idempotents $e_j$. The group $\mathcal{G}(k)$, acting on $\Lambda$ via automorphisms, operates on the set $E := \{e_1, \ldots, e_n\}$, and after suitable renumbering we may assume that the orbits are given by $\mathcal{O}_i := \mathcal{G}(k)e_i \quad 1 \leq i \leq \ell$. For each index $i \in \{1, \ldots, \ell\}$ we consider the isotropy group $G_i := \mathcal{G}(k)_{e_i}$. Then $\Lambda_i := \bigoplus_{e \in \mathcal{O}_i} \Lambda e$ is a $\mathcal{G}(k)$-invariant subalgebra, and we have a decomposition

$$\Lambda[\mathcal{G}(k)] = \bigoplus_{i=1}^{\ell} \Lambda_i[\mathcal{G}(k)]$$

of $H(\mathcal{G})$ into ideals. Owing to [13, (1.6)] each constituent $\Lambda_i[\mathcal{G}(k)]$ is Morita equivalent to $\mathcal{B}_i[G_i]$. This readily yields our second assertion.

In order to verify (a) we consider the principal block $\Lambda_0 = \Lambda e_0$ of $\Lambda$, and observe that for every $g \in \mathcal{G}(k)$ we have

$$\varepsilon(g e_0 g^{-1}) = 1.$$

Consequently, $\mathcal{G}(k)$ fixes the central primitive idempotent $e_0$.

Finally, if $\mathcal{G}^0$ is multiplicative, then we have $\mathcal{B}_i \cong k$ for every $i \in \{1, \ldots, \ell\}$. □

*Remark.* The representation-finite blocks of the group algebras of finite groups are well understood. Let $\mathcal{B}$ be such a block with cyclic defect group $D_{\mathcal{B}}$ and inertial index $e$. Then $\mathcal{B}$ is a Brauer tree algebra with $e$ simple modules $e \cdot |D_{\mathcal{B}}|$ indecomposables. Moreover, $\mathcal{B}$ is stably equivalent to a Nakayama algebra $\Lambda(e, |D_{\mathcal{B}}|)$. It follows that $\Gamma_s(\mathcal{B}) \cong \mathbb{Z}[A_{|D_{\mathcal{B}}|-1}]/(\tau^e)$. In contrast to infinitesimal groups, we have $e|(p-1)$ (cf. [2, (6.5.5)] and [1, (X.3)] for details).

**Proposition 5.2** *Suppose that $H(\mathcal{G})$ has finite representation type with $p \not| ord(\mathcal{G}(k))$. Let $P$ be a principal indecomposable $H(\mathcal{G})$-module. Then there exists a principal indecomposable $H(\mathcal{G}^0)$-module $Q$ such that $\ell(P) = \ell(Q)$.*

*Proof.* A consecutive application of [7, (4.1)], [7, (3.6)], and [13, (1.3(g))] shows that $\Lambda$ and $H(\mathcal{G})$ are Nakayama algebras.

As in the proof of (5.1) we may decompose $\Lambda := H(\mathcal{G}^0)$ into a direct sum of $\mathcal{G}(k)$-invariant ideals $\Lambda = \Lambda_1 \oplus \cdots \oplus \Lambda_r$, so that

$$H(\mathcal{G}) = \Lambda_1[\mathcal{G}(k)] \oplus \cdots \oplus \Lambda_r[\mathcal{G}(k)]$$

is a direct sum of ideals. Consequently, $P$ is a direct summand of some $\Lambda_j[\mathcal{G}(k)]$. By construction, $\Lambda_j$ is a sum of isomorphic blocks. Since $\Lambda_j[\mathcal{G}(k)]|_\Lambda \cong \bigoplus_{g \in \mathcal{G}(k)} \Lambda_j$, it follows from (2.4) that all direct summands of $P|_\Lambda$ have the same length. In view of [13, (1.3(e)] any indecomposable summand $Q$ of $P|_\Lambda$ has the desired property. □

We have seen before that the number of simple modules in a block of a representation-finite $k$-group $\mathcal{G}$ is a $p$-power if $\mathcal{G}$ is infinitesimal, and a divisor of $p-1$ if $\mathcal{G}$ is constant. Our concluding example shows that in the general case, there is no restriction on the number of simples.

**Example.** Let $a = p^m q$, where $p \not| q$. Since $\mathcal{AUT}({}_p\mathcal{W}(n)_k)$ contains a copy of $\mu_k$, the direct product $\mathcal{D} := {}_{p^m}\mu_k \times \mathbb{Z}/(q) \subset \mu_k$ acts faithfully on ${}_p\mathcal{W}(n)_k$ via automorphisms. We consider the semidirect product $\mathcal{G} := {}_p\mathcal{W}(n)_k \times ({}_{p^m}\mu_k \times \mathbb{Z}/(q))$. The subgroup $\mathcal{G}^0 = {}_p\mathcal{W}(n)_k \times {}_{p^m}\mu_k$ is representation-finite (cf. [7, (2.7)]), trigonalizable, and has exactly one block with $p^m$ simple modules (cf. [7, (2.4)]). According to [13, (1.3)] the algebra

$$H(\mathcal{G}) = H(\mathcal{G}^0)[\mathbb{Z}/(q)]$$

has finite representation type. Note that the subgroup $\mathcal{D}$ is diagonalizable. Thus, $H(\mathcal{D})$ has $a = p^m q$ simple modules $k_\lambda$, indexed by the elements of $G(\mathcal{O}(\mathcal{D}))$. Consequently, $P(\lambda) := H(\mathcal{G}) \otimes_{H(\mathcal{D})} k_\lambda$ is projective and of dimension $p^n$. This implies that $P(\lambda)|_{H({}_p\mathcal{W}(n)_k)}$ is free of rank 1 and hence indecomposable. Since

$$H(\mathcal{G}) \cong H(\mathcal{G}) \otimes_{H(\mathcal{D})} H(\mathcal{D}) \cong \bigoplus_\lambda P(\lambda),$$

it follows that the $P(\lambda)$ form a complete set of representatives of the principal indecomposable $H(\mathcal{G})$-modules. The arguments of [7, (2.4)] show that $H(\mathcal{G})$ has exactly one block.

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